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APPROXIMATIONS OF THE ULTIMATE RUIN PROBABILITY IN THE CLASSICAL RISK MODEL USING THE BANACH'S FIXED-POINT THEOREM AND THE CONTINUITY OF THE RUIN PROBABILITY

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In this paper, we show two applications of the Banach's Fixed-Point Theorem: first, to approximate the ultimate ruin probability in the classical risk model or Cramér-Lundberg model when claim sizes have some arbitrary continuous distribution and second, we propose an algorithm based in this theorem and some conditions to guarantee the continuity of the ruin probability with respect to the weak metric (Kantorovich). In risk theory literature, there is no methodology based in the Banach's Fixed-Point Theorem to calculate the ruin probability. Numerical results in this paper, guarantee a good approximation to the analytic solution of the ruin probability problem.

Finally, we present numerical examples when claim sizes have distribution light and heavy-tailed.

Keywords: Banach's Fixed-Point Theorem, classical risk model, continuity of ruin probability, probabilistic metric, ultimate ruin probability.

1. INTRODUCTION

This paper shows an application of the properties of contracting operators and Banach's Fixed-Point Theorem (BFPT), see this theorem in Appendix A to obtain an approximation to ultimate ruin probability (or ruin probability) for an insurance company using the Cramér-Lundberg (CL) risk model. The results obtained in this paper by using a certain contractive operator provide favorable results compared to known cases where there is an analytical solution. This contractive operator is used to generate an algorithm to test the continuity of the ruin probability.

In the CL model, for studying the ruin probability, the surplus process of the insurance company, $\boldsymbol{C} = \{C_t\}_{t\geq 0}$, is the continuous stochastic process that models the time evolution of the reserves at time t is given by

$$C_t = u + ct - \sum_{i=1}^{N_t} Y_i, \tag{1}$$

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where $u \geq 0$ is the initial reserve, $c > 0$ is the premium per unit of time, the number of claims until time t is a homogeneous Poisson process $\boldsymbol{N} = \{N_t\}_{t\geq 0}$ with parameter $\lambda > 0$, the claim sizes $\{Y_i\}_{i\geq 1}$ is a sequence of independent and identically distributed (iid) non-negative random variables with continuous distribution function F and independent of $\boldsymbol{N}$. The surplus model when arrival times are modeled by a Hawkes process was studied in [16].

The ruin time is defined as

$$\tau := \inf\{t > 0 | C_t < 0\}, \qquad \text{where} \qquad \inf\{\emptyset\} = \infty.$$

We will be interested in the ruin probability (or ruin probablity) given by

$$\psi(u) = \mathbb{P}(\tau < \infty | C_0 = u). \tag{2}$$

The ruin probability (2) can be found analytically only in particular cases, such as in those cases when claim sizes have exponential, mixture of exponential, phase-type (PH) or one-point degenerate distributions (see [2], [7] or [19]). In cases where the inverse of the Laplace transform of the functions of the risk probability as well as that of survival can be calculated, then an analytical solution can also be obtained (see [2]).

If the adjustment coefficient R exists (defined as the first positive root of $\lambda(M_Y(r) - 1) - cr = 0$, assuming that the moment generating function $M_Y(r)$ exist), then it is possible to have an upper bound for the ruin probability in terms of R, in particular, we have that $\psi(u) < e^{-Ru}$ (where $u \geq 0$), called inequality of Cramér-Lundberg. However, the adjustment coefficient does not exist for heavy-tailed claim size distributions. In [3], the authors propose a PH distribution of infinite dimension with a finite number of parameters to model heavy-tailed distributions. In particular, the authors prove that the PH distribution complies with having an exact formula for the ruin probability, however, a calibration algorithm difficult to implement should be applied.

On the other hand, in the literature we can find different methods to approximate the ruin probability. Each of them presents different requirements, for example, the existence of the adjustment coefficient, or the existence of the second, third, or moments of superior order of the distribution function F; others methods require that the initial reserve u to be very large, etc., for a summary of the methods and their requirements, see [21]. Moreover, there are several approaches to estimate the ruin probability using, for example, Monte Carlo Methods (see [1]) via a simulation of the surplus process. Finally, in [15] the robustness properties of certain estimators are considered to approximate the ruin probability using the non-parametric plug-in estimators of the claims distribution function.

As far as we investigate there is no application of BFPT as a method to estimate the ruin probability in the CL model.

A more general method to approximate the ruin probability was proposed in [6] through the so-called "continuity problem". To achieve the continuity of ruin probability (2), in this work, we demonstrate that by using appropriate probabilistic metrics that guarantee weak convergence and mean convergence for the distribution of claim sizes, we can get continuity for the ruin probability.

In this work the properties of the contracting operators are used, which makes possible to use the BFPT to find an approximation of the ruin probability (2). The advantage of using the BFPT is that it only requires Assumption 3.1.

The principal contribution of this work is to solve (using a contracting operator and BFPT) a risk problem and then give conditions to get their continuity under the Kantorovich metric to obtain a new method of approximation of the ruin probability in the CL model.

This paper is organized as follows. Section 2 presents the concept of continuity for the ruin probability and its present previous results. In Section 3,we introduce the contracting operator, how use it to obtain the approximation (using BFPT) to the ruin probability, and we show the results obtained from one numerical estimate for approximating the ruin probability. Section 4 presents the main result of this work, Theorem 4.2, which establishes the conditions to achieve an estimate of the continuity of the ruin probability, as an application of Theorem 4.2, the Algorithm 1 is presented to obtain an approximation to the ruin probability for the case in which the distribution of claim sizes is modeled by an Erlang mixture and its corresponding PH representation, the results obtained from such numerical estimates support the Theorem 4.2 with examples of light and heavy-tailed distributions of the claim sizes. Some concluding remarks are given in Section 5. Finally, Appendix A containts the BFPT and, in Appendix B, the mathematical induction test is presented for the iterative sequence of the approximation of the ruin probabilities under the use of the contracting operator.

2. CONTINUITY OF THE RUIN PROBABILITY

To introduce the problem of continuity of the ruin probability, we suppose that the risk process (1) is governed by the parameter vector $\theta = (\lambda, c, F)$, denoting this process by $\boldsymbol{C}_t(\theta)$, and that we can not find the corresponding ruin probability $\psi(\theta|u)$ (to simplify the notation we will omit u) . If Θ denotes the parameter space of θ, the ruin probability can be seen as the following mapping

$$\psi(\theta) : \Theta \to \Psi,$$

where Ψ is the functional space of all possible functions $\psi(\theta)$. Assume that Θ and Ψ are metric spaces with metrics δ and ν respectively, then if we can find the other appropriate vector parameter $\tilde{\theta} = (\lambda, c, \tilde{F})$ that governs the risk process

$$C_t(\tilde{\theta}) = u + ct - \sum_{i=1}^{N_t} \tilde{Y}_i,$$

where $\tilde{F}$ is the distribution function of $\tilde{Y}$ and such that we can calculate the ruin probability corresponding $\psi_{\tilde{\theta}}$. Then, the continuity for the ruin probability $\psi(\theta)$ on θ is defined as

$$\text{if} \quad \delta(\theta, \tilde{\theta}) \to 0, \ \text{then} \ \ \nu(\psi(\theta), \psi(\tilde{\theta})) \to 0, \tag{3}$$

for $\theta, \tilde{\theta} \in \Theta$. If we can find an inequality as following

$$\nu(\psi(\theta), \psi(\tilde{\theta})) \leq \bar{K}\phi(\delta(\theta, \tilde{\theta})), \tag{4}$$

where $\bar{K} \in [0, \infty)$ is in terms of the vector parameter θ (called continuity constant), ϕ is a non-negative function such that $\phi(0) = 0$ and $\phi(s)$ converges to 0 when s converges to 0, then inequality (4) is called a continuity estimate and provides the possibility of limiting $\nu(\psi_\theta, \psi_{\tilde{\theta}})$ in terms of a distance $\delta(\theta, \tilde{\theta})$.

Inequalities as (4) have been found in [6] to the Sparre-Andersen model and based on the representation of the ruin probability as the stationary probability for a reversible Markov chain. In [5], for the Sparre-Andersen model with the inclusion of interest on the surplus, the authors find exponential upper bounds to the ruin probability using martingales and recursive techniques, but without the use of distance as in (4).

In [9] the authors find an upper bound as given in (4) (using the Kantorovich and weighted metrics) and under the assumptions that λ and F in (1) are unknown. However, there is no general method to approximate the ruin probability, unlike this work in which the BFPT is used to propose such an approximation of the ruin probability and to obtain continuity in the terms defined in (3).

In [21], the authors introduced an additional assumption about the convergence of means ($|\mu - \tilde{\mu}| < \epsilon$, $\epsilon > 0$) and they found a similar upper bound to (4): $\nu(\psi_\theta, \psi_{\tilde{\theta}}) \leq k_1\delta(\theta, \tilde{\theta})u + k_2\epsilon$, where k_1, k_2 are explicitly calculated constants in terms of the parameters of the model.

In this work, we demonstrate that using appropriate probabilistic metrics for δ and ν, we can guarantee continuity for the ruin probability as defined in (3).

3. RUIN PROBABILITY APPROXIMATION USING BANACH'S FIXED-POINT THEOREM

In this section, we define the operator T and demonstrate that there is a contracting operator on a certain Banach space; then based on the BFTP, operator T will be used to obtain an approximation of the ruin probability (2). An immediate result obtained by the application of the BFTP to the risk model (1), is the deduction of the Pollaczeck–Khinchine formula. Finally, we apply the method presented in one numerical example.

To obtain the results presented in this work to approximate the ruin probability, we only must assume that the parameters of the surplus process (1) satisfy only the following conditions.

Assumption 3.1. The parameters of the surplus process (1) satisfy

1. $\mu := \mathbb{E}[Y_1] < \infty$ [1],

2. $F(0) = 0$,

3. $\alpha := \frac{\lambda\mu}{c} < 1$ called security loading [2].

One important result to find the ruin probability (2) in the classical risk model, which will be used to find the approximations in this work, is the following lemma.

[1] The expected value must be finite because otherwise no insurance company would insure such a risk.

[2] If $\alpha \geq 1$, then for all $u \geq 0$ it holds that $\psi(u) = 1$.

Lemma 3.2. Let $\psi(u)$ be the ruin probability in the CL model with claim sizes having distribution function F. Under Assumption 3.1, $\psi(u)$ satisfies

$$\psi(u) = \frac{\lambda}{c}\left(\int_u^\infty \bar{F}(x)dx + \int_0^u \psi(u-x)\bar{F}(x)dx\right), \qquad u \geq 0, \tag{5}$$

where $\bar{F}(x) := 1 - F(x)$ is the survival function.

For a proof of Lemma 3.2 see [8].

3.1. Contracting operator

In order to obtain the first approximation for the ruin probability (2), we define the following.

Definition 3.3. For all $u \geq 0$, we define the following measurable functions set

$$\mathfrak{Y} := \{g(u) : [0,\infty) \to [0,1]\}.$$

Remark 3.4. For all $g \in \mathfrak{Y}$ we have that $||g||_\infty := \sup_{u\geq 0}|g(u)| < 1$ (which is called the supremum norm).

Definition 3.5. For each $g \in \mathfrak{Y}$, we consider the operator $T : \mathfrak{Y} \to \mathfrak{Y}$, defined by

$$Tg(u) := \frac{\lambda}{c}\left(\int_u^\infty \bar{F}(x)dx + \int_0^u g(u-x)\bar{F}(x)dx\right), \qquad u \geq 0. \tag{6}$$

Definition 3.6. The uniform metric (also called Kolmogorov metric) ρ defined on $\mathfrak{Y}$, given $u \geq 0$, is

$$\rho(g_1, g_2) := \sup_{u\geq 0}|g_1(u) - g_2(u)|, \qquad g_1, g_2 \in \mathfrak{Y}.$$

The space of functions of Definition 3.3 is a Banach space with the uniform metric, i. e., $(\mathfrak{Y}, \rho)$ is a complete metric space (see [12]).

The following result shows that the operator of Definition 3.5 represents a contraction mapping on the metric space $(\mathfrak{Y}, \rho)$ with module α.

Lemma 3.7. Under Assumption 3.1, for all $g, g_1, g_2 \in \mathfrak{Y}$ and the operator T given in (6), we have:

1. $Tg \in \mathfrak{Y}$,
2. $\rho(Tg_1, Tg_2) \leq \alpha\rho(g_1, g_2)$, where $\alpha = \frac{\lambda\mu}{c}$.

P r o o f.

1. Given $u \geq 0$ and for all $g \in \mathfrak{Y}$ we have that

$$\begin{aligned} Tg(u) &= \frac{\lambda}{c}\left(\int_u^\infty \bar{F}(x)dx + \int_0^u g(u-x)\bar{F}(x)dx\right) \\ &\leq \frac{\lambda}{c}\left(\int_u^\infty \bar{F}(x)dx + \int_0^u \bar{F}(x)dx\right) \\ &= \frac{\lambda\mu}{c} = \alpha < 1. \end{aligned}$$

   That is, $T\mathfrak{Y} \subset \mathfrak{Y}$.

2. For all $u \geq 0$ and $g_1, g_2 \in \mathfrak{Y}$, by Definition 3.6 we have

$$\begin{aligned} \rho(Tg_1, Tg_2) &= \sup\nolimits_{u\geq 0}|Tg_1(u) - Tg_2(u)| \\ &= \frac{\lambda}{c}\sup\nolimits_{u\geq 0}\left|\int_0^u g_1(u-x)\bar{F}(x)dx - \int_0^u g_2(u-x)\bar{F}(x)dx\right| \\ &\leq \frac{\lambda}{c}\sup\nolimits_{u\geq 0}\int_0^u |g_1(u-x) - g_2(u-x)|\bar{F}(x)dx \\ &\leq \frac{\lambda}{c}\sup\nolimits_{u\geq 0}\int_0^u \sup\nolimits_{w\in[0,u]}|g_1(w) - g_2(w)|\bar{F}(x)dx \\ &\leq \frac{\lambda}{c}\sup\nolimits_{u\geq 0}\int_0^u \rho(g_1, g_2)\bar{F}(x)dx \\ &= \frac{\lambda\rho(g_1, g_2)}{c}\int_0^\infty \bar{F}(x)dx = \frac{\lambda\mu\rho(g_1, g_2)}{c} = \alpha\rho(g_1, g_2). \end{aligned}$$

□

Hence, by the BFTP for contraction mappings (see Appendix A), there exists a unique function (fixed point) $\psi \in \mathfrak{Y}$, such that $T\psi = \psi$ and ψ is the limit of the functions

$$\psi_n := T\psi_{n-1} = T^n\psi_0 \qquad n \geq 1, \tag{7}$$

for some arbitrary $\psi_0 \in \mathfrak{Y}$ and T given in (6).

3.2. Approximations of the ruin probability with the contracting operator method

To apply the BFPT to find an approximation ruin probability (2), consider the iterative sequence of ruin probabilities defined in (7), then

$$\psi_n(u) := \frac{\lambda}{c}\left(\int_u^\infty \bar{F}(x)dx + \int_0^u \psi_{n-1}(u-x)\bar{F}(x)dx\right), \qquad u \geq 0, \ \ n \geq 1, \tag{8}$$

and ψ_0 is some arbitrary function of $\mathfrak{Y}$.

The following corollary is an immediate consequence of Lemma 3.7.

Corollary 3.8. Let $\{\psi_n\}_{n\geq 0}$ be the functions iterative sequence defined in (8). Under Assumption 3.1 we have

$$\rho(\psi_n, \psi) \to 0,$$

where ψ is the ruin probability (2).

Remark 3.9. An advantage of using the BFPT to obtain an approximation to ruin probability ψ is that the BFPT provides inequalities, which describe the speed of convergence. The speed of convergence on the space metric $(\mathfrak{Y}, \rho)$ is given by

1. $\rho(\psi_n, \psi) \leq \frac{\alpha^n}{1-\alpha}\rho(\psi_1, \psi_0)$.
2. $\rho(\psi_n, \psi) \leq \frac{\alpha}{1-\alpha}\rho(\psi_n, \psi_{n-1})$.
3. $\rho(\psi_n, \psi) \leq \alpha^n \rho(\psi_0, \psi)$.

Where α is defined in Assumption 3.1 (for a demostration see [11] and [20]). We will use the third speed of convergence in the numeric example, which is presented in the next section (see Table 2).

We define the equilibrium probability density function of non-negative random variables Y_i's of claim sizes for the surplus process (1) as

$$f_e(y) := \frac{1 - F(y)}{\mu}, \qquad y > 0, \tag{9}$$

and the corresponding equilibrium distribution function is denoted by $F_e(x)$.

We can rewrite the contractive operator (6) in terms of the equilibrium probability density function (9) as

$$T\psi(u) = \alpha \left(\int_u^\infty f_e(y)dy + \int_0^u \psi(u-y) f_e(y)dy \right).$$

Now, the iterative sequence of ruin probabilities defined in (8) can be rewritten as

$$\psi_n(u) = \alpha \left(1 - F_e(u) + \int_0^u \psi_{n-1}(u-y) f_e(y)dy \right), \qquad u \geq 0, n \geq 1. \tag{10}$$

The following result provides an analytical formula to calculate the elements of the sequence $\{\psi_n\}_{n\geq 0}$ of Corollary 3.8.

Lemma 3.10. Under Assumption 3.1, if $\psi_0(u) = k$ ($k \in [0,1]$) and for $u \geq 0$, then

$$\psi_n(u) = \begin{cases} \alpha - (1-k)\alpha^n F_e^{*(n)}(u) - (1-\alpha)\sum_{i=1}^{n-1} \alpha^i F_e^{*(i)}(u) & \text{if} \quad n \geq 2, \\ \alpha - (1-k)\alpha F_e(u) & \text{if} \quad n = 1. \end{cases}$$

Where $F_e^{*(i)}$ is the ith convolution power of $Y_{e,i}$'s iid random variables with probability density function given by (9).

P r o o f. The result can be demonstrated by mathematical induction. For $n+1$ applying (10) with $\psi_0(u) = k$, we have

$$\begin{aligned}\psi_1(u) &= T\psi_0(u) = Tk \\ &= \alpha\left(1 - F_e(u) + \int_0^u k f_e(x)dx\right) \\ &= \alpha - (1-k)\alpha F_e(u). \end{aligned} \tag{11}$$

Now, to obtain the second iteration of approximation, we use again (10) with $\psi_1(u)$ given in (11)

$$\begin{aligned}\psi_2(u) &= T\psi_1(u) = T(\alpha - (1-k)\alpha F_e(u)) \\ &= \alpha\left(1 - F_e(u) + \int_0^u (\alpha - (1-k)\alpha F_e(u))f_e(x)dx\right) \\ &= \alpha - (1-k)\alpha^2(F_e(u) + F_e^{*(2)}(u)).\end{aligned}$$

Suppose that the result holds for $n = m \geq 2$ and for the next iteration we have

$$\begin{aligned}\psi_{m+1}(u) &= T\psi_m(u) \\ &= \alpha\left(1 - F_e(u) + \int_0^u \left[\alpha - (1-k)\alpha^m F_e^{*(m)}(u-x) - (1-\alpha)\sum_{i=1}^{m-1}\alpha^i F_e^{*(i)}(u-x)\right] f_e(x)dx\right) \\ &= \alpha - (1-k)\alpha^{m+1}F_e^{*(m+1)}(u) - (1-\alpha)\alpha F_e(u) - (1-\alpha)\sum_{i=1}^{m}\alpha^{i+1}F_e^{*(i+1)}(u) \\ &= \alpha - (1-k)\alpha^{m+1}F_e^{*(m+1)}(u) - (1-\alpha)\sum_{i=1}^{m}\alpha^i F_e^{*(i)}(u).\end{aligned}$$

The last equality shows that the statement holds for $n = m+1$ and $u \geq 0$ (fixed). □

In the limit as $n \to \infty$, we obtain the Pollaczeck-Khinchine formula or the Beekman's convolution formula (Corollary 3.11) as an immediate consequence of BFPT and Lemma 3.10.

Corollary 3.11. Under Assumption 3.1, for the CL model (1) with ruin probability $\psi(u)$ and claim sizes with distribution function F. If $\psi_0(u) = \alpha$, then

$$\psi(u) = \lim_{n\to\infty}\psi_n(u) = (1-\alpha)\sum_{i=1}^{\infty}\alpha^i(1 - F_e^{*i}(u)), \qquad u \geq 0.$$

In particular $\psi(0) = \alpha$.

3.3. Numerical example: Claim sizes with exponential distribution

For the surplus process (1), we suppose that $Y_i \sim \text{Exp}(\beta)$, $i \geq 1$. Here, we have that Y satisfies the conditions of Assumption 3.1 if $\beta \in (\lambda/c, \infty)$. Since in this case the analytical solution can be found using Lemma 3.2 ($\psi(u) = \alpha e^{-u(\beta - \frac{\lambda}{c})}$), it will be used to calibrate the accuracy of our method.

For this case, the approximation for the ruin probability can be obtained by applying the BFPT and the corresponding iterative sequence of ruin probabilities defined in (8), which is

$$\psi_n(u) = \alpha^n k + \alpha e^{-\beta u} \sum_{j=0}^{n-1} \frac{(\lambda u/c)^j}{j!} - \alpha^n e^{-\beta u} k \sum_{j=0}^{n-1} \frac{(\beta u)^j}{j!} \qquad n \geq 1, \tag{12}$$

where $\psi_0 = k$ for some arbitrary constant $k \in [0, 1]$. A proof for the statement (12) is in Appendix B. Since that $\alpha < 1$, by Corollary 3.8, we have that

$$\psi(u) = \lim_{n \to \infty} \psi_n(u) = \alpha e^{-u(\beta - \frac{\lambda}{c})}. \tag{13}$$

The right side of equation (13) is the unique solution of equation (5) when the claim sizes follow an exponential distribution.

3.3.1. Numerical results

Here, we show the results obtained when the claim sizes have exponential distribution with parameter $\beta = 2$, and $u = \lambda = c = 1$, then $\alpha = 0.5$. Here, the exact ruin probability is given by (13), i. e., $\psi(1) = 0.5e^{-1} \approx 0.1839397$.

The numerical results to approximate the ruin probability (13) for initial values $\psi_0 := k = 0.0, 0.1, \ldots, 1.0$ and the first five iterations of expression (12) are presented in Table 1 and the corresponding values at each iteration are plotted in Figure 1.

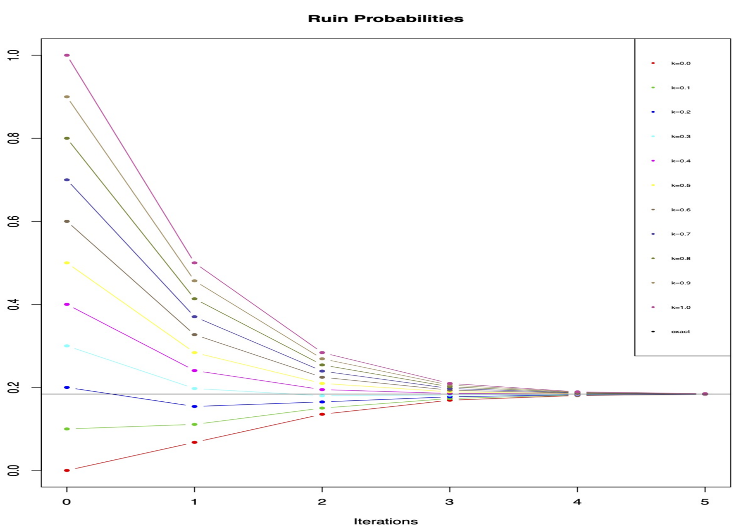

Fig. 1. The first five approximation values to ruin probability when $Y_i \sim \text{Exp}(2)$, $u = \lambda = c = 1$ and the true value $\psi(1) = 0.5e^{-1} \approx 0.1839397$.

n	$\psi_0 = 0.0$	$\psi_0 = 0.1$	$\psi_0 = 0.2$	$\psi_0 = 0.3$	$\psi_0 = 0.4$	$\psi_0 = \alpha = 0.5$	$\psi_0 = 0.6$	$\psi_0 = 0.7$	$\psi_0 = 0.8$	$\psi_0 = 0.9$	$\psi_0 = 1.0$
1	0.06766764	0.1109009	0.1541341	0.1973673	0.2406006	0.2838338	0.3270671	0.3703003	0.4135335	0.4567668	0.500000
2	0.13533528	0.1501851	0.1650350	0.1798848	0.1947347	0.2095846	0.2244344	0.2392843	0.2541341	0.2689840	0.2838338
3	0.16916910	0.1732106	0.1772522	0.1812937	0.1853353	0.1893768	0.1934184	0.1974599	0.2015015	0.2055430	0.2095846
4	0.18044704	0.1813400	0.1822330	0.1831260	0.1840190	0.1849119	0.1858049	0.1866979	0.1875909	0.1884838	0.1893768
5	0.18326653	0.1834311	0.1835956	0.1837602	0.1839247	0.1840892	0.1842538	0.1844183	0.1845829	0.1847474	0.1849119

Tab. 1. The first five approximation values to ruin probability when $Y_i \sim \text{Exp}(2)$, $u = \lambda = c = 1$ and initial values $\psi_0 = 0.0, 0.1, \ldots, 1.0$.

Therefore, in this example, we know the exact ruin probability, we can observe in 5 iterations we have a good approximation (there is convergence) to ruin probability for all initial values. However, when the exact ruin probability is unknown it is important to have an idea about the speed of convergence.

To illustrate the speed of convergence given by part 3 of Remark 3.9, the upper bounds are reported in Tables 2 and plotted in Figure 2.

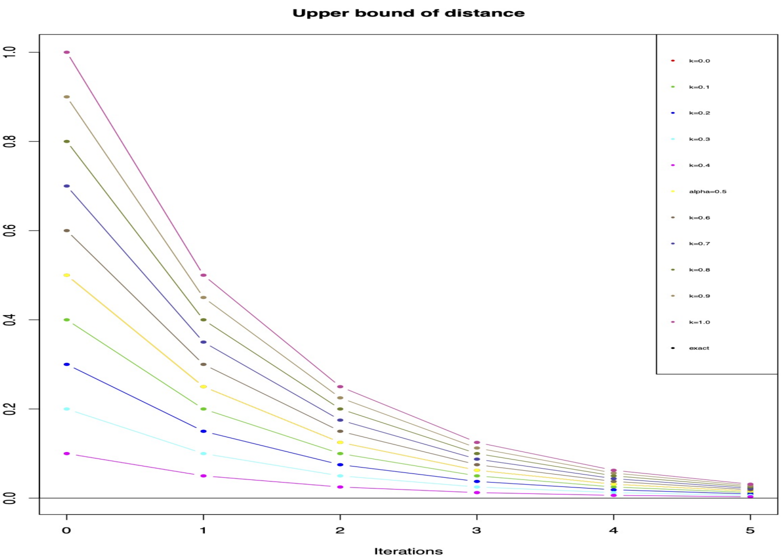

Fig. 2. The first five upper bounds values $\alpha^n(\psi_0(1), \psi(1))$ when $Y_i \sim \text{Exp}(2)$ and $\alpha = 0.5$.

n	$\psi_0 = 0.0$	$\psi_0 = 0.1$	$\psi_0 = 0.2$	$\psi_0 = 0.3$	$\psi_0 = 0.4$	$\psi_0 = \alpha = 0.5$	$\psi_0 = 0.6$	$\psi_0 = 0.7$	$\psi_0 = 0.8$	$\psi_0 = 0.9$	$\psi_0 = 1.0$
0	0.500000	0.4000	0.300000	0.20000	0.100000	0.500000	0.60000	0.700000	0.800	0.900000	1.00000
1	0.250000	0.2000	0.150000	0.10000	0.050000	0.250000	0.30000	0.350000	0.400	0.450000	0.50000
2	0.125000	0.1000	0.075000	0.05000	0.025000	0.125000	0.15000	0.175000	0.200	0.225000	0.25000
3	0.062500	0.0500	0.037500	0.02500	0.012500	0.062500	0.07500	0.087500	0.100	0.112500	0.12500
4	0.031250	0.0250	0.018750	0.01250	0.006250	0.031250	0.03750	0.043750	0.050	0.056250	0.06250
5	0.015625	0.0125	0.009375	0.00625	0.003125	0.015625	0.01875	0.021875	0.025	0.028125	0.03125

Tab. 2. The first five upper bounds values $\alpha^n(\psi_0(u), \psi(u))$ when $Y_i \sim \text{Exp}(2)$, $u = 1$, $\alpha = 0.5$ and initial values $\psi_0 = k = 0.0, 0.1, \ldots, 1.0$.

Finally, we generated random values of the equilibrium distribution of the exponential distribution with parameter $\beta = 2$. The corresponding estimators via Monte Carlo and its 95% confidence intervals are reported in Table 3. The fast speed of the variance reduction of the Monte Carlo estimators can be observed, based on the speed of the reduction of the confidence intervals.

n	MC Estimator	Lower limit CI	Upper limit CI
1	0.2905000	-0.09094541	0.6719454
2	0.2147500	0.02402730	0.4054727
3	0.1943125	0.09895115	0.2896739
4	0.1900312	0.14235057	0.2377119
5	0.1892344	0.16539404	0.2130747

Tab. 3. Estimators via Monte Carlo and its 95% confidence intervals using a sample of size 1000.

In the Figure 3 the Monte Carlo estimators and its 95% confidence intervals are plotted.

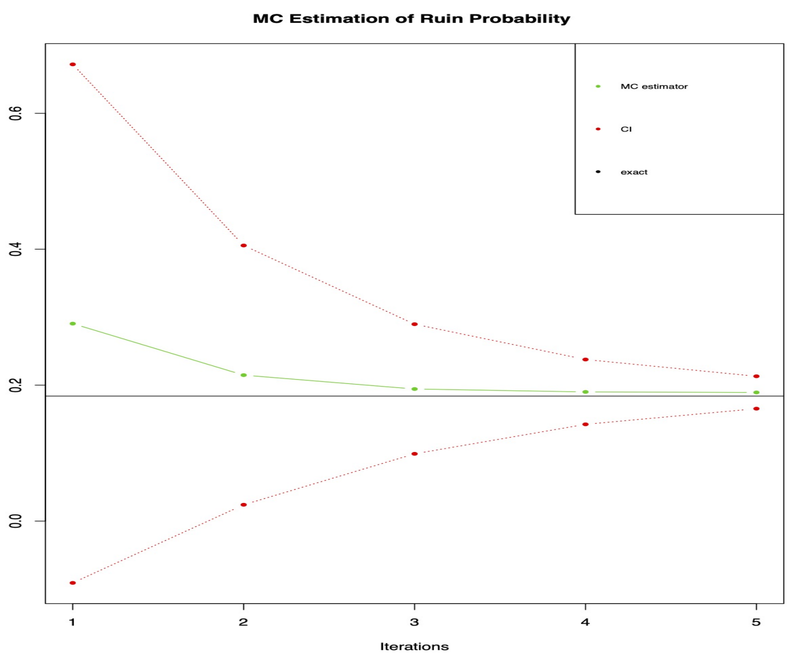

Fig. 3. Iterations of ruin probabilities when the claim sizes are Exp(2).

4. RUIN PROBABILITY APPROXIMATION USING THE CONTINUITY OF THE RUIN PROBABILITY

In this section, we will study the conditions for obtaining the continuity of the ruin probability in the context defined in (3). The result obtained is stated in Theorem 4.2. Moreover, the results obtained in this paper generate an upper bound such as that given in (4), under the fact that the function ϕ in (4) is the identity function, i. e., $\phi(s) = s$ and the probabilistic metric δ in (4) is the Kantorovich probabilistic metric in our result. Such upper bound can be used as an estimate for the difference between the ruin probability (2) and its approximate obtained by the BFPT.

Moreover, we study continuity conditions of the ruin probability between surplus processes as it was presented in (3). We use the continuity to establish that a certain type of convergence between the claim sizes distribution implies convergence, in some sense, of the ruin probability.

4.1. Continuity of the ruin probability

In order to obtain an approximation of the ruin probability (2), we propose that if the distribution function F of the claim sizes of the surplus process (1) can be approximated (in a certain sense) by a sequence of distribution functions $\{F_m\}_{m\geq 1}$, then we can approximate to the ruin probability (2) by the sequence of ruin probabilities of the surplus processes associated with the $\{F_m\}_{m\geq 1}$, these are the distribution functions of the corresponding claim sizes.

For each element of the sequence $\{F_m\}_{m\geq 1}$, we consider a sequence of surplus processes $\boldsymbol{C}^m = \{C^m\}_{m\geq 1}$ ($\boldsymbol{C}^m = \{C_t^m\}_{t\geq 0}$) defined by

$$C_t^m := u + ct - \sum_{i=1}^{N_t} Y_i^m, \tag{14}$$

where u, c and $\{N_t\}_{t\geq 0}$ is as in (1), for each m fix, the claim sizes $\{Y_i^m\}_{i\geq 1}$ is a sequence of non-negative, iid random variables with distribution functions $\{F_m\}_{m\geq 1}$. The ruin time for mth surplus process (14) is denoted by τ_m and the corresponding ruin probability by ψ_m. Moreover, we suppose that for all $m \geq 1$ the parameters corresponding to the surplus processes (14) satisfy Assumption 3.1 with $\mu_m = E(Y^m)$ and $\alpha_m := \frac{\lambda \mu_m}{c}$.

To obtain the continuity of the ruin probability we should establish conditions to guarantee that the sequence of ruin probabilities $\{\psi_m\}_{m\geq 1}$ converges (in a certain sense) to the ruin probability (2). To get these conditions, we define the following sequence of maps $\{T_m\}_{m\geq 1}$ in the functions set $\mathfrak{Y}$ equipped with the metric Kolmogorov ρ (see Definition 3.6).

Definition 4.1. For all $g \in \mathfrak{Y}$, each $m \geq 1$, $u \geq 0$, and the surplus process (14), the contraction mapping (6) becomes in the next operator

$$T_m g(u) := \frac{\lambda}{c}\left(\int_u^\infty \bar{F}_m(x)dx + \int_0^u g(u-x)\bar{F}_m(x)dx\right).$$

It is easy to prove that for each $m \geq 1$, the sequence $\{T_m\}_{m\geq 1}$ from Definition 4.1 satisfies the Lemma 3.7, i. e.:

1. If $g \in \mathfrak{Y}$ then $T_m g \in \mathfrak{Y}$.
2. For $g_1, g_2 \in \mathfrak{Y}$ then $\rho(T_m g_1, T_m g_2) \leq \alpha_m \rho(g_1, g_2)$, where $\alpha_m = \frac{\lambda \mu_m}{c}$.

Hence, by the BFPT for contraction mappings we have that for each $m \geq 1$, there exists a unique function $\psi_m \in \mathfrak{Y}$ for the mth iteration such that $T_m \psi_m = \psi_m$ and ψ_m is the limit of the functions

$$\psi_{m,n} := T_m \psi_{m,n-1} = T_m^n \psi_{m,0} \qquad n \geq 1,$$

for some arbitrary initial solution $\psi_{m,0} \in \mathfrak{Y}$ and T_m from Definition 4.1.

Finally, the BFPT shows that for each $m \geq 1$: ψ_m is the unique solution to the integral equation given in Definition 4.1.

In order to apply the BFPT to approximate the ruin probability ψ given in (2), using the ruin probability ψ_m associated with the surplus process (14), we will proceed analogous to the iterative sequence defined in (8). For each $m \geq 1$, consider the iteration functions (approximate ruin probabilities) defined as

$$\psi_{m,n}(u) := \frac{\lambda}{c} \left(\int_u^\infty \bar{F}_m(x)dx + \int_0^u \psi_{m,n-1}(u-x)\bar{F}_m(x)dx \right), \qquad u \geq 0, n \geq 1, \quad (15)$$

where $\psi_{m,0}$ is some arbitrary initial solution of $\mathfrak{Y}$. Then, we have that

$$\lim_{m \to \infty} \lim_{n \to \infty} \psi_{m,n} = \lim_{m \to \infty} \psi_m = \psi, \qquad (16)$$

where ψ_m is the ruin probability for the surplus process (14).

The main result of this section is the next theorem, which shows that the ruin probability ψ, see (2), can be approximated by the ruin probabilities defined in (15) and that the expression (16) is satisfied.

Theorem 4.2. Let $\psi, \psi_m \in \mathfrak{Y}$ be the ruin probabilities for the surplus processes (1) and (14) respectively. If $F_m \xrightarrow{d} F$ (convergence in distribution) and under Assumptions 3.1, then we have

$$\lim_{m \to \infty} \rho(\psi, \psi_m) = 0,$$

where ρ is the Kolmogorov metric.

To prove Theorem 4.2 we will use the properties of the Kantorovich metric κ (see Definition 4.3) and some results (see propositions 3.4-3.6), which will apply to the sequence of the random variables $\{Y_i^m\}_{m \geq 1}$ (i fixed) given in (14).

Let $\mathfrak{X}$=$\mathfrak{X}(\mathbb{R})$ be the space of all real-valued non-negative random variables X with $\mathbb{E}(|X|) < \infty$, and $\mathfrak{F}$ the set of their distribution functions.

Definition 4.3. For each F_X,$F_Y$$\in \mathfrak{F}$, the Kantorovich metric $\kappa(F_X, F_Y)$ is defined by

$$\kappa(F_X, F_Y) := \int_{\mathbb{R}} |F_X(x) - F_Y(x)|dx, \qquad X, Y \in \mathfrak{X}.$$

Proposition 4.4. Suppose that $X \in \mathfrak{X}$ and that $\epsilon > 0$, then there is a constant that depens on X, $K(X) \in [0, \infty)$ such that $\mathbb{E}(|X|\mathbf{1}_{\{|X|>K(X)\}}) < \epsilon$.

For a proof see [22].

Proposition 4.5. Let $\{X_n\}_{n\geq 1}$ be a sequence of non-negative random variables uniformly integrable (UI) such that X_n converges in distribution to X, then $\lim_{n\to\infty}\mathbb{E}(X_n) = \mathbb{E}(X) < \infty$.

For a proof see [13].

Proposition 4.6. The Kantorovich metric κ metricizes convergence in distribution, and the convergence of the first moments in the space $\mathfrak{X}$, i. e., $\lim_{n\to\infty} \kappa(F_n, F) = 0$ if and only if

1. $F_n \xrightarrow{d} F$, and
2. $\lim_{n\to\infty} \mathbb{E}|X_n| = \mathbb{E}|X|$.

For a proof see [18].

Now, we give a proof of Theorem 4.2.

P r o o f. For some arbitrary $m \geq 1$ fixed, let ψ, ψ_m be the corresponding ruin probabilities of surplus processes (1) and (14). Since that T and T_m given in Definition 3.5, and Definition 4.1 are contraction mappings, by the BPFT, we have that $\psi = T\psi$ and $\psi_m = T_m\psi_m$, then

$$\rho(\psi, \psi_m) = \rho(T\psi, T_m\psi_m)$$

by the triangle inequality, we have

$$\rho(\psi, \psi_m) \leq \rho(T\psi, T\psi_m) + \rho(T\psi_m, T_m\psi_m),$$

since T is a contractive operator

$$\begin{aligned}
\rho(\psi, \psi_m) &\leq \alpha\rho(\psi, \psi_m) + \rho(T\psi_m, T_m\psi_m), \\
(1-\alpha)\rho(\psi, \psi_m) &\leq \rho(T\psi_m, T_m\psi_m), \\
\rho(\psi, \psi_m) &\leq \frac{1}{1-\alpha}\rho(T\psi_m, T_m\psi_m). \qquad (17)
\end{aligned}$$

where α is given in Assumption 3.1.

The next step is to find an upper bound for the inequality (17). By the Definition 3.6 we have that

$$\begin{aligned}
\rho(T\psi_m, T_m\psi_m) &= \sup_{u>0}|T\psi_m(u) - T_m\psi_m(u)| \\
&= \sup_{u>0}\left|\frac{\lambda}{c}\left(\int_u^\infty (\bar{F}(x) - \bar{F}_m(x))dx + \int_0^u \psi_m(u-x)(\bar{F}(x) - \bar{F}_m(x))dx\right)\right| \\
&\leq \frac{\lambda}{c}\sup_{u>0}\left(\int_u^\infty |\bar{F}(x) - \bar{F}_m(x)|dx + \int_0^u |\psi_m(u-x)||(\bar{F}(x) - \bar{F}_m(x))|dx\right) \\
&\leq \frac{\lambda}{c}\sup_{u>0}\left(\int_0^\infty |\bar{F}(x) - \bar{F}_m(x)|dx\right) \\
&\leq \frac{\lambda}{c}\int_0^\infty |F_m(x) - F(x)|dx. \qquad (18)
\end{aligned}$$

Using inequalities (17) and (18) we have

$$\rho(\psi, \psi_m) \quad \leq \quad \frac{\lambda}{c(1-\alpha)} \int_0^\infty |F_m(x) - F(x)| dx. \tag{19}$$

Taking the limit as m approaches infinity

$$\lim_{m\to\infty} \rho(\psi, \psi_m) \leq \frac{\lambda}{c(1-\alpha)} \lim_{m\to\infty} \int_0^\infty |F_m(x) - F(x)| dx.$$

By Definition 4.3, we have

$$\lim_{m\to\infty} \rho(\psi, \psi_m) \leq \frac{\lambda}{c(1-\alpha)} \lim_{m\to\infty} \kappa(F_m, F).$$

Since $\{Y^m\}_{m\geq 1} \in L^1$ for all $m \geq 1$, by Proposition 4.4 the sequence $\{Y^m\}_{m\geq 1}$ are UI. Moreover, since $Y^m \xrightarrow{d} Y$ (convergence in distribution), then by Proposition 4.5 we have that $\mathbb{E}(Y_m) \to \mathbb{E}(Y) < \infty$.

Finally, as $\lim_{m\to\infty} F_m = F$ and $\lim_{m\to\infty} \mathbb{E}(Y_m) = \mathbb{E}(Y) < \infty$, by the Proposition 4.6 we have that $\lim_{m\to\infty} \kappa(F_m, F) = 0$, then

$$\lim_{m\to\infty} \rho(\psi, \psi_m) = 0.$$

□

Remark 4.7. If we eliminate the assumption that F_m converges to F in Theorem 4.2, then the expression (19)

$$\rho(\psi, \psi_m) \leq \frac{\lambda}{c(1-\alpha)} \int_0^\infty |F_m(x) - F(x)| dx,$$

provides an inequality for the continuity estimate for the ruin probability, in the context defined in (4), i. e., we can rewrite the equation (19) in terms of the Kantorovich metric (see Definition 4.3) as

$$\rho(\psi, \psi_m) \leq \frac{\lambda}{c(1-\alpha)} \kappa(F_m, F) = \bar{K} \kappa(F_m, F),$$

where $\bar{K} = \frac{\lambda}{c(1-\alpha)} > 0$ is the continuity constant, then $\rho(\psi, \psi_m) \to 0$ if $\kappa(F_m, F) \to 0$.

4.2. Approximation of the ruin probability using the continuity via Erlang mixture and PH distributions

In this section, we use the representation PH for the Erlang mixture (ErM) distributions to calculate the ruin probability when the claim sizes are ErM distributed. Moreover, using Theorem 4.2 we show that the ruin probability (2) for a CL model with any distribution of the claim sizes can be seen as a limit of ruin probabilities of a CL model with ErM distributions of the claim sizes and the corresponding PH representation.

4.2.1. Erlang mixture and PH distributions

We start introducing some definitions and results that we will use to develop our method to approximate the ruin probability.

Definition 4.8. The random variable Y has an Erlang distribution with parameters $k \in \mathbb{N}$, and $\eta > 0$ ($Y \sim Er(k, \eta)$) if its density function is

$$f_Y(y) = \frac{\eta^k y^{k-1} e^{-\eta y}}{\Gamma(k)}, \qquad y > 0.$$

Definition 4.9. We say that a random variable Y has an Erlang mixture distribution with parameters $\boldsymbol{\pi} = (\pi_1, \pi_2, \ldots)$ for $\pi_k \geq 0$, $\sum_{k=1}^{\infty} \pi_k = 1$, and $\eta > 0$ (we write $Y \sim ErM(\pi, \eta)$) if its density function is

$$f_Y(y) = \sum_{k=1}^{\infty} \pi_k f_E(y; k, \eta), \qquad y > 0,$$

where $f_E(y; k, \eta)$ is the density function a random variable with Erlang distribution of parameters k and η.

The next proposition shows that ErM distributions are dense in the space of continuous distributions having support in $[0, \infty)$. For a proof of this result see [14].

Proposition 4.10. Let F be a distribution function with support in $[0, \infty)$. Let $\{F_n\}_{n \geq 1}$ be a sequence of distribution functions defined by

$$F_n(x) := \sum_{m=1}^{\infty} \pi_{m,n} \mathrm{F}_E(x; m, n),$$

where $\mathrm{F}_E(x; m, n)$ is the Erlang distribution function with parameters (m, n) evaluated in $x \in \mathbb{R}$ and

$$\pi_{m,n} = F(\frac{m}{n}) - F(\frac{m-1}{n}).$$

Then

$$\lim_{n \to \infty} F_n(x) = F(x).$$

If F has bounded support, then convergence is uniform.

On the other hand, let $\{X_t\}_{t \geq 0}$ be a Markov jump process (MJP) (for details of MJP see [4]) with finite-state space $E = \{1, 2, \ldots, n, n+1\}$, initial distribution $\boldsymbol{\omega} = (\omega_1, \omega_2, \ldots, \omega_{n+1})$, $\sum_{k=1}^{n} \omega_k = 1$, and intensity matrix $\boldsymbol{\Lambda}$. If we assume that the state $\{n+1\}$ is the unique absorbing state, then the infinitesimal generator has the following structure

$$\boldsymbol{\Lambda} = \begin{pmatrix} \boldsymbol{Q} & \boldsymbol{r} \\ \boldsymbol{0} & 0 \end{pmatrix},$$

where $\boldsymbol{Q}$ (phase-type generator or subintensity matrix) is a square matrix of order n, $\boldsymbol{r} = -\boldsymbol{Q}\boldsymbol{e}$ (exit vector) ($\boldsymbol{e}$ is a n-dimensional colum vector of ones) and $\boldsymbol{0}$ is a row vector of dimension n with all its entries zero.

Definition 4.11. Let Y be the random variable which models the time it takes for the process $\{X_t\}_{t\geq 0}$ to reach the absorbing state $\{n+1\}$, i. e.

$$Y := \inf\{t \geq 0; X_t = n+1\},$$

we say that Y has PH distribution with initial distribution $\boldsymbol{\omega} = (\omega_1, \omega_2, \ldots, \omega_n)$ and subintensity matrix $\boldsymbol{Q}$. We write $Y \sim PH(\boldsymbol{\omega}, \boldsymbol{Q})$.

Proposition 4.12. Phase-type distribution forming a dense class of distribution within the class of distributions on $[0, \infty)$.

For a proof of Proposition 4.12 see [4].

Corollary 4.13. If $Y \sim ErM(\boldsymbol{\pi}, \eta)$ with $\sum_{k=1}^{n} \pi_k = 1$, then Y has a PH distribution with representation $\boldsymbol{\omega} = (\boldsymbol{\omega}_1, \boldsymbol{\omega}_2, \ldots, \boldsymbol{\omega}_n)$ where $\boldsymbol{\omega}_k$ is a row vector of dimension k with the first enter equals to π_k and all others entries zero and

$$\boldsymbol{Q} = \begin{pmatrix} -\eta & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & -\eta & \eta & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & -\eta & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & -\eta & \eta & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & -\eta & \eta & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & -\eta & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & -\eta & \eta & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & -\eta \end{pmatrix}$$

is a square matrix of order $\frac{n(n+1)}{2}$.

The next proposition gives an analytic solution of the ruin probability (2) when the claim sizes are distributed PH.

Proposition 4.14. Let $\boldsymbol{C}$ be the surplus process (1) where $c = 1$ and the claim sizes $Y_i \sim PH(\boldsymbol{\omega}, \boldsymbol{Q})$, then the ruin probability is

$$\psi(u) = \boldsymbol{\gamma} e^{(\boldsymbol{Q}+\boldsymbol{r}\boldsymbol{\gamma})u} \boldsymbol{e},$$

where $\gamma = \lambda \boldsymbol{\omega} \boldsymbol{Q}^{-1}$.

For a proof of Proposition 4.14 see [2].

4.2.2. Method to approximate the ruin probability using the continuity of Theorem 4.2

Since the ruin probability (2) can be calculated if the claim sizes have PH distribution (see Proposition 4.14), we can find the phase-type representation of an Erlang mixture distribution (see Corollary 4.13) and if Y is a random variable with support in $[0, \infty)$ there is a succession of Erlang mixture distribution that converges to Y in distribution (see Proposition 4.10). Then, based on Theorem 4.2, we propose the algorithm following to approximate the ruin probability (2) for any distribution of the claim sizes.

Let $\boldsymbol{C}$ be the surplus process (1) where F is the distribution function of the claim sizes $\{Y_i\}_{i\geq 1}$. Let $\{F_n\}_{n=1}$ be the succession of distribution functions such that $\lim_{n\to\infty} F_n(x) = F(x)$ (see Proposition 4.10).

Definition 4.15. Given $\epsilon > 0$, we define the finite sequence $\{F_{n,k}\}_{n=1}^{k}$ by

$$F_{n,k}(x) := \sum_{m=1}^{k} \pi_{m,n} \mathrm{F}_E(x; m, n),$$

where $\pi_{m,n} := F(\frac{m}{n}) - F(\frac{m-1}{n})$, and $k \in \mathbb{N}$ such that $1 - \sum_{m=1}^{k} \pi_{m,n,k} < \epsilon$. Moreover, we define $\boldsymbol{\pi}_{n,k} := (\pi_{1,n}, \pi_{2,n}, \ldots, \pi_{k,n})$.

The algorithm to approximate to ruin probability (2) works as follows.

Algorithm 1 Approximation of ruin probability

1: Choose $\epsilon > 0$.
2: Find $n \in \mathbb{N}$ such that $F_n(x) \approx F(x)$ for all $x \in \mathbb{R}$, where F_n is the nth distribution function of the Proposition 4.10. We calculate the percentiles of both distributions and we choose the n such that 95% of the difference of the absolute value of the percentiles is less than ϵ.
3: Find $k \in \mathbb{N}$ such that $1 - \sum_{m=1}^{k} \pi_{m,n} < \epsilon$ and that $|\mu_{n,k} - \mu| < \epsilon$, where $\mu_{n,k} := \mathbb{E}[Y_{n,k}]$ and $Y_{n,k}$ is the random variable with distribution function $F_{n,k}$ (see Definition 4.15).
4: Find the PH distribution ($PH(\boldsymbol{\omega}, \boldsymbol{Q})$) for the random variable with distribution $F_{n,k}$, see Corollary 4.13.
5: $\psi(u) \doteq \boldsymbol{\gamma} e^{(\boldsymbol{Q}+\boldsymbol{r}\boldsymbol{\gamma})u}\boldsymbol{e}$, where $\boldsymbol{\gamma}$ and $\boldsymbol{r}$ are as in Proposition 4.14 and $\boldsymbol{\omega}$ and $\boldsymbol{Q}$ are the parameters of the Step 4.

4.3. Numerical examples

Example 4.16. Exponential distribution. Consider the surplus process (1) and we suppose that the claim sizes have exponential distribution. In particular, $Y_i \sim \mathrm{Exp}(10)$ $(i \geq 1)$, and $u = \lambda = c = 1$, then $\alpha = 0.1$.

To apply the Algorithm 1, if we choose, $\epsilon = 0.0001$ then $n = 142$ (step 2) and the approximation of ruin probability is 0.3309169. In this case, the exact ruin probability is $\psi(1) = 0.3310915$. In Figure 4 the approximations to the ruin probability for n from 39 to 200 are plotted.

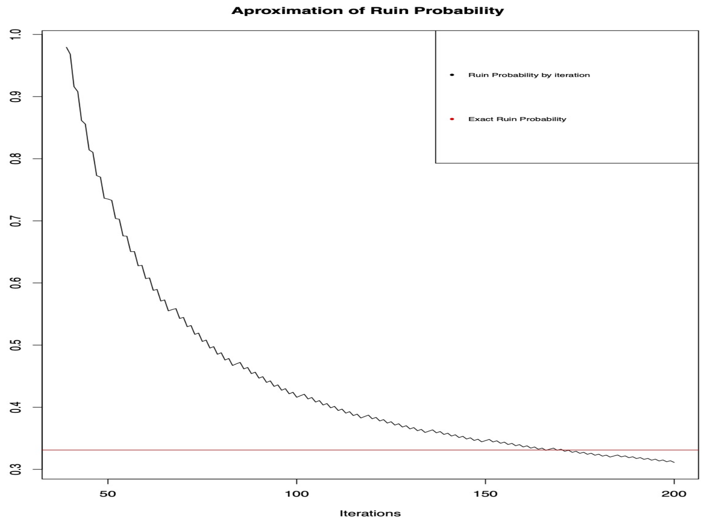

Fig. 4. The iterations from 39 to 200 of approximation values to ruin probability when $Y_i \sim \text{Exp}(10)$, $u = \lambda = c = 1$ and the true value $\psi(1) = 0.3310915$.

On the other hand, in Figure 5 the approximations to the mean of claim sizes for n from 39 to 200 are plotted.

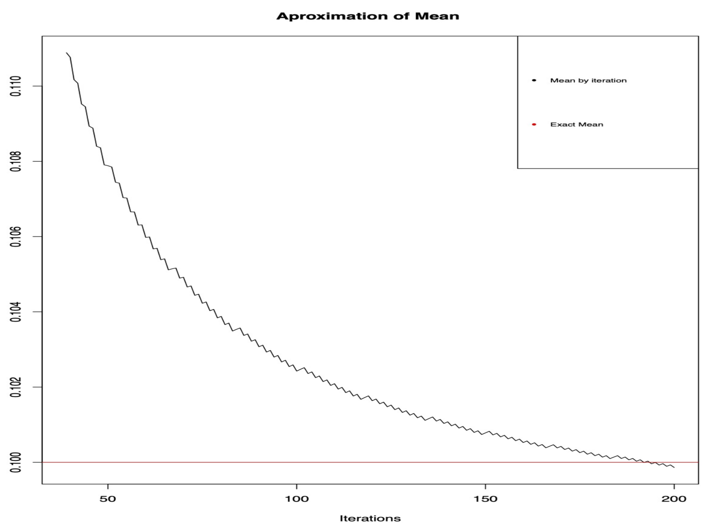

Fig. 5. The iterations from 39 to 200 of approximation values to mean of claim sizes when $Y_i \sim \text{Exp}(10)$, $\mu = \lambda = c = 1$ and the true value $\alpha = 0.1$.

Example 4.17. Weibull distribution. We consider that the claim sizes have Weibull distribution with parameters $r, \beta > 0$, i.e., the distribution function is given by $F_Y(y) = 1 - e^{-(\beta y)^r}$, $y > 0$. When $0 < r < 1$ this distribution is heavy-tailed, the adjustment does not exist coefficient and there is not known explicit formula for the ruin probability. We consider the case when $r = 0.5, \beta = 1, u = 10, \lambda = 0.5, c = 1.1$ and then $\alpha = 0.9090$.

In order to apply Algorithm 1 we chose $\epsilon = 0.001$. We used the approximation of Panjer (see [17]) as the exact value of this probability ($\psi_p(10) = 0.7507$). Table 4 shows the first five approximations using Algorithm 1.

n	k	$\psi_n(10)$	$\lvert\psi_n(10) - \psi_p(10)\rvert$	μ_n	$\lvert\mu_n - \mu\rvert$
1	39	1	0.2493	2.565646	0.565646
2	78	1	0.2493	2.209561	0.209561
3	116	0.8484282	0.0977282	2.096162	0.096162
4	155	0.7811964	0.0304964	2.04296	0.04296
5	194	0.75475	0.00405	2.011808	0.011808

Tab. 4. The first five approximation values to ruin probability when $Y_i \sim \text{Weibull}(0.5, 1), r = 0.5, \beta = 1, u = 10, \lambda = 0.5, c = 1.1$, and initial values k = number of phases.

5. CONCLUSIONS

In this paper, two new methods were proposed to approximate the ruin probability for the Cramér-Lundberg model, under quite flexible assumptions of this model. Using the properties of the contracting operators and the Banach's Fixed-Point Theorem it was possible to obtain a succession of iterative functions that approximate to this ruin probability very well. Another contribution is that we have been able to establish conditions for the distribution of the claim sizes that guarantee the continuity of the ruin probability. Particularly, we used the continuity of the ruin probability to propose the second method of approximation in the Algorithm 1.

The numerical results support the theoretical results mentioned above. In the first method, the proposed approximation for the ruin probability in the case where the distribution of the claim sizes is exponential, using the iterative functions provided by the Banach's Fixed-Point Theorem, provide a fairly acceptable convergence, in addition, the method we used is efficient regardless of the initial solution that we chose, i.e., if the initial solution is very close (or far) from the analytical solution, the method converges very fast (or very slow). One of the limitations of this method is that, in general, it is not very easy to have an expression for the nth function in the Banach's Fixed-Point Theorem for some arbitrary distribution of claim size.

On the other hand, in order to use Theorem 4.2, a fairly efficient algorithm is presented. We presented examples of distributions of claim sizes with light and heavy-tailed. Finally, these new methods are very easy to implement.

A. APPENDIX

In this appendix Banach's Fixed-Point Theorem is introduced.

Definition A.1. A fixed point of a function $T : X \to X$ is an element of X that is mapped to itself by the function T, that is

$$Tx = x.$$

Definition A.2. Let (X, d) be a metric space. A mapping $T : X \to X$ is a contraction on X if there exists a positive constant $\alpha < 1$ such that $d(Tx, Ty) \leq \alpha d(x, y)$ for all $x, y \in X$.

Theorem A.3. Banach's Fixed-Point Theorem. Let (X, d) be a complete metric space and let $T : X \to X$ be a contraction on X. Then T has a unique fixed point $x^* \in X$, and $\lim_{n\to\infty} T^n x = x^*$ for each $x \in X$, where $T^n x = T(T^{n-1}x) = T(T(T^{n-2}x)) = \cdots = T(T \cdots (Tx))$ for $n = 1, 2, \ldots$.

For a proof see [10].

B. APPENDIX

We will prove the statement of expression (12) by mathematical induction.

Proof. **Base case:** To obtaint ψ_1 we apply the operator T defined in (8) to $\psi_0 = k$, then

$$\begin{aligned}
\psi_1(u) &= T\psi_0(u) \\
&= \frac{\lambda}{c}\left(\int_u^\infty e^{-\beta x}dx + \int_0^u ke^{-\beta x}dx\right) \\
&= \alpha\left(k + e^{-\beta x} - ke^{-\beta x}\right).
\end{aligned}$$

The last equality shows that the statement holds for $n = 1$, and $u \geq 0$ (fixed) in expression (12).

Inductive step: We suppose that expression (12) holds for $n = m$. Now we have

$$\begin{aligned}
\psi_{m+1}(u) &= T\psi_m(u) \\
&= \frac{\lambda}{c}\left(\int_0^u [\alpha^m k + e^{-\beta(u-x)}(\alpha\sum_{j=0}^{m-1}\frac{(\lambda(u-x)/c)^j}{j!} - \alpha^n k\sum_{j=0}^{m-1}\frac{(\beta(u-x))^j}{j!})]e^{-\beta x}dx + \frac{e^{-\beta u}}{\beta}\right) \\
&= \alpha^{m+1}k + \alpha e^{-\beta u}\sum_{j=0}^{m}\frac{(\lambda u/c)^j}{j!} - \alpha^n e^{-\beta u}k\sum_{j=0}^{m}\frac{(\beta u)^j}{j!}
\end{aligned}$$

The last equality shows that the statement holds for $n = m + 1$, and $u \geq 0$ (fixed) in expression (12). □

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