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NON-STATIONARY DEPARTURE PROCESS IN A BATCH-ARRIVAL QUEUE WITH FINITE BUFFER CAPACITY AND THRESHOLD-TYPE CONTROL MECHANISM

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Non-stationary behavior of departure process in a finite-buffer $M^X/G/1/K$-type queueing model with batch arrivals, in which a threshold-type waking up N-policy is implemented, is studied. According to this policy, after each idle time a new busy period is being started with the Nth message occurrence, where the threshold value N is fixed. Using the analytical approach based on the idea of an embedded Markov chain, integral equations, continuous total probability law, renewal theory and linear algebra, a compact-form representation for the mixed double transform (probability generating function of the Laplace transform) of the probability distribution of the number of messages completely served up to fixed time t is obtained. The considered queueing system has potential applications in modeling nodes of wireless sensor networks (WSNs) with battery saving mechanism based on threshold-type waking up of the radio. An illustrating simulational and numerical study is attached.

Keywords: departure process, finite-buffer queue, N-policy, power saving, transient state, wireless sensor network (WSN)

Classification: 60K25, 90B22

1. INTRODUCTION

Queueing models with a finite buffer capacity for accumulating incoming messages are widely used nowadays. First of all, they can be used in the analysis of the message processing process in computer and telecommunication network nodes (such as Internet network routers, base stations in wireless communication, etc.), in which messages are naturally queued and delays occur. This is a consequence of the fluctuation of the input stream intensity (most often being a superposition of streams coming from many sources), as well as the speed of message processing, which depends on technical parameters of the switch and/or the throughput of the output link. Particularly noteworthy are queueing models in which the mechanism of managing the process of restarting the service station after a period of inactivity is introduced, or other discipline, which results in temporary suspension of messages processing despite their presence in the accumulating buffer. Various types of mechanisms limiting the access to service stations for some time

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are used in modeling, among others energy saving mode in wireless communication, e. g. wireless sensor networks (WSNs). The extensive use of sensor networks to monitor the state of the natural environment, e. g. detecting the risk of fire, controlling the state of air pollution, water pollution, etc. is connected with the need for trouble-free operation of the network for long time. On the other hand, sensors that are powered by batteries are often placed in hard-to-reach places where battery replacement can be problematic. In the queueing theory, many models have been proposed in which the service station remains for some time inaccessible to handling the input traffic, despite the potential presence of messages in the accumulating buffer. In particular, these are mechanisms like single and multiple vacations, in which the server takes, respectively, exactly one or a number of repeated vacations during which the processing of messages is blocked; T-policy in which the service station is being activated T time units after completing the last busy period; and a threshold-type N-policy in which the server restarts the processing if there are N messages present in the system. In [7], and [31] one can find detailed studies devoted to various types of models with vacation policies.

Queueing models with a control mechanism based on the N-policy were initiated by the fundamental paper [32]. Currently, they have extensive literature. However, analytical results concern mainly the stationary state of the system. Generally, there is quite a lack of new results for transient state of such systems. A lot of publications consider systems with the N-policy implemented with another vacation mechanism, like e. g. single or multiple vacation policy. One of the first analysis of the batch-arrival system with N-policy can be found in [22] (see also [24]), where the decomposition property and the optimal threshold value N under a linear cost structure where derived for the $M^X/G/1$-type queue. Results for similar system but with additional server repeated vacations (multiple vacation policy) are obtained in [23] and [25]. For more complex modifications of the classical N-policy control mechanism applied for infinite-buffer queues see [3] and [4], where the grand vacation process was considered in the case of single and batch Poisson arrivals, respectively, and [2, 11] and [21], where setup times were added. The $M^x/G/1$-type queueing system with two phases of heterogeneous service under N-policy was studied in [5]. Analytical results about finite-buffer queueing models with the N-policy control mechanism can be found e. g. in [30] and [33], where the $M/G/1/N$ and $GI/M/1/N$-type systems were analyzed, respectively. In [12] a recursive method for the $G/M/1$ queueing system with finite capacity operating under the N-policy is proposed.

One can find in [10] and [27] an interesting application of the queueing model with the N-policy in modeling the energy saving control mechanism in wireless networks. In the approach, after each busy period (the period of uninterrupted transmission) the radio transmitter/receiver of the network node (e. g. a sensor in wireless sensor network) is being turned off, and becomes active if the number of messages accumulated and waiting in the buffer queue reaches the fixed level N. One can find new transient-state results for an infinite-buffer queueing model in [13], [14] and [18]. Queue-size distribution in the model with batch Poisson arrivals is studied in [16] and [17], where a threshold-type N-policy and the single vacation policy is implemented, respectively (see also [16]). The departure counting process is analyzed in [15] and [19] in the system with infinite and finite buffer capacity, respectively.

In this paper, we investigate the $M^X/G/1/K$-type queue with batch arrivals and the N-policy in the transient state. We are interested in departure counting process that at fixed time t takes on a random value equal to the number of messages (jobs, packets, customers etc.) completely processed up to time t. As it seems, the departure process is a stochastic characteristic rather occasionally analyzed. However, as it was noted in [28], the investigation of the transient departure process is an integral part of analyzing time-dependent queues in queueing networks. Indeed, the composite nodal arrival process rather rarely can be approximated by a concrete probability distribution or family of distributions. Moreover, the approximation of the arrival stream to a "transitional" network node by using a classical Poisson process can lead to inaccurate results (see e. g. [9]). Hence, using the transient departure process as the approximate composite arrival process to downstream node(s) in a network of tandem queues seems to be a good solution of the problem. In real network devices, the input stream contains messages of different sizes measured in bytes, therefore, considering the group arrivals allows for better modeling the actual input stream. Finally, we investigate the system in the transient (non-stationary state). As it seems, in some situations transient analysis is recommended or is even necessary. In particular, it can be used

— in the case of the observation of the system just after the starting moment, when the steady state is not reached yet;

— after the application of a control mechanism that destabilizes the system operation temporarily;

— in the situation of frequently changing arrival intensity; e. g. the offered IP traffic entering the Internet node: only on short time periods the input flow can be well approximated by a Poisson process with constant rate;

— in wireless sensor networks in which the traffic load is often not very large, hence the stabilization of the system is relatively long.

In this article, we study the transient departure process in the considered queueing model, applying analytical approach based on the paradigm of an embedded Markov chain, integral equations, continuous total probability law, renewal theory and linear algebra. The departure process is analyzed separately on a single buffer loading period, during which the queue reaches the prefixed level N, and on a busy period. The main theoretical contribution of the paper is a compact-form representation for the mixed double transform (probability generating function of the Laplace transform) of the probability distribution of the number of messages completely served up to fixed time t (departure process). Numerical utility of analytical formulae is visualized in numerical examples motivated by the operation of a hypothetical network node. Two different simple probability distributions of batch sizes are considered. The impact of the threshold level N and the offered load ϱ on the mean number of departures up to fixed time t and on the transient loss ratio and the power saving ratio is examined.

The remaining part of the paper is organized as follows. In the next Section 2, the considered queueing model is described mathematically in details. In Section 3, we find the formula for the Laplace transform of the departure process during the buffer loading period and give the representation for the probability distribution of the duration of this period. In Section 4, we analyze the departure process during a single busy period. Section 5 contains analytical results for the busy period in the considered model. The representations for probability distributions of its duration and the number of messages completely processed during the busy period are derived there. The main result, namely the formula for the mixed double transform of the departure process is presented in Section 6. Section 7 contains numerical examples and the last Section 8 contains a short conclusion.

2. QUEUEING MODEL

In this article, we deal with the $M^X/G/1/K$-type queueing model with batch arrivals and finite buffer capacity. The arriving batches occur according to a Poisson process with rate λ. Sizes of successive batches are independent random variables with the same probability mass function, and p_k denotes the probability that an arriving batch consists of k messages, $\sum_{k=1}^{\infty} p_k = 1$. Messages are processed individually with a CDF (=cumulative distribution function) $F(\cdot)$ of the processing time, according to the FIFO service discipline. However, the results obtained in the paper are independent on the order of processing of incoming messages, i. e. they are of a more general nature. By $f(\cdot)$ we denote the LST (=Laplace–Stieltjes transform) of the CDF $F(\cdot)$. The number of messages present simultaneously in the system is bounded by a non-random value K, i. e. we have $K-1$ places in the accumulating buffer and one place at the service station.

It is assumed that the system starts the evolution being empty and the service process is being initialized simultaneously with the arrival of batch consisting the Nth message (threshold-type waking up N-policy), where $1 \le N \le K$ is fixed. After each idle time a new busy period begins in the same way, namely it is preceded by a buffer loading period during which messages accumulate in the buffer up to the level N. During a busy period all accumulated messages are processed, one by one. Evidently, each busy period begins with a number of messages being less than or equal to K due to a finite buffer capacity, namely until the number of messages present in the system equals K (the buffer is saturated and the service station "seat" is occupied) all next arriving messages are lost. Hence, the operation of the system can be observed during successive buffer loading periods $L_1, L_2, \ldots$ followed by busy periods $B_1, B_2, \ldots$, during which the system becomes empty. By virtue of the memoryless property of exponential distribution of interarrival times, start epochs of successive buffer loading periods and busy periods are Markov moments, so (L_k) and (B_k), $k = 1, 2, \ldots$, are sequences of totally independent random variables with the same CDFs in each sequence considered separately. In the article we usually identify a particular period in the evolution of the system (a buffer loading or busy period) with its duration.

In the article, $\delta_{i,j}$ stands for the Kronecker delta function and p_i^{j*} denotes the ith term of the j-fold convolution of the sequence (p_k) with itself, namely

$$p_i^{0*} = \delta_{i,0}, \quad p_i^{j*} = \sum_{k=0}^{i} p_{i-k} p_k^{(j-1)*}, \ \ j \ge 1.$$

So, p_i^{j*} expresses the probability that exactly i messages arrive in j batches.

Denote by $h(t)$ the number of messages completely processed until the moment t. The stochastic process $\{h(t),\, t \geq 0\}$ is called the departure process. We are interested in the closed-form representation for the double transform of the probability $\mathbf{P}\{h(t) = m\}$, i. e. for the functional

$$\widehat{h}(s, z) \stackrel{def}{=} \sum_{m=0}^{\infty} z^m \int_0^{\infty} e^{-st} \mathbf{P}\{h(t) = m\} \, \mathrm{d}t, \tag{1}$$

where $|z| < 1$ and Re(s) > 0.

In the next sections, using the idea of embedded Markov chain and the continuous version of the total probability law, and applying integral equations and linear algebra, we derive the formulae for the mixed double transform of $h(t)$ on a single buffer loading period and during a busy period. Next we obtain the representation for the PGF (=probability generating function) of the number of messages completely processed during the busy period, conditioned by the initial buffer state. Finally, using the renewal-theory approach, we get the formula for $\widehat{h}(s, z)$ defined in (1).

3. DEPARTURE PROCESS DURING BUFFER LOADING PERIOD

In this section, we analyze the behavior of departure process $h(t)$ on the first buffer loading period L_1 that starts at time $t = 0$. Note that the following equation is true:

$$\mathbf{P}\{\big(h(t) = m\big) \cap (t \in L_1)\} = \delta_{m,0} \sum_{i=0}^{N-1} \sum_{j=0}^{i} p_i^{j*} \frac{(\lambda t)^j}{j!} e^{-\lambda t}. \tag{2}$$

Moreover, observe that $\sum_{i=0}^{N-1} \sum_{j=0}^{i} p_i^{j*} \frac{(\lambda t)^j}{j!} e^{-\lambda t}$ corresponds to the probability that $L_1 > t$.

Introducing the following notation:

$$\widehat{h}^L(s, z) \stackrel{def}{=} \sum_{m=0}^{\infty} z^m \int_0^{\infty} e^{-st} \mathbf{P}\{\big(h(t) = m\big) \cap (t \in L_1)\} \, \mathrm{d}t, \tag{3}$$

where $|z| < 1$ and Re(s) > 0, leads to

$$\widehat{h}^L(s, z) = \widehat{h}^L(s) = \sum_{i=0}^{N-1} \sum_{j=0}^{i} p_i^{j*} \int_0^{\infty} e^{-(s+\lambda)t} \frac{(\lambda t)^j}{j!} \, \mathrm{d}t = \sum_{i=0}^{N-1} \sum_{j=0}^{i} p_i^{j*} \frac{\lambda^j}{(\lambda + s)^{j+1}}. \tag{4}$$

Besides, since each buffer loading period L_k, $k \geq 1$, completes simultaneously with the arrival of the batch consisting the Nth message, we obtain the following formula for the LST of the CDF of a single buffer loading period duration:

$$\begin{aligned} \widehat{d}^L(s) \stackrel{def}{=} \int_0^{\infty} e^{-st} \, \mathrm{d}\mathbf{P}\{L_r < t\} &= \sum_{k=1}^{N} \sum_{i=0}^{N-1} p_i^{(k-1)*} \sum_{j=N-i}^{\infty} p_j \int_0^{\infty} e^{-st} \frac{\lambda^k}{(k-1)!} t^{k-1} e^{-\lambda t} \, \mathrm{d}t \\ &= \sum_{k=1}^{N} \left(\frac{\lambda}{\lambda + s} \right)^k \sum_{i=0}^{N-1} p_i^{(k-1)*} \sum_{j=N-i}^{\infty} p_j, \end{aligned} \tag{5}$$

where k denotes here the number of the batch in which the Nth message arrives.

4. DEPARTURE PROCESS DURING BUSY PERIOD

In this section we investigate the departure process during the first busy period B_1 and find the formula for the double mixed transform of the probability distribution of $h(t)$, conditioned by the number of messages accumulated in the buffer queue at the starting moment $t = 0$.

Assume temporarily that the system may initialize the busy period with any possible number of messages $1 \le n \le K$. Introduce the following notation for the conditional probability distribution of departure process $h(t)$:

$$H_n^B(t,m) \stackrel{def}{=} \mathbf{P}\{\big(h(t)=m\big) \cap \big(t \in B_1\big) |\, X(0)=n\}, \tag{6}$$

where $t > 0$, $0 \le m \le K$ and $X(t)$ denotes the number of messages present in the system at the moment t. Since successive departure epochs are Markov moments in the $M/G/1$-type model (see e. g. [6]), then applying the continuous total probability law with respect to the first departure moment y after $t = 0$, we obtain the following system of integral equations:

$$\begin{aligned} H_1^B(t,m) = & I\{m \ge 1\}\Bigg(\sum_{i=1}^{K-2}\int_0^t \sum_{j=0}^{i} p_i^{j*}\frac{(\lambda y)^j}{j!}e^{-\lambda y}H_i^B(t-y,m-1)\,\mathrm{d}F(y) \\ & + \sum_{i=K-1}^{\infty}\sum_{j=0}^{i}\int_0^t p_i^{j*}\frac{(\lambda y)^j}{j!}e^{-\lambda y}H_{K-1}^B(t-y,m-1)\,\mathrm{d}F(y)\Bigg) + \overline{F}(t)\delta_{m,0}, \end{aligned} \tag{7}$$

and, for $2 \le n \le K$,

$$\begin{aligned} H_n^B(t,m) = & I\{m \ge 1\}\Bigg(\sum_{i=0}^{K-n-1}\sum_{j=0}^{i}\int_0^t p_i^{j*}\frac{(\lambda y)^j}{j!}e^{-\lambda y}H_{n+i-1}^B(t-y,m-1)\,\mathrm{d}F(y) \\ & + \sum_{i=K-n}^{\infty}\sum_{j=0}^{i}\int_0^t p_i^{j*}\frac{(\lambda y)^j}{j!}e^{-\lambda y}H_{K-1}^B(t-y,m-1)\,\mathrm{d}F(y)\Bigg) + \overline{F}(t)\delta_{m,0}. \end{aligned} \tag{8}$$

The notation $I\{\mathbb{A}\}$ stands for the indicator of random event $\mathbb{A}$ and $\overline{F}(t) \stackrel{def}{=} 1 - F(t)$.

Comment shortly the formulae (7)–(8). Indeed, if the first message leaves the system at time $y < t$ and the number of messages which arrive until y equals $0 \le i \le K-n-1$, then the system "renews" the operation at time y with exactly $n+i-1$ messages present and must serve $m-1$ messages during time period of length $t-y$ (the first summand on the right side of (7)–(8)). If the system becomes overloaded before y then after departure at time y, then it contains exactly $K-1$ messages (second summand on the right side of (7)–(8)). If the first departure occurs after t (third summand) then the only possibility is $m = 0$.

After introducing the following notations:

$$\widehat{h}_n^B(s,z) \stackrel{def}{=} \sum_{m=0}^{\infty} z^m \int_0^{\infty} e^{-st}H_n^B(t,m)\,\mathrm{d}t \tag{9}$$

and

$$a_i(s,z) \stackrel{def}{=} z\int_0^\infty e^{-(s+\lambda)t}\sum_{j=0}^{i} p_i^{j*}\frac{(\lambda t)^j}{j!}\,\mathrm{d}F(t), \tag{10}$$

where $\mathrm{Re}(s) > 0$, $|z| < 1$, we can transform the system (7)–(8) as follows:

$$\widehat{h}_1^B(s,z) = \sum_{i=1}^{K-2} a_i(s,z)\widehat{h}_i^B(s,z) + \widehat{h}_{K-1}^B(s,z)\sum_{i=K-1}^{\infty} a_i(s,z) + \frac{1-f(s)}{s}, \tag{11}$$

$$\widehat{h}_n^B(s,z) = \sum_{i=0}^{K-n-1} a_i(s,z)\widehat{h}_{n+i-1}^B(s,z) + \widehat{h}_{K-1}^B(s,z)\sum_{i=K-n}^{\infty} a_i(s,z) + \frac{1-f(s)}{s}, \tag{12}$$

where $2 \le n \le K$. Let us note that $a_i(s,z)$ is the Laplace Transform (LT for short) of the probability that i messages enter the system during a service time, multiplied by z.

To obtain the solution of the system (11)–(12) in a compact form we must firstly rewrite it in another way.

Denoting

$$g_n(s,z) = \widehat{h}_{K-n}^B(s,z), \quad 0 \le n \le K-1, \tag{13}$$

we transform the equation (12) as follows:

$$\sum_{i=-1}^{n} a_{i+1}(s,z)g_{n-i}(s,z) - g_n(s,z) = \varphi_n(s,z), \tag{14}$$

where $0 \le n \le K-2$, and

$$\varphi_n(s,z) \stackrel{def}{=} a_{n+1}(s,z)g_0(s,z) - g_1(s,z)\sum_{i=n+1}^{\infty} a_i(s,z) - \frac{1-f(s)}{s}. \tag{15}$$

Similarly, from (11) we get

$$g_{K-1}(s,z) = \sum_{i=1}^{K-2} a_i(s,z)g_{K-i}(s,z) + g_1(s,z)\sum_{i=K-1}^{\infty} a_i(s,z) + \frac{1-f(s)}{s}. \tag{16}$$

In [20] the following linear system with infinite number of equations was considered:

$$\sum_{i=-1}^{n} a_{i+1}g_{n-i} - g_n = \varphi_n, \quad n \ge 0, \tag{17}$$

where $(g_n)_{n=0}^\infty$ is the sequence of unknowns, and $(a_n)_{n=0}^\infty$ and $(\varphi_n)_{n=0}^\infty$ are, respectively, the sequences of coefficients and free terms, where $a_0 \ne 0$. As it was proved in [20], each solution of (17) can be written in the following form:

$$g_n = CR_{n+1} + \sum_{i=0}^{n} R_{n-i}\varphi_i, \quad n \ge 0, \tag{18}$$

where C is a constant and the sequence $(R_n)_{n=0}^{\infty}$, called potential, is defined recursively by means of coefficients of the system, namely

$$R_0 = 0, \;\; R_1 = a_0^{-1}, \;\; R_{n+1} = R_1\big(R_n - \sum_{i=0}^{n} a_{i+1}R_{n-i}\big), \tag{19}$$

where $n \geq 1$.

Comparing (14) to (17), one can observe that the representation (18) can be used for $g_n(s,z)$, $n \geq 0$, where the sequences $(a_n)_{n=0}^{\infty}$, $(R_n)_{n=0}^{\infty}$, $(\varphi_n)_{n=0}^{\infty}$ and C will be now, in general, dependent on s and z. Since the number of equations in (14) is finite, the relationship (16) may be utilized as a boundary condition allowing for finding the formulae for $C(s,z)$, and $g_0(s,z), g_1(s,z)$ occurring in the definition of $\varphi_n(s,z)$ (see (15)).

Indeed, let us start with substituting $n = 0$ into (18), where the role of g_0 plays $g_0(s,z)$. Referring to (19), we obtain

$$C(s,z) = a_0(s,z)g_0(s,z). \tag{20}$$

Similarly, substituting $n = 0$ into (14) leads to

$$g_1(s,z) = a_0^{-1}(s,z)\big[\varphi_0(s,z) + g_0(s,z)\big(1 - a_1(s,z)\big)\big]. \tag{21}$$

Hence we get $g_1(s,z)$ in a function of $g_0(s,z)$, namely

$$g_1(s,z) = \big[zf(s)\big]^{-1}\Big(g_0(s,z) - \frac{1-f(s)}{s}\Big). \tag{22}$$

Introducing now (20)–(21) into (18), we can write $g_n(s,z)$ for $n \geq 0$ in a function of $g_0(s,z)$ in the following way:

$$g_n(s,z) = \gamma_n(s,z)g_0(s,z) + \theta_n(s,z), \tag{23}$$

where $n \geq 0$ and the functional sequences $\big(\gamma_n(s,z)\big)_{n=0}^{\infty}$ and $\big(\theta_n(s,z)\big)_{n=0}^{\infty}$ are defined as follows:

$$\gamma_n(s,z) \stackrel{def}{=} a_0(s,z)R_{n+1}(s,z) + \sum_{i=0}^{n} R_{n-i}(s,z)\left[a_{i+1}(s,z) - \big(zf(s)\big)^{-1}\sum_{j=i+1}^{\infty} a_j(s,z)\right] \tag{24}$$

and

$$\begin{aligned}\theta_n(s,z) &\stackrel{def}{=} \sum_{i=0}^{n} R_{n-i}(s,z)\left[\frac{1-f(s)}{zsf(s)}\sum_{j=i+1}^{\infty} a_j(s,z) - \frac{1-f(s)}{s}\right] \\ &= \frac{1-f(s)}{s}\sum_{i=0}^{n} R_{n-i}(s,z)\left[\big(zf(s)\big)^{-1}\sum_{j=i+1}^{\infty} a_j(s,z) - 1\right],\end{aligned} \tag{25}$$

where

$$f(s) \stackrel{def}{=} \int_0^{\infty} e^{-st}\,\mathrm{d}F(t), \quad \mathrm{Re(s)} > 0. \tag{26}$$

Introducing (23) in (16) allows for eliminating $g_0(s,z)$ in the form

$$g_0(s,z) = T_1(s,z)T_2^{-1}(s,z), \tag{27}$$

where

$$T_1(s,z) = \sum_{i=1}^{K-2} a_i(s,z)\theta_{K-i}(s,z) - \theta_{K-1}(s,z) - \frac{1-f(s)}{zsf(s)}\Big(\sum_{i=K-1}^{\infty} a_i(s,z) - zf(s)\Big) \tag{28}$$

and

$$T_2(s,z) = \gamma_{K-1}(s,z) - \sum_{i=1}^{K-2} a_i(s,z)\gamma_{K-i}(s,z) - \frac{1}{zf(s)}\sum_{i=K-1}^{\infty} a_i(s,z). \tag{29}$$

Collecting (13), (23)–(25) and (27)–(29), we can formulate the following theorem that states the representation for the mixed double transform (PGF of LT) $\widehat{h}_n^B(s,z)$ of the probability distribution of the number of messages successfully processed up to the fixed time t on the first busy period of the $M^X/G/1/K$-type queue:

Theorem 4.1. For Re(s) > 0, $|z| < 1$ and $0 \le n \le K$ the following representation is true:

$$\widehat{h}_n^B(s,z) = \gamma_{K-n}(s,z)T_1(s,z)T_2^{-1}(s,z) + \theta_{K-n}(s,z), \tag{30}$$

where the formulae for $\gamma_j(s,z)$, $\theta_j(s,z)$, $T_1(s,z)$ and $T_2(s,z)$ are given explicitly in (24), (25), (28) and (29), respectively.

5. BUSY PERIOD PERFORMANCE MEASURES

In this section we give the representations for the performance measures of a single busy period in the considered $M^X/G/1/K$-type system with threshold server's waking up, namely the formulae for the LT of the busy period duration and for the PGF of the number of packets served during the busy period.

Let us denote now by $\widehat{d}_n^B(s) \overset{def}{=} \mathbf{E}[e^{-sB_k} \,|\, X(0) = n]$, $k \ge 1$, the LST of CDF of busy period duration in the system that starts working with $1 \le n \le K$ messages present in the buffer queue. It is easy to note that for $\widehat{d}_1^B(s), \ldots, \widehat{d}_K^B(s)$ the following system of equations can be written:

$$\widehat{d}_1^B(s) = \sum_{i=1}^{K-2} a_i(s,1)\widehat{d}_i^B(s) + \widehat{d}_{K-1}^B(s)\sum_{i=K-1}^{\infty} a_i(s,1) + f(\lambda+s), \tag{31}$$

$$\widehat{d}_n^B(s) = \sum_{i=0}^{K-n-1} a_i(s,1)\widehat{d}_{n+i-1}^B(s) + \widehat{d}_{K-1}^B(s)\sum_{i=K-n}^{\infty} a_i(s,1), \quad 2 \le n \le K. \tag{32}$$

Indeed, the first terms on the right sides of (31) and (32) relate to the case in that there is no buffer saturation before the first service completion after $t = 0$; the second terms present the situation in which the buffer becomes full before the first departure.

Note that the last term on the right side of (31) corresponds to the case where there is no arrival before the first service completion epoch t; in such a case the busy period duration equals exactly t, so we have $\int_0^\infty e^{-st}e^{-\lambda t}\,\mathrm{d}F(t) = f(\lambda+s)$.

Observe now that to obtain the solution of the system (31)–(32), we may use the step-by-step procedure described in the Section 4. In fact, since in the original system with threshold-type policy each busy period begins with exactly N packets present, the only representation we need is the formula for $\widehat{d}_N^B(s)$. The exact solution of (31)–(32) can be found in [19]:

$$\widehat{d}^B(s) \stackrel{def}{=} \widehat{d}_N^B(s) = \gamma_{K-N}(s,1)\widetilde{\Pi}_1(s)\Pi_2^{-1}(s) + \widetilde{\eta}_{K-N}(s), \quad n \geq 0, \tag{33}$$

where

$$\widetilde{\eta}_n(s) \stackrel{def}{=} \sum_{i=0}^{n} R_{n-i}(s,1)\Big[\frac{1}{f(s)}\sum_{j=i+1}^{\infty} a_j(s,1)\widetilde{\theta}_K(s) - \widetilde{\theta}_{K-i}(s)\Big],$$

$$\Pi_2(s) \stackrel{def}{=} \gamma_{K-1}(s,1) - \sum_{i=1}^{K-2} a_i(s,1)\gamma_{K-i}(s,1) - \frac{1}{f(s)}\sum_{i=K-1}^{\infty} a_i(s,1),$$

$$\widetilde{\theta}_n(s) \stackrel{def}{=} \begin{cases} f(\lambda+s), & n=1, \\ 0, & n \geq 2, \end{cases}$$

and

$$\widetilde{\Pi}_1(s) \stackrel{def}{=} \sum_{i=1}^{K-2} a_i(s,1)\widetilde{\eta}_{K-i}(s) - \frac{\widetilde{\theta}_K(s)}{f(s)}\sum_{i=K-1}^{\infty} a_i(s,1) + \widetilde{\theta}_1(s) - \widetilde{\eta}_{K-1}(s).$$

Next, let $\epsilon(B_1)$ be the (random) number of messages completely processed during the first busy period of the ordinary $M^X/G/1/K$-type queueing system (without N-policy). If we denote

$$q_n(z) = \mathbf{E}[z^{\epsilon(B_1)} \,|\, X(0) = n], \quad 1 \leq n \leq K,\ |z| < 1,$$

then we can write the following system of equations (compare (7)–(8)), applying the total probability law with respect to the first departure epoch $y > 0$:

$$\begin{aligned} q_1(z) = {} & z\sum_{i=1}^{K-2} q_i(z)\int_0^\infty \sum_{j=0}^{i} p_i^{j*}\frac{(\lambda y)^j}{j!}e^{-\lambda y}\,\mathrm{d}F(y) \\ & + zq_{K-1}(z)\sum_{i=K-1}^{\infty}\int_0^\infty \sum_{j=0}^{i} p_i^{j*}\frac{(\lambda y)^j}{j!}e^{-\lambda y}\,\mathrm{d}F(y) + zf(\lambda) \end{aligned} \tag{34}$$

and

$$\begin{aligned} q_n(z) = {} & z\sum_{i=0}^{K-n-1} q_{n+i-1}(z)\int_0^\infty \sum_{j=0}^{i} p_i^{j*}\frac{(\lambda y)^j}{j!}e^{-\lambda y}\,\mathrm{d}F(y) \\ & + zq_{K-1}(z)\sum_{i=K-n}^{\infty}\int_0^\infty \sum_{j=0}^{i} p_i^{j*}\frac{(\lambda y)^j}{j!}e^{-\lambda y}\,\mathrm{d}F(y), \end{aligned} \tag{35}$$

where $2 \le n \le K$. Indeed, the first terms on the right sides of (34)-(35) relate to the case in which the buffer does not become full before the first service completion epoch after the starting time, while the second ones - to the case of buffer saturation occurring before the first departure time. The last term on the right side of (35) corresponds to the case in which there is no arrival before the first service completion epoch (with probability $\int_0^\infty e^{-\lambda t}\, \mathrm{d}F(t) = f(\lambda)$); hence during a busy period exactly one packet is processed (z).

Utilizing (10), the system (34)–(35) can be rewritten in the form (compare (11)–(12))

$$q_1(z) = \sum_{i=1}^{K-2} a_i(0,z) q_i(z) + q_{K-1}(z) \sum_{i=K-1}^{\infty} a_i(0,z) + z f(\lambda) \tag{36}$$

and

$$q_n(z) = \sum_{i=0}^{K-n-1} a_i(0,z) q_i(z) + q_{K-1}(z) \sum_{i=K-n}^{\infty} a_i(0,z), \;\; 2 \le n \le K. \tag{37}$$

Substituting

$$r_n(z) = q_{K-n}(z), \quad 0 \le n \le K-1, \tag{38}$$

we get from (36)–(37) (compare (14) and (16))

$$\sum_{i=-1}^{n} a_{i+1}(0,z) r_{n-i}(z) - r_n(z) = \psi_n(z), \quad 0 \le n \le K-2,$$

where

$$\psi_n(z) = a_{n+1}(0,z) r_0(z) - r_1(z) \sum_{i=n+1}^{\infty} a_i(0,z),$$

and

$$r_{K-1}(z) = \sum_{i=1}^{K-2} a_i(0,z) r_{K-i}(z) + r_1(z) \sum_{i=K-1}^{\infty} a_i(0,z) + z f(\lambda).$$

Using the same procedure as in the case of the system (14) and (16), we obtain

$$r_1(z) = z^{-1} r_0(z)$$

and a more general identity

$$r_n(z) = \gamma_n(0,z) r_0(z), \quad n \ge 0.$$

Finally, we eliminate $r_0(z)$ in the form

$$r_0(z) = \frac{z f(\lambda)}{\gamma_{K-1}(0,z) - \sum_{i=1}^{K-2} a_i(0,z)\gamma_{K-i}(0,z) - z^{-1}\sum_{i=K-1}^{\infty} a_i(0,z)}.$$

Thus, taking into consideration (38), we can formulate the following

Lemma 5.1. For $|z|<1$ and $1 \le n \le K$ the following formula holds true:

$$q_n(z) = r_{K-n}(z) = \frac{zf(\lambda)\gamma_{K-n}(0,z)}{\gamma_{K-1}(0,z) - \sum_{i=1}^{K-2} a_i(0,z)\gamma_{K-i}(0,z) - z^{-1}\sum_{i=K-1}^{\infty} a_i(0,z)}, \tag{39}$$

where the formulae for $a_i(0,z)$ and $\gamma_i(0,z)$ can be found in (10) and (24), respectively.

Remark 5.2. Let us observe that, if we denote by $q(z)$ the PGF of the number of messages processed during a busy period in the original system with threshold server's waking up (N-policy), then we have

$$q(z) = \sum_{i+j=N}^{K-1} \widetilde{p}_{i+j} q_{i+j}(z) + q_K(z) \sum_{i+j=K}^{\infty} \widetilde{p}_{i+j}, \tag{40}$$

where $\widetilde{p}_{i+j}$ denotes the probability that a busy period starts with exactly $i+j$ packets present, where j denotes the capacity of the last batch for which the number of accumulated packets reaches N, and i is the number of packets being accumulated before this "last" batch. So we have

$$\widetilde{p}_{i+j} = \begin{cases} \sum_{i=0}^{N-1} \sum_{r=0}^{i} p_i^{r*} \sum_{j=N-i}^{K-i-1} p_j & \text{for } N \le i+j \le K-1, \\ \sum_{i=0}^{N-1} \sum_{r=0}^{i} p_i^{r*} \sum_{j=K-i}^{\infty} p_j & \text{for } i+j=K. \end{cases} \tag{41}$$

6. DEPARTURE PROCESS IN GENERAL CASE

In this section, by using the renewal-theory approach, as the main result, we derive a compact-form representation for the CDF of LT of departure process in general case, i. e. at fixed time epoch t. So, we have

Theorem 6.1. The mixed double transform $\widehat{h}(s,z)$ of departure process in the $M^X/G/1/K$-type queueing system with threshold server's waking up mechanism (N-policy) can be written as follows:

$$\widehat{h}(s,z) = \sum_{m=0}^{\infty} z^m \int_0^{\infty} e^{-st}\mathbf{P}\{h(t)=m\}\,\mathrm{d}t = \frac{\widehat{h}^L(s) + \widehat{d}^L(s)\widehat{h}_N^B(s,z)}{1-\widehat{d}^L(s)\widehat{d}^B(s)q(z)}, \tag{42}$$

where Re(s) >0, $|z|<1$, and the formulae for $\widehat{h}^L(s)$, $\widehat{d}^L(s)$, $\widehat{h}_N^B(s,z)$, $\widehat{d}^B(s)$ and $q(z)$ were found in (4), (5), (30), (33) and (39), respectively.

P r o o f. Let $F^L(\cdot)$ and $F^B(\cdot)$ be, respectively, CDFs of arbitrary buffer loading period L_i and busy period B_i ($i \ge 1$) durations in the $M^X/G/1/K$-type queue with N-policy. Besides, let $q_i = \mathbf{P}\{\epsilon(B_1) = i \,|\, X(0)=N\}$, $i \ge 1$, so determines the probability distribution of the number of messages successfully processed during the first busy period of the system. Firstly, observe that

$$\mathbf{P}\{h(t)=m\} = \sum_{i=1}^{\infty}\Big(\mathbf{P}\{\big(h(t)=m\big)\cap\big(t\in L_i\big)\} + \mathbf{P}\{\big(h(t)=m\big)\cap\big(t\in B_i\big)\}\Big) \tag{43}$$

and, since $(L_i)_{i=1}^{\infty}$ and $(B_i)_{i=1}^{\infty}$ are, separately, sequences of independent and identically distributed random variables, we get

$$\begin{aligned}&\mathbf{P}\{(h(t)=m)\cap(t\in L_i)\}\\&=I\{m\ge i-1\}q_m^{(i-1)*}\int_0^t \mathbf{P}\{(h(t-y)=0)\cap(t-y\in L_1)\}\\&\quad\times \mathrm{d}\big(F^L * F^B\big)^{(i-1)*}(y)\end{aligned}\tag{44}$$

and

$$\begin{aligned}&\mathbf{P}\{(h(t)=m)\cap(t\in B_i)\}\\&=I\{m\ge i-1\}\sum_{k=i-1}^{m} q_k^{(i-1)*}\int_0^t \mathbf{P}\{(h(t-y)=m-k)\cap(t-y\in B_1)\}\\&\quad\times \mathrm{d}\big[\big(F^L\big)^{i*} * \big(F^B\big)^{(i-1)*}\big](y),\end{aligned}\tag{45}$$

where the notation $\big(F^L\big)^{j*}$ (or, similarly, $\big(F^B\big)^{j*}$) stands for the $j-$fold Stieltjes convolution of the appropriate CDF with itself. The symbol q_k^{j*} denotes the kth term of the $j-$fold convolution of the sequence $(q_i)_{i=1}^{\infty}$ with itself.

The representations (44)–(45) lead to

$$\sum_{i=1}^{\infty}\sum_{m=0}^{\infty} z^m \int_0^{\infty} e^{-st}\mathbf{P}\{(h(t)=m)\cap(t\in L_i)\}\,\mathrm{d}t=\frac{\widehat{h}^L(s)}{1-\widehat{\mathrm{d}}^L(s)\widehat{\mathrm{d}}^B(s)q(z)}\tag{46}$$

and, similarly,

$$\sum_{i=1}^{\infty}\sum_{m=0}^{\infty} z^m \int_0^{\infty} e^{-st}\mathbf{P}\{(h(t)=m)\cap(t\in B_i)\}\,\mathrm{d}t=\frac{\widehat{\mathrm{d}}^L(s)\widehat{h}_N^B(s,z)}{1-\widehat{\mathrm{d}}^L(s)\widehat{\mathrm{d}}^B(s)q(z)}.\tag{47}$$

Now, collecting (46) and (47) and referring to (43), we obtain the conclusion (42). □

Remark 6.2. The probability distribution of the departure process $h(t)$ can be obtained by the usage of one of algorithms of numerical inversion of the mixed double transform. In the next section we use the algorithm proposed in [1], where the LT is inverted applying the Bromwich integral and Euler's summation formula, and for the PGF the Cauchy integral summation formula is used.

Remark 6.3. According to the property of the probability generating functions, the Laplace transform of the mean number of packets served up to the fixed time t can be expressed as

$$\int_0^{\infty} e^{-st}\mathbf{E}[h(t)]\,\mathrm{d}t=\frac{\partial}{\partial z}\left(\sum_{m=0}^{\infty} z^m\int_0^{\infty} e^{-st}\mathbf{P}\{h(t)=m\}\,\mathrm{d}t\right)\bigg|_{z=1}.\tag{48}$$

Remark 6.4. Having the intensity of $\lambda(t)$ of sending messages in a function of time, we can estimate the transient loss ratio $LR(t)$ function in following way

$$LR(t)\approx 1-\frac{\mathbf{E}[h(t)]}{\mathbf{E}[\lambda(t)]}.\tag{49}$$

7. NUMERICAL STUDY

In this section, we show the utility of analytical results in numerical examples. Let us consider the system of size $K = 8$. Assume that a stream of messages of average sizes 500 B arrive at the WSN node with the threshold-type mechanism according to a compound Poisson process with intensity 450 kb/s. Let us take into consideration two different simple batch distributions:

- $P_1 : p_1 = 0.8, p_2 = 0.2, p_k = 0, k > 2,$
- $P_2 : p_1 = 0.2, p_2 = 0.8, p_k = 0, k > 2,$

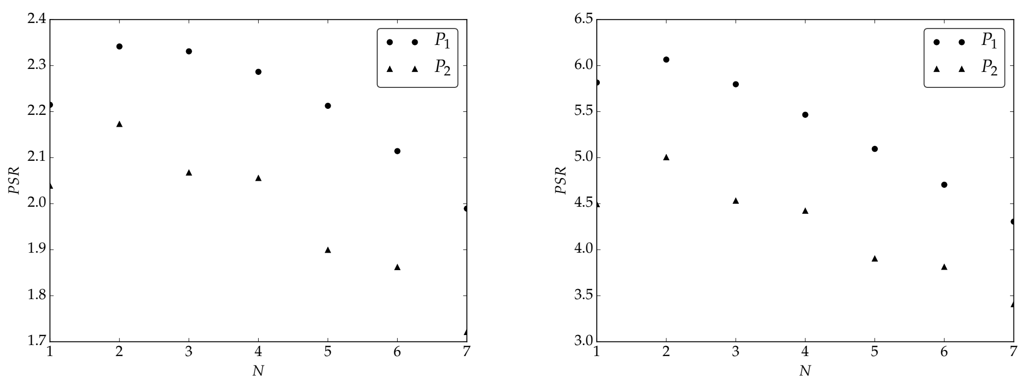
which gives following arrival rate parameters $\lambda_1 = 375$ and $\lambda_2 = 250$, respectively. Let messages be transmitted with rates 2.4 and 1.8 Mb/s, respectively. Assuming the exponential service time distribution, these rates correspond to the mean processing times 1.6 and 2.2 ms (milliseconds), respectively. Under the assumptions about arrival and serving rates the occupation rate ρ of the system equals to 0.75 or 1.00.

Notice that the mean busy and loading period duration can be expressed as

$$\mathbf{E}[e^{-sL}] = -\frac{\partial \widehat{d}^L(s)}{\partial s}\bigg|_{s=0}, \quad \mathbf{E}[e^{-sB}] = -\frac{\partial \widehat{d}^B(s)}{\partial s}\bigg|_{s=0}, \tag{50}$$

respectively. Moreover, let us introduce the power-saving ratio as

$$PSR \stackrel{def}{=} \frac{\mathbf{E}[e^{-sB}]}{\mathbf{E}[e^{-sL}]}. \tag{51}$$

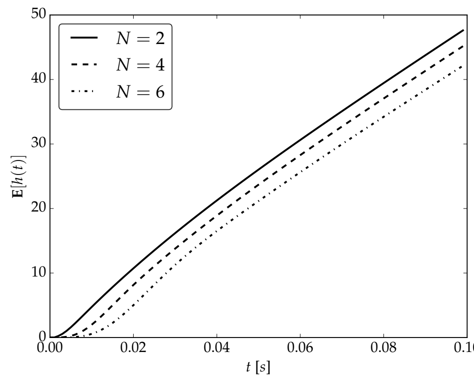


(a) $\rho = 0.75$. (b) $\rho = 1$.

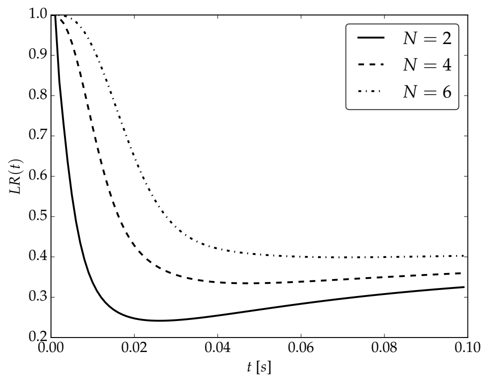
Fig. 1. PSR in dependence on threshold N for P_1, P_2 and $\rho = 0.75, \rho = 1$.

Since, typically, the power consumption in the buffer loading mode is lower than during busy period, then the minimization of the PSR is desired. Figure 1 shows the values of PSR in dependence on the threshold level N for occupation rates $0.75, 1.00$ and batch distributions P_1, P_2. The obtained results show that the PSR decreases essentially with increasing of the threshold N.

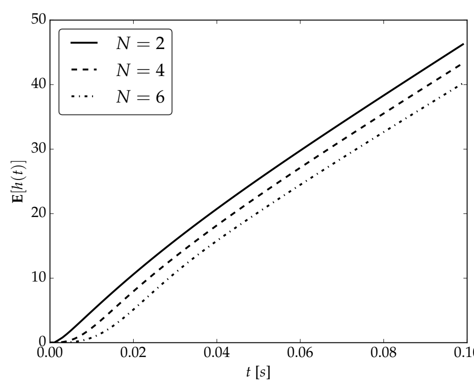
Next, we are interested in the transient distribution of the number of messages completely processed until the fixed time t (departure process) and in the mean number of departures up to the same time. Transient mean number $\mathbf{E}[h(t)]$ of completely processed messages (see 6.3) and transient loss ratio $LR(t)$ (see 6.4) for rates $\rho = 0.75, 1.00$ and batch distributions P_1, P_2 are presented in Fig. 2 and Fig. 3. Observe that the results show that increasing of the threshold level N leads to decreasing of the number of processed messages up to time t. However, the differences in the number of departured messages are not too high.



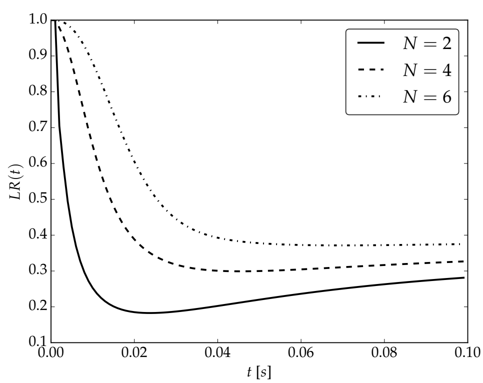
(a) $\mathbf{E}[h(t)]$ for P_1 and $\rho = 0.75$.



(b) $LR(t)$ for P_1 and $\rho = 0.75$.

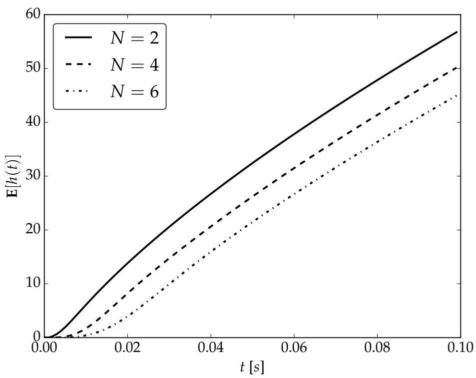


(c) $\mathbf{E}[h(t)]$ for P_2 and $\rho = 0.75$.

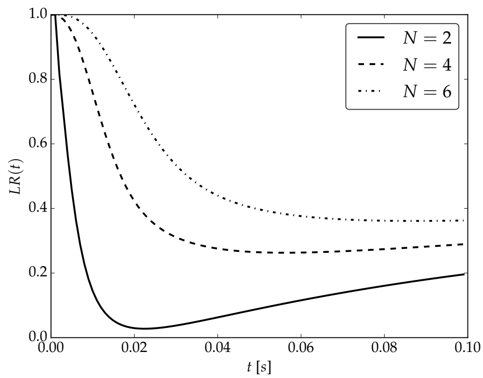


(d) $LR(t)$ for P_2 and $\rho = 0.75$.

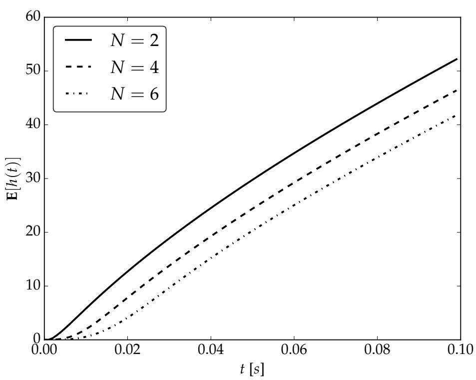
Fig. 2. $\mathbf{E}[h(t)]$ and $LR(t)$ for P_1, P_2 and $\rho = 0.75$.



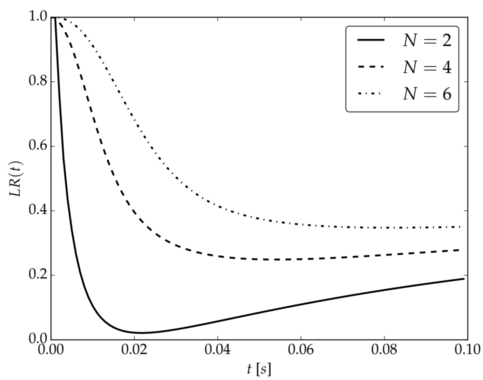
(a) $\mathbf{E}[h(t)]$ for P_1 and $\rho = 1$.



(b) $LR(t)$ for P_1 and $\rho = 1$.



(c) $\mathbf{E}[h(t)]$ for P_2 and $\rho = 1$.



(d) $LR(t)$ for P_2 and $\rho = 1$.

Fig. 3. $\mathbf{E}[h(t)]$ and $LR(t)$ for P_1, P_2 and $\rho = 1$.

8. CONCLUSION

In the paper the explicit representation for the mixed double transform of departure process in a batch-arrival $M^X/G/1/K$-type queue with N-policy is obtained. The analytic approach based on the idea of embedded Markov chain, renewal theory and linear algebra was applied. Numerical utility of theoretical results is shown via computational examples.

This kind of a queueing system can be used in modelling the operation of a wireless sensor network with a threshold-type power saving mechanism (waking up of nodes). The obtained representations can be useful in performance evaluation of such a network or a single node under different "input" parameters (like arrival rate, service speed, buffer size or threshold level). In particular, transient mean number of messages completely processed up to the fixed time , loss ratio and power saving ratio can be estimated.

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