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A NEW APPROACH FOR KM-FUZZY PARTIAL METRIC SPACES

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The main purpose of this paper is to give a new approach for partial metric spaces. We first provide the new concept of KM-fuzzy partial metric, as an extension of both the partial metric and KM-fuzzy metric. Then its relationship with the KM-fuzzy quasi-metric is established. In particularly, we construct a KM-fuzzy quasi-metric from a KM-fuzzy partial metric. Finally, after defining the notion of partial pseudo-metric systems, a one-to-one correspondence between partial pseudo-metric systems and KM-fuzzy partial pseudo-metrics is constructed. Furthermore, a fuzzifying topology τ_P on X deduced from KM-fuzzy partial metric is established and some properties of this fuzzifying topology are discussed.

Keywords: partial metric, KM-fuzzy metric, KM-fuzzy partial metric, partial pseudo-metric system, fuzzy neighborhood system

Classification: 54A40, 46S40

1. INTRODUCTION

Metrics were originally introduced by Fréchet [1] at the beginning of set-theoretic topology. Since then, several generalizations of metrics have already studied by topological researchers. In the beginning, Menger [8] gave the concept of statistical space, which was further developed by Schweizer and Sklar [13, 14] and renamed as probabilistic metric space. Later, Kramosil and Michalek [6] extended the notion of probabilistic metric to the fuzzy setting, and introduced the concept of fuzzy metric (usually called KM-fuzzy metric). George and Veeramani [2, 3] introduced the notion of GV-fuzzy metric, which strengthens the the KM-fuzzy metric. Moreover, Shi gave the notion of (L, M)-fuzzy metric as a more generalized kind of fuzzy metric [10, 16, 17, 22].

Another generalization of the usual metric space was due to Matthews [7]. It was introduced the notion of partial metric space, for which the self-distance is not necessarily zero. Partial metric has received much attention for its applications in computer science (refer to [9, 11, 12]). These years, many different works have tried to extend the notion of partial metric space to the fuzzy setting. Yue and Gu [23] gave a concept of fuzzy partial metric using a continuous minimum t-norm. They also investigated the relationship between partial pseudo-metric and fuzzy pseudo-metric. Besides, Yue [19] discussed formal balls in fuzzy partial metric spaces. Recently, by using the residuum operator

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$\to_*$ to associate with a continuous t-norm, Gregori et al. [15] presented the notions of fuzzy partial metric and GV-fuzzy partial metric, as extensions of KM-fuzzy metric and GV-fuzzy metric respectively. They also defined the concept of open ball induced by (GV-)fuzzy partial metric space and studied its properties. Some other works on partial metric can also be seen in [4, 10, 18, 24].

No matter Yue [23] or Gregori [15], the notions of fuzzy partial metric provided by them always lack monotonicity for the third variable. Moreover, partial metric is closely connected with classic metrics. One of the most important results is that

(A1) if (X, p) is a partial metric space, then the assignment $d_p : X \times X \longrightarrow [0, +\infty)$, defined by $d_p(x, y) = p(x, y) - p(x, x)$, is a quasi-metric.

The above result reveals the interlink between the metric and partial metric. However, as an important result, the validity of (A1) in the fuzzy setting has received very little attention in the existing works. Based above statements, a new approach to the concept of KM-fuzzy partial metric in the fuzzy setting is proposed. One advantage of this approach is that we provide a method of constructing a KM-fuzzy quasi-metric from a KM-fuzzy partial metric, which can be regarded as a parallel result of (A1). The structure of this paper is as follows: In Section 2, we give some basic notions and results on t-norms and fuzzy metric. In Section 3, a new concept of KM-fuzzy partial metric by using the continuous t-norm and the residuum operator is provided. In addition, the relationships between KM-fuzzy partial metric spaces and KM-fuzzy metric spaces are studied. In the last section, the notion of partial pseudo-metric system is given, and it is proved that there is a one-to-one correspondence between partial pseudo-metric systems and KM-fuzzy partial pseudo-metrics. In addition, the fuzzifying topology induced by KM-fuzzy partial metric is introduced and some properties of this fuzzifying topology are obtained.

2. PRELIMINARIES

Throughout this paper, X always denotes a nonempty set. In the following, we recall some basic notations and results related to triangular norms, patrial pseudo-metric and KM-fuzzy quasi-metric.

A triangular norm $*$ on $I = [0, 1]$ is a binary operation on I that is commutative, associative, order-preserving and has neutral element 1. A binary operation $\to_*$ is called a $*$-*residuum* if for any $a, b, c \in [0, 1]$, we have

$$a * b \le c \text{ if and only if } a \le b \to_* c.$$

If $\to_*$ exists, one can easily deduce that $a \to_* b = \bigvee\{c \in [0, 1] : a * c \le b\}$.

The following properties on left-continuous t-norms will be used in the sequel.

Proposition 2.1. For a left-continuous t-norm $*$, we have the following results:

(1) $0 \to_* 0 = 1$ and $1 \to_* 0 = 0$;

(2) $a * (a \to b) \le b$;

(3) $a * 0 = 0$;

(4) $a * \bigvee_{i\in I} a_i = \bigvee_{i\in I} a * a_i$;

(5) $\left(\bigvee_{i\in I} a_i\right) \to_* b = \bigwedge_{i\in I} a_i \to_* b$.

In what follows, we use the notation a' for $a \to_* 0$. The following properties on a' will be used frequently.

Remark 2.2. Let $*$ be a continuous t-norm. Then the following statements hold for all $a, b \in [0, 1]$ and $\{a_i : i \in I\} \subseteq [0, 1]$:

(1) $0' = 1$ and $1' = 0$;

(2) $a * a' = 0$;

(3) $a \leq b$ implies $b' \leq a'$;

(4) $\left(\bigvee_{i\in I} a_i\right)' = \bigwedge_{i\in I} a_i'$.

Definition 2.3. (Matthews [7]) A function $p : X \times X \longrightarrow [0, +\infty)$ is called a *patrial pseudo-metric* if it satisfies the following conditions: $\forall x, y, z \in X$,

(PM1) $p(x, x) \leq p(x, y)$;

(PM2) $p(x, y) = p(y, x)$;

(PM3) $p(x, z) \leq p(x, y) + p(y, z) - p(y, y)$.

A partial pseudo-metric is called a *partial metric* if it also satisfies the following condition:

(PM4) $x = y$ if and only if $p(x, y) = p(x, x) = p(y, y)$.

Definition 2.4. (Kramosil and Michalek [6]) A *KM-fuzzy quasi-metric*, is an ordered pair $(M, *)$ such that $*$ is a continuous t-norm and M is a function $M : X \times X \times [0, +\infty) \longrightarrow [0, 1]$, satisfying the following conditions: $\forall x, y \in X$, $\forall s, t > 0$,

(FKM1) $M(x, y, 0) = 0$;

(FKM2) $x = y$ if and only if $M(x, y, r) = M(y, x, r) = 1$ for all $r > 0$;

(FKM3) $M(x, y, s) * M(y, z, t) \leq M(x, z, s + t)$;

(FKM4) The assignment $M_{x,y} : (0, +\infty] \longrightarrow [0, 1]$, defined by $M_{x,y}(t) = M(x, y, t)$, is a left-continuous function.

A KM-fuzzy quasi-metric $(M, *)$ is called a *KM-fuzzy metric* if it also satisfied

(FKM5) $M(x, y, t) = M(y, x, t)$.

The triple $(X, M, *)$ is called a *KM-fuzzy (quasi-)metric space* if $(M, *)$ is a KM-fuzzy (quasi-)metric on X.

For all x, y in KM-fuzzy (quasi-)metric space $(X, M, *)$, let $M_{x,y}(t) = M(x, y, t)$, it is clear $M_{x,y}(t)$ is an increasing function on $[0, +\infty)$.

Definition 2.5. (Yue and Shi [22]) Let $(X, M_1, *_1)$ and $(Y, M_2, *_1)$ be two *KM-fuzzy metric* spaces. The mapping $f : X \to Y$ is called continuous at x if each $t > 0$, there exist $t_x > 0$ such that $M_2(f(x), f(y), t) \geq M_1(x, y, t_x)$. f is called continuous if f is continuous at x for all $x \in X$.

Definition 2.6. (Xu [20]) A fuzzifying neighborhood system is a set $\mathcal{N} = \{\mathcal{N}_x : x \in X\}$ of functions $\mathcal{N}_x : 2^X \longrightarrow [0, 1]$ such that

(FN1) $\mathcal{N}_x(X) = 1, \mathcal{N}_x(\emptyset) = 0$;

(FN2) $\mathcal{N}_x(U) > 0 \Rightarrow x \in U$;

(FN3) $\mathcal{N}_x(U \cap V) = \mathcal{N}_x(U) \wedge \mathcal{N}_x(V)$;

(FN4) $\mathcal{N}_x(U) = \bigvee_{x \in V \subset U} \mathcal{N}_x(V) \wedge \bigwedge_{y \in V} \mathcal{N}_y(U)$.

Definition 2.7. (Ying [21]) Let (X, τ) be a fuzzifying topological space. The value $[C_I(X, \tau)] = \bigwedge_{x \in X} \bigvee_{\mathcal{B}_x \vdash \mathcal{N}_x^\tau} [FC(\mathcal{B}_x)]$ is called the degree to which (X, τ) is first countable, where $\mathcal{B}_x \vdash \mathcal{N}_x^\tau$ means $\mathcal{B}_x$ is a mapping $2^X \to [0, 1]$ satisfying $\mathcal{N}_x^\tau(U) = \bigvee_{V \subseteq U} \mathcal{B}_x(V)$, and $[FC(\mathcal{B}_x)] = 1 - \wedge\{\alpha \in [0, 1] : (\mathcal{B}_x)_\alpha \text{ is countable }\}$, $(\mathcal{B}_x)_\alpha = \{A \in 2^X : \mathcal{B}_x(A) > \alpha\}$.

Definition 2.8. (Ying [21]) Let $(X, \tau), (Y, \tau_Y)$ be fuzzifying topological spaces. The mapping $f : X \to Y$ is called fuzzifying continuous if $\mathcal{N}_{f(x)}(W) \leq \mathcal{N}_x(f^{\leftarrow}(W))$ for all $W \in 2^Y$, $x \in X$. For each sequence $\{x_n\} \subseteq X, x \in X$, the value $[x_n \to x] = \bigwedge_{\{x_n\} \not\sqsubseteq A}(1 - \mathcal{N}_x(A))$ is called the degree to which $\{x_n\}$ converges to x, where the notation $\{x_n\} \not\sqsubseteq A$ means $\{x_n\}$ is not " almost in " A, that is, for every $n \in \mathbb{N}$, there is $m(n) > n$ such that $x_{m(n)} \notin A$.

Definition 2.9. (Ying [21]) Let (X, τ) be a fuzzifying topological space. The value $[T_0(X, \tau)] = \bigwedge_{x \neq y}\Big(\bigvee_{y \notin A} \mathcal{N}_x(A) \bigvee \bigvee_{x \notin B} \mathcal{N}_y(B)\Big)$ is called the degree to which (X, τ) satisfies T_0 separation.

3. RELATIONSHIPS BETWEEN KM-FUZZY PARTIAL METRIC AND KM-FUZZY QUASI-METRIC

In this section, a new concept of KM-fuzzy partial metrics is introduced and relationships between KM-fuzzy partial metric and KM-fuzzy quasi-metric are discussed.

Definition 3.1. A *KM-fuzzy partial pseudo-metric* on X is an ordered pair $(P, *)$ such that $*$ is a continuous t-norm and P is a function $P : X \times X \times [0, +\infty) \longrightarrow [0, 1]$ satisfying the following conditions: $\forall x, y \in X,\ s, t > 0$,

(FPKM1) $P(x, y, 0) = 0$;

(FPKM2) $P(x,y,t) \le P(x,x,t)$;

(FPKM3) $P(x,y,t) = P(y,x,t)$;

(FPKM4) $\forall r_1 \ge s,\ r_2 \ge t$,

$P(x,y,r_1) * P(x,x,r_1-s)' * P(y,z,r_2) * P(y,y,r_2-t)' \le \bigvee_{r\ge s+t} P(x,z,r) * P(x,x,r-s-t)'$;

(FPKM5) $P(x,y,t) = \bigvee_{r<t} P(x,y,r)$.

A *KM-fuzzy partial pseudo-metric* $(P,*)$ is called a *KM-fuzzy partial metric* on X if it satisfies the following condition: $\forall x,y \in X$,

(FPKM6) $x = y$ if and only if $\bigvee_{r\ge t} P(x,y,r) * P(x,x,r-t)' = \bigvee_{r\ge t} P(y,x,r) * P(y,y,r-t)' = 1$ for all $t > 0$.

The ordered triple $(X,P,*)$ is called a *KM-fuzzy partial (pseudo-)metric space*, if $(P,*)$ is a *KM-fuzzy partial (pseudo-)metric* on X.

For a *KM-fuzzy partial (pseudo-)metric space* $(X,P,*)$ and each pair $x,y \in X$, define $P_{x,y} : (0,+\infty) \longrightarrow [0,1]$ as follows:

$$P_{x,y}(t) = P(x,y,t).$$

From (FPKM5), one can easily observe that $P_{x,y}$ is an increasing and left-continuous function.

Remark 3.2. By the definition of *KM-fuzzy partial metric*, one can easily observe that every KM-fuzzy metric is a *KM-fuzzy partial metric* and this new definition is different from [15] and [23].

Proposition 3.3. Let (X,p) be a partial metric space, $*$ is a continuous t-norm and define $P : X \times X \times [0,+\infty) \longrightarrow [0,1]$ as follows: $\forall x,y \in X,\ t \ge 0$,

$$P(x,y,t) = \begin{cases} 1, & p(x,y) < t \\ 0, & p(x,y) \ge t. \end{cases}$$

Then $(X,P,*)$ is a *KM-fuzzy partial metric space.*

P r o o f. We only need to prove (FPKM4) and (FPKM6) since the other conditions hold trivially.

(FPKM4). Let $x,y,z \in X$, and $\lambda = P(x,y,r_1)*P(x,x,r_1-s)'*P(y,z,r_2)*P(y,y,r_2-t)'$. We need to verify

$$\lambda \le \bigvee_{r\ge s+t} P(x,z,r) * P(x,x,r-s-t)'.$$

The above inequality holds trivially whenever $\lambda = 0$. Now we assume $\lambda = 1$, note that $\lambda \in \{0,1\}$. Then $P(x,y,r_1) * P(x,x,r_1-s)' = 1$, which implies that $P(x,y,r_1) = 1$

and $P(x,x,r_1-s)'=1$, hence $P(x,x,r_1-s)=0$. It follows that $p(x,y)<r_1$ and $p(x,x)\geq r_1-s$, implying that $p(x,y)-p(x,x)<s$. Similarly, we can obtain that $p(y,z)-p(y,y)<t$. By (PM3), we have that

$$p(x,z)-p(x,x)\leq p(x,y)-p(x,x)+p(y,z)-p(y,y)<s+t.$$

Let $r_0=p(x,x)+s+t$. Then $p(x,z)<r_0$ and $p(x,x)=r_0-s-t$, which means that $P(x,z,r_0)=1$ and $P(x,x,r_0-s-t)=0$, i.e., $P(x,x,r_0-s-t)'=1$. Thus we have that

$$\bigvee_{r\geq s+t} P(x,z,r)*P(x,x,r-s-t)'\geq P(x,z,r_0)*P(x,x,r_0-s-t)'=1*1=1.$$

Hence, $\lambda\leq\bigvee_{r\geq s+t} P(x,z,r)*P(x,x,r-s-t)'$.

(FPKM6). Suppose for all $t>0$, $\bigvee_{r\geq t} P(x,y,r)*P(x,x,r-t)'=\bigvee_{r\geq t} P(y,x,r)*P(y,y,r-t)'=1$. Then there exists $r_1\geq t$, such that $P(x,y,r_1)*P(x,x,r-t)'=1$, i.e., $P(x,y,r_1)=1$ and $P(x,x,r_1-t)=0$. Thus $p(x,y)<r_1$ and $p(x,x)\geq r_1-t$, which means that $p(x,y)-p(x,x)<t$. From the arbitrariness of $t>0$, it holds that $p(x,y)-p(x,x)\leq\bigwedge_{t>0} t=0$, showing that $p(x,y)=p(x,x)$. Similarly, we can obtain that $p(y,x)=p(y,y)$. By (PM2) and (PM4), it follows that $x=y$. □

The result of Proposition 3.3 illustrates that every partial (pseudo-)metric space (X,p) can be regarded as a KM-fuzzy partial (pseudo-)metric space $(X,P,*)$ with the property that $P(x,y,t)\in\{0,1\}$ for all $x,y\in X$ and $t>0$. Therefore, the notion of *KM-fuzzy partial (pseudo-)metric* is a natural approach to the fuzzy context of the concept of partial metric space.

Proposition 3.4. Let (X,p) be a partial metric space and define $P:X\times X\times[0,+\infty)\longrightarrow[0,1]$ as follows: $\forall x,y\in X$,

$$P(x,y,r)=\frac{r}{r+p(x,y)}.$$

Then $(X,P,\wedge)$ is a *KM-fuzzy partial pseudo-metric space.*

P r o o f. We only prove (FPKM4) since the other conditions hold trivially.

(FPKM4). Suppose $x,y\in X$ and $r_1,r_2,s,t\in[0,+\infty)$ such that $r_1\geq s, r_2\geq t$. Let $\lambda=P(x,y,r_1)\wedge P(x,x,r_1-s)'\wedge P(y,z,r_2)\wedge P(y,y,r_2-t)'$. We need to prove

$$\lambda\leq\bigvee_{r\geq s+t} P(x,z,r)\wedge P(x,x,r-s-t)'.$$

First, note that

$$a'=a\rightarrow_\wedge 0=\begin{cases}0, & a>0\\ 1, & a=0.\end{cases}$$

Thus if $P(x,x,r_1-s)>0$ or $P(y,y,r_2-t)>0$, then we have that

$$\lambda=0\leq\bigvee_{r\geq s+t} P(x,z,r)\wedge P(x,x,r-s-t)'.$$

Now assume $P(x,x,r_1-s)=P(y,y,r_2-t)=0$. This implies that $r_1=s$ and $r_2=t$. Then it follows that

$$\lambda = P(x,y,r_1)\wedge P(x,x,r_1-s)'\wedge P(y,z,r_2)\wedge P(y,y,r_2-t)' = P(x,y,s)\wedge P(y,z,t).$$

We claim that $\lambda \le P(x,z,s+t)$. In fact, let $k=\dfrac{p(x,y)}{s}\vee\dfrac{p(y,z)}{t}$. Then we have

$$\begin{aligned} k &= \frac{p(x,y)}{s}\vee\frac{p(y,z)}{t} = \frac{s}{s+t}\cdot k+\frac{t}{s+t}\cdot k \\ &\ge \frac{s}{s+t}\cdot\frac{p(x,y)}{s}+\frac{t}{s+t}\cdot\frac{p(y,z)}{t} = \frac{p(x,y)+p(y,z)}{s+t}. \end{aligned}$$

Thus we have that

$$\begin{aligned} P(x,y,s)\wedge P(y,z,t) &= \frac{s}{s+p(x,y)}\wedge\frac{t}{t+p(y,z)} = \frac{1}{1+\frac{p(x,y)}{s}}\wedge\frac{1}{1+\frac{p(y,z)}{t}} \\ &= \frac{1}{1+\frac{p(x,y)}{s}\vee\frac{p(y,z)}{t}} \le \frac{1}{1+\frac{p(x,y)+p(y,z)}{s+t}} = \frac{s+t}{s+t+p(x,y)+p(y,z)} \\ &\le \frac{s+t}{s+t+p(x,z)+p(y,y)} \le \frac{s+t}{s+t+p(x,z)} = P(x,z,s+t). \end{aligned}$$

It follows that $\lambda = P(x,y,s)\wedge P(y,z,t)\le P(x,z,s+t)=P(x,z,s+t)\wedge P(x,x,0)'\le \bigvee_{r\ge s+t} P(x,z,r)\wedge P(x,x,r-s-t)'$, which gives the proof of (FPKM4). □

Following the same idea of the proceeding proof, one can prove the following proposition.

Proposition 3.5. Let (X,d) be a partial metric space and define $P: X\times X\times[0,+\infty)\longrightarrow[0,1]$ as follows: $\forall x,y\in X$,

$$P(x,y,r)=e^{-\frac{d(x,y)}{r}}.$$

Then $(X,P,\wedge)$ is a *KM-fuzzy partial pseudo-metric space.*

By Proposition 3.3, Proposition 3.4 and Proposition 3.5, the following example is obviously.

Example 3.6. Let $R^+=[0,+\infty)$, $d(a,b)=\max\{a,b\}$ for all $x,y\in R^+$. Put $P_1(x,y,r)=\frac{r}{r+d(x,y)}$, $P_2(x,y,r)=e^{-\frac{d(x,y)}{r}}$ and $P_3(x,y,r)=\begin{cases}1, & d(x,y)<r\\ 0, & d(x,y)\ge r.\end{cases}$,

Then $(R^+,P_1,\wedge),(R^+,P_2,\wedge),(R^+,P_3,*)$ are *KM-fuzzy partial pseudo-metric spaces.*

Next, we study the relationships between *KM-fuzzy partial metric spaces* and KM-fuzzy metric spaces.

Theorem 3.7. Let $(X, P, *)$ be a *KM-fuzzy partial metric space*. Then the following statements are equivalent:

(1) $(X, P, *)$ is a KM-fuzzy metric space;

(2) $\forall x, y \in X$ and $\forall t \in [0, +\infty)$, $P(x, x, t) = 1$.

Proof. That (1) implies (2) is trivial by the definition of KM-fuzzy metric space. Now we prove that (2) implies (1).

We only need to check (FKM2) and (FKM3) since the other conditions hold trivially.

(FKM2). Suppose $P(x, y, t) = P(y, x, t) = 1$ for all $t > 0$. From the above proof, one can observe that

$$\bigvee_{r \geq t} P(x, y, r) * P(x, x, r - t)' = P(x, y, t) = 1,$$

and

$$\bigvee_{r \geq t} P(y, x, r) * P(y, y, r - t)' = P(y, x, t) = 1.$$

By (FPKM6), we deduce that $x = y$.

(FKM3). For any $x, y, z \in X$ and $s, t > 0$, since $P(x, x, 0) = P(y, y, 0) = 1$ by (FPKM1), we have that $P(x, x, 0)' = P(y, y, 0)' = 0' = 1$. By (FPKM4) and our assumption (2), it holds that

$$\begin{aligned} P(x, y, s) * P(y, z, t) &= P(x, y, s) * P(x, x, 0)' * P(y, z, t) * P(y, y, 0)' \\ &\leq \textstyle\bigvee_{r \geq s+t} P(x, z, r) * P(x, x, r - s - t)' \\ &= (P(x, z, s + t) * P(x, x, 0)') \bigvee \left(\textstyle\bigvee_{r > s+t} P(x, z, r) * P(x, x, r - s - t)'\right) \\ &= (P(x, z, s + t) * 0') \bigvee (\textstyle\bigvee_{r > s+t} P(x, z, r) * 1') \\ &= (P(x, z, s + t) * 1) \bigvee 0 = P(x, z, s + t). \end{aligned}$$

Hence (FKM3) holds. □

The following theorem generalizes result (A1) pointed out at the start of this section.

Theorem 3.8. Let $(X, P, *)$ be a *KM-fuzzy partial metric space*, and define $M_P : X \times X \times [0, +\infty) \longrightarrow [0, 1]$ as follows: $\forall x, y \in X$, $\forall t \in [0, +\infty)$,

$$M_P(x, y, t) = \bigvee_{r \geq t} P(x, y, r) * P(x, x, r - t)'.$$

Then $(X, M_P, *)$ is a KM-fuzzy quasi-metric space.

Proof. At first we give a assertion.

(a1) $M_P(x,y,t) \geq P(x,y,t)$.

From condition (FPKM1), it holds that

$M_P(x,y,t) = \bigvee_{r\geq t} P(x,y,r) * P(x,x,r-t)' \geq P(x,y,t) * P(x,x,0)' = P(x,y,t) * 0' = P(x,y,t) * 1 = P(x,y,t)$.

It gives the proof of (a1).

Next, we verify conditions (FKM1) – (FKM4) one by one.

(FKM1). By (FPKM2), we have that

$$M_P(x,y,0) = \bigvee_{r\geq 0} P(x,y,r) * P(x,x,r)' \leq \bigvee_{r\geq 0} P(x,x,r) * P(x,x,r)' = 0.$$

(FKM2). Suppose $M_P(x,y,t) = M_P(y,x,t) = 1$ for all $t > 0$. From the above definition, one can observe that

$$\bigvee_{r\geq t} P(x,y,r) * P(x,x,r-t)' = M_P(x,y,t) = 1,$$

and

$$\bigvee_{r\geq t} P(y,x,r) * P(y,y,r-t)' = M_P(y,x,t) = 1.$$

By (FPKM6), we deduce that $x = y$.

(FKM3). By (FPKM4), we have that

$M_P(x,y,s) * M_P(y,z,t) = \bigvee\_r \geq sP(x,y,r) * P(x,x,r-s)' * \bigvee_{l\geq t} P(y,z,l) * P(y,y,l-t)'$

$$= \bigvee_{r\geq s} \bigvee_{l\geq t} P(x,y,r) * P(x,x,r-s)' * P(y,z,l) * P(y,y,l-t)'$$
$$\leq \bigvee_{r\geq s+t} P(x,z,r) * P(x,x,r-s-t)' = M_P(x,z,s+t).$$

(FKM4). First, take an arbitrary $r_0 < t$. Then for each $r \geq t$, since $r - t \leq r - r_0$, we have that $P(x,y,r-t) \leq P(x,y,r-r_0)$, which follows that $P(x,y,r-r_0)' \leq P(x,y,r-t)'$. By (a1), we have that

$$\begin{aligned} M_P(x,y,r_0) &= \bigvee_{r\geq r_0} P(x,y,r) * P(x,x,r-r_0)' \\ &= \Big(\bigvee_{r\geq t} P(x,y,r) * P(x,x,r-r_0)'\Big) \bigvee \Big(\bigvee_{r_0\leq r<t} P(x,y,r) * P(x,x,r-r_0)'\Big) \\ &\leq \Big(\bigvee_{r\geq t} P(x,y,r) * P(x,x,r-t)'\Big) \bigvee P(x,y,t) \\ &= M_P(x,y,t) \bigvee P(x,y,t) = M_P(x,y,t). \end{aligned}$$

By the arbitrariness of r_0, we obtain that$\bigvee_{r<t} M_P(x,y,r) \leq M_P(x,y,t)$.

To prove the reverse inequality, it suffices to prove $P(x,y,r_0) * P(x,x,r_0-t)' \leq \bigvee_{r<t} M_P(x,y,r)$ for all $r_0 \geq t$. We consider the following two cases:

Case 1: $r_0 = t$. By (FPKM1), (FPKM5) and (a1), we have that

$$\begin{aligned} P(x,y,r_0) * P(x,x,r_0-t)' &= P(x,y,t) * P(x,x,0)' = P(x,y,t) * 1 \\ &= P(x,y,t) = \bigvee_{r<t} P(x,y,r) \leq \bigvee_{r<t} M_P(x,y,r). \end{aligned}$$

Case 2: $r_0 > t$. Suppose $k < P(x,y,r_0) * P(x,x,r_0-t)'$. Then $k < P(x,y,r_0)$ and $k < P(x,x,r_0-t)'$. On one hand, by (FPKM5), we have

$$k < P(x,y,r_0) = \bigvee_{r<r_0} P(x,y,r) = \bigvee_{t<r<r_0} P(x,y,r).$$

Then there exists r_1, such that $t < r_1 < r_0$ and $k \leq P(x,y,r_1)$. On the other hand, we have that

$$k < P(x,x,r_0-t)' = \left(\bigvee_{r<r_0-t} P(x,x,r)\right)' = \bigwedge\_r < r_0 - tP(x,x,r)'.$$

It implies the following result

$$\forall r < r_0 - t,\ k < P(x,x,r)'. \qquad (\star)$$

Now let $r_2 = t - \frac{r_0-r_1}{2}$. Then $r_2 < t$ and $r_1 - r_2 = r_1 - t + \frac{r_0-r_1}{2} = \frac{r_0+r_1}{2} - t < r_0 - t$. By $(\star)$, we have that $k < P(x,x,r_1-r_2)'$. Note that $r_1 > t > r_2$, then we have that

$$\begin{aligned} k &< P(x,y,r_1) * P(x,x,r_1-r_2)' \leq \textstyle\bigvee_{r \geq r_2} P(x,y,r) * P(x,x,r-r_2)' \\ &= M_P(x,y,r_2) \leq \textstyle\bigvee_{r<t} M_P(x,y,r). \end{aligned}$$

From the arbitrariness of k, we have that

$$P(x,y,r_0) * P(x,x,r_0-t)' \leq \bigvee_{r<t} M_P(x,y,r)$$

for all $r_0 > t$. Therefore $M_P(x,y,t) = \bigvee_{r \geq t} P(x,y,r) * P(x,x,r-t)' \leq \bigvee_{r<t} M_P(x,y,r)$. □

4. PARTIAL PSEUDO-METRIC SYSTEMS

This section is devoted to study the relationships between *KM-fuzzy partial pseudo-metrics* and partial pseudo-metrics. Results show that every *KM-fuzzy partial pseudo-metric* can be determined by a partial pseudo-metric system.

In this section, we consider the case that $* = \wedge$ and $a' = 1 - a$ for all $a \in [0,1]$, and the corresponding *KM-fuzzy partial metric space* is denoted by $(X, P, \wedge)$.

Let $\mathcal{M}(X)$ denote the set of functions from $X \times X$ to $[0, +\infty)$.

Definition 4.1. Let $\Phi : [0,1) \longrightarrow \mathcal{M}(X)$ be a function. The family $\{\Phi_\varepsilon : \varepsilon \in [0,1)\}$ is called a *partial pseudo-metric system* on X if it satisfies the following conditions: $\forall \varepsilon \in [0,1)$,

(PS1) $\Phi_\varepsilon(x,x) \leq \Phi_\varepsilon(x,y)$;

(PS2) $\Phi_\varepsilon(x,y) = \Phi_\varepsilon(y,x)$;

(PS3) $\Phi_\varepsilon(x,y) = \bigwedge_{\delta>\varepsilon} \Phi_\delta(x,y)$;

(PS4) $\forall \delta < \varepsilon,\ \Phi_\delta(x,z) - \Phi_{1-\delta}(x,x) \leq \Phi_\varepsilon(x,y) - \Phi_{1-\varepsilon}(x,x) + \Phi_\varepsilon(y,z) - \Phi_{1-\varepsilon}(y,y)$.

Theorem 4.2. Let $(X, P, \wedge)$ be a *KM-fuzzy partial pseudo-metric space.* For each $\varepsilon \in [0,1)$, define $\Phi_\varepsilon : X \times X \longrightarrow [0,+\infty)$ as follows: $\forall x, y \in X$,

$$\Phi_\varepsilon(x,y) = \bigwedge\{r > 0 : P(x,y,r) > \varepsilon\}.$$

Then $\{\Phi_\varepsilon : \varepsilon \in [0,1)\}$ is a partial pseudo-metric system on X.

Proof. Before proving the theorem, we first give two useful results.

Claim 1: $\Phi_\varepsilon(x,y) < r \Leftrightarrow P(x,y,r) > \varepsilon$.

On one hand, we have that

$\Phi_\varepsilon(x,y) < r \Leftrightarrow \exists r_0 < r,\ P(x,y,r_0) > \varepsilon \Rightarrow P(x,y,r) \geq P(x,y,r_0) > \varepsilon \Rightarrow P(x,y,r) > \varepsilon$.

On the other hand, by (FPKM5) we have that

$$P(x,y,r) = \bigvee_{s<r} P(x,y,s) > \varepsilon \Rightarrow \exists r_0 < r,\ P(x,y,r_0) > \varepsilon \Rightarrow \Phi_\varepsilon \leq r_0 < r.$$

Thus Claim 1 holds.

Claim 2: As a direct consequence of Claim 1, we have the following result:

$$r \leq \Phi_{1-\varepsilon}(x,x) \Leftrightarrow P(x,x,r) \leq 1-\varepsilon.$$

Next, we prove that $\{\Phi_\varepsilon : \varepsilon \in [0,1)\}$ is a partial pseudo-metric system. Since (PS1) and (PS2) are trivial, we only verify (PS3) and (PS4).

(PS3). It is easy to find that $\Phi_\varepsilon(x,y) \leq \Phi_\delta(x,y)$ whenever $\varepsilon < \delta$, which implies that $\Phi_\varepsilon(x,y) \leq \bigwedge_{\delta>\varepsilon} \Phi_\delta(x,y)$. To show the reverse inequality, let $t > \Phi_\varepsilon(x,y)$. Then by Claim 1, we have that $P(x,y,t) > \varepsilon$. It follows that there exists $\delta_0 > 0$, such that $P(x,y,t) > \delta_0 > \varepsilon$. By Claim 1, we have $\Phi_{\delta_0}(x,y) < t$, which implies $\bigwedge_{\delta>\varepsilon} \Phi_\delta(x,y) \leq \Phi_{\delta_0}(x,y) < t$. From the arbitrariness of t, we have $\bigwedge_{\delta>\varepsilon} \Phi_\delta(x,y) \leq \Phi_\varepsilon(x,y)$. Hence, $\bigwedge_{\delta<\varepsilon} \Phi_\delta(x,y) = \Phi_\varepsilon(x,y)$.

(PS4). Suppose $\delta < \varepsilon$. Let $s, t > 0$ such that

$$\Phi_\varepsilon(x,y) - \Phi_{1-\varepsilon}(x,x) < s \text{ and } \Phi_\varepsilon(x,y) - \Phi_{1-\varepsilon}(x,x) < t.$$

Take $s_1 = \Phi_{1-\varepsilon}(x,x) + s$ and $t_1 = \Phi_{1-\varepsilon}(y,y) + t$. Then we have that

$$\Phi_\varepsilon(x,y) < s_1 \text{ and } \Phi_{1-\varepsilon}(x,x) = s_1 - s;$$
$$\Phi_\varepsilon(y,z) < t_1 \text{ and } \Phi_{1-\varepsilon}(y,y) = t_1 - t.$$

Invoking Claim 1 and Claim 2, we have

$$\varepsilon < P(x,y,s_1) \text{ and } \varepsilon \leq 1 - P(x,x,s_1 - s);$$
$$\varepsilon < P(x,y,t_1) \text{ and } \varepsilon \leq 1 - P(x,x,t_1 - t).$$

By (FPKM4), it holds that

$\varepsilon \leq P(x,y,s_1) \wedge (1-P(x,x,s_1-s)) \wedge P(y,z,t_1) \wedge (1-P(y,y,t_1-t))$
$\leq \bigvee_{r\geq s+t} P(x,z,r) \bigwedge (1-P(x,x,r-s-t))$.

Since $\delta < \varepsilon$, it follows that

$$\delta < \bigvee_{r\geq s+t} P(x,z,r) \bigwedge (1-P(x,x,r-s-t)).$$

Then there exists $r_\delta \geq s+t$ such that $\delta < P(x,z,r_\delta)$ and $\delta < 1-P(x,x,r_\delta-s-t)$. By Claim 1 and Claim 2, we have that $\Phi_\delta(x,z) < r_\delta$ and $\Phi_{1-\delta}(x,x) \geq r_\delta - s - t$, which implies that $\Phi_\delta(x,z) - \Phi_{1-\delta}(x,x) < s+t$. From the arbitrariness of s and t, we have $\Phi_\delta(x,z) - \Phi_{1-\delta}(x,x) \leq \Phi_\varepsilon(x,y) - \Phi_{1-\varepsilon}(x,x) + \Phi_\varepsilon(y,z) - \Phi_{1-\varepsilon}(y,y)$. □

Theorem 4.3. Let $\{\Phi_\varepsilon : \varepsilon \in [0,1)\}$ be a partial pseudo-metric system and define $P : X \times X \times [0,+\infty) \longrightarrow [0,1)$ as follows: $\forall x,y \in X$ and $\forall r \in [0,+\infty)$,

$$P(x,y,r) = \bigvee \{\varepsilon \in [0,1) : \Phi_\varepsilon(x,y) < r\}.$$

Then P is a *KM-fuzzy partial pseudo-metric.*

Proof. Before proving the theorem, we first give three useful results.

Claim 3: $\lambda < 1-P(x,x,r) \Rightarrow \Phi_{1-\lambda}(x,x) \geq r$.
In fact, we have that

$\lambda < 1-P(x,x,r) \Leftrightarrow P(x,x,r) < 1-\lambda \Leftrightarrow \forall\varepsilon > 0, \Phi_\varepsilon(x,y) < r \Rightarrow \varepsilon < 1-\lambda \Rightarrow \forall\varepsilon \geq 1-\lambda, \Phi_\varepsilon(x,y) \geq r$.

In particular, $\Phi_{1-\lambda}(x,x) \geq r$.

Claim 4: $\Phi_{1-\lambda}(x,x) \geq r \Rightarrow \lambda \leq 1-P(x,x,r)$.
It suffices to prove $p(x,x,r) = \bigvee\{\varepsilon \in [0,1) : \Phi_\varepsilon(x,x) < r\} \leq 1-\lambda$. Suppose $\varepsilon \in [0,1)$ such that $\Phi_\varepsilon(x,x) < r$. Since $r \leq \Phi_{1-\lambda}(x,x)$, it follows that $\Phi_\varepsilon(x,x) < \Phi_{1-\lambda}(x,x)$, which implies that $\varepsilon < 1-\lambda$ (otherwise, if $\varepsilon \geq 1-\lambda$, then $\Phi_\varepsilon(x,x) \geq \Phi_{1-\lambda}(x,x)$, a contradiction). This proves that $p(x,x,r) \leq 1-\lambda$.

Claim 5. $\lambda < P(x,y,r) \Leftrightarrow \Phi_\lambda(x,y) < r$.
On one hand, we have that

$\lambda < P(x,y,r) \Leftrightarrow \exists\varepsilon > \lambda,\ \Phi_\varepsilon(x,y) < r \Rightarrow \Phi_\lambda(x,y) \leq \Phi_\varepsilon(x,y) < r \Rightarrow P(x,y,r) \geq \varepsilon > \lambda$.

On the other hand, we have that

$$\Phi_\lambda(x,y) < r \Rightarrow \bigwedge_{\delta>\lambda} \Phi_\delta(x,y) = \Phi_\lambda(x,y) < r \Rightarrow \exists\delta_0 > \lambda, \Phi_{\delta_0}(x,y) < r$$
$$\Rightarrow \exists\delta_0 > \lambda, P(x,y,r) \geq \delta_0 \Rightarrow P(x,y,r) > \lambda,$$

This completes the proof of Claim 5.

Next, we prove (FPKM1), (FPKM4) and (FPKM5) since the other conditions hold trivially.

(FPKM1). It holds since $P(x, y, 0) = \bigvee \emptyset = 0$.

(FPKM4). Suppose $r_1 \geq s, r_2 \geq t, x, y, z \in X$. Let δ be an arbitrary mumber of $[0, 1)$, such that

$$\delta < P(x, y, r_1) \wedge (1 - P(x, x, r_1 - s)) \wedge P(y, z, r_2) \wedge (1 - P(y, y, r_2 - t)).$$

Then there exists $\lambda \in [0, 1)$, such that

$$\delta < \lambda < P(x, y, r_1) \wedge (1 - P(x, x, r_1 - s)) \wedge P(y, z, r_2) \wedge (1 - P(y, y, r_2 - t)).$$

We have that

$$\lambda < P(x, y, r_1) \text{ and } \lambda < 1 - P(x, x, r_1 - s);$$
$$\lambda < P(y, z, r_2) \text{ and } \lambda < 1 - P(x, x, r_2 - t).$$

From Claim 3 and Claim 5, it follows that

$$\Phi_\lambda(x, y) < r_1 \text{ and } \Phi_{1-\lambda}(x, x) \geq r_1 - s;$$
$$\Phi_\lambda(y, z) < r_2 \text{ and } \Phi_{1-\lambda}(y, y) \geq r_2 - t.$$

Since $\delta < \lambda$, by (PS4), it holds that

$$\Phi_\delta(x, z) - \Phi_{1-\delta}(x, x) \leq \Phi_\lambda(x, y) - \Phi_{1-\lambda}(x, x) + \Phi_\lambda(y, z) - \Phi_{1-\lambda}(y, y) < s + t.$$

Let $r_\delta = \Phi_{1-\delta}(x, x) + s + t$. Then $r_\delta \geq s + t$ and we have that

$$\Phi_\delta(x, z) < r_\delta \text{ and } \Phi_{1-\delta}(x, x) = r_\delta - s - t.$$

By Claim 4 and Claim 5, we have that

$$P(x, z, r_\delta) > \delta \text{ and } 1 - P(x, x, r_\delta - s - t) \geq \delta.$$

Hence, $\delta \leq P(x, z, r_\delta) \wedge (1 - P(x, x, r_\delta - s - t)) \leq \bigvee_{r \geq s+t} P(x, z, r) \wedge (1 - P(x, x, r - s - t))$. From the arbitrariness of δ, it follows that

$$P(x, y, r_1) \wedge (1 - P(x, x, r_1 - s)) \wedge P(y, z, r_2) \wedge (1 - P(y, y, r_2 - t)) \leq \bigvee_{r \geq s+t} P(x, z, r) \wedge (1 - P(x, x, r - s - t)).$$

(FPKM5). That $P(x, y, t) \geq \bigvee_{r<t} P(x, y, r)$ holds trivially. For the converse, let $\lambda < P(x, y, t)$. By Claim 5, we have that $\Phi_\lambda(x, y) < t$. Then exists $r_0 < t$ such that $\Phi_\lambda(x, y) < r_0 < t$. By Claim 5, we have that

$$\lambda < P(x, y, r_0) \leq \bigvee_{r<t} P(x, y, r).$$

From the arbitrariness of λ, it follows that $P(x, y, t) \leq \bigvee_{r<t} P(x, y, r)$. Therefore, $P(x, y, t) = \bigvee_{r<t} P(x, y, r)$.

For a given partial pseudo-metric system $\{\Phi_\varepsilon : \varepsilon \in [0, 1)\}$, denote by P_Φ, the *KM-fuzzy partial pseudo-metric* induced by $\{\Phi_\varepsilon : \varepsilon \in [0, 1)\}$ as in Theorem 4.3.

For a given *KM-fuzzy partial pseudo-metric* $(X, P, \wedge)$, denote by $\Phi_P = \{\Phi_\varepsilon : \varepsilon \in [0, 1)\}$, the partial pseudo-metric system induced by $(X, P, \wedge)$ as in Theorem 4.2. □

Theorem 4.4. Let P be a *KM-fuzzy partial pseudo-metric* on X, then $P_{\Phi_P} = P$.

Proof. In fact, we have that

$$\begin{aligned} P_{\Phi_P}(x,y,t) &= \bigvee\{\varepsilon : \Phi_\varepsilon(x,y) < t\} = \bigvee\{\varepsilon : \bigwedge\{r > 0 : P(x,y,r) > \varepsilon\} < t\} \\ &= \bigvee\{\varepsilon : \exists r < t, P(x,y,r) > \varepsilon\} = \bigvee \bigcup_{r<t}\{\varepsilon : P(x,y,r) > \varepsilon\} \\ &= \bigvee_{r<t} \bigvee\{\varepsilon : P(x,y,r) > \varepsilon\} = \bigvee_{r<t} P(x,y,r) = P(x,y,t). \end{aligned}$$ □

Theorem 4.5. Let $\{\Phi_\varepsilon : \varepsilon \in [0,1]\}$ be a partial pseudo-metric system on X, then $(\Phi_{P\Phi})_\varepsilon = \Phi_\varepsilon$.

Proof. In fact, we have that

$$\begin{aligned} (\Phi_{P\Phi})_\varepsilon(x,y) &= \bigwedge\{r > 0 : P(x,y,r) > \varepsilon\} = \bigwedge\{r > 0 : \bigvee\{\delta : \Phi_\delta(x,y) < r\} > \varepsilon\} \\ &= \bigwedge\{r > 0 : \exists \delta > \varepsilon, \Phi_\delta(x,y) < r\} = \bigwedge \bigcup_{\delta>\varepsilon}\{r > 0 : \Phi_\delta(x,y) < r\} \\ &= \bigwedge_{\delta>\varepsilon} \bigwedge\{r > 0 : \Phi_\delta(x,y) < r\} = \bigwedge_{\delta>\varepsilon} \Phi_\delta(x,y) = \Phi_\varepsilon(x,y). \end{aligned}$$ □

From the above result, we can deduce the following corollary.

Corollary 4.6. There is a bijective relationship between *KM-fuzzy partial pseudo-metrics* and partial pseudo-metric systems.

Next, we define a fuzzifying topology which is deduced from a *KM-fuzzy partial metric*.

Theorem 4.7. Let P be a *KM-fuzzy partial metric* on X, for each $x \in X$ and $U \subseteq X$, define $\mathcal{N}_x(U)$ as follows:

$$\mathcal{N}_x(U) = \bigvee_{\varepsilon>0} \bigwedge_{y\notin U} (1 - M_P(x,y,\varepsilon)).$$

Then $\mathcal{N}_x(U)$ is a fuzzifying neighborhood system.

Proof. By Theorem 5.2 in [23], it remains to prove the following statements: $\mathcal{N}_x(\emptyset) = 0$, and $x \notin U$ implies $\mathcal{N}_x(U) = 0$. In fact, $\mathcal{N}_x(\emptyset) = \bigvee_{\varepsilon>0} \bigwedge_{y\in X}(1 - M_P(x,y,\varepsilon)) \leq \bigvee_{\varepsilon>0}(1 - M_P(x,x,\varepsilon)) = \bigvee_{\varepsilon>0}(1-1) = 0$. Moreover, if $x \notin U$, then we have that

$\mathcal{N}_x(U) = \bigvee_{\varepsilon>0} \bigwedge_{y\notin U}(1-M_P(x,y,\varepsilon)) \leq \bigvee_{\varepsilon>0}(1-M_P(x,x,\varepsilon)) = \bigvee_{\varepsilon>0}(1-1) = 0.$ □

In the following, we will use the notation τ_P stands for the fuzzifying topology which is determined by $\tau_P(U) = \bigwedge_{x\in U} \mathcal{N}_x(U)$ for all $U \in 2^X$.

Theorem 4.8. Let P be a KM-fuzzy partial metric and τ_P be a fuzzifying topology on X. Then $[C_I(X, \tau_P)] = 1$.

Proof. Let $x \in X$ and put the set $\mathscr{A}_x = \{B_P(x, \frac{1}{n}, r) \mid n \in \mathbb{N}, r \in \mathbb{Q}\bigcap[0,1]\}$. Here $B_P(x, \frac{1}{n}, r) = \{y \mid M_P(x, y, \frac{1}{n}) > 1 - r\}$. Now we define the mapping $\mathcal{B}_x : 2^X \to [0,1]$ as follows:

$$\mathcal{B}_x(U) = \begin{cases} \mathcal{N}_x(U), & U \in \mathscr{A}_x \\ 0, & \text{others.} \end{cases}.$$

Since $\mathscr{A}_x$ is a countable set, $(\mathcal{B}_x)_0$ is also a countable set. It remains to prove $\mathcal{N}_x(U) = \bigvee_{V \subseteq U} \mathcal{B}_x(V)$ for all $U \in 2^X$. It is obvious $\mathcal{N}_x(U) \geq \bigvee_{V \subseteq U} \mathcal{B}_x(V)$, we only need to prove $\mathcal{N}_x(U) \leq \bigvee_{V \subseteq U} \mathcal{B}_x(V)$.

In fact, for each $r \in \mathbb{Q}\bigcap[0,1]$ satisfying $r < \mathcal{N}_x(U) = \bigvee_{\varepsilon>0} \bigwedge_{y \notin U}(1 - M_P(x, y, \varepsilon)) = \bigvee_{n \in \mathbb{N}} \bigwedge_{y \notin U}(1 - M_P(x, y, \frac{1}{n}))$, then there exists $n \in \mathbb{N}$ such that $1 - M_P(x, y, \frac{1}{n}) > r$ for all $y \notin U$. Equivalently, $B_P(x, \frac{1}{n}, r) \subseteq U$. Furthermore,

$$\begin{aligned} \mathcal{B}_x(B_P(x, \tfrac{1}{n}, r)) = \mathcal{N}_x(B_P(x, \tfrac{1}{n}, r)) &= \bigvee_{\varepsilon>0} \bigwedge_{y \notin B_P(x, \frac{1}{n}, r)}(1 - M_P(x, y, \varepsilon)) \\ &\geq \bigwedge_{y \notin B_P(x, \frac{1}{n}, r)}(1 - M_P(x, y, \tfrac{1}{n})) \geq r. \end{aligned}$$

So, $r \leq \bigvee_{V \subseteq U} \mathcal{B}_x(V)$. Hence $\mathcal{N}_x(U) \leq \bigvee_{V \subseteq U} \mathcal{B}_x(V)$. The proof is completed. □

Theorem 4.9. Let $(X, P_1, \wedge)$ and $(Y, P_2, \wedge)$ be two *KM-fuzzy partial metric spaces.* Then the mapping $f : X \to Y$ is fuzzifying continuous if and only if for each sequence $\{x_n\} \subseteq X, x \in X$, $[x_n \to x] \leq [f(x_n) \to f(x)]$.

Proof. Necessity. Suppose that the mapping $f : X \to Y$ is fuzzifying continuous, then for any sequence $\{x_n\} \subseteq X, x \in X$, if $\alpha \in (0, [x_n \to x])$, we need to prove that $t \leq [f(x_n) \to f(x)]$. In fact, for every $W \subseteq Y$ with $\{f(x_n)\} \sqsubseteq W$ not holding, we have $\{x_n\} \sqsubseteq f^{\leftarrow}(W)$ does not hold. Thus $\alpha < 1 - \mathcal{N}_x(f^{\leftarrow}(W))$. From the fact $\mathcal{N}_x(f^{\leftarrow}(W)) \geq \mathcal{N}_{f(x)}(W)$, so $\alpha < 1 - \mathcal{N}_{f(x)}(W)$. Hence $\alpha \leq \bigwedge_{\{f(x_n)\} \not\sqsubseteq W} 1 - \mathcal{N}_{f(x)}(W) = [f(x_n) \to f(x)]$. By the arbitrariness of α, we have $[x_n \to x] \leq [f(x_n) \to f(x)]$ holds.

Sufficiency. If the mapping $f : X \to Y$ is not fuzzifying continuous, then there exist $x \in X$ and $W \in 2^Y$ such that $\mathcal{N}_x(f^{\leftarrow}(W)) < \mathcal{N}_{f(x)}(W)$. Taking $\mathcal{N}_x(f^{\leftarrow}(W)) < \lambda < \mathcal{N}_{f(x)}(W)$, then $f^{\leftarrow}(W) \notin (\mathcal{N}_x)_\lambda$. By Theorem 4.7, we have $(\mathcal{B}_x)_\lambda = \{U_n : U_{n+1} \subseteq U_n, n \in \mathbb{N}\}$ which satisfies the following:

For each $V \in (\mathcal{N}_x)_\lambda$, there exists $U_n \in (\mathcal{B}_x)_\lambda$ such that $U_n \subseteq V$.

Thus for each $U_n \in (\mathcal{B}_x)_\lambda$, $U_n \not\subseteq f^{\leftarrow}(W)$. So there exists $x_n \in U_n$ and $x_n \notin f^{\leftarrow}(W)$. It is easy to check that the sequences $\{x_n\} \sqsubseteq V$ for each $V \in (\mathcal{N}_x)_\lambda$. That is to say $\{x_n\} \not\sqsubseteq A$ implies $\mathcal{N}_x(A) \leq \lambda$. Then we have

$$[x_n \to x] = \bigwedge_{\{x_n\} \not\sqsubseteq A}(1 - \mathcal{N}_x(A)) \geq 1 - \lambda.$$

On the other hand, since $\{f(x_n)\} \not\sqsubseteq W$, we have

$$[f(x_n) \to f(x)] = \bigwedge_{\{f(x_n)\} \not\sqsubseteq B}(1 - \mathcal{N}_{f(x)}(B)) \leq 1 - \mathcal{N}_{f(x)}(W) < 1 - \lambda.$$

It deduces that $[x_n \to x] > [f(x_n) \to f(x)]$. This contradicts the assumption of sufficiency. Hence f is a fuzzifying continuous mapping. □

Theorem 4.10. Let $(X, P_1, \wedge)$, $(Y, P_2, \wedge)$ be two *KM-fuzzy partial metric spaces*, and the mapping $f : X \to Y$ be continuous, Then for each sequence $\{x_n\} \subseteq X, x \in X$, $[x_n \to x] \leq [f(x_n) \to f(x)]$.

Proof. For each $\{x_n\} \subseteq X, x \in X$, if $\alpha \in (0, [x_n \to x])$, it remains to prove $\alpha \leq [f(x_n) \to f(x)]$. In fact, for all $B \in 2^Y$ satisfying $\{f(x_n)\} \not\sqsubseteq B$, clearly, $\{x_n\} \not\sqsubseteq f^{\leftarrow}(B)$. Then $1 - \mathcal{N}_x(f^{\leftarrow}(B)) > \alpha$. So we have $\mathcal{N}_x(f^{\leftarrow}(B)) = \bigvee_{\varepsilon>0} \bigwedge_{y \notin f^{\leftarrow}(B)} (1 - M_{P_1}(x, y, \varepsilon)) < 1 - \alpha$. On the other hand, since the mapping f is continuous, then for any $t > 0$, there exists $t_x > 0$ such that $M_{P_1}(x, y, t_x) \leq M_{P_2}(f(x), f(y), t)$. Then we have $y \notin f^{\leftarrow}(B)$ satisfying $1 - M_{P_1}(x, y, t_x) < 1 - \alpha$ for above $t_x > 0$. This implies $1 - M_{P_2}(f(x), f(y), t) \leq 1 - M_{P_1}(x, y, t_x) < 1 - \alpha$. Thus $\mathcal{N}_{f(x)}(B) = \bigvee_{t>0} \bigwedge_{z \notin B} (1 - M_{P_2}(f(x), z, t)) \leq \bigvee_{t>0} (1 - M_{P_2}(f(x), f(y), t)) \leq 1 - \alpha$. Furthermore, $[f(x_n) \to f(x)] = \bigwedge_{\{f(x_n)\} \not\sqsubseteq B} (1 - \mathcal{N}_{f(x)}(B)) \geq \alpha$. Since α is arbitrary, we have completed the proof. □

Theorem 4.11. Let P be a *KM-fuzzy partial metric* on X such that $M_P(x, y, 0+) = 0$ or $M_P(y, x, 0+) = 0$ holds for all $x \neq y \in X$. Then $[T_0(X, \tau_P)] = 1$.

Proof. For all $x \neq y$, $M_P(x, y, 0+) = 0$ or $M_P(y, x, 0+) = 0$. If $M_P(x, y, 0+) = 0$, then for any $n \in \mathbb{N}$, there exists $t_n > 0$ such that $M_P(x, y, t_n) < \frac{1}{n}$. Let $A_n = \{z : M_P(x, z, t_n) \geq \frac{1}{n}\}$, clearly, $y \notin A_n$. Thus $\bigvee_{y \notin A} \mathcal{N}_x(A) \geq \mathcal{N}_x(A_n) = \bigvee_{\varepsilon>0} \bigwedge_{z \notin A_n} (1 - M_P(x, z, \varepsilon)) \geq \bigwedge_{z \notin A_n} (1 - M_P(x, z, t_n)) \geq 1 - \frac{1}{n}$.

From the arbitrariness of n, we have $\bigvee_{y \notin A} \mathcal{N}_x(A) \geq 1$. So

$$[T_0(X, \tau_P)] = \bigwedge_{x \neq y} \Big(\bigvee_{y \notin A} \mathcal{N}_x(A) \bigvee \bigvee_{x \notin B} \mathcal{N}_y(B) \Big) = 1.$$ □

5. CONCLUSIONS AND FUTURE WORK

In the present paper, a new approach for fuzzy partial metric spaces is given. This new concept of KM-fuzzy partial metric is an extension of both the partial metric and KM-fuzzy metric. The relationship between KM-fuzzy partial metric with the KM-fuzzy quasi-metric is established. By defining the notion of partial pseudo-metric systems, a one-to-one correspondence between partial pseudo-metric systems and KM-fuzzy partial pseudo-metrics is constructed. Furthermore, a fuzzifying topology τ_P on X determined by KM-fuzzy partial metric is obtained and some interesting properties of this fuzzifying topology are discussed.

A direction worthy of future work is to establish the systemic theory of KM-fuzzy partial metric spaces with respect the fuzzifying topology, such as fuzzifying boundedness, fuzzifying compactness, et al.. Also, studying the fixed point theorems and Ekeland's variational principle in KM-fuzzy partial metric spaces are of interest.

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