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ON THE DIRECT PRODUCT OF UNINORMS ON BOUNDED LATTICES

Emel Aşici and Radko Mesiar

In this paper, we study on the direct product of uninorms on bounded lattices. Also, we define an order induced by uninorms which are a direct product of two uninorms on bounded lattices and properties of introduced order are deeply investigated. Moreover, we obtain some results concerning orders induced by uninorms acting on the unit interval $[0, 1]$.

Keywords: uninorm, direct product, partial order

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1. INTRODUCTION

Uninorms were introduced by Yager and Rybalov [28]. Uninorms are a special kind of aggregation functions that generalize both t-norms and t-conorms [8, 11]. In [20], two first classes of uninorms were introduced: uninorms in $\mathcal{U}_{\min}$ and in $\mathcal{U}_{\max}$, and representable uninorms. Then, some papers were published dealing with uninorms, especially from the point of view of their applications. Thus, uninorms have proved to be useful in a wide range of fields like aggregation of information, expert systems, neural networks, fuzzy system modeling, pseudo-analysis and measure theory, fuzzy mathematical morphology, fuzzy sets and fuzzy logic, approximate reasoning and so on. Since uninorms on $[0, 1]$ are always conjunctive or disjunctive, they have been extensively studied in the framework of logical connectives and they have been used in fuzzy sets theory and fuzzy logic [26].

In [16], direct product of triangular norms on product lattices was introduced and some of the algebraic properties were investigated.

In particular, Karaçal and Mesiar [22] studied uninorms on bounded lattices. They showed the existence of uninorms with neutral element e for an arbitrary element $e \in L \setminus \{0, 1\}$ with underlying t-norms and t-conorms on an arbitrary bounded lattice. Also, they introduced the smallest and the greatest uninorm on a bounded lattice.

In recent years, the orders induced by uninorms, nullnorms and triangular norms have been studied widely. In this sense, in [23], T- partial order, denoted $\preceq_T$, defined by means of t-norms on a bounded lattice was introduced.

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Based on these previous studies, in [1, 18] U-partial order and F-partial order obtained from the uninorm and nullnorm, respectively, were introduced and some properties of these orders were investigated.

In this paper, we study on the direct product of uninorms on bounded lattices. The present paper consists of four main parts. Firstly, we give in preliminaries some necessary definitions we will work with. In Section 3, we define an order induced by uninorms which are a direct product of two uninorms on bounded lattices. In Section 4, we define the set of comparable and incomparable elements with respect to the $U_1 \times U_2$-partial order, denoted $\preceq_{U_1 \times U_2}$ and we obtain some interesting results related to direct product of uninorms on $[0,1]^2$. In Section 5, some concluding remarks are added.

2. PRELIMINARIES

A lattice [9] is a partially ordered set $(L, \leq)$ in which each two element subset $\{x, y\}$ has an infimum, denoted as $x \wedge y$, and a supremum, denoted as $x \vee y$. A bounded lattice $(L, \leq, 0, 1)$ is a lattice that has the bottom and top elements written as 0 and 1, respectively. Given a bounded lattice $(L, \leq, 0, 1)$ and $a, b \in L$, if a and b are incomparable, in this case, we use the notation $a \parallel b$.

Definition 1. (De Baets and Mesiar [16]) Let $(L_1, \leq_1, 0_1, 1_1)$ and $(L_2, \leq_2, 0_2, 1_2)$ be bounded lattices. Then, $L_1 \times L_2 = (L_1 \times L_2, \leq, (0_1, 0_2), (1_1, 1_2))$ is a bounded lattice with partial order relation $\leq$, $\wedge$ and $\vee$ defined by

$$(x_1, y_1) \leq (x_2, y_2) \Leftrightarrow x_1 \leq_1 x_2 \quad \text{and} \quad y_1 \leq_2 y_2,$$

$$(x_1, y_1) \wedge (x_2, y_2) = (x_1 \wedge_1 x_2, y_1 \wedge_2 y_2),$$

$$(x_1, y_1) \vee (x_2, y_2) = (x_1 \vee_1 x_2, y_1 \vee_2 y_2).$$

In this study, for short, we use the L_1 instead of $(L_1, \leq_1, 0_1, 1_1)$, L_2 instead of $(L_2, \leq_2, 0_2, 1_2)$ and $L_1 \times L_2$ instead of $(L_1 \times L_2, \leq, \wedge, \vee, (0_1, 0_2), (1_1, 1_2))$.

Definition 2. (Aşıcı and Karaçal [4], Çaylı [13], Saminger [27]) Let L be a bounded lattice. A *triangular norm* T (briefly t-norm) is a binary operation on L that is commutative, associative, monotone and has neutral element 1.

Example 1. (Aşıcı and Mesiar [5], Klement et al. [25]) The four basic t-norms T_M, T_P, T_L and T_D on $[0,1]$ are given by:
$T_M(x, y) = \min(x, y)$,
$T_P(x, y) = x.y$,
$T_L(x, y) = \max(x + y - 1, 0)$,
$$T_D(x, y) = \begin{cases} 0 & (x, y) \in [0, 1)^2, \\ \min(x, y) & \text{otherwise.} \end{cases}$$

Definition 3. (Aşıcı [3], Çaylı [14], Saminger [27]) Let L be a bounded lattice. A *triangular conorm* S (briefly t-conorm) is a binary operation on L that is commutative, associative, monotone and has neutral element 0.

Example 2. (Aşıcı and Mesiar [6], Klement et al. [25]) The four basic t-conorms S_M, S_P, S_L and S_D on $[0,1]$ are given by:
$S_M(x,y) = \max(x,y)$,
$S_P(x,y) = x + y - x.y$,
$S_L(x,y) = \min(x+y,1)$,
$$S_D(x,y) = \begin{cases} 1 & (x,y) \in (0,1]^2, \\ \max(x,y) & \text{otherwise.} \end{cases}$$

Extremal t-norms $T_\wedge$ and T_W are defined on a bounded lattice as follows, respectively:
$T_\wedge(x,y) = x \wedge y$
$$T_W(x,y) = \begin{cases} x & \text{if } y = 1, \\ y & \text{if } x = 1, \\ 0 & \text{otherwise.} \end{cases}$$
Similarly, the t-conorms $S_\vee$ and S_W can be defined.

Especially we have obtained $T_W = T_D$ and $T_\wedge = T_M$ for $L = [0,1] \subset R$.

Definition 4. (De Baets and Mesiar [16]) Let L_1 and L_2 be bounded lattices and T_1 and T_2 be t-norms on L_1 and L_2, respectively. Then, the direct product $T_1 \times T_2$ of T_1 and T_2, defined by

$$T_1 \times T_2((x_1,y_1),(x_2,y_2)) = (T_1(x_1,x_2), T_2(y_1,y_2))$$

is a t-norm on the product lattice $L_1 \times L_2$.

Definition 5. (Casasnovas and Mayor [12]) A t-norm T on L is *divisible* if the following condition holds:

$$\forall\, x,y \in L \quad \text{with} \quad x \leq y \quad \text{there is a} \quad z \in L \quad \text{such that} \quad x = T(y,z).$$

Definition 6. (Aşıcı and Mesiar [7], Calvo et al. [10], Çaylı [14]) Let L be a bounded lattice. An operation $U : L^2 \to L$ is called a uninorm on L, if it is commutative, associative, monotone and has a neutral element $e \in L$.

We denote by $\mathcal{U}(e)$ the set of all uninorms on L with the neutral element $e \in L$. Also, we denote by $A(e) = L^2 \setminus ([0,e]^2 \cup [e,1]^2)$ and $I(U) = \{x \in L \mid U(x,x) = x\}$.

Theorem 1. (Fodor et al. [19]) Let $U : [0,1]^2 \to [0,1]$ be a uninorm with neutral element $e \in (0,1)$. Then the sections $x \mapsto U(x,1)$ and $x \mapsto U(x,0)$ are continuous in each point except perhaps for e if and only if U is given by one of the following formulas.
(a) If $U(0,1) = 0$, then

$$U(x,y) = \begin{cases} eT(\frac{x}{e}, \frac{y}{e}) & (x,y) \in [0,e]^2, \\ e + (1-e)S(\frac{x-e}{1-e}, \frac{y-e}{1-e}) & (x,y) \in [e,1]^2, \\ \min(x,y) & (x,y) \in A(e), \end{cases} \tag{1}$$

where T is a t-norm and S is a t-conorm.
(b) If $U(0,1) = 1$, then the same structure holds, changing minimum by maximum in $A(e)$.

The class of uninorms as in case (a) will be denoted by $\mathcal{U}_{\min}$ and the class of uninorms as in case (b) by $\mathcal{U}_{\max}$. We will denote a uninorm U in $\mathcal{U}_{\min}$ with underlying t-norm T, underlying t-conorm S and neutral element e by $U \equiv \langle T, e, S\rangle_{\min}$ and in a similar way, a uninorm in $\mathcal{U}_{\max}$ by $U \equiv \langle T, e, S\rangle_{\max}$.

Proposition 1. (Kalina [21]) Let L_1 and L_2 be bounded lattices and U_1 be a uninorm on L_1 with neutral element e_1 and U_2 be a uninorm on L_2 with neutral element e_2. Then the direct product $U_1 \times U_2$ of U_1 and U_2, defined by

$$U_1 \times U_2((x_1, y_1), (x_2, y_2)) = (U_1(x_1, x_2), U_2(y_1, y_2))$$

is a uninorm on the product lattice $L_1 \times L_2$ with neutral element (e_1, e_2).

Definition 7. (Karaçal and Kesicioğlu [23]) Let L be a bounded lattice, T be a t-norm on L. The order defined as follows is called a $T-$ *partial order* (triangular order) for t-norm T:

$$x \preceq_T y :\Leftrightarrow T(\ell, y) = x \text{ for some } \ell \in L.$$

Definition 8. (Ertuğrul et al. [18]) Let L be a bounded lattice, S be a t-conorm on L. The order defined as follows is called an $S-$ *partial order* for t-conorm S:

$$x \preceq_S y :\Leftrightarrow S(\ell, x) = y \text{ for some } \ell \in L.$$

Definition 9. (Ertuğrul et al. [18]) Let L be a bounded lattice and U be a uninorm with neutral element e on L. Define the following relation, for $x, y \in L$, as

$$x \preceq_U y :\Leftrightarrow \begin{cases} \text{if } x, y \in [0, e] \text{ and there exist } k \in [0, e] \text{ such that } U(y, k) = x \text{ or,} \\ \text{if } x, y \in [e, 1] \text{ and there exist } \ell \in [e, 1] \text{ such that } U(x, \ell) = y \text{ or,} \\ \text{if } (x, y) \in L^* \text{ and } x \leq y, \end{cases} \quad (2)$$

where $I_e = \{x \in L \mid x \parallel e\}$ and $L^* = [0, e] \times [e, 1] \cup [0, e] \times I_e \cup [e, 1] \times I_e \cup [e, 1] \times [0, e] \cup I_e \times [0, e] \cup I_e \times [e, 1] \cup I_e \times I_e$.

Proposition 2. (Ertuğrul et al. [18]) The relation $\preceq_U$ defined in (2) is a partial order on L.

Note: The partial order $\preceq_U$ in (2) is called U-partial order on L.

3. $\preceq_{U_1 \times U_2}$-PARTIAL ORDER

In this section, we define an order induced by uninorms which are a direct product of two uninorms on bounded lattices.

Definition 10. Let L_1 and L_2 be bounded lattices, U_1 be a uninorm on L_1 with neutral element e_1 and U_2 be a uninorm on L_2 with neutral element e_2 and consider their direct product $U_1 \times U_2$ on $L_1 \times L_2$. Let $\preceq_{U_1}$ and $\preceq_{U_2}$ are partial orders induced by uninorms U_1 and U_2, respectively. Then, the relation $\preceq_{U_1 \times U_2}$ is defined by

$$(x,y) \preceq_{U_1\times U_2} (z,t) \Leftrightarrow x \preceq_{U_1} z \textit{ and } y \preceq_{U_2} t$$

for all $(x,y),(z,t) \in L_1 \times L_2$.

Example 3. Consider the lattice $(L_1 = L_2 = \{0,a,k,e,b,p,m,1\}, \leq, 0, 1)$ given in Figure 1 and the uninorms U_1 and U_2 on $L_1 = L_2$ defined Table 1 and Table 2, respectively.

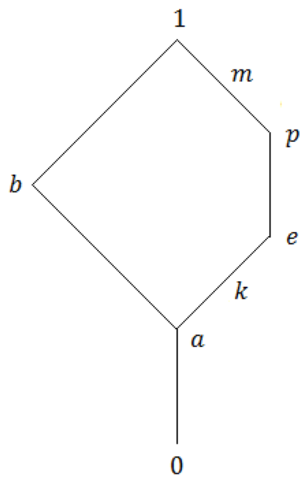

Fig. 1. The lattice $L_1 = L_2$.

U_1	0	a	k	e	b	p	m	1
0	0	0	0	0	b	p	m	1
a	0	a	a	a	b	p	m	1
k	0	a	k	k	b	p	m	1
e	0	a	k	e	b	p	m	1
b	b	b	b	b	1	1	1	1
p	p	p	p	p	1	1	1	1
m	m	m	m	m	1	1	1	1
1	1	1	1	1	1	1	1	1

Tab. 1. The uninorm U_1 on $L_1 = L_2$.

Since $U_1(a,k) = a$ and $U_2(a,k) = a$, then we obtain that $a \preceq_{U_1} k$ and $a \preceq_{U_2} k$. So, it is obtained $(a,a) \preceq_{U_1\times U_2} (k,k)$ by Definition 10. Also, we want to show that $(p,p) \npreceq_{U_1\times U_2} (m,m)$. We assume that $(p,p) \preceq_{U_1\times U_2} (m,m)$. Then, it must be $p \preceq_{U_1}$

U_2	0	a	k	e	b	p	m	1
0	0	0	0	0	b	p	m	1
a	0	a	a	a	b	p	m	1
k	0	a	k	k	b	p	m	1
e	0	a	k	e	b	p	m	1
b	b	b	b	b	b	1	1	1
p	p	p	p	p	1	p	m	1
m	m	m	m	m	1	m	m	1
1	1	1	1	1	1	1	1	1

Tab. 2. The uninorm U_2 on $L_1 = L_2$.

m and $p \preceq_{U_2} m$ by Definition 10. Then, there exist elements $s, \ell \in [e, 1]$ such that $U_1(p, s) = m$ and $U_2(p, \ell) = m$. According to the Table 1, we obtain $U_1(p, s) = m$, a contradiction. Because there does not exist an element $s \in [e, 1]$ such that $U_1(p, k) = m$. So, it must be $p \not\preceq_{U_1} m$. So, it is clear that $(p, p) \not\preceq_{U_1 \times U_2} (m, m)$ by Definition 10.

Proposition 3. Let U_1 be a uninorm on L_1 with neutral element e_1 and U_2 be a uninorm on L_2 with neutral element e_2 and consider their direct product $U_1 \times U_2$ on $L_1 \times L_2$. Then, the relation $\preceq_{U_1 \times U_2}$ defined in Definition 10 is a partial order on $L_1 \times L_2$.

Proof. Since $\preceq_{U_1}$ and $\preceq_{U_2}$ are partial orders on L_1 and L_2, respectively, it is clear that $\preceq_{U_1 \times U_2}$ is a partial order on $L_1 \times L_2$. □

Proposition 4. Let L_1 and L_2 be bounded lattices, U_1 be a uninorm on L_1 with neutral element e_1 and U_2 be a uninorm on L_2 with neutral element e_2 and consider their direct product $U_1 \times U_2$ on $L_1 \times L_2$. Then, $L_1 \times L_2$ is a bounded partially ordered set with respect to the $\preceq_{U_1 \times U_2}$ partial order.

Proof. It is clear that $L_1 \times L_2$ is a partially ordered set with respect to the $\preceq_{U_1 \times U_2}$ partial order. Since $0_1 \preceq_{U_1} x$, $0_2 \preceq_{U_2} y$ and $x \preceq_{U_1} 1_1$, $y \preceq_{U_2} 1_2$, then it is obtained that $(0_1, 0_2) \preceq_{U_1 \times U_2} (x, y)$ and $(x, y) \preceq_{U_1 \times U_2} (1_1, 1_2)$ for all $(x, y) \in L_1 \times L_2$ by Definition 10. □

Note: From Definition 1, it is clear that $\leq_i$ is a partial order on U_i.

Remark 1. Let L_1 and L_2 be bounded lattices, U_1 be a uninorm on L_1 with neutral element e_1 and U_2 be a uninorm on L_2 with neutral element e_2 and consider their direct product $U_1 \times U_2$ on $L_1 \times L_2$. Then,

$$(x, y) \preceq_{U_1 \times U_2} (z, t) \Rightarrow x \leq_1 z \textit{ and } y \leq_2 t$$

for all $(x, y), (z, t) \in L_1 \times L_2$.

Lemma 1. Let L_1 and L_2 be bounded lattices, T_1 be a t-norm on L_1 and T_2 be a t-norm on L_2 and consider their direct product $T_1 \times T_2$ on $L_1 \times L_2$. $T_1 \times T_2$ is divisible if and only if T_1 and T_2 are divisible.

Proof. Let $T_1 \times T_2$ is divisible t-norm on $L_1 \times L_2$. Let $x \leq_1 y$ and $z \leq_2 t$ for $x, y \in L_1$ and $z, t \in L_2$. Then, we have $(x, z) \leq (y, t)$ from Definition 1. Since $T_1 \times T_2$ is divisible, then we have $(x, z) = T_1 \times T_2((y, t), (k, m))$ for $(k, m) \in L_1 \times L_2$. Then, it is clear that $(x, z) = (T_1(y, k), T_2(t, m))$ by Definition 4. Since $x = T_1(y, k)$ and $y = T_2(t, m)$, then we have that T_1 and T_2 are divisible t-norms on L_1 and L_2, respectively. Similarly, if T_1 and T_2 are divisible, then it can be shown that $T_1 \times T_2$ is divisible. □

Lemma 2. Let L_1 and L_2 be bounded lattices, S_1 be a t-conorm on L_1 and S_2 be a t-conorm on L_2 and consider their direct product $S_1 \times S_2$ on $L_1 \times L_2$. $S_1 \times S_2$ is divisible if and only if S_1 and S_2 are divisible.

Proof. It can be proved using similar arguments in the proof of Lemma 1. □

Proposition 5. Let L_1 and L_2 be bounded lattices, U_1 and U_2 be uninorms on L_1 and L_2 with neutral elements e_1 and e_2, respectively, T_1 and T_2 be t-norms on $[0, e_1]$ and $[0, e_2]$, respectively and S_1 and S_2 be t-conorms on $[e_1, 1]$ and $[e_2, 1]$, respectively. Consider direct products $U_1 \times U_2$ on $L_1 \times L_2$, $T_1 \times T_2$ on $[0, e_1] \times [0, e_2]$ and $S_1 \times S_2$ on $[e_1, 1] \times [e_2, 1]$. Then, $T_1 \times T_2$ and $S_1 \times S_2$ are divisible if and only if $\preceq_{U_1 \times U_2} = \leq$.

Proof. The proof can be obtained from Lemma 1 and Lemma 2. □

Proposition 6. (De Baets and Mesiar [16]) Let T_1 and T_2 be t-norms on $[0, 1]$ and their direct product $T_1 \times T_2$ on $[0, 1]^2$. $T_1 \times T_2$ is divisible if and only if $T_1 \times T_2$ is continuous.

Proposition 7. (De Baets and Mesiar [16]) Let S_1 and S_2 be t-conorms on $[0, 1]$ and their direct product $S_1 \times S_2$ on $[0, 1]^2$. $S_1 \times S_2$ is divisible if and only if $S_1 \times S_2$ is continuous.

Corollary 1. Let U_1 and U_2 be uninorms on $[0, 1]$ with neutral elements e_1 and e_2, respectively, T_1 and T_2 be t-norms on $[0, e_1]$ and $[0, e_2]$, respectively and S_1 and S_2 be t-conorms on $[e_1, 1]$ and $[e_2, 1]$, respectively. Consider direct products $U_1 \times U_2$ on $L_1 \times L_2$, $T_1 \times T_2$ on $[0, e_1] \times [0, e_2]$ and $S_1 \times S_2$ on $[e_1, 1] \times [e_2, 1]$. Then, $T_1 \times T_2$ and $S_1 \times S_2$ are continuous if and only if $\preceq_{U_1 \times U_2} = \leq$.

4. SOME INVESTIGATIONS ON THE SET OF COMPARABLE AND INCOMPARABLE ELEMENTS WITH RESPECT TO THE $\preceq_{U_1 \times U_2}$-PARTIAL ORDER

In this section, we investigate some properties of direct product of uninorms on bounded lattices. We define comparable and incomparable elements with respect to the $U_1 \times U_2$ partial order on bounded lattices. By using these definitions, we obtain some interesting results for direct product of uninorms on $[0, 1]^2$.

Definition 11. Let L be a bounded lattice and U be a uninorm L. The set C_U is defined as follows:

$$C_U = \{x \in L| \quad there \quad exist \quad y, y' \in L \setminus \{0, x, 1\}, \quad x \preceq_U y \quad \text{and} \quad y' \preceq_U x\}.$$

Remark 2. It is clear that $\{0, 1\} \notin C_U$. If we take $y, y' \in \{0, x, 1\}$, then it is trivial that all elements in L satisfy the condition of Definition 11. So, we have to take $y, y' \notin \{0, x, 1\}$ in Definition 11.

Example 4. Consider the lattice $(L = \{0, a, e, b, r, p, s, 1\}, \leq, 0, 1)$ which is depicted by Hasse diagram in Figure 2. and consider any uninorm on L.

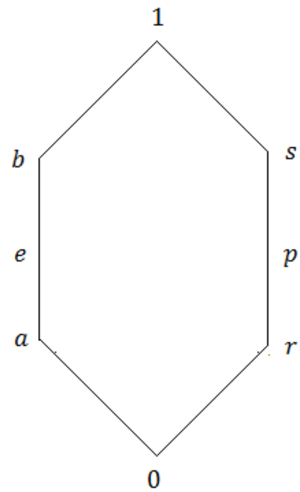

Fig. 2. The lattice L.

Since $U(e, a) = a$ and $U(b, e) = b$, then it is obtained that $a \preceq_U e$ and $e \preceq_U b$. So, $e \in C_U$. Since $p \parallel e$, $r \parallel e$ and $r < p$, then we have $r \preceq_U p$ by the definition of $\preceq_U$. Similarly, it is obtained that $p \preceq_U s$. So, $p \in C_U$. Thus, it is obtained that $C_U = \{e, p\}$.

Example 5. Consider the uninorm $\overline{U_{\frac{1}{3}}} : [0, 1]^2 \to [0, 1]$ with neutral element $\frac{1}{3}$ defined by

$$\overline{U_{\frac{1}{3}}}(x, y) = \begin{cases} \min(x, y) & (x, y) \in [0, \frac{1}{3}]^2, \\ 1 & (x, y) \in (\frac{1}{3}, 1]^2, \\ \max(x, y) & \text{otherwise.} \end{cases}$$

Then, $C_{\overline{U_{\frac{1}{3}}}} = (0, \frac{1}{3}]$. Now, let us show that this claim. Let $x \in (0, \frac{1}{3}]$. Since $\overline{U_{\frac{1}{3}}}(x, y) = y$, then it must be $y \preceq_{\overline{U_{\frac{1}{3}}}} x$ for all $y \in [0, x]$. Since $\overline{U_{\frac{1}{3}}}(x, y') = y'$, then it must be $x \preceq_{\overline{U_{\frac{1}{3}}}} y'$

for all $y' \in [\frac{1}{3}, 1]$. So, it is obtained that $x \in C_{\overline{U_{\frac{1}{3}}}}$, i.e., $(0, \frac{1}{3}] \subseteq C_U$. Conversely let $x \in C_{\overline{U_{\frac{1}{3}}}}$. We want to show that $x \in (0, \frac{1}{3}]$. Suppose that $x \notin (0, \frac{1}{3}]$. Then, it must be $x = 0$ or $x \in (\frac{1}{3}, 1]$. According to the Remark 2, it can not be $x = 0$. So, it must be $x \in (0, \frac{1}{3}]$. Since $x \in C_{\overline{U_{\frac{1}{3}}}}$, there exist elements $y, y' \in (0,1) \setminus \{x\}$ such that $x \preceq_{\overline{U_{\frac{1}{3}}}} y$ and $y' \preceq_{\overline{U_{\frac{1}{3}}}} x$. Let $x \preceq_{\overline{U_{\frac{1}{3}}}} y$. Then, there exist $k \in (\frac{1}{3}, 1]$ such that $\overline{U_{\frac{1}{3}}}(x, k) = y$. Since $x, k \in (\frac{1}{3}, 1]$, by the definition of $\overline{U_{\frac{1}{3}}}$, it is obtained that $\overline{U_{\frac{1}{3}}}(x, k) = 1 = y$, a contradiction. So, it must be $x \not\preceq_{\overline{U_{\frac{1}{3}}}} y$, i.e., $x \notin C_{\overline{U_{\frac{1}{3}}}}$. Thus, it must be $x \in (0, \frac{1}{3}]$, $C_U \subseteq (0, \frac{1}{3}]$.

Proposition 8. Let L_1 and L_2 be bounded lattices, U_1 be a uninorm on L_1 with neutral element e_1 and U_2 be a uninorm on L_2 with neutral element e_2. If $\preceq_{U_1} \subseteq \preceq_{U_2}$, then $C_{U_1} \subseteq C_{U_2}$.

Proof. Let $\preceq_{U_1} \subseteq \preceq_{U_2}$. We assume that $C_{U_1} \nsubseteq C_{U_2}$. Then, it must be $x \in C_{U_1}$ and $x \notin C_{U_2}$. Since $x \in C_{U_1}$, there exist $y, y' \in L \setminus \{0, x, 1\}$ such that $x \preceq_{U_1} y$ and $y' \preceq_{U_1} x$. Since $\preceq_{U_1} \subseteq \preceq_{U_2}$, then we obtain that $x \preceq_{U_2} y$ and $y' \preceq_{U_2} x$. So, it is obtained that $x \in C_{U_2}$, a contradiction. Thus, it must be $C_{U_1} \subseteq C_{U_2}$. □

Corollary 2. Let L_1 and L_2 be bounded lattices, U_1 be a uninorm on L_1 with neutral element e_1 and U_2 be a uninorm on L_2 with neutral element e_2. If $\preceq_{U_1} = \preceq_{U_2}$, then $C_{U_1} = C_{U_2}$.

Remark 3. The converse of Corollary 2 may not be true. Here is an example illustrating such a case.

Example 6. Consider the uninorm $\underline{U_{\frac{1}{2}}} : [0,1]^2 \to [0,1]$ with neutral element $\frac{1}{2}$ defined by

$$\underline{U_{\frac{1}{2}}}(x,y) = \begin{cases} 0 & (x,y) \in [0, \frac{1}{2})^2, \\ \max(x,y) & (x,y) \in [\frac{1}{2}, 1]^2, \\ \min(x,y) & \text{otherwise,} \end{cases}$$

and consider the uninorm $U := U_{\min(T^{nM}, S_M, \frac{1}{2})} : [0,1]^2 \to [0,1]$ with neutral element $\frac{1}{2}$ defined as follows:

$$U_{\min(T^{nM}, S_M, \frac{1}{2})}(x,y) = \begin{cases} 0 & (x,y) \in [0, \frac{1}{2}]^2 \text{ and } x + y \le \frac{1}{2}, \\ \max(x,y) & (x,y) \in [\frac{1}{2}, 1]^2, \\ \min(x,y) & \text{otherwise.} \end{cases}$$

It can be shown that $C_U = [\frac{1}{2}, 1)$ and $C_{\underline{U_{\frac{1}{2}}}} = [\frac{1}{2}, 1)$. That is, $C_U = C_{\underline{U_{\frac{1}{2}}}}$. But, it does not need to be $\preceq_U = \preceq_{\underline{U_{\frac{1}{2}}}}$.

Now, we will show that this claim. Since $U(\frac{1}{3}, \frac{1}{4}) = \frac{1}{4}$, it must be $\frac{1}{4} \preceq_U \frac{1}{3}$. On the other hand $\frac{1}{4} \not\preceq_{\underline{U_{\frac{1}{2}}}} \frac{1}{3}$. We suppose that $\frac{1}{4} \preceq_{\underline{U_{\frac{1}{2}}}} \frac{1}{3}$. Then, there exists an element

$k \in [0, \frac{1}{2}]$ such that $\underline{U_{\frac{1}{2}}}(\frac{1}{3}, k) = \frac{1}{4}$. Since $k \in [0, \frac{1}{2}]$, it must be $\underline{U_{\frac{1}{2}}}(\frac{1}{3}, k) = \frac{1}{4} = 0$, a contradiction. So, $\frac{1}{4} \not\preceq_{\underline{U_{\frac{1}{2}}}} \frac{1}{3}$. Consequently, $\preceq_U \neq \preceq_{\underline{U_{\frac{1}{2}}}}$.

The set C_U allows us to introduce the next equivalence relation on the class of all uninorms on bounded lattices.

Definition 12. Define a relation δ on the class of all uninorms on bounded lattices by $U_1 \delta U_2$

$$U_1 \delta U_2 :\Leftrightarrow C_{U_1} = C_{U_2}.$$

Lemma 3. The relation δ given in Definition 12 is an equivalence relation.

Definition 13. For a given uninorm U on bounded lattice L, we denote by $\overline{U}$ the δ equivalence class linked to U, i.e,

$$\overline{U} = \{U' | \quad U' \delta U\}.$$

If we take $L = [0, 1]$, then we obtain the following Proposition 9 and Proposition 10.

Proposition 9. The set $[0, 1]/\delta$ of all equivalence classes of all uninorms on the unit interval $[0, 1]$ under δ, is uncountably infinite.

Proof. Let $e_1, e_2 \in (0, 1)$ be arbitrary but fixed two elements and $e_1 \neq e_2$. We assume that $e_1 < e_2$.

Consider the uninorms on the unit interval $[0, 1]$ with neutral elements e_1 and e_2, respectively defined as follows:

$$\underline{U_{e_1}}(x, y) = \begin{cases} 0 & (x, y) \in [0, e_1)^2, \\ \max(x, y) & (x, y) \in [e_1, 1]^2, \\ \min(x, y) & \text{otherwise,} \end{cases}$$

and

$$\underline{U_{e_2}}(x, y) = \begin{cases} 0 & (x, y) \in [0, e_2)^2, \\ \max(x, y) & (x, y) \in [e_2, 1]^2, \\ \min(x, y) & \text{otherwise.} \end{cases}$$

It can be shown that $C_{\underline{U_{e_1}}} = [e_1, 1)$ and $C_{\underline{U_{e_2}}} = [e_2, 1)$. Since $e_1 < e_2$, then we have that the uninorms $\underline{U_{e_1}}$ and $\underline{U_{e_2}}$ are not equivalent under the relation δ. So, we obtain that $\overline{\underline{U_{e_1}}} \neq \overline{\underline{U_{e_2}}}$.

Define the mapping $\alpha : (0, 1) \to [0, 1]/\delta$ by

$$\alpha(e) = \overline{\underline{U_e}}.$$

We showed that $\alpha(e_1) \neq \alpha(e_2)$ for $e_1 \neq e_2$. So, α is an injective function, and it is obtained that $|(0, 1)| \leq |[0, 1]/\delta|$. So, the set $[0, 1]/\delta$ has uncountably infinite cardinality. □

Proposition 10. Let $e \in [0,1]$. If $U \in \mathcal{U}(e)$, then

$$U(x,y) = \begin{cases} T_U(x,y) & (x,y) \in [0,e]^2, \\ S_U(x,y) & (x,y) \in [e,1]^2, \\ D(x,y) & (x,y) \in A(e), \end{cases}$$

where T_U is a t-norm on $[0,e]$, S_U is a t-conorm on $[e,1]$ and $D : A(e) \to [0,1]$ is increasing and fulfills

$\min(x,y) \le D(x,y) \le \max(x,y)$ for $(x,y) \in A(e)$ by [17].

If T_U and S_U are continuous t-norm and t-conorm, respectively, then $C_U = (0,1)$.

Proof. Let T_U and S_U are continuous t-norm and t-conorm, respectively. Then, it is obtained that $C_{T_U} = (0,e]$ and $C_{S_U} = [e,1)$ from Corollary 1. So, we have that $C_U = (0,1)$. □

Example 7. Let $e \in [0,1]$. Consider the uninorms $U^{\min}$ and $U^{\max}$ as unique idempotent uninorm $U_e^{\min}$ and $U_e^{\max}$, respectively:

$$U^{\min}(x,y) = \begin{cases} \max(x,y) & (x,y) \in [e,1]^2, \\ \min(x,y) & \text{otherwise}, \end{cases}$$

$$U^{\max}(x,y) = \begin{cases} \min(x,y) & (x,y) \in [0,e]^2, \\ \max(x,y) & \text{otherwise}. \end{cases}$$

Then, it is obtained that $C_{U^{\min}} = (0,1)$ and $C_{U^{\max}} = (0,1)$.

The next example shows the importance of continuity in Proposition 10.

Example 8. Consider the uninorm $U := U_{\min(T^{nM}, S_M, \frac{1}{2})} : [0,1]^2 \to [0,1]$ with neutral element $\frac{1}{2}$ defined in Example 6. Since T^{nM} is left continous t-norm, it need not be $C_U = (0,1)$. Also, it is clear that $C_U = [\frac{1}{2}, 1)$.

Proposition 11. Let L_1 and L_2 be bounded lattices, U_1 be a uninorm on L_1 with neutral element e_1 and U_2 be a uninorm on L_2 with neutral element e_2 and consider their direct product $U_1 \times U_2$ on $L_1 \times L_2$. Then,

$$C_{U_1 \times U_2} = C_{U_1} \times C_{U_2}.$$

Proof. Let $(x,y) \in C_{U_1 \times U_2}$. Then there exist (z,t) and (k,ℓ) in $L^2 \setminus \{(0_1,0_2),(x,y),(1_1,1_2)\}$ such that $(x,y) \preceq_{U_1 \times U_2} (z,t)$ and $(k,\ell) \preceq_{U_1 \times U_2} (x,y)$. Then we obtain $x \preceq_{U_1} z$, $y \preceq_{U_2} t$ and $k \preceq_{U_1} x$, $\ell \preceq_{U_2} y$ by Definition 10. So, we have $x \preceq_{U_1} z$, $k \preceq_{U_1} x$ and $y \preceq_{U_2} t$, $\ell \preceq_{U_2} y$. Then, it is obtained that $x \in C_{U_1}$ and $y \in C_{U_2}$. Thus, $(x,y) \in C_{U_1} \times C_{U_2}$.

Conversely, let $(x,y) \in C_{U_1} \times C_{U_2}$. Then, it must be $x \in C_{U_1}$ and $y \in C_{U_2}$. Since $x \in C_{U_1}$, there exist elements $z, z' \in L \setminus \{0_1, x, 1_1\}$ such that $x \preceq_{U_1} z$ and $z' \preceq_{U_1} x$. Similarly, since $y \in C_{U_2}$, there exist elements $k, k' \in L \setminus \{0_2, y, 1_2\}$ such that $y \preceq_{U_2} k$ and $k' \preceq_{U_2} y$. So, $(x,y) \preceq_{U_1 \times U_2} (z,k)$ and $(z',k') \preceq_{U_1 \times U_2} (x,y)$. Then, it is obtained that $(x,y) \in C_{U_1 \times U_2}$. Consequently, we have $C_{U_1 \times U_2} = C_{U_1} \times C_{U_2}$. □

Definition 14. (Kesicioğlu et al. [24]) Let L be a bounded lattice, U be a uninorm on L and let K_U be defined by

$$K_U = \{x \in L \setminus \{0,1\} \mid \text{for some } y \in L \setminus \{0,1\},\ [x < y \text{ and } x \not\preceq_U y] \text{ or } [y < x \text{ and } y \not\preceq_U x] \text{ or } x \parallel y\}.$$

Proposition 12. Let L_1 and L_2 be bounded lattices, U_1 be a uninorm on L_1 with neutral element e_1 and U_2 be a uninorm on L_2 with neutral element e_2 and consider their direct product $U_1 \times U_2$ on $L_1 \times L_2$. Then,

$$K_{U_1} \times K_{U_2} \subseteq K_{U_1 \times U_2}.$$

Proof. Let $(x,y) \in K_{U_1} \times K_{U_2}$. It must be $x \in K_{U_1}$ and $y \in K_{U_2}$. Then, there exist elements $z, t \in L$ such that $[x \not\preceq_{U_1} z$ or $z \not\preceq_{U_1} x]$ and $[y \not\preceq_{U_2} t$ or $t \not\preceq_{U_2} y]$. In this case, it is obtained that $(x,y) \not\preceq_{U_1 \times U_2} (z,t)$ or $(z,t) \not\preceq_{U_1 \times U_2} (x,y)$. So, we have that $(x,y) \in K_{U_1 \times U_2}$. Consequently, $K_{U_1} \times K_{U_2} \subseteq K_{U_1 \times U_2}$. □

Remark 4. The converse of the Proposition 12 may not be true. Here is an example illustrating such a case.

Example 9. Consider the greatest uninorm $\overline{U_{\frac{1}{2}}} : [0,1]^2 \to [0,1]$ with neutral element $\frac{1}{2}$ defined by

$$\overline{U_{\frac{1}{2}}}(x,y) = \begin{cases} \min(x,y) & (x,y) \in [0, \frac{1}{2}]^2, \\ 1 & (x,y) \in (\frac{1}{2}, 1]^2, \\ \max(x,y) & \text{otherwise}, \end{cases}$$

and the smallest uninorm $\underline{U_{\frac{1}{2}}} : [0,1]^2 \to [0,1]$ with neutral element $\frac{1}{2}$ defined in Example 6.

Now, we want to show that it need not to be $K_{\underline{U_{\frac{1}{2}}} \times \overline{U_{\frac{1}{2}}}} \subseteq K_{\underline{U_{\frac{1}{2}}}} \times K_{\overline{U_{\frac{1}{2}}}}$. We will show that $(\frac{3}{4}, \frac{3}{4}) \in K_{\underline{U_{\frac{1}{2}}} \times \overline{U_{\frac{1}{2}}}}$ and $(\frac{3}{4}, \frac{3}{4}) \notin K_{\underline{U_{\frac{1}{2}}}} \times K_{\overline{U_{\frac{1}{2}}}}$. We claim that $\frac{3}{4} \not\preceq_{\overline{U_{\frac{1}{2}}}} \frac{5}{6}$. Suppose that $\frac{3}{4} \preceq_{\overline{U_{\frac{1}{2}}}} \frac{5}{6}$. Then, there exists an element $k \in [\frac{1}{2}, 1]$ such that $\overline{U_{\frac{1}{2}}}(\frac{3}{4}, k) = \frac{5}{6}$. If $k = \frac{1}{2}$, then it is obtained that $\overline{U_{\frac{1}{2}}}(\frac{3}{4}, k) = \frac{5}{6} = \frac{3}{4}$, a contradiction. If $k \in (\frac{1}{2}, 1]$, then we have that $\overline{U_{\frac{1}{2}}}(\frac{3}{4}, k) = \frac{5}{6} = 1$, a contradiction. So, it holds $\frac{3}{4} \not\preceq_{\overline{U_{\frac{1}{2}}}} \frac{5}{6}$. Thus, $(\frac{3}{4}, \frac{3}{4}) \not\preceq_{\underline{U_{\frac{1}{2}}} \times \overline{U_{\frac{1}{2}}}} (x, \frac{5}{6})$ for all $x \in [\frac{3}{4}, 1]$. So, $(\frac{3}{4}, \frac{3}{4}) \in K_{\underline{U_{\frac{1}{2}}} \times \overline{U_{\frac{1}{2}}}}$. On the other side, since $K_{\underline{U_{\frac{1}{2}}}} = (0, \frac{1}{2})$ by Aşıcı (see [2]), then we have that $\frac{3}{4} \notin K_{\underline{U_{\frac{1}{2}}}}$. So, $(\frac{3}{4}, \frac{3}{4}) \notin K_{\underline{U_{\frac{1}{2}}}} \times K_{\overline{U_{\frac{1}{2}}}}$. Consequently, $K_{\underline{U_{\frac{1}{2}}} \times \overline{U_{\frac{1}{2}}}} \subseteq K_{\underline{U_{\frac{1}{2}}}} \times K_{\overline{U_{\frac{1}{2}}}}$ does not hold.

Remark 5. If we take the uninorms U_1 and U_2 to be equal, then the converse of the Proposition 12 is true, i. e., equality is satisfied.

Remark 6. The converse of the Proposition 12 may be true for some special uninorms on the unit interval $[0, 1]$. Here is an example illustrating such a case.

Example 10. Consider the uninorm $U_1 : [0, 1]^2 \to [0, 1]$ with neutral element $\frac{1}{2}$ defined as follows:

$$U_1(x, y) = \begin{cases} 0 & (x, y) \in [0, \frac{1}{2}]^2 \text{ and } x + y \leq \frac{1}{2} \text{ and } (x, y) \neq (\frac{1}{4}, \frac{1}{4}), \\ \frac{1}{4} & (x, y) = (\frac{1}{4}, \frac{1}{4}), \\ \max(x, y) & (x, y) \in [\frac{1}{2}, 1]^2, \\ \min(x, y) & \text{otherwise}, \end{cases}$$

and consider the uninorm $U := U_{\min(T^{nM}, S_M, \frac{1}{2})} : [0, 1]^2 \to [0, 1]$ with neutral element $\frac{1}{2}$ defined in Example 6. We know that $K_U = (0, \frac{1}{2})$ by Aşıcı (see [2]). Similarly, it can be shown that $K_{U_1} = (0, \frac{1}{2})$. Also, it is clear that $K_{U \times U_1} = (0, \frac{1}{2}) \times (0, \frac{1}{2}) = K_U \times K_{U_1}$.

Definition 15. Let L be a bounded lattice, U be a uninorm on L with neutral element e and $K_U^\star$ defined by

$$K_U^\star = \{x \in K_U \mid \text{for some } y, y' \in L \setminus \{0, 1\},\ [x < y \text{ but } x \not\preceq_U y] \\ \text{and } [y' < x \text{ but } y' \not\preceq_U x]\}.$$

Proposition 13. Let L_1 and L_2 be bounded lattices, U_1 be a uninorm on L_1 with neutral element e_1 and U_2 be a uninorm on L_2 with neutral element e_2 and consider their direct product $U_1 \times U_2$ on $L_1 \times L_2$. Then,

$$K_{U_1}^\star \times K_{U_2}^\star \subseteq K_{U_1 \times U_2}^\star.$$

P r o o f. Let $(x, y) \in K_{U_1}^\star \times K_{U_2}^\star$. Then, we have $x \in K_{U_1}^\star$ and $y \in K_{U_2}^\star$. Then for some $k, k' \in L$ and $\ell, \ell' \in L$ such that $[x < k, x \not\preceq_{U_1} k$ and $k' < x, k' \not\preceq_{U_1} x]$ and $[y < \ell, y \not\preceq_{U_2} \ell$ and $\ell' < y, \ell' \not\preceq_{U_2} y]$. Then, we have that $[(x, y) < (k, \ell)$ but $(x, y) \not\preceq_{U_1 \times U_2} (k, \ell)]$ and $[(k', \ell') < (x, y)$ but $(k', \ell') \not\preceq_{U_1 \times U_2} (x, y)]$. So, it is obtained that $(x, y) \in K_{U_1 \times U_2}^\star$. □

Remark 7. The converse of the Proposition 13 may not be true. Here is an example illustrating such a case.

Example 11. Consider the uninorm $\underline{U_{\frac{1}{2}}} : [0, 1]^2 \to [0, 1]$ with neutral element $\frac{1}{2}$ defined in Example 6 and consider the uninorm $\overline{U_{\frac{1}{2}}} : [0, 1]^2 \to [0, 1]$ with neutral element $\frac{1}{2}$ defined in Example 9.

We want to show that $K_{\underline{U_{\frac{1}{2}}} \times \overline{U_{\frac{1}{2}}}}^\star \nsubseteq K_{\underline{U_{\frac{1}{2}}}}^\star \times K_{\overline{U_{\frac{1}{2}}}}^\star$. We will show that $(\frac{1}{5}, \frac{1}{5}) \in K_{\underline{U_{\frac{1}{2}}} \times \overline{U_{\frac{1}{2}}}}^\star$ and $(\frac{1}{5}, \frac{1}{5}) \notin K_{\underline{U_{\frac{1}{2}}}}^\star \times K_{\overline{U_{\frac{1}{2}}}}^\star$. We claim that $\frac{1}{5} \preceq_{\underline{U_{\frac{1}{2}}}} \frac{1}{4}$. Suppose that $\frac{1}{5} \preceq_{\underline{U_{\frac{1}{2}}}} \frac{1}{4}$. Then, there exists an element $k \in [0, \frac{1}{2}]$ such that $\underline{U_{\frac{1}{2}}}(\frac{1}{4}, k) = \frac{1}{5}$. Since $k \in [0, \frac{1}{2}]$, then we have $\underline{U}_{\frac{1}{2}}(\frac{1}{4},k)=\frac{1}{5}=0$, a contradiction. So, $\frac{1}{5}\not\preceq_{\underline{U}_{\frac{1}{2}}}\frac{1}{4}$. So, $(\frac{1}{5},\frac{1}{5})\not\preceq_{\underline{U}_{\frac{1}{2}}\times\overline{U}_{\frac{1}{2}}}(\frac{1}{4},y)$ for some $y\in[\frac{1}{5},1]$. Also we want to show that $\frac{1}{6}\not\preceq_{\underline{U}_{\frac{1}{2}}}\frac{1}{5}$. Suppose that $\frac{1}{6}\preceq_{\underline{U}_{\frac{1}{2}}}\frac{1}{5}$. Then, there exists an element $\ell\in[0,\frac{1}{2}]$ such that $\underline{U}_{\frac{1}{2}}(\frac{1}{5},\ell)=\frac{1}{6}$. Since If $\ell\in[0,\frac{1}{2}]$, then we have $\underline{U}_{\frac{1}{2}}(\frac{1}{5},\ell)=\frac{1}{6}=0$, a contradiction. So, $\frac{1}{6}\not\preceq_{\underline{U}_{\frac{1}{2}}}\frac{1}{5}$. So, $(\frac{1}{6},y')\not\preceq_{\underline{U}_{\frac{1}{2}}\times\overline{U}_{\frac{1}{2}}}(\frac{1}{5},\frac{1}{5})$ for some $y'\in[0,\frac{1}{5}]$. Consequently, we have that $(\frac{1}{5},\frac{1}{5})\in K^{\star}_{\underline{U}_{\frac{1}{2}}\times\overline{U}_{\frac{1}{2}}}$. On the other side, since $K^{\star}_{\overline{U}_{\frac{1}{2}}}=(\frac{1}{2},1)$ by Aşıcı (see [2]), we have that $\frac{1}{5}\notin K^{\star}_{\overline{U}_{\frac{1}{2}}}$. So, $(\frac{1}{5},\frac{1}{5})\notin K^{\star}_{\underline{U}_{\frac{1}{2}}}\times K^{\star}_{\overline{U}_{\frac{1}{2}}}$. Consequently, $K^{\star}_{\underline{U}_{\frac{1}{2}}\times\overline{U}_{\frac{1}{2}}}\subseteq K^{\star}_{\underline{U}_{\frac{1}{2}}}\times K^{\star}_{\overline{U}_{\frac{1}{2}}}$ does not hold.

Remark 8. If we take the uninorms U_1 and U_2 to be equal, then the converse of the Proposition 13 is true, i. e., equality is satisfied.

Remark 9. The converse of the Proposition 13 may be true for some special uninorms on the unit interval $[0,1]$. Here is an example illustrating such a case.

Example 12. Consider the uninorm $U_1:[0,1]^2\to[0,1]$ with neutral element $\frac{1}{2}$ defined in Example 10 and consider the uninorm $U:=U_{\min(T^{nM},S_M,\frac{1}{2})}:[0,1]^2\to[0,1]$ with neutral element $\frac{1}{2}$ defined in Example 6. We know that $K^{\star}_{U}=(0,\frac{1}{4})$ by Aşıcı (see [2]). Similarly, it can be shown that $K^{\star}_{U_1}=(0,\frac{1}{4})$. Also, it is clear that $K^{\star}_{U\times U_1}=(0,\frac{1}{4})\times(0,\frac{1}{4})=K^{\star}_{U}\times K^{\star}_{U_1}$.

5. CONCLUDING REMARKS

We have introduced and studied uninorms on bounded lattices. On one side, we have developed several new results in the domain of uninorms acting on bounded lattices, including the direct products of bounded lattices, on the other side, we have introduced several new results for classical uninorms acting on the real unit interval $[0,1]$. Among others, we have studied new partial orderings induced by uninorms on bounded lattices. As already stressed in introduction, uninorms (together with related operations such as t-norms, t-conorms) have important applications in several domains such as decision making in fuzzy environment, general measure and integral theory (again with application in ordinal decision making), image processing, etc.

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Emel Aşıcı, Corresponding author. Department of Software Engineering, Faculty of Technology, Karadeniz Technical University, 61830 Trabzon. Turkey.
e-mail: emelkalin@hotmail.com

Radko Mesiar, Department of Mathematics and Descriptive Geometry, Faculty of Civil Engineering, Slovak University of Technology, Radlinského 11, 81005 Bratislava. Slovak Republic.
e-mail: mesiar@math.sk