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STRUCTURAL IDENTIFIABILITY ANALYSIS OF NONLINEAR TIME DELAYED SYSTEMS WITH GENERALIZED FREQUENCY RESPONSE FUNCTIONS

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In this paper a novel method is proposed for the structural identifiability analysis of nonlinear time delayed systems. It is assumed that all the nonlinearities are analytic functions and the time delays are constant. We consider the joint structural identifiability of models with respect to the ordinary system parameters and time delays by including delays into a unified parameter set. We employ the Volterra series representation of nonlinear dynamical systems and make use of the frequency domain representations of the Volterra kernels, i. e. the Generalized Frequency Response Functions (GFRFs), in order to test the unique computability of the parameters. The advantage of representing nonlinear systems with their GFRFs is that in the frequency domain representation the time delay parameters appear explicitly in the exponents of complex exponential functions from which they can be easily extracted. Since the GFRFs can be symmetrized to be unique, they provide us with an exhaustive summary of the underlying model structure. We use the GFRFs to derive equations for testing structural identifiability. Unique solution of the composed equations with respect to the parameters provides sufficient conditions for structural identifiability. Our method is illustrated on non-linear dynamical system models of different degrees of non-linearities and multiple time delayed terms. Since Volterra series representation can be applied for input-output models, it is also shown that after differential algebraic elimination of unobserved state variables, the proposed method can be suitable for identifiability analysis of a more general class of non-linear time delayed state space models.

Keywords: structural identifiability, Volterra series, generalized frequency response

Classification: 93B30

1. INTRODUCTION

Several dynamical systems of physical, chemical and biological importance can be modeled by means of continuous differential equations [28, 56]. An important step in constructing appropriate mathematical models is estimating the model parameters [26, 44, 51, 55]. Accurate estimation of parameters is of paramount importance especially if the purpose is to make predictions or design a controller based on the identified model.

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A related problem of parameter estimation is structural identifiability (also called prior, theoretical or qualitative identifiability): a given model structure is said to be structurally identifiable with respect to a set of parameters, if the exact parameter values of the set can be uniquely determined in theory, assuming unlimited, noise-free observational data [2, 32, 47]. Structural identifiability is a property depending on the underlying model structure and possibly on the initial conditions. It is independent of the quality and amount of available observed data. Examining structural identifiability of model structures is especially important if the system parameters are endowed with physical, chemical or biological significance.

Structural identifiability was introduced among the first by Bellman and Astrom in the framework of linear systems theory [6]. They used the parameterized transfer function of the system as exhaustive summary in order to obtain necessary and sufficient conditions on structural identifiability. In the context of linear systems the similarity transformation approach is based on finding invertible transformations of the state space realization matrices [52]. Assuming joint observability and controllability, by Kalman's algebraic rank condition the similarity transformation approach provides necessary and sufficient conditions for structural identifiability. In the case of non-linear systems, deciding structural identifiability is computationally much more complicated. For uncontrolled autonomous systems the direct test approach provides a conceptually simple, but limited method by equating the system non-linearities of different parameterizations [14]. The similarity transformation based approach was extended to non-linear systems by means of the local state isomorphism theorem [48, 49]. For locally reduced systems (structural controllability and observability conditions have to be fulfilled), it seeks for state variable transformations and leads to solving a set of partial differential equations [19]. The Taylor series approach expands the model output in Taylor series around $t = 0$ [37]. The coefficients of the resulting power series are unique and provide an exhaustive summary for testing structural identifiability. Conceptually similar to the Taylor series based expansion, the generating series approach employs the Fliess series expansion of the observables [53]. The coefficients of the series, which are unique descriptors of the input-output behavior, are the Lie derivatives of the model output. Both Taylor series and generating series expansions provide sufficient conditions of structural identifiability, but the algebraic expressions obtained by the generating series approach are often simpler [11, 54]. The differential algebra based approach reformulates the system equations so that the non-measured state variables are eliminated in order to obtain an equivalent set of differential algebraic equations containing only the model inputs, outputs and system parameters [27]. Elimination of unobserved variables can be performed by Ritt's pseudodivision algorithm [7]. The resulting set of equations is called the input-output map or characteristic set, which provides a Grbner basis of the model equations [38]. Necessary and sufficient conditions on structural identifiability can be obtained by the characteristic set [27]. The implicit function theorem approach also eliminates the unobservable variables [60]. It determines a matrix composed of the derivatives of non-linearities with respect to the system parameters. Then structural identifiability testing translates to the non-singularity of the obtained matrix. Assuming constant input excitation, testing structural identifiability can be performed by means of the concept of non-linear observability [57, 58]. Viewing the constant parameters as state variables with zero dynamics, structural identifiability can be examined through the observability of the extended state vector involving the system parameters. Then identifiability can be examined through the rank of the extended non-linear observability matrix.

In practice, many dynamical processes involve time delayed terms [9, 22, 30, 31, 43], and this is true for biological models as well. For example, metabolic regulatory networks may contain delayed signaling pathways, e. g. delayed feedback loops, which imply specific qualitative dynamical phenomena [12, 20, 23, 24]. Time delayed signals are also proven to be useful for controlling dynamics of biological networks [34]. Time delayed models are often employed to model the dynamical behavior of systems in population biology and epidemiology [13, 22, 25]. Delayed terms are also used in models of chemical kinetics (e. g. if the dynamics is partially known or certain intermediates are omitted from the description for simplicity) [15, 39].

A related problem arising in modeling with time delayed differential equations is to examine the possibility of uniquely determining the constant delay parameters. Structural identifiability of time delayed systems – including the delays as parameters – has received less attention. Compared to ordinary system parameters, time delays appear implicitly in the inputs, outputs and internal state variables, which makes the respective identifiability problem more involved. Identifiability of delayed systems is typically analyzed in the context of dynamical systems of some specific structure. For linear time delayed systems sufficient condition on joint identifiability of ordinary parameters and delays can be obtained [33]. Furthermore, it is shown that under sufficiently exciting input signals, weak controllability provides necessary and sufficient condition for identifiability [5]. For linear retarded functional differential equations the unique identifiability of coefficients, delays and initial conditions were also examined, necessary and sufficient conditions for identifiability are available [29]. In case of non-linear systems, assuming constant integer time delays, identifiability was examined in [63], however, identifiability analysis was restricted to the ordinary parameters not including the delays. In [1] authors examined the class of nonlinear systems with a single constant time delay and considered the identifiability problem of the delay parameter. It is shown that identifiability of the time delay parameter is a necessary condition of observability of state variables and identifiability of ordinary system parameters. The authors also showed that by state elimination, the resulting input-output relations can be used to decide the identifiability of the delay parameter. Linear algebraic conditions are also obtained to eliminate explicit calculation of the input-output relations for testing identifiability. In [64] the case of delay identifiability in nonlinear systems with unknown inputs was examined. The proposed approach is based on the deduction of an output-delay equation, which is proven to be related to identifiability. Assuming the existence of the output-delay equation in the single delay case, necessary and sufficient conditions were derived for delay identifiability. The results are shown to be necessary for the more general case of systems with multiple delays. For the class of kinetic systems (a subclass of non-negative polynomial systems) polynomial time algorithms were proposed for finding delayed system realizations with different parameterizations [46]. This way an efficient computational method was obtained that can be employed to test (local) structural identifiability. However, the delays are assumed to be known, hence they cannot be involved in the identifiability analysis.

In this paper a novel approach is proposed for testing structural identifiability of non-linear time delayed systems. Compared to existing identifiability methods, the proposed approach can be used to examine structural identifiability of delay parameters and ordinary system parameters jointly with arbitrary analytic non-linearities in the model structures. It is assumed that the non-linearities are analytic functions and the delays are constants. It is important that the proposed method does not prescribe upper limit on the number of (different) delays in the model structure. Making use of the Volterra series representation of nonlinear systems, sufficient conditions are provided for structural identifiability. The generalized frequency response functions (GFRFs), i. e. the Fourier transforms of the Volterra kernel functions, are used to construct identifiability conditions. It is shown that structural identifiability of delayed systems can be translated to solving a set of algebraic equations with respect to the ordinary parameters and time delays.

The paper is organized as follows. Section 2 introduces the definition of structural identifiability for time delayed systems in which the delays are fixed constants. In Section 3.1 the Volterra series representation is summarized for non-linear parametric input-output models. Section 3.2 outlines the harmonic probing method for computation of the GFRFs, which is extensively used in the sequel. In Section 4 the proposed method for testing structural identifiability is discussed. Section 5 provides computational examples for illustrating the proposed approach.

2. STRUCTURAL IDENTIFIABILITY OF TIME DELAYED SYSTEMS

We consider a continuous SISO input-output dynamical system model

$$M(t,\theta,u,y,p)=0, \tag{1}$$

where u and y are the input excitation and the respective output of the system. The signals u and y can involve arbitrary number of constant time delays, i. e.

$$\begin{aligned} u &= \big\{u(t),u(t-T_1^u),\dots\ u(t-T_{k_u}^u)\big\},\\ y &= \big\{y(t),y(t-T_1^y),\dots\ y(t-T_{k_y}^y)\big\}, \end{aligned} \tag{2}$$

with k_u, k_y denoting the number of delayed components with respect to u and y. θ denotes the set of parameters, which includes the time delays $T_1^u,\dots T_{k_u}^u,T_1^y,\dots T_{k_y}^y\in\mathbb{R}$. θ is assumed to be independent of the initial conditions and the input signals. p denotes the differentiation operator, which is omitted in the sequel. The system operator $M(.)$ is assumed to be analytic. Moreover, we assume zero initial conditions for $M(\cdot)$ in Eq. (1) (see, e. g. [10]). We note that the analytic assumption on $M(.)$ is not restrictive as it is satisfied by several important systems, e. g. systems of polynomial non-linearities, which are widely used to model physical, chemical and biological systems. $M(.)$ may also involve non-linearities in terms of θ.

The input-output model structure defined by Eq. (1) may be obtained from non-linear state space models by differential algebraic elimination of unobserved state variables [18, 27].

Example 2.1. In order to illustrate the mapping $M(.)$ we use the following simple system model:

$$\dot{y}(t)=ay(t-T_1^y)+bu(t-T_1^u)+cy^3(t-T_2^y), \tag{3}$$

where $y^n(.)$ denotes the nth power of $y(.)$. The parameter vector is $\theta = \{a, b, c, T_1^u, T_1^y, T_2^y\}$, where a, b and c are ordinary system parameters, while T_1^u , T_1^y and T_2^y are constant time delay parameters. In this particular case $M(.)$ takes the form

$$M(t, u, y, \theta) = \dot{y}(t) - ay(t - T_1^y) - bu(t - T_1^u) - cy^3(t - T_2^y). \tag{4}$$

Prior to any parameter estimation procedure performed on a model of E.q (1), it is useful to examine whether it is theoretically possible to uniquely determine the system parameters.

Definition 2.2. The model of Eq. (1) is said to be structurally globally identifiable (s.g.i.), if

$$y(\theta) = y(\hat{\theta}) \;\Rightarrow\; \theta = \hat{\theta} \tag{5}$$

for any measurable value of θ, where $y(\theta)$ denotes the output of the system Eq. (1) parameterized with θ.

If Eq. (5) is valid only in a bounded neighborhood $\mathcal{V}(\theta)$ of θ, then the system is said to be structurally locally identifiable (s.l.i.) around θ. If the system is not identifiable, it is called structurally unidentifiable (s.u.i.). If the identfiability definitions are restricted to a subset $\overline{\theta} \subset \theta$, then the respective parameters $\overline{\theta}$ are said to be s.g.i, s.l.i and s.u.i.

By involving the time delays in the parameter set θ, structural identifiability is considered jointly for the ordinary system parameters and the time delays.

Structural identifiability is a model property depending on the underlying model structure and possibly on the initial conditions. It is independent of the amount and quality of data available about the system dynamics.

3. INPUT-OUTPUT REPRESENTATION FOR IDENTIFIABILITY ANALYSIS

In this section the Volterra series representation of non-linear dynamic input-output models is discussed. We consider the Volterra kernels and their respective frequency domain representations, which provide unqie description of the underlying input-output behavior. A computational method is also reviewed for determining the frequency domain representations. Later we employ the obtained frequency domain descriptors to derive sufficient conditions for structural identifiability of ordinary system parameters and constant time delays. The notations and results of this section in the context of Volterra series are based on the references [45, 62].

3.1. Volterra series representation for non-linear input-output models

Here we provide the Volterra series representation for non-linear SISO system models. The equivalent frequency domain representation is also detailed which will be extensively used in the sequel for identifiability analysis.

The Volterra series representation of a dynamical system of the form of Eq. (1) can be written as [10, 59]:

$$y(t) = y_0(t) + \sum_{i=1}^{\infty} y_i(t), \tag{6}$$

where $y_n(t)$ is the nth-order non-linearity, which is represented by a series of generalized convolutional integrals:

$$y_n(t) = \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} h_n(\tau_1, \dots \tau_n) \prod_{j=1}^{n} u(t - \tau_j)\, \mathrm{d}\tau_j, \tag{7}$$

where $h_n(\tau_1, \dots \tau_n)$ is the nth-order Volterra kernel. $h_n(\tau_1, \dots \tau_n)$ is a so-called generalized impulse response function. Clearly, for a linear mapping $M(.)$, $h_1(\tau_1)$ is the impulse response function known from linear system theory [40]. Necessary and sufficient conditions for the existence of Volterra series representation of a nonlinear dynamical system are derived in [35]. If all the non-linearities in Eq. (1) are analytic (e. g. polynomial systems) or can be approximated by polynomials with arbitrary precision, then the existence of input-output representation of the form Eq. (6) is guaranteed. Note that the introduction of time delays in the input and output does not affect the analytic property of the system model, hence Volterra series expansion is also available for time delayed systems [45].

The frequency domain description of non-linear systems through the Volterra series representation can be obtained by the multidimensional Fourier transformation of the Volterra kernels [17]:

$$H_n(j\omega_1, \dots j\omega_n) = \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} h_n(\tau_1, \dots \tau_n) e^{-j(\omega_1\tau_1 + \dots \omega_n\tau_n)} \, \mathrm{d}\tau_1 \dots \mathrm{d}\tau_n. \tag{8}$$

$H_n(j\omega_1, \dots j\omega_n)$ is called the n-order Generalized Frequency Response Function (GFRF) or simply the nth-order transfer function. Observe that for $n = 1$ $H_n(j\omega_1)$ is the known linear transfer function. Since $h_n(\tau_1, \dots \tau_n)$ and $H_n(j\omega_1, \dots j\omega_n)$ are related through the multivariable Fourier transform, the n-order output can be expressed by the GFRFs:

$$y_n(t) = \frac{1}{(2\pi)^n} \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} H_n(j\omega_1, \dots j\omega_n) \prod_{k=1}^{n} U(j\omega_k) e^{j(\omega_1 + \dots + \omega_n)t} \mathrm{d}\omega_k. \tag{9}$$

$h_n()$ and $H_n()$, $n \geq 1$ are independent of the input signal and they uniquely describe of the input-output behavior of underlying non-linear system [3, 36]. However, since the change of the order in the arguments $\tau_1, \dots \tau_n$ may result in different kernels without affecting the input-output behavior, $h_n()$ and $H_n()$ are not necessarily unique descriptors. Hence it is common to introduce the symmetrized GFRFs as follows:

$$H_n^{sym}(j\omega_1, \dots j\omega_n) = \frac{1}{n!} \sum_{\substack{all \ permutations \\ of \omega_1, \dots \omega_n}} H_n(j\omega_1, \dots j\omega_n). \tag{10}$$

The symmetric GFRFs are independent of the order of arguments and provide unique representations for analytic SISO system models of the from Eq. (1) [45].

3.2. Computation of the GFRFs

In this section computation of the GFRFs for SISO systems is reviewed using the harmonic probing method and the extraction operator [4, 8, 62]. Let us consider an arbitrary non-linear input-output dynamical system model described by Eq. (1) for which a Volterra series representation of Eq (6) exists. Since the output of the system can be expressed by the GFRFs according to Eq. (9), Eq. (1) becomes

$$M(t, \theta, u, \mathcal{H}) = 0, \tag{11}$$

with $\mathcal{H} = \{H_n(j\omega_1, \ \dots j\omega_n)\}_{n \geq 1}$. Expressing $H_n(.)$ using Eq. (11) may lead to complicated integral equations, which make the problem computationally intractable. In order to remedy this problem, special input excitations can be employed. The harmonic probing technique applies a multi-tone input of R distinct frequency components:

$$u(t) = \sum_{i=1}^{R} e^{j\omega_i t}, \tag{12}$$

where $\omega_1, \ldots \omega_R$ are arbitrarily chosen, non-zero different frequencies. Then the respective Fourier transform $U(j\omega)$ is

$$U(j\omega) = \sum_{i=1}^{R} 2\pi\delta(j\omega - j\omega_r). \tag{13}$$

By applying the input Eq. (12) on the system Eq. (1), the output becomes

$$\begin{aligned} y(t) = \sum_{n=1}^{\infty} \sum_{r_1,\ldots r_n=1}^{R} H_n(j\omega_{r_1}, \ldots j\omega_{r_n}) e^{j(\omega_{r_1}+\ldots+\omega_{r_n})t} = \\ \sum_{n=1}^{\infty} \sum_{\substack{[all\ combinations\\ of\ R\ frequencies\\ taken\ n\ at\ a\ time]}} \sum_{\substack{[all\ permutations\\ of\ \omega_{r_1},\ldots\omega_{r_n}]}} H_n(j\omega_{r_1}, \ldots j\omega_{r_n}) e^{j(\omega_{r_1}+\ldots+\omega_{r_n})t}. \end{aligned} \tag{14}$$

In order to determine $H_n(.)$, it is convenient to choose $n = R$ so that there is only one non-repetitive combination of frequencies $\{\omega_1, \ldots \omega_R\}$. Then the nth-order output can be written as

$$y_n(t) = n! H_n^{sym}(j\omega_1, \ldots j\omega_n) e^{j(\omega_1+\ldots+\omega_n)t} + [\text{terms of repetitive frequency components}]. \tag{15}$$

By substituting Eq. (12) and Eq. (14) into Eq. (1) one obtains

$$M(t, \theta, u(\Omega_R), \mathcal{H}) = 0, \tag{16}$$

where $u(\Omega_R)$ denotes the harmonic input of Eq. (12) and $\Omega_R = \{\omega_1, \ldots \omega_R\}$ indicates the set of R distinct frequency components. Since $M(t, \theta, u(\Omega_R), \mathcal{H})$ is a linear combination of distinct exponential basis functions, Eq. (16) is satisfied only with the trivial linear combination, i. e. all the coefficients of the exponential terms must be equal to zero. Then $H_n^{sym}(.)$ can be determined as the coefficient of the exponential term of the non-repetitive combination of the n distinct frequency components.

The harmonic probing technique is performed by means of the extraction operator $\varepsilon_n[.]$ [62].

Definition 3.1. The extraction operator $\varepsilon_n[f]$ on a differential algebraic expression f is defined by the following consecutive steps:

1. Substitute Eq. (12) and Eq. (14) (with $R = n$) to the given expression f.
2. Extract the coefficient of $e^{j(\omega_1+\ldots+\omega_n)t}$.

Clearly, the extraction operator $\varepsilon_n[.]$ returns the coefficient of the complex exponential basis function containing n distinct frequency components.

Making use of the extraction operator, the equation

$$\varepsilon_n[M(t, \theta, u, y)] = 0 \tag{17}$$

can be used to determine the nth-order GFRF $H_n^{sym}(.)$.

We emphasize that the harmonic probing technique requires n distinct, arbitrarily chosen frequency components in order to compute the nth order (symmetrized) GFRF. Clearly, the symmetrized GFRFs compose a unique descriptor of the input-output behavior of the underlying dynamical system model regardless of the chosen (distinct) frequency components $\{\omega_1, \ldots \omega_n\}$. The only requirements are to guarantee that the chosen frequency components are different and non-zero. It is also important to emphasize that the harmonic probing method can be employed regardless of parametric non-linearities in the dynamical input-output system model $M(.)$, hence the method is not restricted to systems which are linear in parameters. We also note that the discussed probing method is suitable for determining the GFRFs for non-linear time delayed systems as well. Another important approach for computing GFRFs is the application of recursive algorithms [45, 62].

Now a computational example is provided to illustrate the harmonic probing method and the related extraction operator.

Example 3.2. Let us consider the following SISO input-output model:

$$\ddot{y}(t) = \theta_1 \dot{y}(t) + \theta_2 e^{-\theta_3 t} u(t) + \theta_4 u(t) y(t). \tag{18}$$

In order to determine the 1st order GFRF, $H_1(j\omega_1)$, we employ the input $u(t) = e^{j\omega_1 t}$ so that $\omega_1 \neq 0$. Then, according to Eq. (14), the respective output is of the form $y(t) = H_1(j\omega_1)e^{j\omega_1 t}$. According to the extraction operator $\varepsilon_1[.]$ we substitute $u(t)$ and $y(t)$ to the given model equation Eq. (18), which gives rise to the following expression:

$$(j\omega_1)^2 H_1(j\omega_1) e^{j\omega_1 t} = \theta_1 (j\omega_1) H_1(j\omega_1) e^{j\omega_1 t} + \theta_2 e^{-t\theta_3} e^{j\omega_1 t} + \theta_4 H_1(j\omega_1) e^{j2\omega_1 t}. \tag{19}$$

Then we take the coefficient of the complex exponential basis function $e^{j\omega_1 t}$, which is known to be zero, that is:

$$(j\omega_1)^2 H_1(j\omega_1) - \theta_1(j\omega_1) - \theta_2 e^{-t\theta_3} = 0. \tag{20}$$

$H_1(j\omega_1)$ can be calculated by solving Eq. (20) with respect to $H_1(j\omega_1)$. Formally, using the notation of the extraction operator $\varepsilon_1[.]$, we solve the following equation with respect to $H_1(j\omega_1)$:

$$\varepsilon_1[M(t,\theta,u,y)] = 0, \tag{21}$$

where

$$M(t,\theta,u,y) = \ddot{y}(t) - \theta_1 \dot{y}(t) - \theta_2 e^{-\theta_3 t} u(t) - \theta_4 u(t) y(t). \tag{22}$$

Finally, the 1st order GFRF is as follows:

$$H_1(j\omega_1) = \frac{\theta_2 e^{-t\theta_3}}{(j\omega_1)^3 + \theta_1 (j\omega_1)}. \tag{23}$$

Now determine the 2nd order symmetrized GFRF. We employ an input signal $u(t) = e^{j\omega_1 t} + e^{j\omega_2 t}$ of two frequency components so that $\omega_1 \neq 0$, $\omega_2 \neq 0$ and $\omega_1 \neq \omega_2$. The respective output of the system can be calculated by means of Eq. (14):

$$\begin{aligned} y(t) = &H_1(j\omega_1)e^{j\omega_1 t} + H_1(j\omega_2)e^{j\omega_2 t} + 2! H_2(j\omega_1, j\omega_2) e^{j(\omega_1+\omega_2)t} \\ &+ [\text{terms of repetitious frequency components}]. \end{aligned} \tag{24}$$

Next we substitute $u(t)$ and $y(t)$ to the given model equation Eq. (18) and extract the coefficient of the two-tone complex exponential basis function $e^{j(\omega_1+\omega_2)t}$. Since the coefficient is equal to zero we have an equation from which $H_2(j\omega_1, j\omega_2)$ can be calculated:

$$H_2(j\omega_1, j\omega_2) = \frac{\theta_4 [H_1(j\omega_1) + H_1(j\omega_2)] e^{j(\omega_1+\omega_2)t}}{2![j(\omega_1+\omega_2)^2] - 2!\theta_1 j(\omega_1+\omega_2)}. \tag{25}$$

Here we note that the above procedure of calculating the 2nd order GFRF can be compactly represented by the extraction operator $\varepsilon_2[.]$ as follows:

$$\varepsilon_2[M(t,\theta,u,y)] = 0, \tag{26}$$

where $M(t,\theta,u,y)$ is defined by Eq. (22).

4. TESTING STRUCTURAL IDENTIFIABILITY WITH GFRFS

Making use of the generalized frequency response functions associated to a Volterra series representation, sufficient conditions for joint structural identifiability of ordinary system parameters and time delays can be derived in the form of algebraic equations. Compared to other structural identifiability tests based on series expansions, the proposed method allows us for directly examining the identifiability of constant time delay parameters appearing in the model structure. This can be performed since the time delay parameters appear in the exponents of complex exponential functions in the GFRFs. From the exponents the delay parameters can be easily extracted.

Assuming that the Volterra series representation exists for a dynamical system model $M(t, \theta, u, y, p) = 0$, the unique GFRFs provide us with an exhaustive summary:

$$s(\theta) = \left\{ H_k^{sym}(j\omega_1, \dots j\omega_k, \theta) \right\}_{k=1}^{\infty}, \tag{27}$$

where $H_k^{sym}(j\omega_1, \dots j\omega_k, \theta)$ denotes the parameterized GFRF of order k and θ stands for the set of parameters including time delays. Then testing structural identifiability translates to the following equations:

$$H_k^{sym}(j\omega_1, \dots j\omega_k, \theta) = H_k^{sym}(j\omega_1, \dots j\omega_k, \hat{\theta}), \quad \forall \omega_1, \dots\ \omega_k,\ k \geq 1. \tag{28}$$

Here $H_k^{sym}(.)$ is of the form of a fraction of complex exponential polynomials and we translate Eq. (28) to the equality of the numerator and denominator polynomials. Clearly, complex exponentials of different exponents are linearly independent of each other, which means that the existence of a particular exponent in one polynomial implies that an exponential function of the same exponent appears in the other polynomial [41]. Then the equality of exponential polynomials is equivalent to the equality of the corresponding coefficients. This way algebraic equations can be obtained for identifiability testing. Algebraic conditions on the time delay parameters are derived as the equality of the respective exponents. We note that the proposed approach can be employed even if the numerators and the denominators are polynomials of multiple variables (i. e. frequencies) as it could be the case of higher order ($k > 1$) GFRFs. In this case the arising complex exponential basis functions in the GFRFs have exponents including sums of different frequency components. Clearly, complex exponential basis functions with the same sums of different frequencies must have the same coefficients in order to guarantee equality of the polynomials.

Joint structural identifiability of ordinary parameters and time delays is guaranteed by the unique solution of Eq. (28) with respect to θ. However, in general case $s(\theta)$ is composed of a countably infinite set of GFRFs, but in practice we are restricted to a finite set of $s(\theta)$. This way sufficient conditions for structural identifiability can be derived. Formally, if the relation

$$H_k^{sym}(j\omega_1, \dots j\omega_k, \theta) = H_k^{sym}(j\omega_1, \dots j\omega_k, \hat{\theta}),\ k \leq K \Longrightarrow \theta = \hat{\theta} \tag{29}$$

holds for some $K \in \mathbb{N}$, then the parameters θ are structurally globally identifiable. Clearly, if the above relation holds for a subset $\overline{\theta} \subset \theta$, the subset $\overline{\theta}$ is s.g.i.

Proposition 4.1. Let us consider a SISO dynamical system model $M(t, \theta, u, y)$. Assuming that $M(.)$ is an analytical function, sufficient condition of structural identifiability is obtained by the GFRFs as follows

$$H_k^{sym}(j\omega_1, \dots j\omega_k, \theta) = H_k^{sym}(j\omega_1, \dots j\omega_k, \hat{\theta}),\ k \leq K \Longrightarrow \theta = \hat{\theta} \tag{30}$$

for any $K \in \mathbb{N}$.

Proof. Since $H_k^{sym}(j\omega_1,\dots j\omega_k,\theta)$ is unique for any $k \geq 0$, any feasible parametrization $\hat{\theta}$ satisfies

$$H_k^{sym}(j\omega_1,\dots j\omega_k,\theta) = H_k^{sym}(j\omega_1,\dots j\omega_k,\hat{\theta}),\ k \leq K \tag{31}$$

for arbitrary $K \in \mathbb{N}$. The number of equations of the form Eq. (31) to be satisfied by a feasible parametrization $\hat{\theta}$ is infinitely countable, hence by a finite subset of K equations sufficient conditions can be obtained. □

Clearly, if Eq. (29) holds in a finite subset $\overline{\Theta} \subset \Theta$ of the parameter space Θ, then sufficient conditions for local structural identifiability are derived in $\overline{\Theta}$.

If the model structure under study is proven to be weakly nonlinear (i. e. a finite number of non-zero GFRFs exist), then the proposed approach provides sufficient and necessary conditions for structural identifiability and structural non-identifiability of model parameters can also be determined.

We note that the proposed method can be viewed as an extension of the transfer function approach applied for identifiability analysis of linear time invariant systems [6]. If the system under study is linear, then the Volterra series expansion is composed of a single kernel function and the associated GFRF coincides with the frequency response function known form linear system theory. It is clear that constant time delay parameters appear explicitly in the exponents of the linear frequency response function. Since $s(\theta)$ involves only a one-dimensional function $H(j\omega)$, Eq. (28) reduces to a necessary and sufficient condition.

We also indicate the relation to the generating series approach. It is based on the Fliess functional series expansion of non-linear systems. In this case an exhaustive summary is obtained as the coefficients of Fliess series expansion, which are the Lie derivatives of the output signals. It can be shown that the Volterra series expansion is a reordering of the Fliess decomposition [16, 21].

5. EXAMPLES

In this section we provide examples for structural identifiability testing of time delayed nonlinear model structures. First the GFRFs are determined using the harmonic probing technique along with the extraction operator. Then structural identifiability of parameters (including time delays) are evaluated trough the equations Eq. (28).

Note that $H_k^{sym}(.)$ is a fraction of sums of complex exponential basis functions. We formulate structural identifiability of the system parameters as the equality of the coefficients of the complex exponential basis functions of the same exponents and identifiability of time delay parameters is algebraically expressed as the equality of the respective exponents.

1. *Duffing oscillator with time delay*

   As a delayed nonlinear input-output model we consider the Duffing oscillator, which is used for modeling Electroencephalography (EEG) time series data [61]. The model is equipped with a time delayed term according to [45]. The differential equation model describing the delayed Duffing oscillator is as follows:

$$\ddot{y}(t) + k_1\dot{y}(t) + k_2\dot{y}(t-T) + c_1 y(t) + k_3 y^3(t) = bu(t), \tag{32}$$

   where $T \in \mathbb{R}$ denotes a constant time-delay parameter. The parameter set of the above system is $\theta = \{k_1, k_2, k_3, b, c_1, T\}$.

   In order to decide whether the model is structurally identifiable with respect to the parameters in θ, first the GFRFs are computed based on the harmonic probing technique. The GFRFs provide a set of algebraic equations in the system parameters so that the uniqueness of the solution is sufficient condition of structural identifiability.

The first GFRF $H_1^{sym}(j\omega)$ can be determined by applying the input signal

$$u(t) = e^{j\omega_1 t}. \tag{33}$$

Then the respective system output – according to Eq. (14) – becomes

$$y(t) = H_1(j\omega_1)e^{j\omega_1 t}. \tag{34}$$

Applying the extraction operator $\varepsilon_1[.]$ on the system Eq. (33) involves substituting the input and the output into the system, and extracting the coefficient of $e^{j\omega_1}$. Then $H_1^{sym}(j\omega_1)$ can be obtained by solving the equation

$$\varepsilon_1[M(t,\theta,u,y)] = 0, \tag{35}$$

where

$$M(t,\theta,u,y) = \ddot{y}(t) + k_1\dot{y}(t) + k_2\dot{y}(t-T) + c_1 y(t) + k_3 y^3(t) - bu(t). \tag{36}$$

For the first order GFRF we obtain

$$H_1^{sym}(j\omega_1) = \frac{b}{(j\omega_1)^2 + k_1(j\omega_1) + k_2(j\omega_1)e^{-j\omega_1 T} + c_1}. \tag{37}$$

The above solution of $H_1^{sym}(j\omega_1)$ provides the following equations for structural identifiability analysis:

$$\begin{aligned} b &= \hat{b} \\ (j\omega_1)^2 + k_1(j\omega_1) + c_1 &= (j\omega_1)^2 + \hat{k}_1(j\omega_1) + \hat{c}_1 \\ k_2(j\omega_1)e^{-j\omega_1 T} &= \hat{k}_2(j\omega_1)e^{-j\omega_1 T}. \end{aligned} \tag{38}$$

Clearly the model is structurally globally identifiable with respect to b and c_1. Depending on T two cases can be distinguished. If $T = 0$, then it follows that

$$k_1 + k_2 = \hat{k}_1 + \hat{k}_2, \tag{39}$$

i. e. k_1 and k_2 are not structurally uniquely identifiable. If $T \neq 0$, then k_1, k_2 and T are structurally globally identifiable. Note that the identifiability of k_3 cannot be analyzed via $H_1(j\omega_1)$, since it does not account for higher order non-linearities, but k_3 is the coefficient of a 3rd-order term in Eq. (33).

Since Eq. (33) has no 2nd-order non-linearities, $H_2^{sym}(j\omega_1, j\omega_2)$ is absent.

To determine $H_3^{sym}(j\omega_1, j\omega_2, j\omega_3)$, the input according to the harmonic probing technique is of the form

$$u(t) = e^{j\omega_1 t} + e^{j\omega_2 t} + e^{j\omega_3 t}. \tag{40}$$

The respective output of the system according to Eq. (14) is as follows:

$$\begin{aligned} y(t) =& H_1(j\omega_1)e^{j\omega_1 t} + H_1(j\omega_2)e^{j\omega_2 t} + H_1(j\omega_3)e^{j\omega_3 t} \\ &+ 2!H_2^{sym}(j\omega_1, j\omega_2)e^{j(\omega_1+\omega_2)t} + 2!H_2^{sym}(j\omega_1, j\omega_3)e^{j(\omega_1+\omega_3)t} \\ &+ 2!H_2^{sym}(j\omega_2, j\omega_3)e^{j(\omega_2+\omega_3)t} + 3!H_3^{sym}(j\omega_1, j\omega_2, j\omega_3)e^{j(\omega_1+\omega_2+\omega_3)t} \\ &+ [\text{terms involving repetitious frequency components}]. \end{aligned} \tag{41}$$

Then applying the extraction operator $\varepsilon_3[.]$ involves substituting $u(t)$ and $y(t)$ to the given dynamical model Eq. (33) and extracting the coefficient of $e^{j(\omega_1+\omega_2+\omega_3)t}$. Then the equation

$$\varepsilon_3[M(t,\theta,u,y,p)] = 0 \tag{42}$$

can be used to express the 3rd-order symmetrized GFRF as follows:

$$\begin{aligned} &H_3^{sym}(j\omega_1, j\omega_2, j\omega_3) \\ &= \frac{-k_3 H_1(j\omega_1)H_1(j\omega_2)H_1(j\omega_3)}{(j\omega_1+j\omega_2+j\omega_3)^2 + (k_1 + k_2 e^{(j\omega_1+j\omega_2+j\omega_3)T})(j\omega_1+j\omega_2+j\omega_3) + c_1}. \end{aligned} \tag{43}$$

The above equation of $H_3^{sym}(.)$ implies

$$k_3 = \hat{k}_3, \tag{44}$$

from which it follows that k_3 can be uniquely determined and therefore the model is globally structurally identifiable with respect to the whole parameter vector θ if the delay is non-zero.

2. *Non-linear state space model with multiple time delays*
Let us consider the following nonlinear delayed system model

$$\begin{aligned} \dot{x}_1(t) &= [\theta_1 x_2(t) + 1]u(t-\tau_1) - \theta_2 x_1(t), \\ \dot{x}_2(t) &= \theta_3 x_1(t), \\ y &= \theta_4 x_2(t) + \theta_5 y(t-\tau_2), \end{aligned} \tag{45}$$

with θ_i, $i = 1, \ldots 4$ are ordinary parameters and τ_1, τ_2 are constant time delays. We wish to decide whether the system is structurally identifiable with respect to the parameters $\theta = \{\theta_1, \theta_2, \theta_3, \theta_4, \tau_1, \tau_2\}$.
Since Eq. (45) involves unobserved state variables, they must be eliminated in order to obtain an equivalent input-output relation containing the parameters θ. Such an input-output representation can be obtained by differential algebraic manipulations of the state space model.
We extended the equations with further derivatives as follows

$$\begin{aligned} \dot{x}_1(t) &= [\theta_1 x_2(t) + 1]u(t-\tau_1) - \theta_2 x_1(t), \\ \dot{x}_2(t) &= \theta_3 x_1(t), \\ y &= \theta_4 x_2(t) + \theta_5 y(t-\tau_2) \\ \ddot{x}_1(t) &= \theta_1 \dot{x}_2 u(t-\tau_1) + \theta_1 x_2(t)\dot{u}(t-\tau_1) + \dot{u}(t-\tau_1) - \theta_2 \dot{x}_1(t) \\ \ddot{x}_2(t) &= \theta_3 \dot{x}_1(t) \\ \dot{y}(t) &= \theta_4 \dot{x}_2(t) + \theta_5 \dot{y}(t-\tau_2) \\ \ddot{y}(t) &= \theta_4 \ddot{x}_2(t) + \theta_5 \ddot{y}(t-\tau_2) \end{aligned} \tag{46}$$

Then Eq. (46) involves 7 equations from which the state variables and their derivatives (6 unobserved variables) should be algebraically eliminated. In this example the elimination was performed by Mathematica and the resulted input-output model associated to Eq. (45) is as follows:

$$\begin{aligned} &-\theta_3\theta_4 u(t-\tau_1) + \theta_1\theta_3\theta_5 u(t-\tau_1)y(t-\tau_2) - \theta_2\theta_5\dot{y}(t-\tau_2) \\ &-\theta_5\ddot{y}(t-\tau_2) + \theta_2\dot{y} + \ddot{y} - \theta_1\theta_3 u(t-\tau_1)y = 0. \end{aligned} \tag{47}$$

Note that the above input-output model of Eq. (47) is not linear in the parameters θ.
Now we can examine structural identifiability of θ based on the GFRFs associated to Eq. (47). First the harmonic probing method is employed to determine the GFRFs. Next, making use of the obtained GFRFs we derive algebraic equations with respect to the model parameters. Unique solution of the calculated algebraic equations for some parameters proves structural identifiability.
In order to determine $H_1(j\omega_1)$ we employed the one-tone input $u(t) = e^{j\omega_1 t}$. According to the extraction operator $\varepsilon_1[.]$ we subsitute the input $u(t)$ and the respective output $y(t) = H_1(j\omega_1)e^{j\omega_1 t}$ to the obtained input-output system model of Eq. (47). Then we take the coefficient of $e^{j\omega_1 t}$. As the extracted coefficient is guaranteed to be zero we result in an equation from which $H_1(j\omega_1)$ can be calculated. Formally, we solve $\varepsilon_1[M(t,\theta,u,y,p)] = 0$ with respect to $H_1(j\omega_1)$, which results in:

$$H_1(j\omega_1) = \frac{\theta_3\theta_4 e^{-j\omega_1\tau_1}}{(j\omega_1)^2 + (j\omega_1)^2\theta_5 e^{-j\omega_1\tau_2} + \theta_2(j\omega_1) - \theta_2\theta_5(j\omega_1)e^{-j\omega_1\tau_2}}. \tag{48}$$

From the numerator we obtain the following equation

$$\theta_3\theta_4 e^{-j\omega_1\tau_2} = \hat{\theta}_3\hat{\theta}_4 e^{-j\omega_1\hat{\tau}_2}. \tag{49}$$

Clearly, complex exponential basis functions are linearly independent, that is

$$\tau_1 = \hat{\tau}_1, \tag{50}$$

and τ_1 is structurally globally identifiable. Then it also holds that

$$\theta_3\theta_4 = \hat{\theta}_3\hat{\theta}_4, \tag{51}$$

from which the structural identifiability of θ_3 and θ_4 does not follow.
From the denominator of $H_1(j\omega_1)$ we have

$$\begin{aligned}&(j\omega_1)^2 + (j\omega_1)^2\theta_5 e^{-j\omega_1\tau_2} + \theta_2(j\omega_1) - \theta_2\theta_5(j\omega_1)e^{-j\omega_1\tau_2}\\ &= (j\omega_1)^2 + (j\omega_1)^2\hat{\theta}_5 e^{-j\omega_1\hat{\tau}_2} + \hat{\theta}_2(j\omega_1) - \hat{\theta}_2\hat{\theta}_5(j\omega_1)e^{-j\omega_1\hat{\tau}_2}.\end{aligned} \tag{52}$$

There are two different cases. If $\tau_2 = 0$, then the following equations hold for the ordinary parameters:

$$\begin{aligned}\theta_5 &= \hat{\theta}_5,\\ \theta_2 - \theta_2\theta_5 &= \hat{\theta}_2 - \hat{\theta}_2\hat{\theta}_5,\end{aligned} \tag{53}$$

from which we have that θ_5 is s.g.i. Regarding θ_5 there are two different cases. If $\theta_5 \neq 1$, then s.g.i of θ_2 is guaranteed, otherwise it is not necessarily s.g.i.
If $\tau_2 \neq 0$, then the linear independence of complex exponentials implies that $\tau_2 = \hat{\tau}_2$, i. e. τ_2 is s.g.i. and the follwing set of equations are arising:

$$\begin{aligned}(j\omega_1)^2 + \theta_2(j\omega_1) &= (j\omega_1)^2 + \hat{\theta}_2(j\omega_1),\\ (j\omega_1)^2\theta_5 - \theta_2\theta_5(j\omega_1) &= (j\omega_1)^2\hat{\theta}_5 - \hat{\theta}_2\hat{\theta}_5(j\omega_1),\end{aligned} \tag{54}$$

from which it can be seen that θ_2 and θ_5 are s.g.i.
In order to examine the identifiability of the remaining parameters we determine the 2nd order GFRF. We employ an input signal $u(t) = e^{j\omega_1 t} + e^{j\omega_2 t}$ of two distinct frequency components. Then the respective output according to Eq. (14) is of the form of

$$\begin{aligned}y(t) = H_1(j\omega_1)e^{j\omega_1 t} + H_1(j\omega_2)e^{j\omega_2 t} + 2!H_2^{sym}(j\omega_1, j\omega_2)e^{j(\omega_1+\omega_2)t}\\ +[\text{terms involving repetituous frequency components}].\end{aligned} \tag{55}$$

We substitute $u(t)$ and $y(t)$ to the obtained input-output form Eq. (47) of the system model. Then the coefficient of $e^{j(\omega_1+\omega_2)t}$ can be extracted. Since the obtained coefficient is known to be zero we result in an equation from which $H_2^{sym}(j\omega_1, j\omega_2)$ can be derived. Formally, we solve $\varepsilon_2[M(t,\theta,u,y,p)] = 0$ with respect to $H_2^{sym}(j\omega_1, j\omega_2)$. Finally, for the 2nd order symmetrized GFRF we have:

$$H_2^{sym}(j\omega_1, j\omega_2) = \frac{-\theta_1\theta_3\theta_5 H_1(j\omega_2)e^{-j\omega_1\tau_1} - \theta_1\theta_3\theta_5 H_1(j\omega_1)e^{-j\omega_2\tau_1} + \theta_1\theta_3 H_1(j\omega_2)e^{-j\omega_1\tau_1} + \theta_1\theta_3 H_1(j\omega_1)e^{-j\omega_2\tau_1}}{-\theta_2\theta_5 2! e^{-j(\omega_1+\omega_2)\tau_2} j(\omega_1+\omega_2) - \theta_5 2! e^{-j(\omega_1+\omega_2)\tau_2} + \theta_2 2! j(\omega_1+\omega_2) + 2![j(\omega_1+\omega_2)]^2}. \tag{56}$$

Clearly, the structure of the denominator of $H_2(j\omega_1, j\omega_2)$ is equivalent to that of $H_1(j\omega)$, hence it does not provide us with further information on identifiability of parameters. The numerator gives rise to the following equations for identifiability testing:

$$-\theta_1\theta_3\theta_5 + \theta_1\theta_3 = -\hat{\theta}_1\hat{\theta}_3\hat{\theta}_5 + \hat{\theta}_1\hat{\theta}_3, \tag{57}$$

which cannot be used to prove identifiability of θ_1 and θ_3. Hence the 2nd-order GFRF does not provide further insight into parameter identifiability.

Finally, based on the 1st-and 2nd-order symmetrized GFRFs, using the proposed approach we obtained structural identifiability for a subset of parameters. We note that higher order GFRFs may provide further information about the identifiability of system parameters.

6. SUMMARY

In this paper a novel approach is introduced to examine joint structural identifiability of ordinary system parameters and constant time delays in nonlinear dynamical system models. The systems under study are assumed to be of the form of input-output models in which all the non-linearities are analytic functions and the time delays are constants. From a practical point of view, the analytic assumption on the non-linearities is not restrictive, for example it involves the class of polynomial systems, which is widely used to model the dynamical behavior of complex physical, chemical and biological processes. We also note that there is no constraint on the number of different delay parameters.

We employed the Volterra series representation of non-linear dynamical systems. In order to test structural identifiability, the GFRFs, i. e. the Fourier transforms of the Volterra kernels were used. The GFRFs have the advantageous property of containing explicitly the time delay parameters of input and output signals. Since the symmetrized GFRFs are unique, they can be used to construct an exhaustive summary of the underlying dynamic input-output model structure. Based on the GFRFs, we derived equations of complex exponential polynomials which are suitable to examine parameter identifiability. Unique solution of the obtained equations with respect to some parameters implies parametric uniqueness, this way sufficient condition of structural identifiability of the respective parameters is guaranteed. Furteher property of the GFRF-based approach is that structural identifiability can be examined regardless of the input signals.

We illustrated the proposed method on time delayed non-linear system models. Among the provided examples we illustrated the case of non-linear state space models with unobserved state variables. Since the Volterra series modeling assumes input-output model structure, first we performed differential algebraic manipulations to eliminate unobserved variables. This way an equivalent input-output model structure can be obtained that contains the parameters. In the provided example the resulting input-output representation was suitable to prove structural identifiability of a subset of the parameters by means of the proposed method.

Our contribution is to propose a novel method which can be used to determine the joint identifiability of ordinary system parameters and constant time delays. The method can also be used to examine how the time delay parameters affect structural identifiability of ordinary system parameters. The proposed approach can be applied to systems of analytical non-linearities which involves many biologically motivated system classes, such as polynomial systems (e. g. Lotka-Volterra systems, epidemiological models). An important aspect of our contribution – compared to many existing results – is that the proposed method does not impose upper limit on the number of constant time delays appearing in the model structures. It has to be noted that the direct application of our method for actual parameter estimation is not straightforward, since the resulting nonlinear equations between observations and parameters often become overly complicated even for relatively simple nonlinear models. Future work will be focused on the application of recursive methods for GFRF computation in the context of identifiability analysis, and on the studying of specific nonlinear system classes.

7. DECLARATION OF CONFLICTING INTEREST

The authors declare that there are no competing interest regarding the content of this document.

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