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SUZUKI TYPE FUZZY $\mathcal{Z}$-CONTRACTIVE MAPPINGS AND FIXED POINTS IN FUZZY METRIC SPACES

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In this paper, we propose the concept of Suzuki type fuzzy $\mathcal{Z}$-contractive mappings, which is a generalization of Fuzzy $\mathcal{Z}$-contractive mappings initiated in the article [S. Shukla, D. Gopal, W. Sintunavarat, A new class of fuzzy contractive mappings and fixed point theorems, Fuzzy Sets and Systems 350 (2018)85-95]. For this type of contractions suitable conditions are framed to ensure the existence of fixed point in G-complete as well as M-complete fuzzy metric spaces. A comprehensive set of examples are furnished to demonstrate the validity of the obtained results.

Keywords: fuzzy metric space, fuzzy $\mathcal{Z}$-contractive mapping, Suzuki type fuzzy $\mathcal{Z}$-contractive mappings, fixed point

Classification: 54H25, 47H10

1. INTRODUCTION

The study of fixed points of mappings satisfying certain contraction conditions has many applications and has been at the centre of various research activities. Fuzzy fixed point theory is a fuzzy extension of fixed point theory. The notion of fuzzy metric was originally introduced by Kramosil and Michálek [11] and later was modified by George and Veeramani [2] in order to obtained a Hausdorff topology. An important and interesting topic in fuzzy metric spaces is the fixed point theory. A lot of fixed point theorems were obtained by introducing various fuzzy contractive mappings [5, 6, 12, 20, 3, 4, 20]. But due to the complexity exhibited in fuzzy metric spaces, researchers need to add various conditions to obtain fixed point theorems in fuzzy metric spaces (see[4, 9, 20, 21]). Recently, Shukla et al. [16] introduced the notion of fuzzy $\mathcal{Z}$-contractive mappings in order to unify different classes of fuzzy contractive mappings. Following this direction of research, we introduce a new concept of Suzuki type fuzzy $\mathcal{Z}$-contractive mappings and formulate the conditions guaranteeing the existence of unique fixed point in G-complete as well as M-complete fuzzy metric spaces. The paper includes a comprehensive set of examples showing the generality of our results and demonstrating that the formulated conditions are significant and cannot be omitted.

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2. PRELIMINARIES

In this section, we state some basic concepts and results which will be needed in the sequel.

Definition 2.1. (Schweizer and Sklar [14]) A mapping $*\colon [0,1]\times[0,1]\to[0,1]$ is called a continuous triangular norm (t-norm for short) if $*$ satisfies the following conditions:

(i) $*$ is commutative and associative, i. e., $a*b=b*a$ and $a*(b*c)=(a*b)*c$, for all $a,b,c\in[0,1]$;

(ii) $*$ is continuous;

(iii) $1*a=a$, for all $a\in[0,1]$;

(iv) $a*b\leq c*d$, whenever $a\leq c$ and $b\leq d$, with $a,b,c,d\in[0,1]$.

Some basic examples of t-norm are the minimum t-norm $*_m$, $a*_m b=\min\{a,b\}$, product t-norm $*_p$, $a*_p b=ab$, the Łukasiewicz t-norm $*_L$, $a*_L b=\max\{a+b-1,0\}$, for all $a,b\in[0,1]$.

If $*$ is a t-norm, then $*^n(t)$ is defined for every $t\in[0,1]$ and $n\in\mathbb{N}\cup 0$ by 1, if $n=0$ and $*(t,*^{n-1}(t))$, if $n\geq 1$. A t-norm $*$ is said to be of Hadžić type if the family of functions $\{*^n(t)\}_{n=1}^{\infty}$ is equicontinuous at $t=1$ [10].

Definition 2.2. (George and Veeramani [2]) A fuzzy metric space (GV-fuzzy metric space, for short) is an ordered triple $(X,M,*)$ such that X is a (nonempty) set, $*$ is a continuous t-norm and M is a fuzzy set on $X\times X\times(0,\infty)$ satisfying the following conditions, for all $x,y,z\in X$ and $t,s>0$;

(GV1) $M(x,y,t)>0$;

(GV2) $M(x,y,t)=1$ if and only if $x=y$;

(GV3) $M(x,y,t)=M(y,x,t)$;

(GV4) $M(x,z,t+s)\geq M(x,y,t)*M(y,z,s)$;

(GV5) $M(x,y,\cdot)\colon(0,\infty)\to(0,1]$ is continuous.

Remark 2.3. Note that in this context, condition (GV2) of Definition 2.2 is equivalent to the following:
$M(x,x,t)=1$ for all $x\in X$ and $t>0$, and $M(x,y,t)<1$ for all $x\neq y$ and $t>0$.

If, in the above definition, the triangular inequality (GV4) is replaced by the following condition:

(NA) $M(x,z,\max\{t,s\})\geq M(x,y,t)*M(y,z,s)$ for all $x,y,z\in X$ and $t,s>0$,

then the triple $(X,M,*)$ is called a non-Archimedean fuzzy metric space. It is easy to check that (NA) implies (GV4), that is, every non-Archimedean fuzzy metric space is itself a fuzzy metric space.

For the topological properties of a fuzzy metric space, the reader is referred to [2].

Definition 2.4. (George and Veeramani [2]) A sequence $\{x_n\}$ in a fuzzy metric space $(X, M, *)$ is said to be M-Cauchy if for each $\varepsilon \in (0,1)$ and each $t > 0$, there is $n_0 \in \mathbb{N}$ such that $M(x_n, x_m, t) > 1 - \varepsilon$, for all $n, m \geq n_0$.

Definition 2.5. (Grabiec [5]) A sequence $\{x_n\}$ in a fuzzy metric space $(X, M, *)$ is said to be G-Cauchy if $\lim_{n\to\infty} M(x_n, x_{n+m}, t) = 1$ for each $m \in \mathbb{N}$ and $t > 0$ or, equivalently, $\lim_{n\to\infty} M(x_n, x_{n+1}, t) = 1$ for all $t > 0$.

The above definitions of Cauchy sequences are different (see [19]).

Definition 2.6. (George and Veeramani [2]) A sequence $\{x_n\}$ in X converges to x if and only if $\lim_{n\to\infty} M(x_n, x, t) = 1$ for all $t > 0$.

We say that the space $(X, M, *)$ is M-complete (resp., G-complete) if every M-Cauchy (resp., G-Cauchy) sequence in X is convergent to some $x \in X$.

Following [16], we denote by $\mathcal{Z}$ the family of all functions $\zeta\colon (0,1] \times (0,1] \to \mathbb{R}$ satisfying the following condition:

$$\zeta(t, s) > s \quad \text{for all} \quad t, s \in (0,1).$$

The paper [16] has many examples of functions ζ. For our purpose, we mentioned the following:

Example 2.7. Consider the function ζ defined from $(0,1] \times (0,1]$ into $\mathbb{R}$ by $\zeta(t,s) = \psi(s)$, where $\psi\colon (0,1] \to (0,1]$ is a function such that $s < \psi(s)$ for all $s \in (0,1)$. Then, $\zeta \in \mathcal{Z}$.

Remark 2.8. By the above definition of function ζ, it is obvious that $\zeta(t,t) > t$ for all $0 < t < 1$.

Definition 2.9. (Shukla et al. [16]) Let $(X, M, *)$ be a fuzzy metric space and $T\colon X \to X$ be a mapping. Suppose, there exists $\zeta \in \mathcal{Z}$ such that

$$M(Tx, Ty, t) \geq \zeta(M(Tx, Ty, t), M(x, y, t)) \tag{1}$$

for all $x, y \in X, Tx \neq Ty, t > 0$. Then T is called a fuzzy $\mathcal{Z}$-contractive mapping with respect to the function $\zeta \in \mathcal{Z}$.

Remark 2.10. In [16] it has been shown that fuzzy contractive, Tirado's contraction, fuzzy $\mathcal{H}$-contractive mapping and fuzzy ψ-contractive mapping are fuzzy $\mathcal{Z}$-contractive mapping but converse is not necessarily true.

3. MAIN RESULTS

In this section, we define Suzuki type fuzzy $\mathcal{Z}$-contractive mappings and discuss corresponding fixed point theorems.

Definition 3.1. Let $(X, M, *)$ be a fuzzy metric space and $T\colon X \to X$ be a mapping. Suppose, there exists $\zeta \in \mathcal{Z}$ and $q \in (0,1)$ such that

$$M(x, Tx, t) > qM(x, y, t) \implies M(Tx, Ty, t) \geq \zeta(M(Tx, Ty, t), M(x, y, t)) \qquad (2)$$

for all $x, y \in X, Tx \neq Ty, t > 0$. Then T is called a Suzuki type fuzzy $\mathcal{Z}$-contractive mapping with respect to the function $\zeta \in \mathcal{Z}$.

The following example shows that a Suzuki type fuzzy $\mathcal{Z}$-contractive mappings may not have a fixed point even in a G-complete fuzzy metric space.

Example 3.2. Let $X = \mathbb{N}$. Define $a *_m b = \min\{a, b\}$ for all $a, b \in [0,1]$ and

$$M(x, y, t) = \left[exp\left(\frac{|x-y|}{t}\right)\right]^{-1}$$

for all $x, y \in X$ and $t \in (0, \infty)$. Then $(X, M, *_m)$ is a G-complete (as well as M-complete) fuzzy metric space. Define the mapping $T : X \to X$ by $Tx = 2x$ for all $x \in X$. Then, in view of remark 2.6 of [16], we conclude that T is a Suzuki type fuzzy $\mathcal{Z}$-contractive mapping with respect to function $\zeta(t, s) = \eta^{-1}(k\eta(s))$ for all $s, t \in (0, 1]$.

Note that T is a fixed point free mapping on X.

The above example motivates us for the consideration of a space having some additional properties so that the existence of fixed point of Suzuki type fuzzy $\mathcal{Z}$-contractive mapping can be ensured.

We need the following properties to prove our main theorems.

Definition 3.3. Let $(X, M, *)$ be a fuzzy metric space, $T\colon X \to X$ a mapping and $\zeta \in \mathcal{Z}$. Then we say that the quadruple (X, M, T, ζ) has the property (P), if for any Picard sequence $\{x_n\}$ with initial value $x \in X$, i. e., $x_n = T^n x$ for all $n \in \mathbb{N}$ such that $\lim_{n\to\infty} M(x_n, x_{n+1}, t) = l$ where $l \in (0, 1]$, we have,

$$\lim_{n\to\infty} \inf_{m>n} \zeta((M(x_{n+1}, x_{m+1}, t), M(x_n, x_m, t)) = 1 \text{ for all } t > 0.$$

Definition 3.4. Let $(X, M, *)$ be a fuzzy metric space, $T\colon X \to X$ a mapping and $\zeta \in \mathcal{Z}$. Then we say that the quadruple (X, M, T, ζ) has the property (N), if for any sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} M(x_n, x^*, t) = 1$, we have

$$M(x_n, Tx_n, t) \geq M(x^*, Tx_n, t)$$

for all $t > 0$ and $x^* \in X$.

Now, we are ready to prove our main theorems.

Theorem 3.5. Let $(X, M, *)$ be a G-complete fuzzy metric space and $T\colon X \to X$ be a Suzuki type fuzzy $\mathcal{Z}$-contractive mapping. If the quadruple (X, M, T, ζ) have the properties (P) and (N) respectively, then T has a unique fixed point $u \in X$.

P r o o f . First, we show that if the fixed point of T exists, then it is unique. Suppose, u, v be two distinct fixed points of T, i. e., $Tu = u$ and $Tv = v$ and there exists $s > 0$ such that $M(u, v, s) < 1$. Since $M(u, Tu, s) > qM(u, v, s)$ then by the condition (2) and definition of ζ, we have

$$M(u, v, s) = M(Tu, Tv, s) \geq \zeta(M(Tu, Tv, s), M(u, v, s)) > M(u, v, s).$$

This contradiction shows that $M(u, v, t) = 1$ for all $t > 0$, and so, $u = v$. It proves the uniqueness.

Now, we shall show the existence of fixed point of T. Let $x_0 \in X$ and define the Picard sequence $\{x_n\}$ by $x_n = Tx_{n-1}$ for all $n \in \mathbb{N}$.

Being M a GV-fuzzy metric, we have $M(x_0, Tx_0, t) > qM(x_0, Tx_0, t)$, for all $t > 0$. Then, by condition (2), we have

$$M(Tx_0, Tx_1, t) \geq \zeta(M(Tx_0, Tx_1, t), M(x_0, x_1, t))$$

i. e.

$$M(x_1, x_2, t) \geq \zeta(M(x_1, x_2, t), M(x_0, x_1, t)) > M(x_0, x_1, t).$$

Similarly, we get

$$M(x_2, x_3, t) > M(x_1, x_2, t).$$

In general, we have

$$M(x_n, x_{n+1}, t) > M(x_{n-1}, x_n, t).$$

Therefore, $\{(M(x_n, x_{n+1}, t))\}$ is a strictly non decreasing sequence of positive real numbers in (0,1] and denote $\lim_{n\to\infty} M(x_n, x_{n+1}, t)$ by l. We claim that $l = 1$. If $l = 1$ we finish, otherwise since $M(x_n, x_{n+1}, t) > qM(x_n, x_{n+1}, t)$, then, by condition (2), we have

$$M(Tx_n, Tx_{n+1}, t) \geq \zeta(M(Tx_n, Tx_{n+1}, t), M(x_n, x_{n+1}, t))$$

i. e.

$$M(x_{n+1}, x_{n+2}, t) \geq \zeta(M(x_{n+1}, x_{n+2}, t), t), M(x_n, x_{n+1}, t))$$

taking limit as $n \to \infty$, and using property (P), we get

$$\lim_{n\to\infty} \inf_{(m=n+1)>n} M(x_{n+1}, x_{m+1}, t) \geq \lim_{n\to\infty} \inf_{m>n} \zeta(M(x_{n+1}, x_{m+1}, t), M(x_n, x_m, t)) = 1$$

i. e., $l \geq 1 \implies l = 1$.

Thus $\{x_n\}$ is a G-Cauchy sequence. Since X is G-complete, there exists $u \in X$ such that $\lim_{n\to\infty} x_n = u$.

We shall show that u is a fixed point of T. First, we have

$$M(x_n, Tx_n, t) = M(x_n, x_{n+1}, t) > M(x_{n-1}, x_n, t) = M(x_{n-1}, Tx_{n-1}, t).$$

By property (N), we have

$$M(x_n, Tx_n, t) > M(x_{n-1}, Tx_{n-1}, t) \geq M(u, Tx_{n-1}, t) > qM(u, x_n, t)$$

and then we have,

$$M(Tx_n, Tu, t) \geq \zeta(M(Tx_n, Tu, t), M(x_n, u, t)) > M(x_n, u, t),$$

i. e.

$$M(x_{n+1}, Tu, t) > M(x_n, u, t).$$

Taking limit as $n \to \infty$, we obtain

$$M(Tu, u, t) \geq 1$$

i. e.

$$M(Tu, u, t) = 1$$

for all $t > 0$ and so $Tu = u$. □

Example 3.6. Let $X = \{1, 2, 4\}$, $a *_m b = \min\{a, b\}$ for all $a, b \in [0, 1]$ and

$$M(x, y, t) = \frac{t}{t + d(x, y)},$$

for all $t > 0$. Then $(X, M, *)$ is a G- complete (as well as M- complete) fuzzy metric space. Define $T : X \to X$ by $T(1) = 1 = T(2), T(4) = 2$. Then T is a Suzuki type fuzzy $\mathcal{Z}$-contractive mapping. Also, the quadruple (X, M, T, ζ) possess properties (P) and (N) trivially and hence all the conditions of the above Theorem satisfied. Thus T has a unique fixed point $u = 1$.

Lemma 3.7. (Altun and Mihet [1]) Each M-complete non-Archimedean fuzzy metric space $(X, M, *)$ with $*$ of Hadžić type is G-complete.

Corollary 3.8. Let $(X, M, *)$ be a M-complete non-Archimedean fuzzy metric space with $*$ of Hadžić type and $T\colon X \to X$ be a Suzuki type fuzzy $\mathcal{Z}$-contractive mapping. If the quadruple (X, M, T, ζ) have the properties (P) and (N) respectively, then T has a unique fixed point $u \in X$.

Now we will try to replace in previous result the Hadžić -typeness condition on $*$ by a new property, called (K1).

Definition 3.9. Let $(X, M, *)$ be a fuzzy metric space, $T\colon X \to X$ a mapping and $\zeta \in \mathcal{Z}$. Then we say that the quadruple (X, M, T, ζ) has the property $(K1)$, if for any Picard sequence $\{x_n\}$ with initial value $x \in X$, i. e., $x_n = T^n x$ for all $n \in \mathbb{N}$ such that $\lim_{n\to\infty} M(x_n, x_{n+1}, t) = 1$, there exists $k_0 \in \mathbb{N}$ and $q \in (0, 1)$ such that

$$M(x_n, x_{n+1}, t) > qM(x_n, x_m, t) \tag{3}$$

for all $m > n \geq k_0$, $t > 0$.

Theorem 3.10. Let $(X, M, *)$ be an M-complete non-Archimedean fuzzy metric space and $T\colon X \to X$ be a Suzuki type fuzzy $\mathcal{Z}$-contractive mapping. If the quadruple (X, M, T, ζ) have the properties (P), $(K1)$ and (N) respectively, then T has a unique fixed point $u \in X$.

Proof. Following the same proof of Theorem 3.5, we obtain a G-Cauchy sequence $\{x_n\}$. Now, we prove that $\{x_n\}$ is a M-Cauchy sequence.

If the sequence $\{x_n\}$ is not M-Cauchy, then there are $\epsilon \in (0,1)$ and $t > 0$ such that for each $k \in \mathbb{N}$, there exist $m(k), n(k) \in \mathbb{N}$ with $m(k) > n(k) \geq k$ and

$$M(x_{m(k)}, x_{n(k)}, t) \leq 1 - \epsilon\ .$$

Let, for each k, $m(k)$ be the least positive integer exceeding $n(k)$ satisfying the above property, that is

$$M(x_{m(k)-1}, x_{n(k)}, t) > 1 - \epsilon \text{ and } M(x_{m(k)}, x_{n(k)}, t) \leq 1 - \epsilon.$$

Then, for each positive integer k,

$$\begin{aligned} 1 - \epsilon &\geq M(x_{m(k)}, x_{n(k)}, t) \\ &\geq M(x_{m(k)}, x_{m(k)-1}, t) * M(x_{m(k)-1}, x_{n(k)}, t) \qquad \text{(by (NA))} \\ &\geq M(x_{m(k)}, x_{m(k)-1}, t) * (1 - \epsilon). \end{aligned}$$

Taking limit as $k \to \infty$, we obtain

$$(1 - \epsilon) \geq \lim_{k\to\infty} M(x_{m(k)}, x_{n(k)}, t) \geq (1 - \epsilon)$$

and therefore

$$\lim_{k\to\infty} M(x_{m(k)}, x_{n(k)}, t) = 1 - \epsilon.$$

Since $\lim_{n\to\infty} M(x_n, x_{n+1}, t) = 1$, using property $(K1)$, there exists $k_0 \in \mathbb{N}$ such that

$$M(x_{n(k)}, Tx_{n(k)}, t) = M(x_{n(k)}, x_{n(k)+1}, t) > qM(x_{n(k)}, x_{m(k)}, t)$$

for all $m(k) > n(k) \geq k_0$, $t > 0$.

Now, for each positive integer $k \geq k_0$, we have

$$\begin{aligned} M(x_{m(k)}, x_{n(k)}, t) &\geq M(x_{m(k)}, x_{m(k)+1}, t) * M(x_{m(k)+1}, x_{n(k)}, t) \\ &\geq M(x_{m(k)}, x_{m(k)+1}, t) * M(x_{m(k)+1}, x_{n(k)+1}, t) \\ &* M(x_{n(k)+1}, x_{n(k)}, t) \\ &\geq M(x_{m(k)}, x_{m(k)+1}, t) * M(Tx_{m(k)}, Tx_{n(k)}, t) * M(x_{n(k)+1}, x_{n(k)}, t) \\ &\geq M(x_{m(k)}, x_{m(k)+1}, t) * \zeta(M(Tx_{m(k)}, Tx_{n(k)}, t), M(x_{m(k)}, x_{n(k)}, t)) \\ &* M(x_{n(k)+1}, x_{n(k)}, t). \qquad \text{(by property (K1))} \end{aligned}$$

Taking limit as $k \to \infty$, we obtain

$$1-\epsilon \geq 1*1*1=1 \qquad \text{(by property (P))}$$

which is a contradiction and so $\{x_n\}$ is a M-Cauchy sequence.

Rest of the proof follows similar to the proof of Theorem 3.5. □

Remark 3.11. It is obvious that the set of our contractions in Theorem 3.10 includes that of the fuzzy $\mathcal{Z}$-contractions. But converse is not true.

The following example illustrate above theorem and also validate remark 3.11.

Example 3.12. Let $X = (0, \frac{1}{2}] \cup \{1, 2\}$, $a *_p b = ab$ for all $a, b \in [0,1]$ and $M(x,y,t) = \frac{\min\{x,y\}}{\max\{x,y\}}$ for all $x, y \in X$ and for all $t > 0$. Clearly, $(X, M, *_p)$ is an M-complete non-Archimedean fuzzy metric space.

Define the mapping $T : X \to X$ by

$$Tx = \begin{cases} 2 & \text{if } x \in (0, \frac{1}{2}], \\ 1 & \text{if } x = 1, 2 \end{cases}$$

for all $t > 0$. It is easy to see that T is a Suzuki type fuzzy $\mathcal{Z}$-contractive mapping with respect to the function $\zeta \in \mathcal{Z}$ defined by $\zeta(t,s) = \psi(s)$. However, T is not a fuzzy $\mathcal{Z}$-contractive mapping for any $\zeta \in \mathcal{Z}$ (for this consider $x \in (0, \frac{1}{2}]$ and $y = 1$).

Also, the quadruple (X, M, T, ζ) possess properties (P), $(K1)$ and (N) trivially and hence all the conditions of the above Theorem satisfied. Thus T has a unique fixed point $u = 1$.

The next examples demonstrates that the properties (P), $(K1)$ and (N) in Theorem 3.10 are not superfluous.

Example 3.13. Let $X = \mathbb{N}$, $a *_p b = ab$ for all $a, b \in [0,1]$ and define the fuzzy set M on $X \times X \times (0, \infty)$ by $M(n,m,t) = \min\left\{\frac{n}{m}, \frac{m}{n}\right\}$ for all $n, m \in X$ and $t > 0$. Then $(X, M, *_p)$ is an M-complete fuzzy metric space. Define a mapping $T \colon X \to X$ by $Tn = n+1$ for all $n \in X$. Then T is a fuzzy $\mathcal{Z}$-contractive mapping and hence Suzuki type fuzzy $\mathcal{Z}$-contractive mapping with respect to the function $\zeta_m \in \mathcal{Z}$ (see [16]) but the quadruple (X, M, T, ζ) does not satisfy the property (P) (for this, consider Picard sequence $x_n = n$ generated by T). Notice that T is a fixed point free mapping on X.

Example 3.14. Let $X = (0, \infty)$, $a *_p b = ab$ for all $a, b \in [0,1]$ and $M(x,y,t) = \frac{\min\{x,y\}}{\max\{x,y\}}$ for all $x, y \in X$ and for all $t > 0$. Clearly, $(X, M, *_p)$ is an M-complete non-Archimedean fuzzy metric space. Define the mapping $T : X \to X$ by

$$Tx = \begin{cases} \sqrt{x} & \text{if } x \neq 1, \\ 2 & \text{if } x = 1, \end{cases}$$

for all $t > 0$. If ζ defined from $(0,1] \times (0,1]$ into $\mathbb{R}$ by $\zeta(t,s) = \psi(s)$, where $\psi \colon (0,1] \to (0,1]$ is a function such that $s < \psi(s)$ for all $s \in (0,1)$. Then, it is easy to see that T

is a Suzuki type fuzzy $\mathcal{Z}$-contractive mapping but the quadruple (X, M, T, ζ) does not satisfy the property (N) (for this, consider $x_n = 2 + \frac{1}{n}, n \in \mathbb{N}$). Notice that T is a fixed point free mapping on X.

In order to establish Theorem 3.10 for general fuzzy metric space and to generalize the main theorem of [16], we introduce the followings:

Definition 3.15. Let $(X, M, *)$ be a fuzzy metric space, $T: X \to X$ a mapping and $\zeta \in \mathcal{Z}$. Then we say that the quadruple (X, M, T, ζ) has the property $(K2)$, if for any Picard sequence $\{x_n\}$ with initial value $x \in X$, i. e., $x_n = T^n x$ for all $n \in \mathbb{N}$ and $t > 0$ such that $M(x_m, x_{m+1}, t) > M(x_n, x_{n+1}, t)$, for all $m > n$, there exists $k_0 \in \mathbb{N}$ and $q \in (0, 1)$ such that

$$M(x_n, x_{n+1}, t) > qM(x_m, x_n, t) \tag{4}$$

for all $m > n \geq k_0$.

Definition 3.16. Let $(X, M, *)$ be a fuzzy metric space, $T: X \to X$ a mapping and $\zeta \in \mathcal{Z}$. Then we say that the quadruple (X, M, T, ζ) has the property (S), if for any Picard sequence $\{x_n\}$ with initial value $x \in X$, i. e., $x_n = T^n x$ for all $n \in \mathbb{N}$ such that $\inf\limits_{m>n} M(x_n, x_m, t) \leq \inf\limits_{m>n} M(x_{n+1}, x_{m+1}, t)$ for all $n \in \mathbb{N}, t > 0$ implies that

$$\lim_{n\to\infty} \inf_{m>n} \zeta(M(x_{n+1}, x_{m+1}, t), M(x_n, x_m, t)) = 1 \text{ for all } t > 0.$$

The following example verifies the fact that condition (P) is weaker than condition (S).

Example 3.17. Let $X = [1, \infty)$, $a *_m b = \min\{a, b\}$ for all $a, b \in [0, 1]$. Define a fuzzy set M on $X \times X \times (0, \infty)$ by:

$$M(x, y, t) = \begin{cases} 1, & \text{if } \quad x = y; \\ \dfrac{1}{1 + \max\{x, y\}}, & \text{otherwise} \end{cases} \quad \text{for all } x, y \in X, t \in (0, \infty).$$

Then, $(X, M, *_m)$ is a fuzzy metric space. Define $T: X \to X$ by $Tx = 2x$ for all $x \in X$. Suppose $\zeta: (0, 1] \times (0, 1] \to \mathbb{R}$ be defined by $\zeta(t, s) = \psi(s)$ for all $t, s \in (0, 1]$, where $\psi \in \Psi$ is such that $\psi(0) = 0$. Then, it is easy to see that the quadruple (X, M, T, ζ) satisfies the condition (P) trivially. On the other hand, the quadruple (X, M, T, ζ) does not satisfy the condition (S). Indeed, for any $x \in X,\ t > 0$ we have

$$\inf_{m>n} M(T^n x, T^m x, t) = \inf_{m>n} M(2^n x, 2^m x, t) = 0 < 1.$$

Therefore, $\inf\limits_{m>n} M(T^n x, T^m x, t) \leq \inf\limits_{m>n} M(T^{n+1} x, T^{m+1} x, t)$ for all $n \in \mathbb{N}, t > 0$. But,

$$\lim_{n\to\infty} \inf_{m>n} \zeta(M(x_{n+1}, x_{m+1}, t), M(x_n, x_m, t)) = \lim_{n\to\infty} \inf_{m>n} \psi(M(x_n, x_m, t)) = 0 \neq 1.$$

Theorem 3.18. Let $(X, M, *)$ be an M-complete fuzzy metric space and $T: X \to X$ be a Suzuki type fuzzy $\mathcal{Z}$-contractive mapping. If the quadruple (X, M, T, ζ) have the properties (S), $(K2)$ and (N) respectively, then T has a unique fixed point $u \in X$.

Proof. Same proof of Theorem 3.5 is valid for uniqueness.

We shall show the existence of fixed point of T. Let $x_0 \in X$ and define the Picard sequence $\{x_n\}$ by $x_n = Tx_{n-1}$ for all $n \in \mathbb{N}$.

If $x_n = x_{n-1}$ for any $n \in \mathbb{N}$, then $Tx_{n-1} = x_n = x_{n-1}$ is a fixed point of T. Therefore, we assume that $x_n \neq x_{n-1}$ for all $n \in \mathbb{N}$, i.e., no consecutive terms of the sequence $\{x_n\}$ are equal. As no consecutive terms of the sequence $\{x_n\}$ are equal and $M(x_n, x_{n+1}, t) > qM(x_n, x_{n+1}, t)$, from (2), we have

$$M(x_{n+1}, x_{n+2}, t) \geq \zeta(M(x_{n+1}, x_{n+2}, t), M(x_n, x_{n+1}, t)) > M(x_n, x_{n+1}, t)$$

i.e., $M(x_n, x_{n+1}, t) < M(x_{n+1}, x_{n+2}, t)$. Fix $n < m$. Similarly one can prove that

$$M(x_n, x_{n+1}, t) < M(x_{n+1}, x_{n+2}, t) < \cdots < M(x_m, x_{m+1}, t). \tag{5}$$

Suppose, now that $x_n = x_m$, we have $x_{n+1} = Tx_n = Tx_m = x_{m+1}$, and so, the above inequality yields a contradiction. Thus, we can assume that $x_n \neq x_m$ for all distinct $n, m \in \mathbb{N}$.

Now, for $t > 0$, let

$$a_n(t) = \inf_{m>n} M(x_n, x_m, t).$$

By (5) and property (K2) there exists $k_0 \in \mathbb{N}$ such that

$$M(x_n, x_{n+1}, t) > qM(x_m, x_n, t)$$

for all $m > n \geq k_0$. Then it follows from (2) and by definition of ζ that

$$\begin{aligned} M(x_{n+1}, x_{m+1}, t) &= M(Tx_n, Tx_m, t) \\ &\geq \zeta(M(Tx_n, Tx_m, t), M(x_n, x_m, t)) \\ &> M(x_n, x_m, t) \end{aligned} \tag{6}$$

for each $t > 0$ and $m > n \geq k_0$. Therefore, we have

$$M(x_n, x_m, t) < M(x_{n+1}, x_{m+1}, t) \text{ for all } k_0 \leq n < m.$$

Taking infimum over $m(> n)$ in the above inequality we obtain

$$\inf_{m>n} M(x_n, x_m, t) \leq \inf_{m>n} M(x_{n+1}, x_{m+1}, t)$$

i.e., $a_n(t) \leq a_{n+1}(t)$ for all $n \geq k_0$. Thus, $\{a_n(t)\}$ is bounded and monotonic for each $t > 0$. Suppose, $\lim_{n\to\infty} a_n(t) = a(t), t > 0$. We claim that $a(t) = 1$ for each $t > 0$. If $s > 0$ and $a(s) < 1$, then, using the fact that the quadruple (X, M, T, ζ) have the property (S), we obtain

$$\lim_{n\to\infty} \inf_{m>n} \zeta(M(x_n, x_m, s), M(x_{n+1}, x_{m+1}, s)) = 1. \tag{7}$$

From inequality (6) we have

$$\inf_{m>n} M(x_{n+1}, x_{m+1}, s) \geq \inf_{m>n} \zeta(M(Tx_n, Tx_m, s), M(x_n, x_m, s)) \geq \inf_{m>n} M(x_n, x_m, s)$$

i. e.,

$$a_{n+1}(s) \geq \inf_{m>n} \zeta(M(Tx_n, Tx_m, s), M(x_n, x_m, s)) \geq a_n(s).$$

Letting $n \to \infty$ and using (7) in the above inequality we obtain

$$\lim_{n\to\infty} \inf_{m>n} M(x_n, x_m, s) = a(s) = 1.$$

This contradiction verifies our claim. By definition of a_n we have $\lim_{n,m\to\infty} M(x_n, x_m, t) = 1$ for all $t > 0$. Hence, $\{x_n\}$ is an M-Cauchy sequence and by M-completeness of X there exists $u \in X$ such that

$$\lim_{n\to\infty} M(x_n, u, t) = 1 \quad \text{for all} \quad t > 0. \tag{8}$$

Finally, same proof of Theorem 3.5 is valid for $u = Tu$.

□

Remark 3.19. In view of Theorem 3.10, Theorem 3.18 and example 3.17 it seems that property($K2$) is weaker than property($K1$). It will be interesting to have an example illustrating this fact.

4. OBSERVATIONS

1. The following definition can be considered as an variant of Suzuki type fuzzy $\mathcal{Z}$-contractive mapping.

Definition 4.1. Let $(X, M, *)$ be a fuzzy metric space and $T\colon X \to X$ be a mapping. Suppose, there exists $\zeta \in \mathcal{Z}$ such that

$$M(x, Tx, t) \geq M(x, y, \frac{t}{2}) \implies M(Tx, Ty, t) \geq \zeta(M(Tx, Ty, t), M(x, y, t)) \tag{9}$$

for all $x, y \in X, Tx \neq Ty, t > 0$. Then T is called a Suzuki type fuzzy $\mathcal{Z}$-contractive mapping with respect to the function $\zeta \in \mathcal{Z}$.

2. The following examples demonstrates that Definition 3.1 and Definition 4.1 are independent of each other.

Example 4.2. Consider the M-complete non-Archimedean fuzzy metric space as defined in 3.14 and define the mapping $T : X \to X$ by

$$Tx = \begin{cases} \sqrt{x} & \text{if } x \neq 1, \\ 5 & \text{if } x = 1 \end{cases}$$

for all $t > 0$. If ζ defined from $(0, 1] \times (0, 1]$ into $\mathbb{R}$ by $\zeta(t, s) = \psi(s)$, where $\psi\colon (0, 1] \to (0, 1]$ is a function such that $s < \psi(s)$ for all $s \in (0, 1)$. Then, it is easy to see that T is a Suzuki type fuzzy $\mathcal{Z}$-contractive mapping with respect to condition (9) but T does not satisfy condition (2)(for this, consider $x = 1$ and $y = 2.6$).

Example 4.3. Let $X = \{0, 1, 2\}$. Define $a * b = ab$ for all $a, b \in [0, 1]$ and

$$M(x, y, t) = \left[exp\left(\frac{|x-y|}{t}\right)\right]^{-1}$$

for all $x, y \in X$ and $t \in (0, \infty)$. Then $(X, M, *)$ is a fuzzy metric space. Define the mapping $T : X \to X$ by

$$Tx = \begin{cases} 2 & \text{if } x = 0, \\ 1 & \text{if } x = \{1, 2\} \end{cases}$$

for all $t > 0$. If ζ defined from $(0, 1] \times (0, 1]$ into $\mathbb{R}$ by $\zeta(t, s) = \psi(s)$, where $\psi\colon (0, 1] \to (0, 1]$ is a function such that $s < \psi(s)$ for all $s \in (0, 1)$. Then, it is easy to see that T is a Suzuki type fuzzy $\mathcal{Z}$-contractive mapping with respect to condition (2) but T does not satisfy condition (9)(for this, consider $t = 1$ and $x = 0, y = 1$).

3. Further, we observe that all the above theorems will remain true if we replace Suzuki type fuzzy $\mathcal{Z}$-contractive mapping with respect to condition (2) by condition (9). For it, we replace (3) and (4) by

$$M(x_n, x_{n+1}, t) > M(x_n, x_m, t/2).$$

4. Example 3.14 and example 4.2 demonstrates that Suzuki type fuzzy $\mathcal{Z}$-contractive mappings need not to be continuous, whereas the known classes of fuzzy contractive mappings [6, 12, 16, 18, 20] are necessarily continuous.

5. It is well known that the contractive mappings given by Suzuki [17] in the sense of classical metric space characterize metric completeness. Thus, it will be interesting to have a kind of Suzuki type fuzzy $\mathcal{Z}$-contractive mapping which could characterize completeness of underline fuzzy metric space.

5. CONCLUSION

Fuzzy fixed point results are more versatile than the regular metric fixed point results. This is due to the flexibility which the fuzzy concepts inherently possess. Even than it is not easy to translate the classical metric contractions and corresponding fixed point theorems in fuzzy setting. Such issues are discussed in [8, 9, 13, 15, 16, 21]. Indeed Grabiec [5] initiated the study of defining Banach contraction in fuzzy metric spaces (in the sense of Kramosil and Michálek [11]) but his method was not appropriate to obtain metric Banach contraction. In 2002 Gregori et al. [7] resolved this problem by introducing fuzzy contractive mappings and obtain extensions of Banach contraction principle, by considering the standard fuzzy metric space, deduced from a metric space. Following this direction of research, we introduce a new concept of Suzuki type fuzzy $\mathcal{Z}$-contractive mappings and formulate the conditions guaranteeing the existence of unique fixed point in G-complete as well as M-complete fuzzy metric spaces. The paper includes a comprehensive set of examples showing the generality of our results and demonstrating that the formulated conditions are significant and cannot be omitted. Further, it will be interesting to formulate a kind of Suzuki type fuzzy $\mathcal{Z}$-contractive mapping which could characterize completeness of underline fuzzy metric space.

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