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DIVISION SCHEMES UNDER UNCERTAINTY OF CLAIMS

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In some economic or social division problems, we may encounter uncertainty of claims, that is, a certain amount of estate has to be divided among some claimants who have individual claims on the estate, and the corresponding claim of each claimant can vary within a closed interval or fuzzy interval. In this paper, we classify the division problems under uncertainty of claims into three subclasses and present several division schemes from the perspective of axiomatizations, which are consistent with the classical bankruptcy rules in particular cases. When claims of claimants have fuzzy interval uncertainty, we settle such type of division problems by turning them into division problems under interval uncertainty.

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1. INTRODUCTION

The classical bankruptcy problem is described as follows: Several individuals hold claims on a finite resource and the total amount is not enough to fulfill all of the claims. We take the example of Aumann and Maschler [1]: a man dies, leaving debts $d_1, d_2, \dots, d_n$ totalling more than his estate E. The key issue is how to distribute the estate to the claimants. For this kind of bankruptcy problem, it always requires an exact knowledge of each term, and has attracted the attention of many scholars, such as Elishakoff [6, 7]. However, in the realistic bankruptcy problems, the debts of claimants are in a possible range and can only be described using some fuzzy words, such as "between" and "around". At this time, the classical bankruptcy rules for crisp bankruptcy problems, such as the CGC rule introduced in Aumann and Maschler [1], are not suitable for solving this kind of uncertain division problems. Based on this, Yager and Kreinovich [16] investigated the uncertainty in the division problems, in which the possible fair proportion of the total estate assigned to each claimant is an interval. They introduced some natural fair properties, which chooses the unique "weights" from these intervals.

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[1] Branzei et al. [3] concentrated on the division problems under interval uncertainty of claims. They deleted the uncertainty of claims by compromising the lower and upper bounds of the claim intervals, and then considered the deterministic division problems with compromise claims.[2]

This paper deals with an extension the classical bankruptcy problem, that we call division problems under uncertainty of claims. Here we mainly pay attention to the interval and fuzzy interval uncertainty of claims, i. e., the claims are intervals or fuzzy intervals. We divide division problems under interval uncertainty of claims (ID-problems for short) into three types: normal interval division problems, strict interval division problems, and trivial interval division problems. For each type, we introduce the corresponding division schemes, in the similar spirit of the classical bankruptcy rules. When the claims have the forms of fuzzy intervals, we transform such type of division problems (FID-problems for short) into ID-problems and then give the corresponding division rules.

It is important to consider division problems with claims under uncertainty, because in various disputes including inheritance, claimants face uncertainty with regard to their effective rights and as a result, individual claims can be expressed in the form of intervals or fuzzy intervals. In such situations, our model offers flexibility to tackle with resource conflict under uncertainty. To get some insight into applications, interpretations and extensions for division problems under uncertainty, the readers can refer to a wide range of papers such as cooperative interval games [2] and stochastic bankruptcy games [8].

The rest of this paper is organized as follows. In Section 2, we briefly review the classical bankruptcy problems. In Section 3, we divide the division problems under interval uncertainty of claims into three subclasses, and propose and characterize the corresponding division schemes. Section 4 further discusses the division schemes for division problems under fuzzy interval uncertainty of claims. Section 5 concludes.

2. CLASSICAL BANKRUPTCY PROBLEMS AND BANKRUPTCY GAMES

Bankruptcy problems originate from the situations that several agents claim portions of a certain amount of estate and the sum of claims is larger than the total estate. The *classical bankruptcy problem* can be modeled by a triple $(N, E; c)$, written as $(E; c)$ for convenience, where E is the non-negative total estate and $c = (c_1, c_2, \ldots, c_n)$ is the vector of claims with c_i ($\forall i \in N$) being the claim of i.

A *cooperative game* is an ordered pair (N, v), where N is a set of players and $v : 2^N \to \mathbb{R}$ is a real-valued function satisying $v(\emptyset) = 0$. The subset S of N is called a *coalition* and $v(S)$ the *worth* of coalition S. The number of elements in S is denoted to $|S|$. A cooperative game is said to be a *convex game* if for any $S, T \subseteq N$, $v(S \cup T) + v(S \cap T) \geq v(S) + v(T)$.

An *imputation* of a game (N, v) is an n-dimensional vector $x = (x_i)_{i \in N} \in \mathbb{R}^N_+$, satisfying $\sum_{i \in N} x_i = v(N)$ and $x_i \geq v(\{i\})$. A *solution concept* for a cooperative game is a mapping that assigns to every game a set of payoff vectors. A single-valued solution

[1] The author focused on division problems under the condition that 1 is between the sum of lower and upper bounds of weights of all claimants.

[2] In this paper, the estate is limited to be less than the sum of all lower bounds of claims.

is called a *value*. The well-known *Shapley value* for cooperative games is defined to be

$$Sh_i(N,v) = \sum_{S\subseteq N; i\in S} \frac{(|N|-|S|)!(|S|-1)!}{|N|!}[v(S)-v(S\backslash\{i\})], \quad \text{for all } i\in N.$$

The *nucleolus* (see Schmeidler [13]) of a game is the set of imputations that minimize the complaint function $\theta(x) = v(S) - \sum_{i\in S} x_i$ in the lexicographic order over the nonempty compact convex imputation set. The another one-point solution concept, called τ value (see Tijs [15]), is defined based on the idea of an upper bound for the core and the excesses with respect to this upper bound.

Any bankruptcy problem $(E;c)$ generates a corresponding cooperative game $(N, v_{E;c})$, called the *bankruptcy game*, whose characteristic form is given by

$$v_{E;c}(S) = \max\left\{0, E - \sum_{i\in N\backslash S} c_i\right\}, \quad \text{for any } S\subseteq N. \tag{1}$$

For classical bankruptcy problems, the adjusted proportional rule (AP-rule) [4], contested garment consistent rule (CGC-rule) [1] and recursive completion rule (RC-rule) [11] are known as specific bankruptcy rules. Curiel et al. [4] showed that these three rules yield the τ value, nucleolus and Shapley value of the corresponding bankruptcy game $(N, v_{E;c})$ successively.

3. ID-PROBLEM

Let $I(\mathbb{R})$ be the set of all closed and bounded intervals in $\mathbb{R}$. We denote by $I(\mathbb{R})^n$ the set of all n-dimensional vectors with components in $I(\mathbb{R})$. Suppose $I, J\in I(\mathbb{R})$, with $I = [I^-, I^+]$, $J = [J^-, J^+]$ and $k\in\mathbb{R}_+$. The interval operations are defined by

$$\begin{aligned} I + J &= [I^- + J^-, I^+ + J^+] \\ kI &= [kI^-, kI^+]. \end{aligned}$$

We say that $I\geq J$ if $I^-\geq J^-$ and $I^+\geq J^+$. When $I\geq J$ and $J\geq I$, we denote $I = J$. Specially, $\overline{0} = [0,0]$.

Let N be a finite set of claimants who face uncertainty regarding his claim. E is the total estate that will be divided among N. We denote the claim vector by $\overline{d}\in I(\mathbb{R})^n$ with $\overline{d}_i = [d_i^-, d_i^+]\geq\overline{0}$ ($\forall i\in N$) meaning the claim interval of claimant i. A *division problem under interval uncertainty of claims* (ID-problem) can be defined by a triple $(N, E; \overline{d})$, often abbreviated to $(E;\overline{d})$. Particularly, when $d_i^- = d_i^+$, the claim interval $\overline{d}_i$ degenerates into a real number $d_i^-(d_i^+)$. Generally, we interpret d_i^- as the lower bound of claim or the least demand of claimant i, similarly d_i^+ the upper bound of claim or the utmost expectation of claimant i. For any $S\subseteq N$, we use $d^+(S)$ and $d^-(S)$ instead of $\sum_{i\in S} d_i^+$ and $\sum_{i\in S} d_i^-$.

Definition 3.1. A division scheme for any ID-problem is a nonnegative mapping g that assigns to each ID-problem $(E;\overline{d})$ a payoff vector $g(E;\overline{d})\in\mathbb{R}_+^n$ and satisfies efficiency if $d^+(N)\geq E$, i. e.,

$$\sum_{i\in N} g_i(E;\overline{d}) = E.$$

The division scheme prescribes somehow a specific division of the total estate E among n claimants. It requires that when the sum of utmost expectations of all claimants is not less than E, the sum of payoff to any claimant is equal to E. Here we divide the ID-problems into three types based on the relationship between E and $\overline{d}$.

Definition 3.2. An ID-problem $(E;\overline{d})$ is said to be a *normal interval division problem* (NID-problem) if

$$d^-(N) \leq E \leq d^+(N).$$

Definition 3.3. An ID-problem $(E;\overline{d})$ is said to be a *strict interval division problem* (SID-problem) if

$$E < d^-(N) \leq d^+(N).$$

Definition 3.4. An ID-problem $(E;\overline{d})$ is said to be a *trivial interval division problem* (TID-problem) if

$$d^-(N) \leq d^+(N) < E.$$

In the next three subsections, we treat the NID-problem, SID-problem and TID-problem respectively, in each one of which the specific division schemes are provided and axiomatized.

Remark 3.5. With regard to the TID-problem, it is clear that each claimant can receive his utmost expectation. As a matter of fact, in the following context we only need to place emphasis on the NID-problem. Because any SID-problem can be converted to an NID-problem by making a compromise to the least demands of claimants.

3.1. NID-problem

In this subsection, we tacitly treat $(E;\overline{d})$ as the NID-problem.

Definition 3.6. Any NID-problem $(E;\overline{d})$ generates a corresponding cooperative game $(N, v_{E;\overline{d}})$ with the characteristic function form

$$v_{E;\overline{d}}(S) = \max\{d^-(S), E - d^+(N\backslash S)\}, \quad \text{for any } S \subseteq N. \tag{2}$$

We say an NID-problem $(E;\overline{d})$ is *lower-normalized* if its corresponding cooperative game $(N, v_{E;\overline{d}})$ satisfies $v_{E;\overline{d}}(\{i\}) = d_i^-$ $(\forall i \in N)$ and *simple* if the utmost expectation of any claimant is not greater than the total estate, i. e., $d_i^+ \leq E$ $(\forall i \in N)$.

Definition 3.7. For any NID-problem $(E;\overline{d})$, the *minimum right* of claimant i is defined as the surplus of the total estate after others fulfill their utmost expectations on the understanding that it exceeds his own least demand, i. e., for all $i \in N$

$$m_i = \max\{d_i^-, E - d^+(N\backslash\{i\})\}.$$

Similarly, the *maximum right* of claimant i is defined as the left part of the estate after others meet their least demands on the understanding that it is less than his own utmost expectation, i. e., for all $i \in N$

$$M_i = \min\{d_i^+, E - d^-(N\backslash\{i\})\}.$$

Conveniently, we denote $m(S) = \sum_{i\in S} m_i$ and $M(S) = \sum_{i\in S} M_i$ for all $S \subseteq N$.

Proposition 3.8. For any NID-problem $(E;\overline{d})$, the following statements hold:

1. $d_i^- \leq m_i \leq M_i \leq d_i^+$, for all $i \in N$,
2. $v_{E;\overline{d}}(S) = \max\{m(S), E - M(N\backslash S)\}$, for all $S \subseteq N$.

Proof. (1) Note that $m_i = \max\{d_i^-, E - d^+(N\backslash\{i\})\} = v_{E;\overline{d}}(\{i\})$ and $v_{E;\overline{d}}(N) - v_{E;\overline{d}}(N\backslash\{i\}) = E - \max\{d^-(N\backslash\{i\}), E - d_i^+\} = \min\{d_i^+, E - d^-(N\backslash\{i\})\} = M_i$. Then, it follows from the convexity[3] of game $(N, v_{E;\overline{d}})$ that

$$d_i^- \leq m_i \leq M_i \leq d_i^+.$$

(2) Put another cooperative game w with the characteristic function

$$w(S) = \max\{m(S), E - M(N\backslash S)\}, \quad \text{for any } S \subseteq N. \tag{3}$$

On one hand, since $m(S) \geq d^-(S)$ and $E - M(N\backslash S) \geq E - d^+(N\backslash S)$, we easily have $w(S) \geq v_{E;\overline{d}}(S)$. On the other hand, by the convexity of $(N, v_{E;\overline{d}})$, we derive that $m(S) = \sum_{i\in S} v_{E;\overline{d}}(\{i\}) \leq v_{E;\overline{d}}(S)$ and $E - M(N\backslash S) = v_{E;\overline{d}}(N) - \sum_{i\in N\backslash S}(v_{E;\overline{d}}(N) - v_{E;\overline{d}}(N\backslash\{i\}))$ $\leq v_{E;\overline{d}}(N) - (v_{E;\overline{d}}(N) - v_{E;\overline{d}}(S)) = v_{E;\overline{d}}(S)$ for all $S \subseteq N$. Further, it follows from (2) and (3) that $w(S) \leq v_{E;\overline{d}}(S)$. Hence, it concludes that

$$w(S) = v_{E;\overline{d}}(S).$$

□

A division scheme what we want to find here is a function g which maps every NID-problem into a vector $g(E;\overline{d}) \in \mathbb{R}^n$ with the i_{th} component varying from the least demand to the utmost expectation of claimant i, i. e., for each $i \in N$,

$$d_i^- \leq g_i(E;\overline{d}) \leq d_i^+,$$

and simultaneously satisfies that

$$g_1(E;\overline{d}) + \cdots + g_n(E;\overline{d}) = E.$$

Naturally, we expect that this mapping has the following properties.

[3]The convexity of $(N, v_{E;\overline{d}})$ is immediately obtained because Curiel [4] showed that the bankruptcy game (1) is a convex game.

- Equal treatment: if there exists $i, j \in N$ such that $\overline{d}_i = \overline{d}_j$, then $g_i(E;\overline{d}) = g_j(E;\overline{d})$.

Equal treatment property indicates that any two claimants with equal claims receive equal outcome.

- Invariance under truncated claim:

$$g_i(E;\overline{d}) = g_i(E;[d_j^-, \min\{d_j^+, E - d^-(N\backslash\{j\})\}]_{j\in N})$$

for all $i \in N$.

This property means that truncating the upper bounds to ensure that other claimants obtain their lower bounds of claims does not influence the final division.

Proposition 3.9. The corresponding cooperative game $(N, v_{E;\overline{d}})$ of each NID-problem $(E;\overline{d})$ coincides with its truncated form, i. e.,

$$v_{E;\overline{d}} = v_{E;[d_j^-,\min\{d_j^+,E-d^-(N\backslash\{j\})\}]_{j\in N}}.$$

Proof. Let $\overline{d^t}_j = [d_j^-, \min\{d_j^+, E - d^-(N\backslash\{j\})\}]$, it suffices to show $v_{E;\overline{d}}(S) = v_{E;\overline{d^t}}(S)$ for all $S \subseteq N$, that is, $\max\{d^-(S), E - d^+(N\backslash S)\} = \max\{d^-(S), E - {d^t}^+(N\backslash S)\}$. We prove it in two cases as follows.

Case 1: if $d_i^+ \leq E - d^-(N\backslash\{i\})$ for any $i \in N\backslash S$, we have that $d^+ = {d^t}^+$ for each $i \in N\backslash S$. Then $\max\{d^-(S), E - d^+(N\backslash S)\} = \max\{d^-(S), E - {d^t}^+(N\backslash S)\}$.

Case 2: if there exists $j \in N\backslash S$ such that $d_j^+ \geq E - d^-(N\backslash\{j\})$, namely $d_j^+ - d_j^- > E - d^-(N)$, we deduce that $E - {d^t}^+(N\backslash S) = E - \sum_{i\in N\backslash S} \min\{d_i^+, E - d^-(N\backslash\{i\})\} = E - d^-(N\backslash S) - \sum_{i\in N\backslash S} \min\{d_i^+ - d_i^-, E - d^-(N)\} \leq E - d^-(N\backslash S) - (E - d^-(N)) = d^-(S)$. Therefore, $E - d^+(N\backslash S) \leq E - {d^t}^+(N\backslash S) \leq d^-(S)$ and then $\max\{d^-(S), E - d^+(N\backslash S)\} = \max\{d^-(S), E - {d^t}^+(N\backslash S)\} = d^-(S)$. $\square$

- Composition of minimum right:
  for all $i \in N$, $g_i(E;\overline{d}) = m_i + g_i(E - m(N); [0, d_j^+ - m_j]_{j\in N})$.

This property implies that each claimant receives at least his minimum right, and also the division result does not change even if each claimant takes his minimum right firstly.

- Divisibility of claims: for any lower-normalized and simple NID-problem $(E;\overline{d})$, if claimant i is replaced by several claimants $i_1, i_2, \ldots, i_k$ with claims $\overline{d}_{i_1}, \overline{d}_{i_2}, \ldots, \overline{d}_{i_k}$ satisfying $\overline{d}_i = \overline{d}_{i_1} + \overline{d}_{i_2} + \cdots + \overline{d}_{i_k}$, then for all $j \in N\backslash\{i\}$,

$$g_j(E;\overline{d}) = g_j(E;\overline{d}_1, \ldots, \overline{d}_{i-1}, \overline{d}_{i_1}, \overline{d}_{i_2}, \ldots, \overline{d}_{i_k}, \overline{d}_{i+1}, \ldots, \overline{d}_n).$$

This property says that for any lower-normalized and simple NID-problem, each claimant has no incentive to split into several claimants whose claims sum up to his initial claim.

- I-bilateral consistency: for any $(E;\overline{d})$ and $S \subseteq N$ with $|S| = 2$, $g_i(N, E; \overline{d}) = g_i(S, E_S; \overline{d}_S)$ $(\forall i \in S)$, where $E_S = \sum_{j\in S} g_j(N, E; \overline{d})$ and $\overline{d}_S = (\overline{d}_j)_{j\in S}$.

I-bilateral consistency property associates a given division problem to its subproblems, that any two claimants become new groups and the amounts accumulated in the original problem become the new amount to be divided.

- Two-person minimal share property: if $N = \{i, j\}$ with $i \neq j$, then $g_i(N, E; \overline{d}) \geq \frac{1}{2}\min\{d_i^- + d_i^+, E + d_i^- - d_j^-\}$.

Two-person minimal share property guarantees a minimal share to each claimant in the two-person NID-problem. Assume $N = \{i, j\}$, this property says that claimant i will receive at least $(d_i^+ - d_i^-)/2$ when $E - d_j^- \geq d_i^+$, otherwise at least $(E - d_i^- - d_j^-)/2$ after he takes d_i^- firstly.

Definition 3.10. Let g be a division scheme of an NID-problem $(E;\overline{d})$. We define the *dual division scheme* of g, denoted by $g^*(E;\overline{d})$, to be

$$g^*(E;\overline{d}) = d^- + d^+ - g(d^+(N) + d^-(N) - E; \overline{d}).$$

- Self-duality: for any $(E;\overline{d})$, $g(E;\overline{d}) = g^*(E;\overline{d})$.

The self-duality property requires that the resulting reward to any claimant in the original problem $(E;\overline{d})$, is equal to the sum of his least demand and his loss with respect to his utmost expectation in the $(d^+(N) + d^-(N) - E;\overline{d})$ problem. The new estate $d^+(N) + d^-(N) - E$ represents the total amount of unfulfilled utmost expectations after they all get their least demands.

Generally, we automatically suppose the division scheme g is continuous, that is, the small change in the endpoints d_i^- or d_i^+ will not dramatically affect the final division result.

Definition 3.11. For any NID-problem $(E;\overline{d})$, the *adjusted utmost expectation* d_i^* $(\forall i \in N)$ is defined as the reduced utmost expectation with respect to the remaining estate $E - m(N)$ after all claimants receive their minimum rights, i. e.,

$$d_i^* = \min\{d_i^+ - m_i, E - m(N)\}.$$

Theorem 3.12. For any NID-problem $(E;\overline{d})$, there exists one unique division scheme g, called the AP-like division scheme, satisfying equal treatment, invariance under truncated claim, composition of minimum right and divisibility of claims. And the general formula is:

$$\begin{aligned} g(E;\overline{d}) &= m + (E - m(N))\frac{d^*}{\sum\limits_{i\in N} d_i^*}, && \text{if } E > m(N); \\ &= m, && \text{if } E = m(N), \end{aligned} \tag{4}$$

where $d^* = (d_1^*, d_2^*, \ldots, d_n^*)$.

P r o o f. For any NID-problem $(E;\overline{d})$, let g be its division scheme possessing the four properties in Theorem 3.12. It is clear that $g(E;\overline{d}) = m$ when $E = m(N)$. In the following, we only need to verify that (4) is true when $E > m(N)$.

If $E > m(N)$, it follows from composition of minimum right and invariance under truncated claim that

$$\begin{aligned} g_i(E;\overline{d}) &= m_i + g_i(E - m(N); [0, d_j^+ - m_j]_{j\in N}) \\ &= m_i + g_i(E - m(N); [0, \min\{d_j^+ - m_j, E - m(N)\}]_{j\in N}) \\ &= m_i + g_i(E - m(N); [0, d_j^*]_{j\in N}). \end{aligned}$$

Next we begin to prove that the division scheme $g(E;\overline{d})$ can be described as the formula (4). To prove it, let us first start by showing that this is true for all claims with rational endpoints. If all endpoints of claims $\overline{d}_i = [d_i^-, d_i^+]$ $(\forall i \in N)$ are rational, so d_i^* for all $i \in N$. We denote this common denominator of d_i^* for all $i \in N$ by C, then each d_i^* has the form α_i/C where α_i is a non-negative integer $(\alpha_i = C \cdot d_i^*)$. Obviously, from $[0, d_i^*] = [0, \alpha_i/C] = \alpha_i \cdot [0, 1/C]$, we know that each interval $[0, d_i^*]$ can be written as a sum of α_i intervals $[0, 1/C]$. Let $(E - (m(N)); \overline{d}'')$ be the division problem which is obtained from $(E - m(N); [0, d_j^*]_{j\in N})$ by replacing all intervals $[0, d_i^*]$ for all $i \in N$ with α_i intervals $[0, 1/C]$. Denote the same value that $(E - (m(N)); \overline{d}'')$ maps all intervals $[0, \frac{1}{C}]$ into by β, by the equal treatment property we can get the mapping

$$\begin{aligned} &\underbrace{[0,1/C],\dots,[0,1/C]}_{\alpha_1},\dots,\underbrace{[0,1/C],\dots,[0,1/C]}_{\alpha_n} \\ &\to \underbrace{\beta,\dots,\beta}_{\alpha_1},\dots,\underbrace{\beta,\dots,\beta}_{\alpha_n}. \end{aligned}$$

Then by efficiency of division scheme, we have the equation $\alpha_1 \cdot \beta + \cdots + \alpha_n \cdot \beta = E - m(N)$, so $\beta = (E - m(N))/\sum_{i\in N} \alpha_i$. Furthermore, we get $g_i(E - m(N); [0, d_j^*]_{j\in N}) = \alpha_i \cdot \beta = (E - m(N))\frac{\alpha_i}{\sum_{i\in N} \alpha_i}$ by the property of divisibility of claims in the NID-problem $(E - m(N); [0, d_j^*]_{j\in N})$. Finally, we obtain the resulting division scheme $g_i(E;\overline{d}) = m_i + (E - m(N))\frac{\alpha_i}{\sum_{i\in N} \alpha_i} = m_i + (E - m(N))\frac{d_i^*}{\sum_{i\in N} d_i^*}$.

Now, the theorem is verified true for all claim intervals with rational endpoints. Since the mapping g is continuous and every interval can be expressed as a limit of intervals with rational endpoints, it concludes that the formula (4) is true for all claim intervals. □

The AP-like division scheme for any NID-problem is carried out as follows: each claimant i first receives his minimum right m_i $(\forall i \in N)$, and then a new classical bankruptcy problem is obtained with the estate $E - m(N)$ and claim vector d^*; after that, this classical bankruptcy problem $(E - m(N); d^*)$ is solved by means of proportional division, i. e., dividing the remaining estate $E - m(N)$ among the claimants proportional to their adjusted utmost expectations d_i^* $(\forall i \in N)$.

Proposition 3.13. The AP-like division scheme for any NID-problem $(E;\overline{d})$ has the expression form

$$g(E;\overline{d}) = d^- + f(E - d^-(N); d^+ - d^-),$$

where f corresponds to the AP rule for classical bankruptcy problems.

P r o o f. When $E = m(N)$, we easily have

$$d^- + f(E - d^-(N); d^+ - d^-) = m = g(E;\overline{d}).$$

When $E > m(N)$, for all $i \in N$,

$$\begin{aligned} &f_i(E - d^-(N); d^+ - d^-) \\ =&AP_i(E - d^-(N); d^+ - d^-) \\ =&m'_i + (E - d^-(N) - m'(N))\frac{d_i^{*\prime}}{\sum\limits_{i\in N} d_i^{*\prime}}, \end{aligned}$$

where $m'_i = \max\{0, E - d^-(N) - \sum_{j\in N\setminus\{i\}}(d_j^+ - d_j^-)\}$, $d_i^{*\prime} = \min\{d_i^+ - d_i^- - m'_i, E - d^-(N) - m'(N)\}$. By simplification, we get that for all $i \in N$

$$m_i = m'_i + d_i^-,\ d_i^{*\prime} = d_i^*.$$

Therefore, $g(E;\overline{d}) = d^- + f(E - d^-(N); d^+ - d^-)$. □

Proposition 3.13 connects the NID-problems with the classical bankruptcy problems. It implies that the AP-like division scheme of each claimant is equal to its least demand plus its allocation by the AP rule in the reduced classical bankruptcy problem $(E - d^-(N); d^+ - d^-)$.

Theorem 3.14. For any NID-problem $(E;\overline{d})$, there exists one unique division scheme g, called the CGC-like division scheme, satisfying I-bilateral consistency, two-person minimal share property and self-duality. And the general formula is

$$g(E;\overline{d}) = d^- + f(E - d^-(N); d^+ - d^-),$$

where f corresponds to the CGC rule for classical bankruptcy problems.

P r o o f. Let g be a division scheme for an NID-problem $(E;\overline{d})$ that satisfies all properties mentioned in Theorem 3.14. According to I-bilateral consistency of g, together with bilateral consistency [10] of f, we only need to prove that the Theorem 3.14 is true when the population of claimants is two, namely $|N| = 2$. Let $(N, E - d^-(N); d^+ - d^-)$ be a bankruptcy problem with $N = \{1, 2\}$ and $d_1^+ - d_1^- \le d_2^+ - d_2^-$. In this setting, $f(E - d^-(N); d^+ - d^-)$ can be expressed as:

$$\begin{aligned} &f(E - d^-(N); d^+ - d^-) \\ =&\begin{cases} \left(\frac{E-d^-(N)}{2}, \frac{E-d^-(N)}{2}\right), & \text{if } E \le d_1^+ + d_2^-; \\ \left(\frac{d_1^+-d_1^-}{2}, E - d^-(N) - \frac{d_1^+-d_1^-}{2}\right), & \text{if } d_1^+ + d_2^- \le E \le d_1^- + d_2^+; \\ \left(d_1^+ - d_1^- - \frac{d_1^++d_2^+-E}{2}, d_2^+ - d_2^- - \frac{d_1^++d_2^+-E}{2}\right), & \text{if } d_1^- + d_2^+ \le E. \end{cases} \end{aligned}$$

There are three cases to be discussed.

Case 1: if $E \leq d_1^+ + d_2^-$, according to two-person minimum share property, we have $g_1(E;\overline{d}) \geq \frac{E-d_2^-+d_1^-}{2}$ and $g_2(E;\overline{d}) \geq \frac{E-d_1^-+d_2^-}{2}$. Together with efficiency of g, we have $g_1(E;\overline{d}) + g_2(E;\overline{d}) = E$ and then $g(E;\overline{d}) = (d_1^-, d_2^-) + f(E - d^-(N); d^+ - d^-)$.

Case 2: if $d_1^+ + d_2^- \leq E \leq d_1^- + d_2^+$, since $d_1^+ + d_2^- \leq E$ and g satisfies two-person minimal share property, we get $g_1(E;\overline{d}) \geq \frac{d_1^+ + d_1^-}{2}$. Meanwhile, $E \leq d_1^- + d_2^+$ is equivalent to $d^+(N) + d^-(N) - E \geq d_1^+ + d_2^-$, which implies that $g_1(d^+(N) + d^-(N) - E;\overline{d}) \geq \frac{d_1^- + d_1^+}{2}$ by two-person minimal share property in the $(d^+(N) + d^-(N) - E;\overline{d})$ division problem. In addition, with self-duality of $g(E;d)$, we immediately obtain that $g_1(E;\overline{d}) = d_1^- + d_1^+ - g_1(d^-(N) + d^+(N) - E;\overline{d}) \leq \frac{d_1^- + d_1^+}{2}$. Thus, $g(E;\overline{d}) = (\frac{d_1^- + d_1^+}{2}, E - \frac{d_1^- + d_1^+}{2}) = (d_1^-, d_2^-) + f(E - d^-(N); d^+ - d^-)$.

Case 3: if $d_1^- + d_2^+ \leq E$, namely $d^-(N) + d^+(N) - E \leq d_1^+ + d_2^-$, we get $g(d^+(N) + d^-(N) - E;\overline{d}) = (d_1^-, d_2^-) + (\frac{d^+(N)-E}{2}, \frac{d^+(N)-E}{2})$. Then, $g(E;\overline{d}) = d^- + d^+ - g(d^-(N) + d^+(N) - E;\overline{d}) = d^- + f(E - d^-(N); d^+ - d^-)$.

Hence, we complete the proof of Theorem 3.14. □

Remark 3.15. When $E \geq d_i^+ + d^-(N\backslash\{i\})$ for all $i \in N$, we can obtain a new division scheme, called the *RC-like* division scheme, for the NID-problem, i. e.,

$$g(E;\overline{d}) = d^- + f(E - d^-(N); d^+ - d^-),$$

where f is the *RC* rule for classical bankruptcy problems.

Definition 3.16. A division scheme g for any NID-problem $(E;\overline{d})$ is called *game theoretical*, if we can find a solution concept ψ for the corresponding cooperative game $(N, v_{E;\overline{d}})$ such that $g(E;\overline{d}) = \psi(N, v_{E;\overline{d}})$.

The following proposition states that the *AP*-like, *CGC*-like and *RC*-like division schemes are game theoretical.

Proposition 3.17. The *AP*-like, *CGC*-like and *RC*-like division schemes agree with the τ value, nucleolus and Shapley value of the corresponding cooperative game $(N, v_{E;\overline{d}})$ respectively.

Proof. According to Definition 3.17, for any $S \subseteq N$,

$$\begin{aligned} v_{E;\overline{d}}(S) &= \max\{d^-(S), E - d^+(N\backslash S)\} \\ &= \max\{d^-(S), E - d^+(N\backslash S) - d^-(S) + d^-(S)\} \\ &= d^-(S) + \max\{0, E - d^+(N\backslash S) - d^-(S)\} \\ &= d^-(S) + \max\{0, E - d^-(N) - (d^+(N\backslash S) - d^-(N\backslash S))\} \\ &= d^-(S) + v_{E-d^-(N);d^+-d^-}(S). \end{aligned}$$

Moreover, it follows Driessen [5] that the *AP* rule, *CGC* rule, and *RC* rule for classical bankruptcy problems $(E - d^-(N); d^+ - d^-)$ agree with the τ value, nucleolus, and the Shapley value of the cooperative game $(N, v_{E-d^-(N);d^+-d^-})$. Therefore, when g is the AP-like, CGC-like, RC-like division schemes respectively, the corresponding f represents the AP rule, CGC rule, RC rule, ψ represents the τ value, nucleolus, and Shapley value,

$$\begin{aligned} g(E;\overline{d}) &= d^- + f(E - d^-(N); d^+ - d^-) \\ &= d^- + \psi(N, v_{E-d^-(N);d^+-d^-}) \\ &= \psi(N, d^- + v_{E-d^-(N);d^+-d^-}) \\ &= \psi(N, v_{E;\overline{d}}) \end{aligned}$$

holds following Proposition 3.13, Theorem 3.14, and Remark 3.15, where the penultimate equation holds because of the relative invariance under S-equivalence of the τ value, nucleolus, and Shapley value.

This proposition is proved. □

Example 3.18. In order to illustrate the division schemes above, we see an example of an NID-problem $(E;\overline{d})$, where $E = 100$ and $\overline{d} = ([20, 24], [30, 32], [40, 48])$. The different division schemes are presented in Table 1.

	Player 1	Player 2	Player 3
AP-like	$22\frac{2}{5}$	$31\frac{1}{5}$	$46\frac{2}{5}$
CGC-like	$22\frac{1}{2}$	31	$46\frac{1}{2}$
RC-like	$22\frac{1}{3}$	$31\frac{1}{3}$	$46\frac{1}{3}$

Tab. 1. Division schemes in Example 3.18.

The comparisons of AP-like, CGC-like and RC-like division schemes are useful in reality to support the choice of the preferred division scheme, for example, CGC-like division scheme is favourable to claimant i with smaller $d_i^+ - d_i^-$ when $E \leq \frac{d^+(N)+d^-(N)}{2}$, conversely beneficial to claimant i with greater $d_i^+ - d_i^-$.

3.2. SID-problem

In this subsection, $(E;\overline{d})$ especially refers to the SID-problem.

Definition 3.19. For any SID-problem $(E;\overline{d})$, the *lower security* of claimant i is defined as the smallest amount he can get when all claimants outside him obtain their lower bounds of claims, i. e., for all $i \in N$,

$$s_i = \max\{0, E - d^-(N\backslash\{i\})\}.$$

Let g be a division scheme for any SID-problem $(E;\overline{d})$, we hope that it has such a property which ensures each claimant get at least his lower security, i. e.,

- Lower security property: $g(E;\overline{d}) = g(E; [s_j, d_j^+]_{j\in N})$.

With the fact that $\sum_{i\in N} s_i = \sum_{i\in N} \max\{0, E - d^-(N\backslash\{i\})\} \leq E \leq d^+(N)$, we get a transformation from the SID-problem to the NID-problem with the aid of lower security property. Therefore, we can handle the SID-problem by the same method as the NID-problem.

Remark 3.20. When the claim $\overline{d}_i$ of any claimant i ($\forall i \in N$) is a degenerated interval, the SID-problem is a particular case as the classical bankruptcy problem, and the division results correspond to the AP rule, CGC rule and RC rule for classical bankruptcy problems respectively if we adopt the AP-like, CGC-like or RC-like division scheme for NID-problems.

Example 3.21. Consider an SID-problem $(E;\overline{d})$, where $E = 100$, and

$$\overline{d} = ([30, 50], [40, 50], [50, 60]).$$

The division results are displayed in Table 2.

	Player 1	Player 2	Player 3
AP-like	26	32	42
CGC-like	$23\ \frac{1}{3}$	$33\ \frac{1}{3}$	$43\ \frac{1}{3}$
RC-like	$26\ \frac{2}{3}$	$31\ \frac{2}{3}$	$41\ \frac{2}{3}$

Tab. 2. Division schemes in Example 3.21.

3.3. TID-problem

In this subsection, $(E;\overline{d})$ is fixed as the TID-problem.

Let g be a division scheme for any TID-problem $(E;\overline{d})$. Since the total estate E is far more than the sum of the utmost expectations of all claimants, it is natural to distribute any claimant his utmost expectation, that is

$$g(E;\overline{d}) = d^+.$$

4. FID-PROBLEM

In the previous context, we present the division schemes for the ID-problems. Actually, in some realistic situations claimants are also likely to declare their claims with vague words just like "about", "around" and so on, which can not be answered simply by "yes" or "no". In order to handle such kind of division problems, we first review some basic concepts about fuzzy intervals.

Different from the ordinary sets that are clear and definite, the fuzzy set is used to represent a set described by some vague concepts. A *fuzzy set* A in $\mathbb{R}$ is characterized by a real-valued function μ_A which assigns to each point in $\mathbb{R}$ a real number in the interval

$[0,1]$. The function μ_A is called the *membership function* of the fuzzy set A. The value $\mu_A(x)$ is called the *grade of membership* of x in A. For any $\lambda \in (0,1]$, the λ-*cut* of the fuzzy set A is defined by $[\mu_A]^\lambda = \{x|\mu_A(x) \geq \lambda, x \in \mathbb{R}\}$. A fuzzy set A defined on $\mathbb{R}$ is said to be a *fuzzy interval*, if

1. there exists $x_0 \in \mathbb{R}$, such that $\mu_A(x_0) = 1$;

2. for any $\lambda \in (0,1]$, $[\mu_A]^\lambda$ is compact and convex.

Remark 4.1. It is worth noting that for each fuzzy interval, its λ-cut is a classical interval and we denote $[\mu_A]^\lambda = [[\mu_A]^{\lambda-}, [\mu_A]^{\lambda+}]$.

A *division problem under fuzzy interval uncertainty* (FID-problem) is defined by a triple $(N, E; D)$, where the total estate E is exactly deterministic while the i_{th} component D_i ($\forall i \in N$) of claim vector $D = (D_1, D_2, \ldots, D_n)$ is a fuzzy interval. Generally, we write $(E; D)$ instead of $(N, E; D)$.

Definition 4.2. A division scheme for any FID-problem is a nonnegative mapping h which assigns to each FID-problem $(E; D)$ a payoff vector $h(E; D) \in \mathbb{R}^n$.

For most of literature on cooperative games with fuzzy payoffs, through the sum and minimum operations of fuzzy intervals, the proposed solution concepts are still fuzzy, such as literature [9, 14, 17]. Although fuzzy intervals can truthfully reflect the fuzzy concepts of objective existence, we often need to obtain the exact outcome when we finally have to make judgment or decision. Now we will focus on the type of division problem $(E; D)$ under fuzzy interval uncertainty of claims. We prepare to deal with it by two steps.

In the first step: for each $\lambda \in (0,1]$ and claim vector $D = (D_1, D_2, \ldots, D_n)$, we can get a sequence of λ-cut intervals $[\mu_{D_1}]^\lambda$, $[\mu_{D_2}]^\lambda$, $\ldots$,$[\mu_{D_n}]^\lambda$. In order to use all data of fuzzy intervals of claims to make the resulting division scheme more fair and reasonable, we convert the fuzzy intervals by applying the thought of integral into a sequence of classical intervals $\overline{d}_1', \overline{d}_2', \ldots, \overline{d}_n'$, namely,

$$D_1, D_2, \ldots, D_n \to \overline{d}_1', \overline{d}_2', \ldots, \overline{d}_n',$$

where $\overline{d}_i' = [\int_0^1 [\mu_{D_i}]^{\lambda-} \mathrm{d}\lambda, \int_0^1 [\mu_{D_i}]^{\lambda+} \mathrm{d}\lambda]$ for any $i \in N$. Then the original FID-problem $(E; D)$ is switched to the ID-problem $(E; \overline{d}')$.

In the second step: we can use the division schemes for the ID-problem referred in the last section to give the division scheme $h(E; D) = g(E; \overline{d}')$ for the FID-problem $(E; D)$, where $\overline{d}' = (\overline{d}_1', \overline{d}_2', \ldots, \overline{d}_n')$.

From the procedure to divide the total estate among claimants under the fuzzy interval uncertainty of claims, we find that the FID-problem can be resolved applying the division schemes for the ID-problem.

Example 4.3. Given an FID-problem $(E;D)$ with $E=100$, $D=(D_1,D_2,D_3)$ and

$$\mu_{D_1}(x)=\begin{cases}1, & \text{if } 20\le x\le 24;\\ 0, & \text{otherwise,}\end{cases}$$

$$\mu_{D_2}(x)=\begin{cases}\frac{1}{2}(x-29), & \text{if } 29\le x\le 31;\\ \frac{1}{2}(33-x), & \text{if } 31< x\le 33;\\ 0, & \text{otherwise,}\end{cases}$$

$$\mu_{D_3}(x)=\begin{cases}\frac{1}{4}(x-38), & \text{if } 38\le x\le 42;\\ 1, & \text{if } 42< x< 46;\\ \frac{1}{4}(50-x), & \text{if } 46\le x\le 50;\\ 0, & \text{otherwise.}\end{cases}$$

We change the fuzzy intervals of claims into classical intervals:

$$\overline{d}_1'=\left[\int_0^1[\mu_{D_1}]^{\lambda-}\mathrm{d}\lambda,\int_0^1[\mu_{D_1}]^{\lambda+}\mathrm{d}\lambda\right]=[20,24],$$

$$\overline{d}_2'=\left[\int_0^1[\mu_{D_2}]^{\lambda-}\mathrm{d}\lambda,\int_0^1[\mu_{D_2}]^{\lambda+}\mathrm{d}\lambda\right]=[30,32],$$

$$\overline{d}_3'=\left[\int_0^1[\mu_{D_3}]^{\lambda-}\mathrm{d}\lambda,\int_0^1[\mu_{D_3}]^{\lambda+}\mathrm{d}\lambda\right]=[40,48].$$

Then, an ID-problem is derived and we take

$$h(E;D)=g(E;\overline{d}')=g(100;([20,24],[30,32],[40,48])).$$

This division problem $(E;\overline{d}')$ is exactly the NID-problem referred in the Example 3.18 and therefore the final division result is the same as the Table 1.

5. CONCLUSIONS

In real life, when faced with uncertainty of claims for bankruptcy problems, we need to give each participant a certain money or entity in the final decision. So, it is necessary to provide division schemes with definite values for division problems under uncertainty of claims. In this paper, we study division problems under interval or fuzzy interval uncertainty of claims. We propose a series of division schemes based on the classical bankruptcy rules. After discussing the division problems under this kind of uncertainty, we naturally imagine that whether we can dispose of extended division problems with other forms of uncertainty, for instance, the claims of claimants may change with the total estate or time variable. It needs further research.

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REFERENCES

[1] R. J. Aumann and M. Maschler: Game theoretic analysis of a bankruptcy problem from the Talmud. J. Econom. Theory *36* (1982), 195–213. DOI:10.1016/0021-9045(82)90041-7

[2] Cooperative interval games: A survey. Cent. Europ. J. Oper. Res. *18* (2010), 397–411. DOI:10.1007/s10100-009-0116-0

[3] R. Branzei, D. Dimitrov, S. Pickl, and S. Tijs: How to cope with division problems under interval uncertainty of claims? Int. J. Uncertain. Fuzz. *12* (2004), 191–200. DOI:10.1142/S021848850400276X

[4] I. J. Curiel, M. Maschler, and S. H. Tijs: Bankruptcy games. Z. Oper. Res. *31* (1987), A143–A159. DOI:10.1007/BF02109593

[5] T. Driessen: Cooperative Games, Solutions and Applications. Kluwer Academic Publishers, 1988.

[6] I. Elishakoff: Resolution of two millennia-old Talmudic mathematical conundrums. BeOr HaTorah *21* (2012), 61–76.

[7] I. Elishakoff and A. Bégin-Drolet: Talmudic bankruptcy problem: special and general solutions. Scientiae Mathematicae Japonicae *69* (2009), 387–403.

[8] H. Habis and P. J. J. Herings: Stochastic bankruptcy games. Int. J. Game Theory *42* (2013), 973–988. DOI:10.1007/s00182-012-0350-x

[9] L. Mallozzi, V. Scalzo, and S. Tijs: Fuzzy interval cooperative games. Fuzzy Set Syst. *165* (2011), 1, 98–105. DOI:10.1016/j.fss.2010.06.005

[10] J. D. Moreno-Ternero and A. Villar: The Talmud rule and the securement of agents' awards. Math. Soc. Sci. *47* (2004), 245–257. DOI:10.1016/s0165-4896(03)00087-8

[11] B. O'Neill: A problem of rights arbitration from the Talmud. Math. Soc. Sci. *2* (1982), 345–371.

[12] M. Pulido, J. Sánchez-Soriano, and N. Llorca: Game theory techniques for university management: an extended bankruptcy model. Ann. Oper. Res. *109* (2002), 129–142. DOI:10.1023/A:1016395917734

[13] D. Schmeidler: The nucleolus of a characeristic function. SIAM J. Appl. Math. *17* (1969), 1163–1170. DOI:10.1137/0117107

[14] W. J. Zhao and J. C. Liu: Interval-valued fuzzy cooperative games based on the least square excess and its application to the profit allocation of the road freight coalition. Symmetry *10* (2018), 709. DOI:10.3390/sym10120709

[15] S. Tijs: Bounds for the core of a game and the t-value. In O. Moeschlin, & D. Pallaschke (Eds.), Game Theory Math. Econom. (1981), pp. 123–132. North-Holland Publishing Company.

[16] R. R. Yager and V. Kreinovich: Fair division under interval uncertainty. Int. J. Uncert. Fuzz. *8* (2000), 611–618.

[17] X. Yu and Q. Zhang: Core for game with fuzzy generalized triangular payoff value. Int. J. Uncert. Fuzz. *27* (2019), 789–813. DOI:10.1142/S0218488519500351

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