GLOBAL OUTPUT FEEDBACK STABILIZATION FOR NONLINEAR FRACTIONAL ORDER TIME DELAY SYSTEMS

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This paper investigates the problem of global stabilization by state and output-feedback for a family of for nonlinear Riemann-Liouville and Caputo fractional order time delay systems written in triangular form satisfying linear growth conditions. By constructing a appropriate Lyapunov-Krasovskii functional, global asymptotic stability of the closed-loop systems is achieved. Moreover, sufficient conditions for the stability, for the particular class of fractional order time-delay system are obtained. Finally, simulation results dealing with typical bioreactor example, are given to illustrate that the proposed design procedures are very efficient and simple.

Keywords: Riemann–Liouville fractional, nonlinear time delay system, observer design, asymptotical stability, Lyapunov functional

Classification: 93C10, 93D15, 93D20

1. INTRODUCTION

Fractional calculus is a concept of integrals and derivatives of arbitrary non-integer order. However, the main reason for describing these systems by fractional model is many physical systems contain by nature non integer derivatives so that these systems are better described by fractional-order equations. In fact, the interest of the fractional order calculation and its applications keeps growing in several areas such as, electromagnetic systems [15], dielectric polarization [28] and economy [19].

The analysis of stabilization of nonlinear systems is one of the most important issues for control theory. The notion of stability is a central issue of control theory, this has a wide range of definitions in [17]. Also, the problem of stabilization of nonlinear time delay systems has been exploited in literature. In practical application, most controlled systems can be modeled as a nonlinear time delay system. In this context, a large number of researchers have been interested in the stability of time-delay systems see e. g. [7, 12, 13, 29] and references therein.

The main reason for describing these systems by fractional model is the much progress has also been made in a set of definitions, theoretical methods and numerical analysis of fractional calculus. The nature of many systems makes that they can be more precisely

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modeled using fractional differential equations. In that sense, the stability of these systems has to be proved using techniques developed for fractional order systems. In these cases, the stability of the whole controlled system has to be analyzed using the fractional order techniques as well. The problem of stability analysis of linear time invariant systems arising in system theory is investigated in [26]. Under Laplace transform and generalized Gronwall inequality, [25] derived a sufficient condition for finite-time Mittag-Leffler stability for fractional-order quaternion-valued memristive neural networks with impulsive effect. For $0 < \alpha < 1$, by a suitable Lyapunov-functional and some fractional inequality techniques, the existence of unique equilibrium point and Mittag-Leffler stability in finite time analysis for considered impulsive fractional-order quaternion-valued memristive neural networks have been is established.

However, the situation of time delay exists frequently in most real control systems, has been the subject of numerous papers and monographs, such as [1] and [23] it comes back from the fact that time delays are often encountered in practical systems, such as chemical, economic models and networks [3]. A study concerning the global stability and the global asymptotic stability independent of the delays of linear time-varying of the real Caputo fractional order having internal point delays is investigated in [9]. Robust stability of fractional-order linear delayed systems with nonlinear perturbations over a finite time interval is achieved under the inequality of Holder and Gronwell in [8]. Under Lyapunov method, [2] derived a sufficient condition for stability the nonlinear Caputo fractional differential equations with variable bounded delays. A sufficient conditions expressed in terms of linear matrix inequalities are presented in [11] to prove a separation principle for a class of nonlinear time-delay fractional differential system and nonlinear time-delay fractional differential system with the Caputo derivative. Recently, by using the combination of non-smooth analysis, set-valued maps, Lyapunov–Krasovskii functional having double integral terms and Kirchhoff's matrix tree theorem, the sufficient conditions to ensure the exponential input-to-state stability are presented in [16].

Stability of Riemann–Liouville fractional singular systems remains an open problem. Since if we compare the stability of singular systems with non-singular systems, we find many main difficulties. It is difficult to satisfy the existence and uniqueness of the solutions given that the initial conditions may not be coherent. It is hard to compute the derivatives of Lyapunov functions and there often happen impulses in the solutions. In this paper, we investigate the problem of asymptotic stability by state and output feedback controller of the class of nonlinear Riemann–Liouville fractional order time delay systems with constant delay with a nominal part written in triangular form. We give a condition on the nonlinearity to cover the time-delay systems considered by [32] for a class of nonlinear time-delay systems. By building the suitable Lyapunov–Krasovskii functional to establish global asymptotic stability of the closed-loop systems, we offer linear state and output feedback controllers. Under, linear state and output feedback controllers, we impose a state and input delay-independent criterion that ensures the stability of the closed-loop system with a state feedback controller.

The remainder of the paper organized as follow. In section 2 some preliminary results are recapitulated and the system model is given. In section 3 presents parameter-dependent linear state and output feedback controllers output feedback controller and gives the main results of this paper. Finally, simulation results and a physical example, are given in section 4 in order to highlight the effectiveness of the obtained results.

2. PROBLEM FORMULATION AND SYSTEM DESCRIPTION

2.1. Basic results

In this section, we give some definitions, lemmas and theorems related to the fractional calculation, for more explanations the readers can see [18] and [24]. In throughout the this paper, we consider the definition of Caputo fractional derivative given by [31].

Definition 2.1. (Podlubny [24]) The Riemann–Liouville fractional integral of order $\alpha > 0$ is defined as:

$$_{t_0}I_t^{\alpha}(x(t)) = \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} x(s)\, \mathrm{d}s,$$

where $x(t)$ is an arbitrary integrable function, $_{t_0}I_t^{\alpha}$ denotes the fractional integral of order α on $[t_0, t]$ and $\Gamma(\alpha) = \int_0^{+\infty} e^{-t} t^{\alpha-1}\, \mathrm{d}t$, is the Gamma function.

Definition 2.2. The Riemann–Liouville and Caputo fractional derivatives of order α on $[t_0, t]$ are defined as, respectively [31] and [24]

$$_{t_0}D_t^{\alpha} x(t) = \frac{1}{\Gamma(n-\alpha)} \frac{\mathrm{d}^n}{\mathrm{d}t^n} \int_{t_0}^{t} \frac{x(s)}{(t-s)^{\alpha+1-n}}\, \mathrm{d}s, \quad (n-1 \leq \alpha < n),$$

$$^{C}_{t_0}D_t^{\alpha} x(t) = \frac{1}{\Gamma(n-\alpha)} \int_{t_0}^{t} \frac{x^{(n)}(s)}{(t-s)^{\alpha+1-n}}\, \mathrm{d}s, \quad (n-1 \leq \alpha < n),$$

where $x(t) \in \mathbb{R}^n$ is an arbitrary differentiable function, $n \in \mathbb{N}$.

When $0 < \alpha \leq 1$, then the Riemann–Liouville and Caputo fractional derivatives of order α of $x(t)$ reduces, respectively to:

$$_{t_0}D_t^{\alpha} x(t) = \frac{1}{\Gamma(1-\alpha)} \frac{\mathrm{d}}{\mathrm{d}t} \left(\int_{t_0}^{t} \frac{x(s)}{(t-s)^{\alpha}}\, \mathrm{d}s), \quad (0 < \alpha \leq 1 \right),$$

and

$$^{C}_{t_0}D_t^{\alpha} x(t) = \frac{1}{\Gamma(1-\alpha)} \int_{t_0}^{t} \frac{x^{(1)}(s)}{(t-s)^{\alpha}}\, \mathrm{d}s, \quad (0 < \alpha \leq 1).$$

Property 2.3. (Baleanu et al. [5]) When $0 < \alpha < 1$, we have

$$^{C}_{t_0}D_t^{\alpha} x(t) =_{t_0} D_t^{\alpha} x(t) - \frac{x(t_0)}{\Gamma(1-\alpha)} (t-t_0)^{-\alpha}. \tag{1}$$

In particular, if $x(t_0) = 0$, we have

$$^{C}_{t_0}D_t^{\alpha} x(t) =_{t_0} D_t^{\alpha} x(t). \tag{2}$$

Property 2.4. (Kilbas et al. [18]) If $\beta > \alpha > 0$, then the formula :

$$_{t_0}D_t^{\alpha}(_{t_0}D_t^{-\beta} x(t)) =_{t_0} D_t^{\alpha-\beta} x(t),$$

holds for all functions $x(t) \in L^1(\mathbb{R}_+)$.

Lemma 2.5. (Liu et al. [21]) Let $\alpha \in (0,1)$ and $x(t) \in \mathbb{R}^n$ be a vector of differentiable function. Then, for any time instant $t \geq t_0$, the following relationship holds

$$\frac{1}{2}\ {}_{t_0}D_t^{\alpha} x^T(t) P x(t) \leq x^T(t) P\ {}_{t_0}D_t^{\alpha} x(t), \tag{3}$$

where $P \in \mathbb{R}^{n\times n}$ is a constant square symmetric positive definite matrix.

Lemma 2.6. (Li et al. [20]) Let $\alpha \in (0,1)$ and $M(0) \geq 0$, then

$${}^{C}_{t_0}D_t^{\alpha} M(t) \leq_{t_0} D_t^{\alpha} M(t);$$

where D and ${}^{C}D$ are the Riemann–Liouville and the Caputo fractional operators, respectively.

2.2. Stability of fractional time delay system

In this section, we investigate stability of fractional time delay system.

Consider the fractional time delay system

$$\begin{cases} {}_{t_0}D_t^{\alpha} x(t) &= f(t, x(t), x(t-\tau)); \quad 0 < \alpha < 1, \\ \quad x(\theta) &= \varphi(\theta); \quad \theta \in [-\tau, 0], \end{cases} \tag{4}$$

where, $\tau > 0$ denotes the time delay, $x(t) \in \mathbb{R}^n$, $f : \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is smooth and satisfies $f(t,0,0) = 0$ and $\varphi \in \mathcal{C}$ denote the Banach space of continuous functions mapping the interval $[-\tau, 0] \to \mathbb{R}^n$. The knowledge of x at time $t = 0$ does not give information of x at time t. The state of equation (4) at time t can be described as a function segment x_t defined by

$$x_t(\theta) = x(t+\theta); \quad \theta \in [-\tau, 0]. \tag{5}$$

The fractional order nonlinear time-delay system can be defined in a special class of fractional differential equation form as [4]:

$${}_{t_0}D_t^{\alpha} x(t) = F(t, x_t) \quad (0 < \alpha < 1) \tag{6}$$

where $F : \mathbb{R}_+ \times \mathcal{C} \to \mathbb{R}^n$. For a function $\varphi \in \mathcal{C}([-\tau, 0], \mathbb{R}^n)$ we define the the supremum-norm:

$$\| \varphi \|_\infty = \max_{\theta \in [-\tau, 0]} \| \varphi(\theta) \|; \quad \forall \varphi \in \mathcal{C}$$

where $\| \ \|$ is the Euclidean-norm. As system (4) is a special case of (6), we consider now system (6). For given initial conditions of the form (5), let $y(t) = y(t, t_0, \varphi(.))$ be a solution of the fractional-order time-delay system (6). The stability of the solution concerns the system's behavior when the system trajectory $x(t)$ deviates from $y(t) = y(t, t_0, \varphi(.))$. In the following, without loss of generality, we will assume that the functional differential equation (6) admits the solution $x(t) = 0$, which will be referred to as the trivial solution. Indeed, if it is desirable to study the stability of a nontrivial solution $y(t)$, then we may resort to the variable transformation $z(t) = x(t) - y(t)$, so that the new system

$${}_{t_0}D_t^{\alpha} x(t) = F(t, z_t + y_t) - F(t, y_t) \tag{7}$$

has the trivial solution $z(t) = 0$. We will recall the definition of asymptotic stability of the origin of system (6). Assume that F is Lipschitz on bounded sets and satisfies $F(t, 0) = 0$. For $\varphi \in \mathcal{C}$, we denote by $x(t, \varphi)$ or shortly $x(t)$ the solution of (6) that satisfies $x_0 = \varphi$. The segment of this solution is denoted by $x_t(\varphi)$ or shortly x_t.

Definition 2.7. For the system described by (6) the trivial solution is called :

- Stable, if for any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\|\varphi\|_\infty < \delta \Rightarrow \|x(t)\| < \varepsilon, \ \forall t \geq 0.$$

- Attractive, if there exists $\sigma > 0$ such that

$$\|\varphi\|_\infty < \sigma \Rightarrow \lim_{t\to+\infty} x(t) = 0. \tag{8}$$

- Asymptotically stable, if it is stable and attractive.
- Globally asymptotically stable, if it is stable and δ can be chosen arbitrarily large for sufficiently large ε, and (8) is satisfied for all $\sigma > 0$.

For a locally Lipschitz functional $V : \mathbb{R}_+ \times \mathcal{C} \to \mathbb{R}_+$ a generalization of the fractional derivative of V along the solutions of (6) is presented as follows:

Definition 2.8. (Sadati et al. [27]) Let $V(t, \varphi)$ be differentiable, and let $x_t(t_0, \varphi)$ be the solution of (6) at time t with initial condition $x_{t_0} = \varphi$. Then the fractional derivative of $V(t, x_t)$ with respect to t and evaluate it at $t = t_0$ is defined as :

$$\begin{aligned} {}_{t_0}D_t^\alpha V(t_0, \varphi) &= {}_{t_0}D_{t_0}^\alpha V(t, x_t(t_0, \varphi))\Big|_{t=t_0, x_t=\varphi} \\ &= \frac{1}{\Gamma(1-\alpha)} \frac{\mathrm{d}}{\mathrm{d}t} \left(\int_{t_0}^t \frac{V(s, x_s)}{(t-s)^\alpha} \, \mathrm{d}s \right) \Big|_{t=t_0, x_t=\varphi}, \quad 0 < \alpha < 1. \end{aligned}$$

2.3. System description

In this paper, we consider the problem of global stabilization for a particular family of nonlinear Riemann–Liouville fractional order time delay systems with known control coefficients as follows:

$$\begin{cases} {}_{t_0}D_t^\alpha x_1(t) &= g_1 x_2 + f_1(t, x(t), x(t-\tau), u(t)), \\ {}_{t_0}D_t^\alpha x_2(t) &= g_2 x_3 + f_2(t, x(t), x(t-\tau), u(t)), \\ \quad \vdots & \\ {}_{t_0}D_t^\alpha x_i(t) &= g_i x_{i+1} + f_i(t, x(t), x(t-\tau), u(t)), \\ \quad \vdots & \\ {}_{t_0}D_t^\alpha x_n(t) &= g_n u(t) + f_n(t, x(t), x(t-\tau), u(t)), \\ \quad y(t) &= x_1(t), \end{cases} \tag{9}$$

where, α is the non-integer differentiation order $(0 < \alpha < 1)$, g_i; $i = 1, \ldots, n$, called control coefficients are non zero known constant parameters, $x = [x_1, \ldots, x_n]^T \in \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}$ is the input of the system, $y \in \mathbb{R}$ is the measured output and τ is a positive known scalar that denotes the time delay affecting the state variables. The mappings $f_i : \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$, $i = 1, \dots, n$ are known functions continuous in the first variable are smooth and satisfy the following assumption.

Assumption 1. There exists two positives parameters θ_1 and θ_2 such that,

$$\big|f_i(t, x(t), x(t-\tau), u(t))\big| \leq \theta_1 \sum_{k=1}^{i} |x_k(t)| + \theta_2 \sum_{k=1}^{i} |x_k(t-\tau)|.$$

Assumption 2. Control coefficients $g_i,\ i = 1, \dots, n$ are of known signs.

Remark 2.9. It should be pointed out that the linear growth condition Assumption 1 is weaker than Lipschitz condition. Compared with the system considered in [11], the system (9) is more general. For $g_i = 1;\ i = 1, \dots, n$ a separation principle is derived by [11].

Remark 2.10. In the case of $\alpha = 1$, for same a class of systems (9), the problem of global stabilization by state feedback and output feedback for a family of nonlinear time-delay systems satisfying linear growth conditions were studied by [7] and [32].

2.4. Scaling transformation

In this section, before to show the global asymptotic stabilization by state and output feedback, we follow the scaling transformation transformation suggested in [30]. For the convenience of notation, denote

$$g_{i\sim n} = \prod_{j=i}^{n} g_j, \quad g = \prod_{i=1}^{n} g_i.$$

To begin with, we introduce the following coordinate transformation:

$$z_i = \frac{1}{g_{i\sim n}} x_i, \quad 1 \leq i \leq n,\ g_i \neq 0. \tag{10}$$

So, the system (9) is equivalent to

$$\begin{cases} {}_{t_0}D_t^{\alpha} z_1(t) &= z_2 + \phi_1(t, x(t), x(t-\tau), u(t)), \\ {}_{t_0}D_t^{\alpha} z_2(t) &= z_3 + \phi_2(t, x(t), x(t-\tau), u(t)), \\ \quad\vdots & \\ {}_{t_0}D_t^{\alpha} x_n(t) &= u + \phi_n(t, x(t), x(t-\tau), u(t)), \\ \quad y(t) &= g z_1(t), \end{cases} \tag{11}$$

where $z = [z_1, \dots, z_n]^T \in \mathbb{R}^n$ and $\phi_i(t, z, z(t-\tau), u) = \frac{1}{g_{i\sim n}} f_i(t, x, x(t-\tau), u)$.

Using (10) and Assumption 1, we have

$$|\phi_i(t, z(t), z(t-\tau), u(t))| \leq c_1 \sum_{k=1}^{i} |z_k| + c_2 \sum_{k=1}^{i} |z_k(t-\tau)|, \tag{12}$$

where

$$\begin{cases} c_1 = \max\{\theta_1|g_{1\sim i-1}|, \theta_1|g_{2\sim i-1}|, \dots, \theta_1|g_{i-1}|, \theta_1\}, \\ c_2 = \max\{\theta_2|g_{1\sim i-1}|, \theta_2|g_{2\sim i-1}|, \dots, \theta_2|g_{i-1}|, \theta_2\}. \end{cases}$$

Remark 2.11. It is not difficult to see that the stabilization of system (9) is implied by that of transformed system (11).

3. MAIN RESULTS

In this paper, we investigate the problem of global output feedback control for a particular family of nonlinear fractional order time delay systems (9) satisfying linear growth conditions, with known control coefficients and constant delay. In matrix form, the nonlinear fractional order time delay systems (9) takes the following form:

$$\begin{cases} {}_{t_0}D_t^{\alpha} x(t) = Ax(t) + Bu(t) + f(t, x(t), x(t-\tau), u(t)) \\ \qquad y(t) = Cx(t), \end{cases} \tag{13}$$

where the matrices A, B and C are given by,

$$A = \begin{bmatrix} 0 & g_1 & 0 & \dots & 0 \\ 0 & 0 & g_2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & g_{n-1} \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ g_n \end{bmatrix}, C = \begin{bmatrix} 1 & 0 & \dots & 0 & 0 \end{bmatrix},$$

and the perturbed term is

$$f(t, x(t), x(t-\tau), u(t)) \triangleq \begin{bmatrix} f_1(t, x(t), x(t-\tau), u(t)) \\ f_2(t, x(t), x(t-\tau), u(t)) \\ \vdots \\ f_n(t, x(t), x(t-\tau), u(t)) \end{bmatrix} \in \mathbb{R}^n.$$

Notation 3.1. Throughout the paper, the time argument is omitted and the delayed state vector $x(t-\tau)$ is noted by x^{τ}. M^T means the transpose of matrix M. $\lambda_{max}(M)$ and $\lambda_{\min}(M)$ denote the maximal and minimal eigenvalue of a matrix M respectively. $P > 0$ means that the matrix P is symmetric positive definite. I is an appropriately dimensioned identity matrix.

3.1. Stabilisation by state feedback

In this subsection, motivated by [14], we establish a delay-independent condition for the global asymptotic state feedback stabilization of the nonlinear fractional order time delay systems (13). The state feedback controller is given by

$$u = K(\theta)x, \tag{14}$$

where $K(\theta) = \left[\begin{array}{c} k_1\theta^n, \dots, k_n\theta \end{array}\right]$ and $K = \left[\begin{array}{c} k_1, \dots, k_n \end{array}\right]$ is selected such that $A_K = A + BK$ is Hurwitz. Since the matrix A_K is Hurwitz, there exist symmetric positive definite $P = P^T \in \mathbb{R}^{n\times n}$ such that P is solution of the Lyapunov equation,

$$A_K^T P + PA_K = -I. \tag{15}$$

Theorem 3.2. Under assumptions (A_1), (A_2) and there exist a positive constant $\theta \geq 1$ such that

$$\frac{\theta}{2} - 2n\theta_1\|P\| - n\theta_2\|P\| \quad > \quad 0 \tag{16}$$

then the closed-loop time-delay system (13) – (14) is globally asymptotically stable.

Proof. Under Equation (14), the closed loop system (13) – (14) becomes

$${}_{t_0}D_t^\alpha x(t) = (A + BK(\theta))x + f(t, x(t), x^\tau, u(t)).$$

For $\theta \geq 1$, let $\Delta_\theta = diag[1, \frac{1}{\theta}, \dots, \frac{1}{\theta^{n-1}}]$. Using the fact that:

$$\Delta_\theta BK(\theta) = \theta BK\Delta_\theta, \quad \Delta_\theta A\Delta_\theta^{-1} = \theta A.$$

Now, let $\chi = \Delta_\theta x$, then we get

$${}_{t_0}D_t^\alpha \chi = \theta(A + BK)\chi + \Delta_\theta f(t, x(t), x^\tau, u(t)). \tag{17}$$

We consider the Lyapunov–Krasovskii functional candidate

$$V(\chi_t) = D^{\alpha-1}\big(\chi^T P\chi\big) + \frac{\theta}{2}\int_{t-\tau}^{t} \|\chi(s)\|^2 \,\mathrm{d}(s) \tag{18}$$

It follow from property (2.4) and Lemma (2.5) the derivative of (18) along the trajectories of (17), can be expressed as :

$$\begin{aligned}
\dot{V}(\chi_t) &= {}_{t_0}D_t^\alpha(\chi^T P\chi) + \frac{\theta}{2}\|\chi\|^2 - \frac{\theta}{2}\|\chi^\tau\|^2 \\
&\leq 2\chi^T P_{t_0}D_t^\alpha\chi + \frac{\theta}{2}\|\chi\|^2 - \frac{\theta}{2}\|\chi^\tau\|^2 \\
&\leq \theta\chi^T(A_K^T P + PA_K)\chi + 2P\chi^T\Delta_\theta f(t, x, x^\tau, u(t)) \\
&+ \frac{\theta}{2}\|\chi\|^2 - \frac{\theta}{2}\|\chi^\tau\|^2 \\
&\leq -\theta\|\chi\|^2 + 2P\chi^T\Delta_\theta f(t, x, x^\tau, u(t)) + \frac{\theta}{2}\|\chi\|^2 - \frac{\theta}{2}\|\chi^\tau\|^2 \\
&\leq -\frac{\theta}{2}\|\chi\|^2 + 2\|P\|\|\chi\|\|\Delta_\theta f(t, x, x^\tau, u(t))\| - \frac{\theta}{2}\|\chi^\tau\|^2.
\end{aligned}$$

Observing assumption **A1** and $\theta \geq 1$, gives, for any i $(i = 1, \ldots, n)$

$$\begin{aligned} \left| \frac{1}{\theta^{i-1}} f_i(t, x, x^\tau, u(t)) \right| &\leq \theta_1 \sum_{k=1}^{n} \frac{|x_k|}{\theta^{k-1}} + \theta_2 \sum_{k=1}^{n} \frac{|x_k^\tau|}{\theta^{k-1}} \\ &\leq \theta_1 \sum_{k=1}^{n} |\chi_k| + \theta_1 \sum_{k=1}^{n} |\chi_k| \\ &\leq \theta_1 \sqrt{n} \|\chi\| + \theta_2 \sqrt{n} \|\chi^\tau\|, \end{aligned}$$

which implies that

$$\|\Delta_\theta f(t, x, x^\tau, u(t))\| \leq n\theta_1 \|\chi\| + n\theta_2 \|\chi^\tau\|. \tag{19}$$

Then, we have that

$$\dot{V}(\chi_t) \leq -\frac{\theta}{2} \|\chi\|^2 + 2n\theta_1 \|P\| \|\chi\|^2 - \frac{\theta}{2} \|\chi^\tau\|^2 + 2n\theta_2 \|P\| \|\chi^\tau\| \|\chi\|. \tag{20}$$

Using the fact that

$$2\|\chi^\tau\| \|\chi\| \leq \|\chi\|^2 + \|\chi^\tau\|^2. \tag{21}$$

By Equation (20) and (21), we obtain,

$$\begin{aligned} \dot{V}(\chi_t) &\leq -\left[\frac{\theta}{2} - 2n\theta_1 \|P\| \right] \|\chi\|^2 + n\theta_2 \|P\| \|\chi\|^2 + n\theta_2 \|P\| \|\chi^\tau\|^2 - \frac{\theta}{2} \|\chi^\tau\|^2 \\ &\leq -\left[\frac{\theta}{2} - 2n\theta_1 \|P\| - n\theta_2 \|P\| \right] \|\chi\|^2 - \left[\frac{\theta}{2} - n\theta_2 \|P\| \right] \|\chi^\tau\|^2. \end{aligned}$$

Therefore,

$$\dot{V}(\chi_t) \leq -a(\theta) \|\chi\|^2 - b(\theta) \|\chi^\tau\|^2 \tag{22}$$

where,

$$\begin{aligned} a(\theta) &= \tfrac{\theta}{2} - 2n\theta_1 \|P\| - n\theta_2 \|P\|, \\ b(\theta) &= \tfrac{\theta}{2} - n\theta_2 \|P\|. \end{aligned} \tag{23}$$

$$\dot{V}(\chi_t) \leq -a(\theta) \|\chi\|^2 - b(\theta) \|\chi^\tau\|^2. \tag{24}$$

One can get

$$\dot{V}(\chi_t) \leq -a(\theta) \|\chi\|^2. \tag{25}$$

Therefore, $\dot{V}(\chi_t)$ is negative definite, this implies that the closed-loop time-delay system (13) – (14) is globally asymptotically stable. □

3.2. Stabilisation by output feedback

In practice, some state variables are not measurable. Thus, it is necessary to study the problem of global stabilization via output feedback. In this subsection, we show that under Assumption 1, it is possible to explicitly build an output return controller. We establish a condition for the asymptotic state feedback stabilization of the nonlinear fractional order time delay systems (13). We propose the following system:

$$ {}_{t_0}D_t^{\alpha}\hat{x}(t) = A\hat{x}(t) + Bu(t) + L(\theta)(C\hat{x} - y), \tag{26} $$

where $L(\theta) = [l_1\theta, \dots, l_n\theta^n]^T$, with $\theta > 0$, and $L = [l_1, \dots, l_n]^T$ is selected such that $A_L = A + LC$ is Hurwitz. Let S be the symmetric positive definite solution of the Lyapunov equation,

$$ A_L^T S + SA_L = -I. \tag{27} $$

The output feedback controller is given by

$$ u = K(\theta)\hat{x}. \tag{28} $$

Theorem 3.3. Suppose that assumptions **A1** and **A2** are satisfied and there exist a positive constant $\theta \geq 1$ such that condition (16) holds and

$$ \frac{\theta}{2} - n\theta_1\|S\| - n\theta_2\|S\| > 0 \tag{29} $$

then the closed-loop time-delay system (13)–(28) is globally asymptotically stable.

Proof. Let $e = x - \hat{x}$, is the estimate error. Then, a simple calculation gives

$$ {}_{t_0}D_t^{\alpha}e = (A + L(\theta)C)e + f(t, x(t), x^{\tau}, u(t)). \tag{30} $$

For $\theta > 0$, let the change of variable $\eta = \Delta_\theta e$ and identities $C = C\Delta_\theta^{-1}$. Therefore, the system (30) can be rewritten as follows:

$$ {}_{t_0}D_t^{\alpha}\eta = \theta A_L\eta + \Delta_\theta f(t, x(t), x^{\tau}, u(t)). \tag{31} $$

Similar to (18), we use the Lyapunov–Krasovskii functional candidate as follow :

$$ W(\eta_t) = D^{\alpha-1}\big(\eta^{\tau} S\eta\big) + \frac{\theta}{2}\int_{t-\tau}^{t} \|\eta(s)\|^2\,\mathrm{d}s. \tag{32} $$

It follow from property (2.4) and Lemma (2.5) the derivative of (32) along the trajectories of (31), can be expressed as :

$$ \begin{aligned} \dot{W}(\eta_t) &= {}_{t_0}D_t^{\alpha}(\eta^T S\eta) + \frac{\theta}{2}\|\eta\|^2 - \frac{\theta}{2}\|\eta^{\tau}\|^2 \\ &\leq 2\eta^T S {}_{t_0}D_t^{\alpha}\eta + \frac{\theta}{2}\|\eta\|^2 - \frac{\theta}{2}\|\eta^{\tau}\|^2 \\ &\leq \theta\eta^T(A_L^T S + SA_L)\eta + 2S\eta^T\Delta_\theta f(t, x, x^{\tau}, u(t)) + \frac{\theta}{2}\|\eta\|^2 - \frac{\theta}{2}\|\eta^{\tau}\|^2 \\ &\leq -\theta\|\eta\|^2 + 2S\eta^T\Delta_\theta f(t, x, x^{\tau}, u(t)) + \frac{\theta}{2}\|\eta\|^2 - \frac{\theta}{2}\|\eta^{\tau}\|^2 \\ &\leq -\frac{\theta}{2}\|\eta\|^2 + 2\|S\|\|\eta\|\|\Delta_\theta f(t, x, x^{\tau}, u(t))\| - \frac{\theta}{2}\|\eta^{\tau}\|^2. \end{aligned} $$

Since $f(t,0,0,u(t))=0$, (19), implies that

$$\|\Delta_\theta f(t,x,x^\tau,u(t))\| \leq n\theta_1\|\chi\| + n\theta_2\|\chi^\tau\|.$$

It is clear that

$$2\|S\|\|\eta\|\|\Delta_\theta f(t,x,x^\tau,u(t))\| \leq 2n\theta_1\|S\|\|\eta\|\|\chi\| + 2n\theta_2\|S\|\|\eta\|\|\chi^\tau\|.$$

Using the fact that

$$2\|\chi\|\|\eta\| \leq \|\chi\|^2 + \|\eta\|^2 \text{ and } 2\|\chi^\tau\|\|\eta\| \leq \|\chi^\tau\|^2 + \|\eta\|^2.$$

So by a simple calculation we get:

$$\dot{W}(\eta_t) \leq -\left\{\frac{\theta}{2} - (\theta_1+\theta_2)n\|S\|\right\}\|\eta\|^2 - \frac{\theta}{2}\|\eta^\tau\|^2 + n\theta_1\|S\|\|\chi\|^2 + n\theta_2\|S\|\|\chi^\tau\|^2.$$

Let now,

$$U(\eta_t,\chi_t) = \beta W(\eta_t) + V(\chi_t)$$

where V is given by (18). It follows from (24) and (23) we get,

$$\begin{aligned}\dot{U}(\eta_t,\chi_t) \leq\ & -\beta\left\{\frac{\theta}{2} - n\theta_1\|S\| - n\theta_2\|S\|\right\}\|\eta\|^2 \\ & -\{a(\theta) - \beta n\theta_1\|S\|\}\|\chi\|^2 - \{b(\theta) - \beta n\theta_2\|S\|\}\|\chi^\tau\|^2.\end{aligned}$$

Finally, we choose β such that:

$$\beta < \min\left(\frac{a(\theta)}{n\theta_1\|S\|}, \frac{b(\theta)}{n\theta_2\|S\|}\right).$$

As a result, the closed-loop time-delay system (13) – (28) is globally asymptotically stable. □

Remark 3.4. It is noted that $a(\theta)$ and $b(\theta)$ tend to ∞ as θ tends to ∞. It can be concluded that there exists $\theta_0 > 1$ such that for all $\theta > \theta_0 > 1$ conditions (16) and (29) are fulfilled.

Remark 3.5. In this article, we use control laws that depend on the parameters. For the linear part, it is assumed that there is a parameter dependent linear feedback that asserts the global asymptotic stability. Then, under the same controller, we take the θ-parameter to establish the global asymptotic stability of the non-linear system. This kind of controllers is widely used in the stabilization of nonlinear time-delay system for $\alpha = 1$, (see e. g. [6, 7, 12, 10]).

Remark 3.6. In this paper, the problem for a class of nonlinear fractional order is achieved. The time-delay is a given constant and known. If τ is a time-varying delay, and satisfies there exist $0 < \beta < 1$ such that $\dot{\tau}(t) \leq 1 - \beta$, the Lyapunov–Krasovskii functional (18), (32) and the controller design (14) and (28) respectively do not need to change.

Remark 3.7. From the Lemma 2.6, we suppose the Caputo fractional derivative takes the place of the Riemann–Liouville derivative in fractional time delay nonlinear system (13) and the assumption **A1** and assumption **A2** remain the same, then the conclusions of Theorem 3.2-3.3 still hold. Indeed, from Lemma 2.6, we have:

$$ {}^{C}_{t_0}D^{\alpha}_{t}V(\chi_t) \leq_{t_0} D^{\alpha}_{t}V(\chi_t). $$

So,

$$ {}_{t_0}D^{\alpha}_{t}V(\chi_t) \leq 0 $$

then

$$ {}^{C}_{t_0}D^{\alpha}_{t}V(\chi_t) \leq 0. $$

4. NUMERICAL EXAMPLE: APPLICATION TO BIOREACTOR

In this section, one shall illustrate the effectiveness of the proposed control law, one will through investigate the problem dealing with a typical bioreactor. We consider a microbial culture of the mass balance for reactors. The reactor is supposed to be continuous with concentration distribution across the reactor. The mathematical dynamical model is constituted by the following system, for $\alpha = 0.5$:

$$ \begin{cases} {}_0D^{\alpha}_{t}x_1(t) = g_1x_2 + x_2\cos x_3 + \frac{2}{3}x_1(t-\tau), \\ {}_0D^{\alpha}_{t}x_2(t) = g_2x_3 + \frac{1}{1+t^2}x_1 + \frac{2}{3}x_3(t-\tau), \\ {}_0D^{\alpha}_{t}x_3(t) = u(t) + x_2\sin x_1, \end{cases} \tag{33} $$

where x_1, x_2 and x_3 respectively denote rate of biomass accumulation, the concentration of the substrate and volume of the stream/substrate, g_1 is rate of reaction and g_2 is a yield coefficient. The reactor is assumed to be continuous with a dilution rate $u(t)$. τ denote the measurements of the biomass concentration. The main objective of the example is to estimate the actual biomass accumulation together with the time variations from the available delayed measurements. It is easy to see that Assumption 1 is satisfied with $\theta_1 = 1$, $\theta_2 = \frac{2}{3}$, control coefficients $g_1 = 2$, $g_2 = 3$. Select $K = [-8 - 12 - 8]$ and $L = [-9 - 20 - 13]^T$, such that A_K and A_L are Hurwitz. For solving the Lyapunov equation (15) and (27), we use Matlab the matrix P and S are given respectively by

$$ P = \begin{bmatrix} 0.6181 & -0.2500 & -0.2847 \\ -0.2500 & 0.4271 & -0.1667 \\ -0.2847 & -0.1667 & 0.5972 \end{bmatrix}, \; S = \begin{bmatrix} 0.0854 & 0.1342 & 0.0385 \\ 0.1342 & 1.400 & 0.7280 \\ 0.0385 & 0.7280 & 0.8380 \end{bmatrix}. $$

So, $\|P\| = 0.9051$ and $\|S\| = 1.9090$. A straightforward computation shows that the condition (16) is satisfied for all $\theta > 25.0836$ and condition (29) is satisfied for all $\theta > 52.9038$. We take the value parameter $\theta = 53$. Simulation results in Figure 1, Figure 2 and Figure 3 show that response of the closed loop with unchanged parameters, for the initial condition for the system are $x(0) = [0, -20, -10]^T$ and $\hat{x}(0) = [10, -10, -10]^T$.

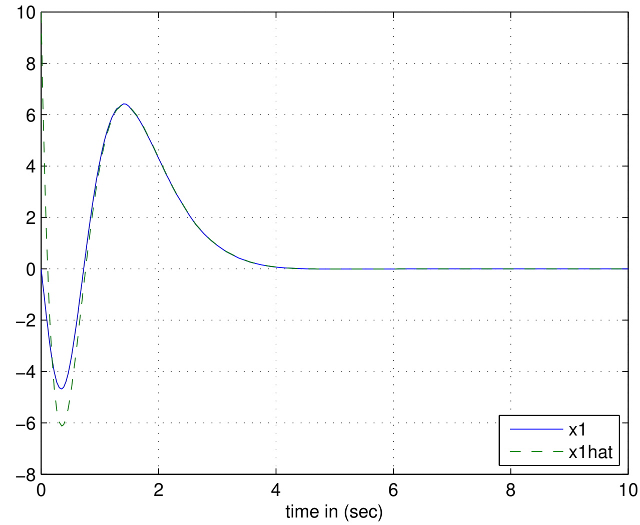

Fig. 1. Trajectories of x_1 and its estimate $\hat{x}_1$.

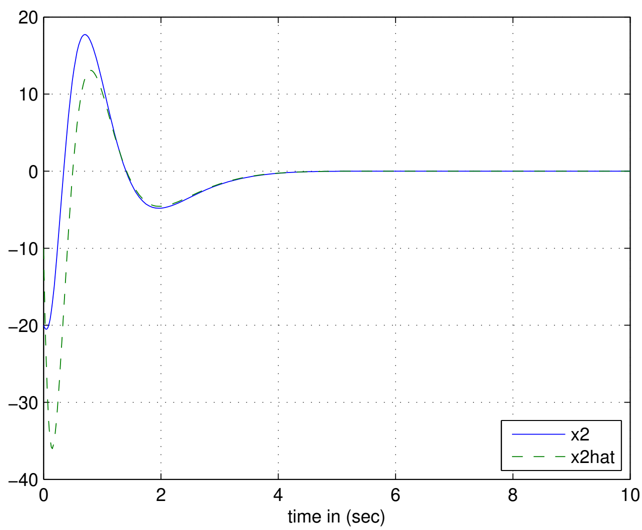

Fig. 2. Trajectories of x_2 and its estimate $\hat{x}_2$.

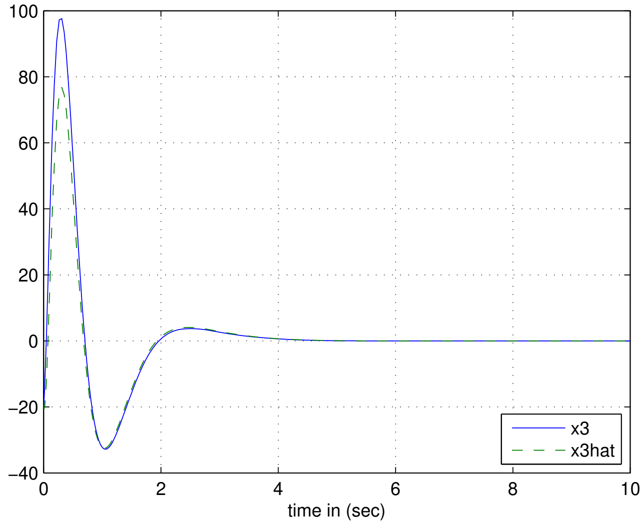

Fig. 3. Trajectories of x_3 and its estimate $\hat{x}_3$.

5. CONCLUSION

The main motivation of this paper, was to present state and output feedback controllers for a particular family of nonlinear Riemann–Liouville fractional order time delay systems with non-integer differentiation order between 0 and 1 where known control coefficients satisfying linear growth condition considered by [32]. Under, Lyapunov–Krasovskii functional donated by (18) and (32), we have proved that the global asymptotic stability by using a linear controller. Extension of the method used in this paper, may be feasible for the more general fractional-order order and nonlinear Caputo fractional order time delay systems, for non-integer differentiation order $\alpha > 0$. The effectiveness of the approach has been provided by numerical result bioreactor example. As a perspective, it is interesting to consider other Lyapunov–Krasovskii functionals we may sharpen Theorem 3.3 and generalize it to get exponential stabilization of system (9).

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