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STABILITY ANALYSIS OF THE FIVE-DIMENSIONAL ENERGY DEMAND-SUPPLY SYSTEM

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In this paper, a five-dimensional energy demand-supply system has been considered. On the one hand, we analyze the stability for all of the equilibrium points of the system. For each of equilibrium point, by analyzing the characteristic equation, we show the conditions for the stability or instability using Routh-Hurwitz criterion. Then numerical simulations have been given to illustrate all of cases for the theoretical results. On the other hand, by introducing the phenomenon of time delay, we establish the five-dimensional energy demand-supply model with time delay. Then we analyze the stability of the equilibrium points for the delayed system by the stability switching theory. Especially, Hopf bifurcation has been considered by showing the explicit formulae using the central manifold theorem and Poincare normalization method. For each cases of the theorems including the Hopf bifurcation, numerical simulations have been given to illustrate the effectiveness of the main results.

Keywords: energy demand-supply, equilibrium points, stability, hopf bifurcation

Classification: 93C15, 34K20

1. INTRODUCTION

With the development of economy, energy has become an essential part in peoples' daily life under which the energy consumption increases greatly. As a result, the pressure in the supply of energy has been remarkably strengthening. It is practically significant to maintain the world energy supply and demand balance. It is known that the presented energy demand-supply system may be originated from the Volterra model which is involved in many ecological or biological applications. The reference [5] shows the stability delay-set for the Lotka–Volterra systems. Correspondingly, the orginal three-dimensional energy demand-supply model has been established on the basis of the relationship between the demand and supply of energy in two regions([16]). The reference [23] considers the delayed three-dimensional energy demand-supply system subsequently. By introducing the renewable energy utilization, a four-dimensional energy demand-supply system has been established in the reference [15]. The linear feedback control has been shown to be effective to suppress chaos to unstable equilibrium or periodic orbits for the four-dimensional energy demand-supply system in the paper [17]. Further on, with energy

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constraints, a five-dimensional energy demand-supply system has been constructed in the reference [22].

Stability switching technique has been widely applied in the stability analysis of ordinary differential equations, especially for the time-delay systems. The reference [14] considers the stability switching hypersurfaces for the linear time invariant systems with time delay. In the reference [7], the eigenvalue crossing directions have been shown for the mechanical systems with time delay. And the reference [2] describes the stability crossing set for the linear time-delay systems. The reference [3] studies the stability crossing curves which consist of all the delays such that the characteristic quasipolynomial has at least one imaginary zero. The paper [1] considers the situations of the spectrum that lies on the imaginary axis for some delays in order to detect the stability of the system. Especially, for the high-dimensional dynamic systems involving a time delay and some unknown parameters, the reference [18] presents a systematic method of stability analysis.

Furthermore, it is well known techniques of applying D-subdivision and computing stability-instability regions in the delay parameter space. For example, the paper [9] analyzes completely the stability of the linear time-invariant systems with two scalar-delay channels by the idea of D-subdivision method. In the paper [6], the discretized Lyapunov-krasovskii functional method is proposed to estimate the maximal stable delay interval accurately. The paper [11] concerns the linear time-invariant systems with uncertain or time-varying delays, and studies the largest range of delay such that there exists the feedback controller capable of stabilizing all the plants for delays within the range. The reference [12] considers the delay margin of a time-delay system such that a controller may exist to stabilize an unstable delay plant for a range of delay values. The paper [10] presents the stability regions in the domain of time delay, and declares the number of unstable characteristic roots by a simplifying substitution for the transcendental characteristic equation. The paper [4] studies the largest delay just for the occurrance of instability and the effect of certain graphs on the delay bound.

In this paper we will generalize the stability properties of the five-dimensional energy demand-supply system, and then obtain the theoretical and numerical results for the system under arbitrary coefficients. Furthermore, we will further analyze the stability for the system with time delay by using the stability switching technique. The energy demand-supply system considered in the paper is seen as following

$$\begin{cases} \dot{x}(t) = a_1 x(t)\left(1 - \dfrac{x(t)}{M}\right) - a_2\left[y(t) + z(t)\right] - d_3 u(t) - k_4 w(t), \\ \dot{y}(t) = -b_1 y(t) - b_2 z(t) + b_3 x(t)\left[N - (x(t) - z(t))\right], \\ \dot{z}(t) = c_1 z(t)\left(c_2 x(t) - c_3\right), \\ \dot{u}(t) = d_1 x(t) - d_2 u(t), \\ \dot{w}(t) = k_1 y(t)\left(\dfrac{y(t)}{T_1} - 1\right) + k_2 w(t)\left(k_3 x(t) - T_2\right), \end{cases} \tag{1-1}$$

where $x(t)$ is the amount of the energy demand shortage of Region A, $y(t)$ is the amount of the energy increment supplied from Region B to Region A, $z(t)$ is the amount of the energy import of Region A, $u(t)$ represents the amount of the self energy supply, and $w(t)$ signifies the amount of the utilization of new energy in region A. All of the constant coefficients are given in the reference [22] where the stability properties of the equilibrium points have been shown under certain given coefficients.

The main contribution of this paper is focused on the stability analysis of the five-dimensional energy demand-supply system. First of all, all of the equilibrium points of the system have been computed. Then the stability of the equilibrium points has been analyzed by using Hurwitz criterion. Next the phenomenon of the time delay has been considered in which the system with the time delay has been established. By the stability switching technique, the stability analysis of the delayed system have been given. For the case of Hopf bifurcation, the explicit formula have been obtained by the center manifold theorem and normal form method. For all of the situations of the stability and instability, numerical simulations have been given which illustrate the effectiveness of the main results. The innovation of this paper lies in the complexity for the stability analysis and the corresponding simulations of the higher dimensional ordinary differential equations. Based on the results of this paper, our next work may be generalizing the theoretic and simulating results to the abstract and general ordinary differential equations with a wide range of applications.

The paper is composed of the following five sections. In the second section we analyze stability properties of the equilibrium points for the five-dimensional energy demand-supply system and give the corresponding numerical simulations respectively. In the third section we construct the system with time delay and analyze stability properties of the equilibrium points for the delayed system. Especially the fourth section is devoted to the properties of Hopf bifurcation expressed by explicit formulae, and numerical simulations illustrating effectiveness of the main results. In the fifth section we take the conclusion of the paper.

2. STABILIZATION OF FIVE-DIMENSIONAL ENERGY SYSTEM

In this section we consider stability properties of the equilibrium points for the system (1-1) with arbitrary coefficients established in the reference [22]. And then numerical simulations are given to show effectiveness of the main results.

2.1. Analysis of equilibrium points

By simple calculations, we compute the equilibrium points as that below

$$S_0 = (0,0,0,0,0), S_1^* = (x_1^*, y_1^*, z_1^*, u_1^*, w_1^*), S_2^* = (x_2^*, y_2^*, z_2^*, u_2^*, w_2^*), \tag{1}$$

where x_1^* satisfies the following polynomial equation

$$\begin{aligned}
&\frac{k_1 b_3^2}{b_1^2 T_1} x_1^{*3} + \left[\frac{k_2 k_3}{k_4}\left(\frac{a_2 b_3}{b_1} - \frac{a_1}{M}\right) - 2\frac{k_1 b_3^2 N}{b_1^2 T_1}\right] x_1^{*2} + \left\{\frac{k_1 b_3}{b_1}\left(1 + \frac{b_3 N^2}{b_1 T_1}\right)\right.\\
&\left.+\frac{k_3}{k_4}\left[k_3\left(a_1 - \frac{a_2 b_3 N}{b_1} - \frac{d_1 d_3}{d_2}\right) - T_2\left(\frac{a_2 b_3}{b_1} - \frac{a_1}{M}\right)\right]\right\} x_1^* \\
&-\left[\frac{k_1 b_3 N}{b_1} + \frac{k_2 T_2}{k_4}\left(a_1 - \frac{a_2 b_3 N}{b_1} - \frac{d_1 d_3}{d_2}\right)\right] = 0,
\end{aligned} \tag{2}$$

and the other variables $y_1^*, z_1^*, u_1^*, w_1^*$ are determined in the following equations respectively

$$\begin{aligned} y_1^* &= \frac{b_3}{b_1} x_1^* \left(N - x_1^*\right), \quad z_1^* = 0, \quad u_1^* = \frac{d_1}{d_2} x_1^*, \\ w_1^* &= \frac{1}{k_4} \left[a_1 x_1^* \left(1 - \frac{x_1^*}{M}\right) - \frac{a_2 b_3}{b_1} x_1^* \left(N - x_1^*\right) - \frac{d_1 d_3}{d_2} x_1^* \right], \end{aligned} \tag{3}$$

furthermore,

$$x_2^* = \frac{c_3}{c_2}, \quad u_2^* = \frac{c_3 d_1}{c_2 d_2}, \tag{4}$$

y_2^* satisfies the following polynomial equation

$$\begin{aligned} &\frac{k_1 k_4}{T_1 k_2 \left(T_2 - \dfrac{k_3 c_3}{c_2}\right)} y_2^{*2} + \left[\frac{a_2 b_1}{\dfrac{b_3 c_3}{c_2} - b_2} - \frac{k_1 k_4}{k_2 \left(T_2 - \dfrac{k_3 c_3}{c_2}\right)} + a_2 \right] y_2^* \\ &- \left[\frac{a_1 c_3}{c_2} \left(1 - \frac{c_3}{c_2 M}\right) - \frac{c_3 d_1 d_3}{c_2 d_2} \right] = 0, \end{aligned} \tag{5}$$

and

$$z_2^* = \frac{b_1 y_1^* + \dfrac{b_3 c_3^2}{c_2^2}}{\dfrac{b_3 c_3}{c_2} - b_2}, \quad w_2^* = \frac{k_1 y_2^* \left(\dfrac{y_2^*}{T_1} - 1\right)}{k_2 \left(T_2 - \dfrac{k_3 c_3}{c_2}\right)}, \tag{6}$$

in the assumption of $\dfrac{b_3 c_3}{c_2} - b_2 \neq 0$ and $T_2 - \dfrac{k_3 c_3}{c_2} \neq 0$.

2.2. Stability of the equilibrium point S_0

The jacobian matrix of the system (1-1)can be computed as that

$$\begin{pmatrix} a_1 \left(1 - \dfrac{2x}{M}\right) & -a_2 & -a_2 & -d_3 & -k_4 \\ b_3 \left[N - (2x - z)\right] & -b_1 & b_3 x - b_2 & 0 & 0 \\ c_1 c_2 z & 0 & c_1 \left(c_2 x - c_3\right) & 0 & 0 \\ d_1 & 0 & 0 & -d_2 & 0 \\ k_2 k_3 w & k_1 \left(\dfrac{2y}{T_1} - 1\right) & 0 & 0 & k_2 \left(k_3 x - T_2\right) \end{pmatrix}. \tag{7}$$

From (7), the jacobian matrix of the equilibrium point S_0 can be naturally given as that

$$\begin{pmatrix} a_1 & -a_2 & -a_2 & -d_3 & -k_4 \\ b_3 N & -b_1 & -b_2 & 0 & 0 \\ 0 & 0 & -c_1 c_3 & 0 & 0 \\ d_1 & 0 & 0 & -d_2 & 0 \\ 0 & -k_1 & 0 & 0 & -k_2 T_2 \end{pmatrix}. \tag{8}$$

Its characteristic equation can be computed as that

$$\begin{aligned}&(\lambda+c_1c_3)\left[-k_1k_4b_3N(\lambda+d_2)+(\lambda+k_2T_2)(d_1d_3(\lambda+b_1)\right.\\&\left.+(\lambda+d_2)((\lambda-a_1)(\lambda+b_1)+a_2b_3N))\right]=0.\end{aligned}\tag{9}$$

Obviously, $\lambda=-c_1c_3$ is a negative real root of the equation (9). Thus we only need to consider the roots of following equation

$$\begin{aligned}&-k_1k_4b_3N(\lambda+d_2)+(\lambda+k_2T_2)\left[d_1d_3(\lambda+b_1)+\right.\\&\left.(\lambda+d_2)((\lambda-a_1)(\lambda+b_1)+a_2b_3N)\right]=0.\end{aligned}\tag{10}$$

The equivalent form of the equation above is

$$\lambda^4+A_1\lambda^3+A_2\lambda^2+A_3\lambda+A_4=0,\tag{11}$$

where

$$\begin{cases}A_1=b_1-a_1+d_2+k_2T_2,\\A_2=d_1d_3-a_1b_1+b_3Na_2+b_1d_2-a_1d_2+(b_1-a_1+d_2)k_2T_2,\\A_3=-k_1k_4b_3N+d_1d_3b_1-a_1b_1d_2+b_3Na_2d_2+\\\qquad(d_1d_3-a_1b_1+b_3a_2N+b_1d_2-a_1d_2)k_2T_2,\\A_4=-k_1k_4b_3Nd_2+(b_1d_1d_3-a_1b_1d_2+b_3a_2Nd_2)k_2T_2.\end{cases}\tag{12}$$

Now define the equation

$$f(x)=x^4+A_1x^3+A_2x^2+A_3x+A_4,\tag{13}$$

then from the reference [8], we take the assumption $H(1)$ as follows

$$H(1):\qquad A_4<0,\quad\text{or}\quad\begin{cases}A_4\geq 0,m_1\geq 0,\\\lambda_1>0,f(\lambda_1)<0,\end{cases}$$

$$\text{or}\quad\begin{cases}A_4\geq 0,m_1<0,\\\exists\lambda_i\in\{\lambda_1,\lambda_2,\lambda_3\},\ \text{such that}\ \lambda_i>0,f(\lambda_i)\leq 0,\ i=1,2,3,\end{cases}$$

where

$$\begin{cases}p=\dfrac{1}{2}A_2-\dfrac{3}{16}A_1^2,\ q=\dfrac{3}{16}A_1^3-\dfrac{1}{8}A_1A_2+A_3,\ m_1=\left(\dfrac{q}{2}\right)^3+\left(\dfrac{p}{3}\right)^3,\ w_1=-\dfrac{1}{2}+\dfrac{\sqrt{3}}{2}i,\\V_1=\sqrt[3]{-\dfrac{q}{2}+\sqrt{m_1}}+\sqrt[3]{-\dfrac{q}{2}-\sqrt{m_1}},\ V_2=w_1\sqrt[3]{-\dfrac{q}{2}+\sqrt{m_1}}+w_1^2\sqrt[3]{-\dfrac{q}{2}-\sqrt{m_1}},\\V_3=w_1^2\sqrt[3]{-\dfrac{q}{2}+\sqrt{m_1}}+w_1\sqrt[3]{-\dfrac{q}{2}-\sqrt{m_1}},\ \lambda_i=V_i-\dfrac{3A_1}{4}(i=1,2,3)\end{cases}$$

and $A_j(j=1,2,3,4)$ are the coefficients of the equation (11).

Theorem 2.1. (i) If the coefficients of the system (1-1) satisfy the following inequalities

$$A_1>0,\ A_4>0,\ A_1A_2-A_3>0,\ A_1A_2A_3-A_3^2-A_1^2A_4>0,$$

the equilibrium point S_0 is stable.

(ii) If the characteristic equation (11) satisfies the assumption $H(1)$, the equilibrium point S_0 is unstable.

Proof. (i) According to the Routh–Hurwitz criterion [20], the sufficient and necessary conditions for the equation (11) that the real parts of all of characteristic roots are negative are as following

$$A_1 > 0,\ A_4 > 0,\ \begin{vmatrix} A_1 & 1 \\ A_3 & A_2 \end{vmatrix} = A_1A_2 - A_3 > 0,\ \begin{vmatrix} A_1 & 1 & 0 \\ A_3 & A_2 & A_1 \\ 0 & A_4 & A_3 \end{vmatrix} = A_1A_2A_3 - A_3^2 - A_1^2A_4 > 0,$$

where $A_j (j = 1, 2, 3, 4)$ are the coefficients of the equation (11).

(ii) According to the lemma in the 12th page of the reference [8], if the coefficients of the equation (11) satisfy the assumption $H(1)$, there exist positive roots for the characteristic equation. □

2.3. Stability of the equilibrium points S_1^* and S_2^*

From (7), the jacobian matrix of the equilibrium points S_1^* and S_2^* can be shown as that

$$\begin{pmatrix} a_1\left(1 - \dfrac{2x^*}{M}\right) & -a_2 & -a_2 & -d_3 & -k_4 \\ b_3[N - (2x^* - z^*)] & -b_1 & -b_2 + b_3x^* & 0 & 0 \\ c_1c_2z^* & 0 & c_1\left(c_2x^* - c_3\right) & 0 & 0 \\ d_1 & 0 & 0 & -d_2 & 0 \\ k_2k_3w^* & k_1\left(\dfrac{2y^*}{T_1} - 1\right) & 0 & 0 & k_2\left(k_3x^* - T_2\right) \end{pmatrix}, \qquad (14)$$

where $(x^*, y^*, z^*, u^*, w^*)$ signifies the equilibrium points S_1^* or S_2^*.

Then we compute the corresponding characteristic equation for the equilibrium points S_1^* and S_2^* as follows

$$\lambda^5 + C_1\lambda^4 + C_2\lambda^3 + C_3\lambda^2 + C_4\lambda + C_5 = 0, \qquad (15)$$

where

$$\begin{cases} C_1 = b_1 + d_2 - P_0 - T_0 - V_0, \\ C_2 = a_2S_0 + a_2Q_0 + T_0P_0 - b_1P_0 - b_1T_0 + d_2\left(b_1 - P_0 - T_0\right) \\ \qquad +d_1d_3 - V_0\left(b_1 + d_2 - P_0 - T_0\right) + k_4E_0, \\ C_3 = a_2\left(S_0R_0 + b_1S_0 - Q_0T_0 + d_2S_0 + d_2Q_0\right) + d_2\left(T_0P_0 - b_1P_0 - b_1T_0\right) \\ \qquad +d_1d_3\left(b_1 - T_0\right) - V_0(a_2S_0 + a_2Q_0 + T_0P_0 - b_1P_0 - b_1T_0 + d_2b_1 - d_2P_0 \\ \qquad -d_2T_0 + d_1d_3) + k_4\left(E_0\left(b_1 + d_2 - T_0\right) + F_0Q_0\right) + b_1T_0P_0, \\ C_4 = d_2\left(a_2S_0R_0 + a_2b_1S_0 + b_1T_0P_0 - a_2Q_0T_0\right) - d_1d_3b_1T_0 - V_0(a_2S_0R_0 + a_2b_1S_0 \\ \qquad +b_1T_0P_0 - a_2Q_0T_0 + d_2a_2S_0 + d_2a_2Q_0 + d_2T_0P_0 - d_2b_1P_0 - d_2b_1T_0 + d_1d_3b_1 \\ \qquad -d_1d_3T_0) + k_4\left(E_0(b_1d_2 - T_0\left(b_1 + d_2\right)) + F_0Q_0\left(d_2 - T_0\right) + F_0S_0R_0\right), \\ C_5 = -V_0\left(d_2\left(S_0R_0a_2 + a_2b_1S_0 + b_1T_0P_0 - a_2Q_0T_0\right) - d_1d_3b_1T_0\right) - b_1d_2k_4E_0T_0 \\ \qquad -d_2k_4F_0Q_0T_0 + d_2k_4F_0S_0R_0, \end{cases} \qquad (16)$$

and

$$\begin{cases} P_0 = a_1\left(1 - \dfrac{2x^*}{M}\right),\ Q_0 = b_3\left[N - (2x^* - z^*)\right],\ R_0 = b_3x^* - b_2,\ S_0 = c_1c_2z^*, \\ T_0 = c_1\left(c_2x^* - c_3\right),\ E_0 = k_2k_3w^*,\ F_0 = k_1\left(\dfrac{2y^*}{T_1} - 1\right),\ V_0 = k_2\left(k_3x^* - T_2\right). \end{cases}$$

According to the Routh-Hurwitz criterion ([20]), we have the following theorem.

Theorem 2.2. If the coefficients of the equation (15) satisfy the following inequalities

$$\begin{cases} C_1 > 0,\ C_5 > 0,\ C_1C_2 - C_3 > 0,\ -C_1^2C_4 + C_1C_5 + C_1C_2C_5 - C_3^2 > 0, \\ C_4\left(-C_1^2C_4 + C_1C_5 + C_1C_2C_5 - C_3^2\right) - C_5\left(C_1C_2^2 - C_2C_3 - C_1C_4 + C_5\right) > 0, \end{cases} \tag{17}$$

where $C_j (j = 1, 2, 3, 4, 5)$ are defined in the equation (16), all of the equilibrium points S_1^* and S_2^* are asymptotically stable.

2.4. Simulations

This section is focused on numerical simulations in order to show effectiveness of the Theorem 2.1 and Theorem 2.2. All of the situations for both of the theorems will be simulated in this section.

First of all, we choose the parameters of the system (1-1) below

$$\begin{cases} a_1 = 0.01,\ a_2 = 0.15,\ b_1 = 0.06,\ b_2 = 0.1,\ b_3 = 0.16,\ c_1 = 0.2,\ c_2 = 0.5,\ c_3 = 0.4, \\ d_1 = 0.1, d_2 = 0.0616, d_3 = 0.08, M = 1.85, N = 0.5, k_1 = 0.68, k_2 = 0.5, k_3 = 0.49, \\ k_4 = 0.09,\ T_1 = 1.95,\ T_2 = 2.8, \end{cases} \tag{18}$$

and the initial values as that

$$x(0) = 0.17,\ y(0) = 0.11,\ z(0) = 0.07,\ u(0) = 0.37,\ w(0) = 0.002.$$

Then we obtain Figure 1 which shows that the equilibrium point S_0 is stable to illustrate the first case of the Theorem 2.1.

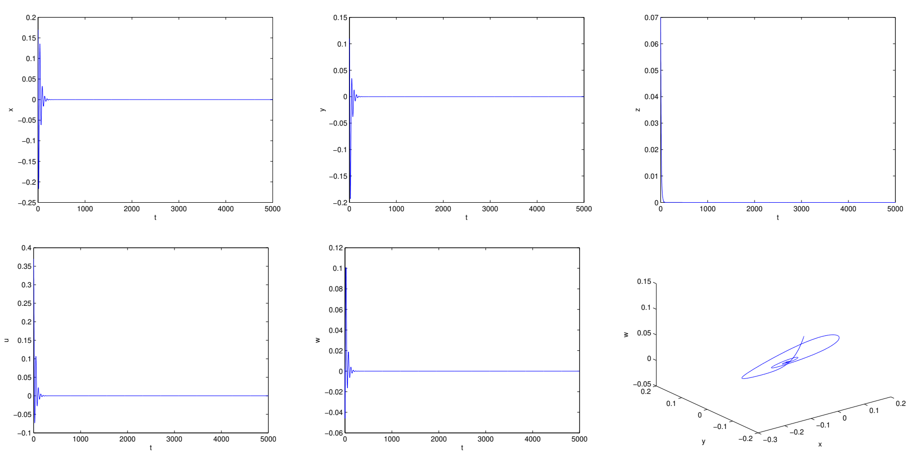

Fig 1. The equilibrium point S_0 is stable.

In order to show the effectiveness for the second case of the Theorem 2.1, we choose the parameters as follows

$$\begin{cases} a_1 = 0.05,\ a_2 = 0.5,\ b_1 = 0.006,\ b_2 = 0.1,\ b_3 = 0.16,\ c_1 = 0.2,\ c_2 = 0.5,\ c_3 = 0.4, \\ d_1 = 0.1,\ d_2 = 0.06,\ d_3 = 0.08,\ M = 1.85,\ N = 1,\ k_1 = 0.68,\ k_2 = 1.59,\ k_3 = 0.5, \\ k_4 = 0.09,\ T_1 = 1.05,\ T_2 = 2.5, \end{cases}$$

and initial values below

$$x(0) = 0.17,\ y(0) = 0.11,\ z(0) = 0.07,\ u(0) = 0.37,\ w(0) = 0.002,$$

then we obtain Figure 2 which means that the equilibrium point S_0 is unstable to illustrate the second case of the Theorem 2.1.

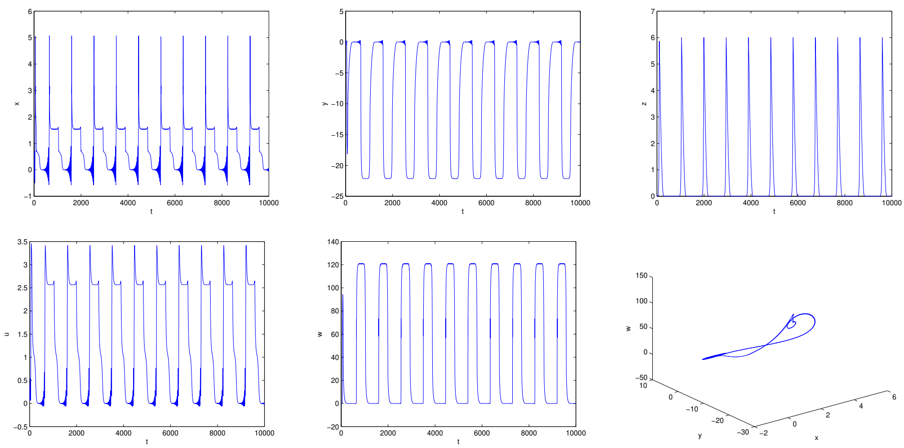

Fig 2. The equilibrium point S_0 is unstable.

In order the show the effectiveness of the Theorem 2.2, we choose parameters as following

$$\begin{cases} a_1 = 0.09, a_2 = 0.077, b_1 = 0.08, b_2 = 5, b_3 = 0.16, c_1 = 0.2, c_2 = 0.5, c_3 = 0.4, d_1 = 0.09, \\ d_2 = 0.06, d_3 = 0.08, M = 1.85, N = 0.6, k_1 = 0.68, k_2 = 1.4, k_3 = 0.49, k_4 = 0.099, \\ T_1 = 1.95, T_2 = 2.7, \end{cases}$$

and initial values as that

$$x(0) = 0.016, y(0) = 0.11, z(0) = 0.07, u(0) = 0.37, w(0) = 0.002,$$

under which the equilibrium point can be computed as that

$$(0.8000, -3.7852, 0.0569, 1.2000, 2.3429).$$

Then we have Figure 3 which indicates that the state of the system is stabilized to the equilibrium point to show the effectiveness of Theorem 2.2.

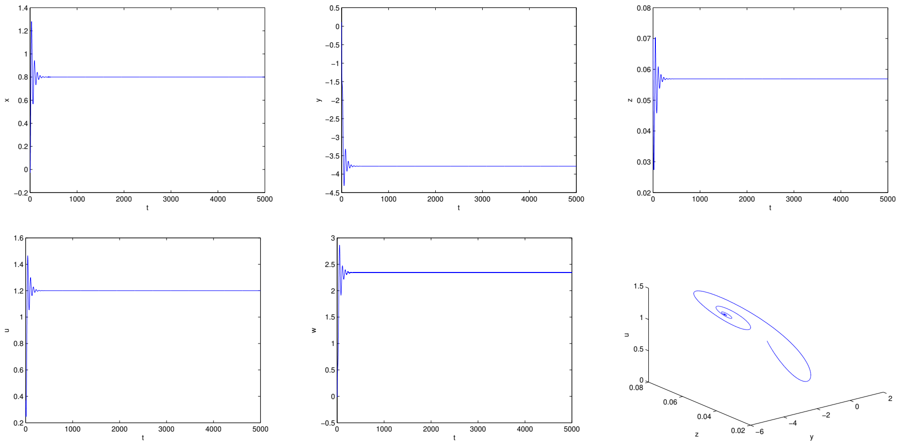

Fig 3. The equilibrium point is stable.

3. STABILIZATION OF FIVE-DIMENSIONAL ENERGY SYSTEM WITH TIME DELAY

In this section, we consider the phenomenon of time delay. Firstly we establish the five-dimensional demand-supply system with time delay, and then we analyze the stability properties of the equilibrium points for the delayed five-dimensional energy demand-supply system.

3.1. The model of five-dimensional energy system with time delay

The phenomenon of time delay is wide and important in reality. It is natural to consider the five-dimensional energy demand-supply system with time delay. The delayed system can be constructed as follows dependent on the actual situations.

$$\begin{cases} \dot{x}(t) = a_1 x(t)\left(1 - \dfrac{x(t-\tau)}{M}\right) - a_2\left[y(t-\tau) + z(t-\tau)\right] - d_3 u(t-\tau) - k_4 w(t-\tau), \\ \dot{y}(t) = -b_1 y(t) - b_2 z(t-\tau) + b_3 x(t)\left[N - (x(t-\tau) - z(t-\tau))\right], \\ \dot{z}(t) = c_1 z(t)\left(c_2 x(t-\tau) - c_3\right), \\ \dot{u}(t) = d_1 x(t) - d_2 u(t), \\ \dot{w}(t) = k_1 y(t)\left(\dfrac{y(t-\tau)}{T_1} - 1\right) + k_2 w(t)\left(k_3 x(t-\tau) - T_2\right), \end{cases} \tag{3-1}$$

where $\tau > 0$ is the time delay, and all of the variables and coefficients can be seen in the system (1-1).

The process of modeling is given as following. In the first equation, the term $(1 - \frac{x(t-\tau)}{M})$ means that the speed of energy demand of A region is positively correlated with the share of energy consumption demand development potential in the previous stage. And the term $-a_2[y(t-\tau)+z(t-\tau)]$ indicates that the energy increment supplying from B region to A region and the energy import in region A are negatively correlated with the speed of energy demand of A region in the previous stage. Furthermore, the term $u(t-\tau)$ and $w(t-\tau)$ indicate the self-sustaining energy and the utilization of new energy in region A are negatively correlated with the speed of energy demand of A region in the previous stage. In the second equation, the energy import in region A is negatively correlated with the energy supply speed from region B to region A in the previous period, which results in the term $-b_2z(t-\tau)$. And the term $b_3x(t)[N-(x(t-\tau)-z(t-\tau))]$ indicates that the energy demand shortage in the previous stage of A region is less than the critical value. The speed of energy supply from B region to A region increases as $x(t)$ increases. But when the energy of A region is large enough, The speed of energy supply from B region to A region decreases as $x(t)$ increases. In the third equation, the term $c_2x(t-\tau)$ indicates the energy demand shortage of A region is proportion to the speed of the energy import of A region in a previous period. In the fifth equation, the term $(\frac{y(t-\tau)}{T_1}-1)$ means that the speed of the utilization of new energy in region A is proportional to the energy supply rate in region A. And the term $(k_3x(t-\tau)-T_2)$ indicates the speed of the utilization of new energy in region A is related to the energy demand shortage of A region and the development cost of new energy in region A in the previous stage.

Next we will analyze the stability of the delayed five-dimensional energy demand-supply system (3-1).

3.2. Stability analysis

Obviously, the equilibrium points of the system (3-1) are S_0, S_1^* and S_2^* which can be seen in the equation (1). Then we consider the stability properties of the equilibrium points respectively.

3.2.1. Stability of the equilibrium point S_0

First of all, the linearization matrix of the system (3-1) can be calculated as that

$$\begin{pmatrix} a_1\left(1-\dfrac{2xe^{-\lambda\tau}}{M}\right) & -a_2e^{-\lambda\tau} & -a_2e^{-\lambda\tau} & -d_3e^{-\lambda\tau} & -k_4e^{-\lambda\tau} \\ b_3\left[N-(2x-z)\,e^{-\lambda\tau}\right] & -b_1 & (b_3x-b_2)\,e^{-\lambda\tau} & 0 & 0 \\ c_1c_2ze^{-\lambda\tau} & 0 & c_1\left(c_2xe^{-\lambda\tau}-c_3\right) & 0 & 0 \\ d_1 & 0 & 0 & -d_2 & 0 \\ k_2k_3we^{-\lambda\tau} & k_1\left(\dfrac{2ye^{-\lambda\tau}}{T_1}-1\right) & 0 & 0 & k_2\left(k_3xe^{-\lambda\tau}-T_2\right) \end{pmatrix}, \tag{19}$$

then the linearization matrix for the equilibrium point S_0 is

$$\begin{pmatrix} a_1 & -a_2e^{-\lambda\tau} & -a_2e^{-\lambda\tau} & -d_3e^{-\lambda\tau} & -k_4e^{-\lambda\tau} \\ b_3N & -b_1 & -b_2e^{-\lambda\tau} & 0 & 0 \\ 0 & 0 & -c_1c_3 & 0 & 0 \\ d_1 & 0 & 0 & -d_2 & 0 \\ 0 & -k_1 & 0 & 0 & -k_2T_2 \end{pmatrix}. \tag{20}$$

Its characteristic equation is

$$\begin{vmatrix} \lambda - a_1 & a_2 e^{-\lambda\tau} & a_2 e^{-\lambda\tau} & d_3 e^{-\lambda\tau} & k_4 e^{-\lambda\tau} \\ -b_3 N & \lambda + b_1 & b_2 e^{-\lambda\tau} & 0 & 0 \\ 0 & 0 & \lambda + c_1 c_3 & 0 & 0 \\ -d_1 & 0 & 0 & \lambda + d_2 & 0 \\ 0 & k_1 & 0 & 0 & \lambda + k_2 T_2 \end{vmatrix} = 0, \tag{21}$$

that is to say,

$$\begin{gathered}(-c_1 c_3 - \lambda)\left[-k_1 k_4 b_3 N (d_2 + \lambda) e^{-\lambda\tau} + (k_2 T_2 + \lambda)(d_1 d_3 (b_1 + \lambda) e^{-\lambda\tau} \right. \\ \left. + (d_2 + \lambda)\left((\lambda - a_1)(\lambda + b_1) + b_3 a_2 N e^{-\lambda\tau}\right))\right] = 0. \end{gathered} \tag{22}$$

Obviously $\lambda = -c_1 c_3$ is a negative real root of the equation (22). Now we only need to consider the roots of the equation below

$$\begin{gathered}[-k_1 k_4 b_3 N (d_2 + \lambda) e^{-\lambda\tau} + (k_2 T_2 + \lambda)(d_1 d_3 (b_1 + \lambda) e^{-\lambda\tau} + \\ (d_2 + \lambda)((\lambda - a_1)(\lambda + b_1) + b_3 a_2 N e^{-\lambda\tau}))] = 0. \end{gathered} \tag{23}$$

It can be calcuated as that

$$\lambda^4 + A\lambda^3 + B\lambda^2 + C\lambda + D + e^{-\lambda\tau}\left(E\lambda^2 + F\lambda + G\right) = 0, \tag{24}$$

where

$$\begin{cases} A = b_1 - a_1 + d_2 + k_2 T_2, \\ B = b_1 d_2 - a_1 b_1 - a_1 d_2 + (b_1 - a_1 + d_2) k_2 T_2, \\ C = (b_1 d_2 - a_1 b_1 - a_1 d_2) k_2 T_2 - a_1 b_1 d_2, \\ D = -a_1 b_1 d_2 k_2 T_2, \\ E = a_2 b_3 N + d_1 d_3, \\ F = b_1 d_1 d_3 + a_2 d_2 b_3 N - k_1 k_4 b_3 N + (d_1 d_3 + a_2 b_3 N) k_2 T_2, \\ G = (b_1 d_1 d_3 + a_2 d_2 b_3 N) k_2 T_2 - d_2 k_1 k_4 b_3 N. \end{cases} \tag{25}$$

Assume that $\lambda = \pm iw (w > 0)$ are a pair of pure imaginary roots of the equation (24), then substituting it into the equation (24) gives that

$$w^4 \mp Aw^3 i - Bw^2 \pm Cwi + D + (\cos(w\tau) \mp i\sin(w\tau))\left(-Ew^2 \pm Fwi + G\right) = 0. \tag{26}$$

Depart the real and imaginary parts of the equation (26), then we have the following two equations

$$\begin{cases} \left(G - Ew^2\right)\cos(w\tau) + Fw\sin(w\tau) = -w^4 + Bw^2 - D, \\ Fw\cos(w\tau) - \left(G - Ew^2\right)\sin(w\tau) = Aw^3 - Cw, \end{cases} \tag{27}$$

which indicate that

$$\begin{gathered} w^8 + \left(A^2 - 2B\right) w^6 + \left(B^2 + 2D - 2AC - E^2\right) w^4 + \left(2EG - F^2 - 2BD + C^2\right) w^2 \\ + D^2 - G^2 = 0. \end{gathered} \tag{28}$$

From the equation (27), we have that

$$\begin{aligned}&\left[\left(Aw^3-Cw\right)\left(G-Ew^2\right)-\left(-w^4+Bw^2-D\right)Fw\right]\cos(w\tau)+\\&\left[\left(Aw^3-Cw\right)Fw+\left(-w^4+Bw^2-D\right)\left(G-Ew^2\right)\right]\sin(w\tau)=0,\end{aligned}\tag{29}$$

which is equivalent to

$$\begin{aligned}&\frac{\left[\left(Aw^3-Cw\right)\left(G-Ew^2\right)-\left(-w^4+Bw^2-D\right)Fw\right]\cos(w\tau)}{\left(G-Ew^2\right)^2+\left(Fw\right)^2}+\\&\frac{\left[\left(Aw^3-Cw\right)Fw+\left(-w^4+Bw^2-D\right)\left(G-Ew^2\right)\right]\sin(w\tau)}{\left(G-Ew^2\right)^2+\left(Fw\right)^2}=0.\end{aligned}\tag{30}$$

Suppose that$(G-Ew^2)^2+(Fw)^2\neq 0$, then the corresponding values of critical time delay are given by

$$\tau_n=\frac{\theta+n\pi}{w},\theta\in[0,2\pi)\,,n=0,1,2,\dots$$

where

$$\begin{cases}\sin\theta=\dfrac{\left(-w^4+Bw^2-D\right)Fw+\left(Cw-Aw^3\right)\left(G-Ew^2\right)}{\left(G-Ew^2\right)^2+\left(Fw\right)^2},\\ \cos\theta=\dfrac{\left(Aw^3-Cw\right)Fw+\left(-w^4+Bw^2-D\right)\left(G-Ew^2\right)}{\left(G-Ew^2\right)^2+\left(Fw\right)^2}.\end{cases}\tag{31}$$

Lemma 3.1. $\operatorname{sign}\left\{\dfrac{(\mathrm{d}Re\lambda)}{\mathrm{d}\tau}\,|_{\lambda=\pm iw}\right\}=\operatorname{sign}\left\{4w^7+3\left(A^2-2B\right)w^5+\right.$ $\left.2\left(B^2+2D-2AC-E^2\right)w^3+\left(C^2-2BD+2EG-F^2\right)w\right\}.$

Proof. Taking the derivative of the equation (24) with τ shows the following equation

$$\begin{aligned}&\left[4\lambda^3+3A\lambda^2+2B\lambda+C+e^{-\lambda\tau}\left(2E\lambda+F\right)-\tau e^{-\lambda\tau}\left(E\lambda^2+F\lambda+G\right)\right]\frac{\mathrm{d}\lambda}{\mathrm{d}\tau}\\&=\lambda e^{-\lambda\tau}\left(E\lambda^2+F\lambda+G\right).\end{aligned}\tag{32}$$

From the equation (24), it can be obtained that

$$e^{-\lambda\tau}=-\frac{\lambda^4+A\lambda^3+B\lambda^2+C\lambda+D}{E\lambda^2+F\lambda+G},\tag{33}$$

which together with (32) give that

$$\begin{aligned}\left(\frac{\mathrm{d}\lambda}{\mathrm{d}\tau}\right)^{-1}&=-\frac{4\lambda^3+3A\lambda^2+2B\lambda+C}{\lambda\left(\lambda^4+A\lambda^3+B\lambda^2+C\lambda+D\right)\left(E\lambda^2+F\lambda+G\right)}\\&\quad+\frac{2E\lambda+F}{\lambda\left(E\lambda^2+F\lambda+G\right)}-\frac{\tau}{\lambda}.\end{aligned}\tag{34}$$

Hence,

$$\operatorname{sign}\left\{\frac{\mathrm{d}(\mathrm{Re}\lambda)}{\mathrm{d}\tau}\,|_{\lambda=\pm iw}\right\} = \operatorname{sign}\left\{4w^7 + 3\left(A^2 - 2B\right)w^5 + \right. \\ \left. 2\left(B^2 + 2D - 2AC - E^2\right)w^3 + \left(C^2 - 2BD + 2EG - F^2\right)w\right\}. \tag{35}$$

□

In the equation (24), define a polynomial function as that

$$Q(\lambda) = E\lambda^2 + F\lambda + G, \tag{36}$$

then according to the stability theorem in the reference [18], we get the following main theorem.

Theorem 3.2. On the assumption that there exists no pure imaginary characteristic roots $\pm i\omega(\omega > 0)$ such that $Q(\pm iw) = 0$, we have the following results.

(i) If there exists no root of the equation (28) and the system (3-1) when $\tau = 0$ is unstable, then the equilibrium point S_0 is unstable for any $\tau > 0$.

(ii) If there exists only one positive root of the equation (28) and the system (3-1) when $\tau = 0$ is asymptotically stable, then the equilibrium point S_0 is asymptotically stable when $\tau \in [0, \tau_0)$ and unstable when $\tau \in (\tau_0, +\infty)$ respectively, and Hopf bifurcation appears when $\tau = \tau_0$.

(iii) If there exists two roots ω_1 and ω_2 of the equation (28) (without loss of generality assume that $\omega_1 > \omega_2$), then we obtain the following two situations, where $\tau_1^{(j)}$ and $\tau_2^{(j)}$ $(j = 0, 1, 2, \ldots)$ represent the time delay corresponding to ω_1 and ω_2 respectively, and n signifies a nonnegative integer.

Situation I: When $\tau_1^0 < \tau_2^0$, the equilibrium point S_0 is unstable for any $\tau > 0$.

Situation II: When $\tau_1^0 > \tau_2^0$, there exists a nonnegative integer l such that $\tau_2^{(l)} < \tau_1^{(l)} < \tau_1^{(l+1)}$. If $\tau \in \bigcup_{j=0}^{m}(\tau_2^{(j)}, \tau_1^{(j)})$, the equilibrium point S_0 is asymptotically stable. If $\tau \in \bigcup_{j=0}^{m}(\tau_1^{(j-1)}, \tau_2^{(j)}) \cup (\tau_1^{(m)}, +\infty)$, the equilibrium point S_0 is unstable, where m is the minimum for all of the values l. Moreover, when $\tau = \tau_1^{(j)}$ or $\tau = \tau_2^{(j)}(j = 0, 1, 2, \ldots, m)$, Hopf bifurcation appears.

Proof. (i) According to Corollary 2.4 in the reference [13], the sum of the orders of the zeros of the characteristic equation in the open right half plane can change only if a zero appears on or crosses in the imaginary axis. Then it is known that, when there exists no root of the characteristic equation (28), the number of roots with positive real parts is equal to the number of that when $\tau = 0$. Since when $\tau = 0$ the system (1-1) is unstable, there always exists a root with a positive real part for the characteristic equation (22). Thus when $\tau > 0$ there always exists a root with a positive real part for the equation (22). Therefore, the equilibrium point S_0 is unstable for any $\tau > 0$.

(ii) From the 218th page of the reference [18], when there exists only one positive root of the equation (28), the following inequality

$$\frac{\mathrm{d}(\mathrm{Re}\lambda)}{\mathrm{d}\tau}\ |_{\lambda=\pm iw}>0 \tag{37}$$

holds. Together with the Corollary 2.4, when the system (1-1) without time delay is stable, as the time delay varies from zero to infinity, there always appears a new pair of conjugate characteristic roots of the characteristic equation (22) of the system (3-1) for each crossing of time delay at $\tau_k (k = 0, 1, \ldots)$. This indicates that the system is asymptotically stable when $\tau \in [0, \tau_0)$ and unstable when $\tau \in (\tau_0, +\infty)$. When $\tau = \tau_0$, there exists only one pair of pure imaginary roots of the characteristic equation (22), while there exist other roots with negative real parts. Furthermore, the inequalities

$$\frac{\mathrm{d}(\mathrm{Re}\lambda)}{\mathrm{d}\tau}\ |_{\lambda=\pm iw}\neq 0, \tag{38}$$

hold. Therefore, Hopf bifurcation appears for the system (3-1) when $\tau = \tau_0$.

(iii) If there exists two positive roots ω_1 and ω_2 of the equation (28), without loss of generality assume that $\omega_1 > \omega_2$ which are corresponding to time delay $\tau_1^{(j)}$ and $\tau_2^{(j)} (j = 0, 1, 2, \ldots)$ respectively. From the reference [18] we have that

$$\frac{\mathrm{d}(\mathrm{Re}\lambda)}{\mathrm{d}\tau}\ |_{\lambda=\pm iw_1,\tau=\tau_1^{(j)}}>0,\quad \frac{\mathrm{d}(\mathrm{Re}\lambda)}{\mathrm{d}\tau}\ |_{\lambda=\pm iw_2,\tau=\tau_2^{(j)}}<0, \tag{39}$$

moreover, the period of $\tau_1^{(j)}$ is less than that of $\tau_2^{(j)}$.

Situation I: The case of $\tau_1^0 < \tau_2^0$.

From the two inequalities (39), it is known that, as the time delay varies from zero to infinity, there always appears a new pair of conjugate characteristic roots with positive real parts of the characteristic equation (22) of the system (3-1), for each crossing of time delay at $\tau_1^{(j)} (j = 0, 1, \ldots)$. But there removes such a pair of conjugate characteristic roots for each crossing of time delay at $\tau_2^{(j)} (j = 0, 1, \ldots)$. Furthermore, $\tau_1^0 < \tau_2^0$ and the period of $\tau_1^{(j)}$ is less than that of $\tau_2^{(j)} (j = 0, 1, \ldots)$. Therefore, since the system (1-1) without time delay is unstable, the system (3-1) is also unstable for any $\tau > 0$.

Situation II: The case of $\tau_1^0 > \tau_2^0$.

When $\tau_1^0 > \tau_2^0$, as the time delay varies from zero to infinity, there always removes a new pair of conjugate characteristic roots of the characteristic equation (22) of the system (3-1) for each crossing of time delay at $\tau_2^{(j)} (j = 0, 1, \ldots)$. Thus the system (3-1) is asymptotically stable when $\tau \in (\tau_2^0, \tau_1^0)$.

When $\tau_1^0 > \tau_2^0$, as the time delay varies from zero to infinity, there always increases a new pair of conjugate characteristic roots of the characteristic equation(22) of the system (3-1) for each crossing of time delay at $\tau_1^{(j)} (j = 0, 1, \ldots)$. Thus the system (3-1) is unstable when $\tau \in (\tau_1^0, \tau_2^1)$.

Since the period of $\tau_1^{(j)}$ is less than that of $\tau_2^{(j)}$, it is seen that there exists a nonnegative integer l such that $\tau_2^{(l)} < \tau_1^{(l)} < \tau_1^{(l+1)}$. Assume that m is the minimum of all values l, similar proof as that above can show that, if $\tau \in \bigcup_{j=0}^{m}(\tau_2^{(j)}, \tau_1^{(j)})$, the system (3-1) is asymptotically stable, and if $\tau \in \bigcup_{j=0}^{m}(\tau_1^{(j-1)}, \tau_2^{(j)}) \cup (\tau_1^{(m)}, +\infty)$, the system (3-1) is unstable.

When $\tau = \tau_1^{(j)}$ or $\tau = \tau_2^{(j)} (j = 0, 1, \ldots, m)$, there exists only one pair of pure imaginary roots while there exists other roots with negative real parts of the characteristic equation (22). Furthermore, the inequalities

$$\frac{\mathrm{d}(\mathrm{Re}\lambda)}{\mathrm{d}\tau} \,|_{\lambda=\pm i w_1} \neq 0, \quad \frac{\mathrm{d}(\mathrm{Re}\lambda)}{\mathrm{d}\tau} \,|_{\lambda=\pm i w_2} \neq 0, \tag{40}$$

hold. Therefore, Hopf bifurcation appears for the system (3-1) when $\tau = \tau_1^{(j)}$ or $\tau = \tau_2^{(j)} (j = 0, 1, \ldots, m)$. □

3.2.2. Stability of the equilibrium points S_1^* and S_2^*

Define $\overline{x}(t) = x(t) - x^*, \overline{y}(t) = y(t) - y^*, \overline{z}(t) = z(t) - z^*, \overline{u}(t) = u(t) - u^*, \overline{w}(t) = w(t) - w^*$, on the assumption that $(x^*, y^*, z^*, u^*, v^*, w^*)$ signifies the equilibrium points S_1^* or S_2^*. Then the system (3-1) can be transformed equivalently as the following system (3-2)

$$\begin{cases} \dot{\overline{x}}(t) = a_1\left(1 - \dfrac{x^* e^{-\lambda\tau}}{M}\right)\overline{x}(t) - \dfrac{a_1 x^*}{M}\overline{x}(t-\tau) - a_2\left[\overline{y}(t-\tau) + \overline{z}(t-\tau)\right] - d_3\overline{u}(t-\tau) \\ \qquad -k_4\overline{w}(t-\tau), \\ \dot{\overline{y}}(t) = b_3\left[N - (x^* - z^*)e^{-\lambda\tau}\right]\overline{x}(t) - b_3 x^*\overline{x}(t-\tau) - b_1\overline{y}(t) + \left(b_3 x^* - b_2\right)\overline{z}(t-\tau), \\ \dot{\overline{z}}(t) = c_1 c_2 z^*\overline{x}(t-\tau) + c_1\left(c_2 x^* e^{-\lambda\tau} - c_3\right)\overline{z}(t), \\ \dot{\overline{u}}(t) = d_1\overline{x}(t) - d_2\overline{u}(t), \\ \dot{\overline{w}}(t) = k_2 k_3 w^*\overline{x}(t-\tau) + k_1\left(\dfrac{y^* e^{-\lambda\tau}}{T_1} - 1\right)\overline{y}(t) + \dfrac{k_1 y^*}{T_1}\overline{y}(t-\tau) \\ \qquad +k_2\left(k_3 x^* e^{-\lambda\tau} - T_2\right)\overline{w}(t). \end{cases} \tag{3-2}$$

The jacobian matrix of the system (3-2) can be obtained as that

$$\begin{pmatrix} a_1\left(1 - \dfrac{2x^* e^{-\lambda\tau}}{M}\right) & -a_2 e^{-\lambda\tau} & -a_2 e^{-\lambda\tau} & -d_3 e^{-\lambda\tau} & -k_4 e^{-\lambda\tau} \\ b_3\left[N - \left(2x^* - z^*\right)e^{-\lambda\tau}\right] & -b_1 & \left(b_3 x^* - b_2\right)e^{-\lambda\tau} & 0 & 0 \\ c_1 c_2 z^* e^{-\lambda\tau} & 0 & c_1\left(c_2 x^* e^{-\lambda\tau} - c_3\right) & 0 & 0 \\ d_1 & 0 & 0 & -d_2 & 0 \\ k_2 k_3 w^* e^{-\lambda\tau} & k_1\left(\dfrac{2y^* e^{-\lambda\tau}}{T_1} - 1\right) & 0 & 0 & k_2\left(k_3 x^* e^{-\lambda\tau} - T_2\right) \end{pmatrix}.$$

Now we choose the coefficients as following

$$\begin{cases} a_1 = 1, a_2 = 0.5, b_1 = 0.006, b_2 = 0.1, b_3 = 0.18, c_1 = 0.6, c_2 = 0.5, c_3 = 0.4, d_1 = 0.1, \\ d_2 = 0.06, d_3 = 0.08, M = 1.8, N = 1, k_1 = 0.68, k_2 = 1.59, k_3 = 0.5, k_4 = 0.09, \\ T_1 = 1.05,\ T_2 = 2, \end{cases} \tag{41}$$

and initial values are given as that

$$x(0) = 0.016,\ y(0) = 0.11,\ z(0) = 0.07,\ u(0) = 0.37,\ w(0) = 0.002, \tag{42}$$

under which the characteristic polynomial for the equilibrium point

$$(0.8000, -3.7852, 0.0569, 1.2000, 2.3429)$$

can be computed simply as

$$\begin{aligned}F(\lambda) &= \lambda^5 + 3.9100\lambda^4 + 0.4876\lambda^3 - 0.0154\lambda^2 - 0.0040\lambda - 0.0001 + e^{-4\lambda\tau}(0.0024\lambda \\ &+0.0001) + e^{-3\lambda\tau}(0.0648\lambda^2 + 0.0100\lambda + 0.0004) + e^{-2\lambda\tau}(0.1355\lambda^3 - 0.0871\lambda^2 \\ &-0.0181\lambda - 0.0008) - e^{-\lambda\tau}(0.5510\lambda^4 + 0.0518\lambda^3 - 0.1044\lambda^2 - 0.0152\lambda - 0.0005).\end{aligned}$$

Obviously, $F(\lambda)$ is a continuous function, by the intermediate value theorem, it is known that there exists at least a real positive root for the equation above, so the equilibrium point S_1^* or S_2^* is unstable.

4. PROPERTIES OF HOPF BIFURCATION

Theorem 3.2 tells us that Hopf bifurcation may appear for the equilibrium point S_0 when $\tau = \tau_i^{(j)} (i = 0, 1, 2, j = 0, 1, \dots, m)$. In this section we analyze the properties of Hopf bifurcation by the center manifold theorem and normal form method ([19, 21]).

For the equilibrium point S_0, after defining functions as follows

$$s_1(t) = x(t\tau),\ s_2(t) = y(t\tau),\ s_3(t) = z(t\tau),\ s_4(t) = u(t\tau),\ s_5(t) = w(t\tau),$$

the system (3-1) can be equivalently transformed into the system below

$$\dot{s}(t) = L_\mu(s_t) + F(\mu, s_t), \mu \in \mathbb{R}, \tag{4-1}$$

where $\dot{s}(t) = (\dot{s}_1(t), \dot{s}_2(t), \dot{s}_3(t), \dot{s}_4(t), \dot{s}_5(t))^T$, $s_t = s(t+\theta) = (s_1(t+\theta), s_2(t+\theta), s_3(t+\theta), s_4(t+\theta), s_5(t+\theta))^T$, $\phi_i(\theta) = s_i(t+\theta) (i = 1, 2, 3, 4, 5)$ and $\mathbb{C} = C([-1, 0], \mathbb{R}^5)$, $\theta \in [-1, 0]$, $L_\mu : \mathbb{C} \to \mathbb{R}^5$, $F : \mathbb{R} \times \mathbb{C} \to \mathbb{R}^5$ are given by

$$L_\mu(\phi) = \tau \begin{bmatrix} a_1 & 0 & 0 & 0 & 0 \\ b_3 N & -b_1 & 0 & 0 & 0 \\ 0 & 0 & -c_1 c_3 & 0 & 0 \\ d_1 & 0 & 0 & -d_2 & 0 \\ 0 & -k_1 & 0 & 0 & -k_2 T_2 \end{bmatrix} \begin{bmatrix} \phi_1(0) \\ \phi_2(0) \\ \phi_3(0) \\ \phi_4(0) \\ \phi_5(0) \end{bmatrix} + \tau \begin{bmatrix} 0 & -a_2 & -a_2 & -d_3 & -k_4 \\ 0 & 0 & -b_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \phi_1(-1) \\ \phi_2(-1) \\ \phi_3(-1) \\ \phi_4(-1) \\ \phi_5(-1) \end{bmatrix},$$

$$F(\mu, \phi) = \tau \begin{bmatrix} -\dfrac{a_1}{M}\phi_1(0)\phi_1(-1) \\ -b_3\phi_1(0)\phi_1(-1) + b_3\phi_1(0)\phi_3(-1) \\ c_1 c_2 \phi_3(0)\phi_1(-1) \\ 0 \\ \dfrac{k_1}{T_1}\phi_2(0)\phi_2(-1) + k_2 k_3 \phi_5(0)\phi_1(-1) \end{bmatrix},$$

$$\phi(\theta) = (\phi_1(\theta), \phi_2(\theta), \phi_3(\theta), \phi_4(\theta), \phi_5(\theta))^T.$$

According to Riesz Representation Theorem, there exists a bounded variation function matrix $\eta(\theta,\mu), \theta\in[-1,0]$, which satisfies that

$$L_\mu(\phi)=\int_{-1}^{0}\mathrm{d}\eta(\theta,\mu)\phi(\theta), \forall\phi\in\mathbb{C}. \tag{43}$$

Next in order to be decomposed in the phase space, the system (4-1) are subsequently transformed into the ordinary differential equations.

For $\phi(\theta)\in\mathbb{C}$, define

$$A(\mu)\phi(\theta)=\begin{cases}\dfrac{\mathrm{d}\phi(\theta)}{\mathrm{d}\theta}, & \theta\in[-1,0),\\ \displaystyle\int_{-1}^{0}\mathrm{d}\eta(\xi,\mu)\phi(\xi), & \theta=0,\end{cases}$$

$$R(\mu)\phi(\theta)=\begin{cases}0, & \theta\in[-1,0),\\ F(\mu,\phi), & \theta=0.\end{cases}$$

Then the system (4-1) can be transformed into the abstract differential equation

$$\dot{s}_t=A(\mu)s_t+R(\mu)s_t, \tag{4-2}$$

where $s_t=s(t+\theta), \theta\in[-1,0]$.

For $\psi(\chi)\in C\left([-1,0],\left(\mathbb{R}^5\right)^*\right)$, define

$$A^*(\mu)\psi(\chi)=\begin{cases}-\dfrac{\mathrm{d}\psi(\chi)}{\mathrm{d}\chi}, & \chi\in(0,1],\\ \displaystyle\int_{-1}^{0}\mathrm{d}\eta^T(t,0)\psi(-t), & \chi=0,\end{cases}$$

where $\eta^T(t,0)$ is defined in (43).

For $\phi\in\mathbb{C}$ and $\psi\in C\left([-1,0],\left(\mathbb{R}^5\right)^*\right)$ define the bilinear function

$$\langle\psi,\phi\rangle=\overline{\psi(0)}\phi(0)-\int_{-1}^{0}\int_{\xi=0}^{\theta}\overline{\psi(\xi-\theta)}\mathrm{d}\eta(\theta,0)\phi(\xi)\mathrm{d}\xi. \tag{44}$$

From the discussions in the section above, we know that $\pm i\omega\tau_i^{(j)}(i=0,1,2,j=0,1,2,\ldots,m)$ are eigenvalues of $A(0)$, and $A^*(0)$ is adjoint operator of $A(0)$. Suppose $q(\theta)=q(0)e^{i\omega\tau_i^{(j)}\theta}$ is an eigenvector of $A(0)$ corresponding to $i\omega\tau_i^{(j)}$ and $q^*(\chi)=Dq^*(0)e^{i\omega\tau_i^{(j)}\chi}$ is the eigenvector of $A^*(0)$ corresponding to $-i\omega\tau_i^{(j)}$, where $q(0)=$

$[1,\alpha,\beta,\gamma,\delta]^T$, $q^*(0)=[1,p,g,k,\varepsilon]$. Moreover $\langle q^*(\chi),q(\theta)\rangle=1$. Then we obtain that

$$\begin{cases}\alpha=\dfrac{b_3N}{b_1+i\omega},\\ \beta=0,\\ \gamma=\dfrac{d_1}{d_2+i\omega},\\ \delta=-\dfrac{b_3k_1N}{(k_2T_2+i\omega)(b_1+i\omega)},\end{cases}\quad\text{and}\quad\begin{cases}p=\dfrac{a_2(i\omega-k_2T_2)+k_1k_4}{(i\omega-b_1)(i\omega-k_2T_2)},\\ q=\dfrac{a_2(i\omega-b_1)(i\omega-k_2T_2)+b_2a_2(i\omega-k_2T_2)+k_1k_4}{(i\omega-c_1c_3)(i\omega-b_1)(i\omega-k_2T_2)},\\ k=\dfrac{d_3}{i\omega-d_2},\\ \varepsilon=\dfrac{k_4}{i\omega-k_2T_2}\\ \overline{D}=\left[1+\alpha\overline{p}+\overline{k}r+\overline{\varepsilon}\delta-\tau_i^{(j)}(a_2\alpha+d_3r+k_4\delta)e^{-i\omega\tau_i^{(j)}}\right]^{-1}.\end{cases}$$

When $\mu=0$, for any solution of the system (4-2) define

$$z(t)=\langle q^*,s_t\rangle,W(t,\theta)=s_t(\theta)-2\operatorname{Re}\{z(t)q(\theta)\} \tag{45}$$

on the center manifold. We have

$$W(t,\theta)=W\left(z(t),\overline{z(t)},\theta\right),$$

where

$$W\left(z(t),\overline{z(t)},\theta\right)=W_{20}(\theta)\frac{z^2(t)}{2}+W_{11}(\theta)z(t)\overline{z(t)}+W_{02}(\theta)\frac{\overline{z(t)}^2}{2}+\cdots, \tag{46}$$

$z(t)$ and $\overline{z(t)}$ are local coordinates for center manifold in the directions of q^* and $\overline{q^*}$.

Then the flow of the system (4-1) on the center manifold can be determined by the following equations

$$\dot{z}(t)=i\omega\tau_i^{(j)}z(t)+\overline{q^*(0)}F(0,W(t,0)+2\operatorname{Re}[z(t)q(0)]).$$

Denote

$$G\left(z(t),\overline{z(t)}\right)=\overline{q^*(0)}F(0,W(t,0)+2\operatorname{Re}[z(t)q(0)]).$$

From

$$W(t,0)+2\operatorname{Re}[z(t)q(0)]=s_t,$$

we get that

$$F(0,W(t,0)+2\operatorname{Re}[z(t)q(0)])=F(0,s_t),$$

$$G\left(z(t),\overline{z(t)}\right)=\tau_i^{(j)}\overline{D}\left(1,\overline{p},\overline{q},\overline{k},\overline{\varepsilon}\right)\begin{bmatrix}-\dfrac{a_1}{M}s_{1t}(0)s_{1t}(-1)\\ -b_3s_{1t}(0)s_{1t}(-1)+b_3s_{1t}(0)s_{3t}(-1)\\ c_1c_2s_{3t}(0)s_{1t}(-1)\\ 0\\ \dfrac{k_1}{T_1}s_{2t}(0)s_{2t}(-1)+k_2k_3s_{5t}(0)s_{1t}(-1)\end{bmatrix}, \tag{47}$$

which can be expressed as power series of $z(t)$ and $\overline{z(t)}$

$$G\left(z(t), \overline{z(t)}\right) = g_{20}\frac{z(t)^2}{2} + g_{11}z(t)\overline{z(t)} + g_{02}\frac{\overline{z(t)}^2}{2} + \cdots, \tag{48}$$

and

$$F(0, s_t) = f_{20}\frac{z(t)^2}{2} + f_{11}z(t)\overline{z(t)} + f_{02}\frac{\overline{z(t)}^2}{2} + \cdots, \tag{49}$$

respectively.

From (45) and (46), we have that

$$\begin{aligned} s_t(\theta) = (1, \alpha, \beta, \gamma, \delta)^T e^{i\omega\tau_i^{(j)}\theta} z(t) + \left(1, \overline{\alpha}, \overline{\beta}, \overline{\gamma}, \overline{\delta}\right)^T e^{-i\omega\tau_i^{(j)}\theta}\overline{z(t)} + \\ W_{20}(\theta)\tfrac{z^2(t)}{2} + W_{11}(\theta) z(t)\overline{z}(t) + W_{02}(\theta)\tfrac{\overline{z(t)}^2}{2} + \cdots, \end{aligned} \tag{50}$$

where $s_t(\theta) = [s_{1t}(\theta), s_{2t}(\theta), s_{3t}(\theta), s_{4t}(\theta), s_{5t}(\theta)]^T$.

Substituting (50) into (47) and comparing their coefficients with that of (48) and (49) respectively gives that

$$\left\{\begin{aligned}
g_{20} &= \overline{q^*(0)}f_{20} = 2\overline{D}\tau_i^{(j)}\left[-\frac{a_1}{M} - \overline{p}b_3 + \left(\frac{k_1}{T_1}\alpha^2 + k_2k_3\delta\right)\overline{\varepsilon}\right]e^{-i\omega\tau_i^{(j)}}, \\
g_{11} &= \overline{q^*(0)}f_{11} = \overline{D}\tau_i^{(j)}\left(-\frac{a_1}{M} - \overline{p}b_3\right)\left(e^{-i\omega\tau_i^{(j)}} + e^{i\omega\tau_i^{(j)}}\right) + \\
&\qquad \left[\left(\frac{k_1}{T_1}\alpha\overline{\alpha} + k_2k_3\overline{\delta}\right)e^{-i\omega\tau_i^{(j)}} + \left(\frac{k_1}{T_1}\alpha\overline{\alpha} + k_2k_3\delta\right)e^{i\omega\tau_i^{(j)}}\right]\overline{D}\tau_i^{(j)}\overline{\varepsilon}, \\
g_{02} &= \overline{q^*(0)}f_{02} = 2\overline{D}\tau_i^{(j)}\left[-\frac{a_1}{M} - \overline{p}b_3 + \left(\frac{k_1}{T_1}\overline{\alpha}^2 + k_2k_3\overline{\delta}\right)\overline{\varepsilon}\right]e^{i\omega\tau_i^{(j)}}, \\
g_{21} &= \overline{q^*(0)}f_{21} \\
&= \left(-\frac{a_1}{M} - \overline{p}b_3\right)\left(W_{11}^{(1)}(-1) + \frac{W_{20}^{(1)}(-1)}{2} + \frac{W_{20}^{(1)}(0)}{2}e^{i\omega\tau_i^{(j)}} + W_{11}^{(1)}(0)e^{-i\omega\tau_i^{(j)}}\right) \\
&\quad \overline{D}\tau_i^{(j)} + \left(W_{11}^{(3)}(0)e^{-i\omega\tau_i^{(j)}} + W_{20}^{(3)}(0)e^{i\omega\tau_i^{(j)}}\right)c_1c_2\overline{q}\overline{D}\tau_i^{(j)} + \\
&\quad \left(W_{11}^{(3)}(-1) + \frac{W_{20}^{(3)}(-1)}{2}\right)b_3\overline{p}\overline{D}\tau_i^{(j)} + \left(W_{20}^{(2)}(-1) + W_{20}^{(2)}(0)e^{i\omega\tau_i^{(j)}}\right) \\
&\quad \overline{\varepsilon\alpha}\frac{k_1}{2T_1}\overline{D}\tau_i^{(j)} + \left(W_{11}^{(2)}(-1) + W_{11}^{(2)}(0)e^{-i\omega\tau_i^{(j)}}\right)\overline{\varepsilon}\alpha\frac{k_1}{T_1}\overline{D}\tau_i^{(j)} + \\
&\quad \left(W_{11}^{(5)}(0)e^{-i\omega\tau_i^{(j)}} + W_{11}^{(1)}(-1)\delta + \frac{W_{20}^{(1)}(-1)}{2}\overline{\delta} + \frac{W_{20}^{(5)}(0)}{2}e^{i\omega\tau_i^{(j)}}\right)k_2k_3\overline{\varepsilon}\overline{D}\tau_i^{(j)}
\end{aligned}\right. \tag{51}$$

where g_{21} is determined by W_{20} and W_{11}.

From the paper [18], we obtain that

$$\left\{\begin{aligned}
W_{20}(\theta) &= \frac{ig_{20}}{w\tau_i^{(j)}}q(0)e^{i\omega\tau_i^{(j)}\theta} + \frac{i\overline{g}_{02}}{3\omega\tau_i^{(j)}}\overline{q(0)}e^{-i\omega\tau_i^{(j)}\theta} + E_1e^{2i\omega\tau_i^{(j)}\theta}, \\
W_{11}(\theta) &= -\frac{ig_{11}}{w\tau_i^{(j)}}q(0)e^{i\omega\tau_i^{(j)}\theta} + \frac{i\overline{g}_{11}}{\omega\tau_i^{(j)}}\overline{q(0)}e^{-i\omega\tau_i^{(j)}\theta} + E_2,
\end{aligned}\right.$$

where

$$E_1 = \left[2i\omega\tau_i^{(j)}I - \int_{-1}^{0} e^{2i\omega\tau_i^{(j)}\theta}\mathrm{d}\eta(\theta, 0)\right]^{-1} f_{20}, \quad E_2 = -\left[\int_{-1}^{0}\mathrm{d}\eta(\theta, 0)\right]^{-1} f_{11}.$$

From $g_{20} = \overline{q^*(0)} f_{20}$ and $g_{11} = \overline{q^*(0)} f_{11}$, we have that

$$f_{20} = 2\tau_i^{(j)} e^{-i\omega\tau_i^{(j)}} \begin{bmatrix} -\dfrac{a_1}{M} \\ -b_3 \\ 0 \\ 0 \\ \left(\dfrac{k_1}{T_1}\alpha^2 + k_2 k_3 \delta\right) \end{bmatrix},$$

and

$$f_{11} = \tau_i^{(j)} \begin{bmatrix} -\dfrac{a_1}{M}\left(e^{i\omega\tau_i^{(j)}} + e^{-i\omega\tau_i^{(j)}}\right) \\ -b_3\left(e^{i\omega\tau_i^{(j)}} + e^{-i\omega\tau_i^{(j)}}\right) \\ 0 \\ 0 \\ \left(\dfrac{k_1}{T_1}\alpha\overline{\alpha} + k_2 k_3 \overline{\delta}\right) e^{-i\omega\tau_i^{(j)}} + \left(\dfrac{k_1}{T_1}\alpha\overline{\alpha} + k_2 k_3 \delta\right) e^{i\omega\tau_i^{(j)}} \end{bmatrix}.$$

In addition, according to the definition of $A(\mu)$, when $\mu = 0$ these two equalities below hold

$$\int_{-1}^{0} e^{2i\omega\tau_i^{(j)}\theta} \mathrm{d}\eta(\theta, 0) = \tau_i^{(j)} \begin{bmatrix} a_1 & -a_2 e^{-2i\omega\tau_i^{(j)}} & -a_2 e^{-2i\omega\tau_i^{(j)}} & -d_3 e^{-2i\omega\tau_i^{(j)}} & -k_4 e^{-2i\omega\tau_i^{(j)}} \\ b_3 N & -b_1 & -b_2 e^{-2i\omega\tau_i^{(j)}} & 0 & 0 \\ 0 & 0 & -c_1 c_3 & 0 & 0 \\ d_1 & 0 & 0 & -d_2 & 0 \\ 0 & -k_1 & 0 & 0 & -k_2 T_2 \end{bmatrix},$$

$$\int_{-1}^{0} \mathrm{d}\eta(\theta, 0) = \tau_i^{(j)} \begin{bmatrix} a_1 & -a_2 & -a_2 & -d_3 & -k_4 \\ b_3 N & -b_1 & -b_2 & 0 & 0 \\ 0 & 0 & -c_1 c_3 & 0 & 0 \\ d_1 & 0 & 0 & -d_2 & 0 \\ 0 & -k_1 & 0 & 0 & -k_2 T_2 \end{bmatrix},$$

$$E_1^{(1)} = \frac{-2e^{-i\omega\tau_i^{(j)}}(2iw + d_2)(2iw + b_1)\left[\dfrac{a_1}{M}(2i\omega + k_2 T_2)e^{2i\omega\tau_i^{(j)}} + \left(\dfrac{k_1}{T_1}\alpha^2 + k_2 k_3 \delta\right)k_4\right]}{(2i\omega + k_2 T_2)\left\{a_2 b_3 N(2iw + d_2) + (2iw + b_1)\left[(2iw - a_1)(2iw + d_2)e^{2i\omega\tau_i^{(j)}} + d_1 d_3\right]\right\} - k_1 k_4 b_3 N(2iw + d_2)} +$$
$$\frac{2b_3(2iw + d_2)e^{-i\omega\tau_i^{(j)}}\left[a_2(2i\omega + k_2 T_2) - k_1 k_4\right]}{(2i\omega + k_2 T_2)\left\{a_2 b_3 N(2iw + d_2) + (2iw + b_1)\left[(2iw - a_1)(2iw + d_2)e^{2i\omega\tau_i^{(j)}} + d_1 d_3\right]\right\} - k_1 k_4 b_3 N(2iw + d_2)},$$

$$E_1^{(2)} = \frac{-2e^{i\omega\tau_i^{(j)}}(2i\omega + k_2 T_2)\left\{\dfrac{a_1}{M}b_3 N(2iw + d_2) + b_3\left[(2iw - a_1)(2iw + d_2) + d_1 d_3 e^{-2i\omega\tau_i^{(j)}}\right]\right\}}{(2i\omega + k_2 T_2)\left\{a_2 b_3 N(2iw + d_2) + (2iw + b_1)\left[(2iw - a_1)(2iw + d_2)e^{2i\omega\tau_i^{(j)}} + d_1 d_3\right]\right\} - k_1 k_4 b_3 N(2iw + d_2)} -$$
$$\frac{2b_3 k_4 N e^{-i\omega\tau_i^{(j)}}(2iw + d_2)\left(\dfrac{k_1}{T_1}\alpha^2 + k_2 k_3 \delta\right)}{(2i\omega + k_2 T_2)\left\{a_2 b_3 N(2iw + d_2) + (2iw + b_1)\left[(2iw - a_1)(2iw + d_2)e^{2i\omega\tau_i^{(j)}} + d_1 d_3\right]\right\} - k_1 k_4 b_3 N(2iw + d_2)},$$

$$E_1^{(3)} = 0,$$

$$E_1^{(4)} = \frac{-2d_1 e^{i\omega\tau_i^{(j)}} \frac{a_1}{M} (2iw + b_1)(2i\omega + k_2T_2) + 2d_1 b_3 e^{-i\omega\tau_i^{(j)}} [a_2 (2iw + k_2T_2) - k_1 k_4]}{(2i\omega + k_2T_2)\left\{a_2 b_3 N(2iw + d_2) + (2iw + b_1)\left[(2iw - a_1)(2iw + d_2)e^{2i\omega\tau_i^{(j)}} + d_1 d_3\right]\right\} - k_1 k_4 b_3 N(2iw + d_2)} -$$

$$\frac{2d_1 k_4 e^{-i\omega\tau_i^{(j)}} \left(\frac{k_1}{T_1}\alpha^2 + k_2 k_3 \delta\right)(2iw + b_1)}{(2i\omega + k_2T_2)\left\{a_2 b_3 N (2iw + d_2) + (2iw + b_1)\left[(2iw - a_1)(2iw + d_2) e^{2i\omega\tau_i^{(j)}} + d_1 d_3\right]\right\} - k_1 k_4 b_3 N (2iw + d_2)},$$

$$E_1^{(5)} = \frac{2e^{-i\omega\tau_i^{(j)}} k_1 b_3 N \left\{\frac{a_1}{M}(2iw + d_2) + b_3 k_1 \left[(2iw - a_1)(2iw + d_2) + d_1 d_3 e^{-2i\omega\tau_i^{(j)}}\right]\right\}}{(2i\omega + k_2T_2)\left\{a_2 b_3 N(2iw + d_2) + (2iw + b_1)\left[(2iw - a_1)(2iw + d_2)e^{2i\omega\tau_i^{(j)}} + d_1 d_3\right]\right\} - k_1 k_4 b_3 N(2iw + d_2)} +$$

$$\frac{2e^{i\omega\tau_i^{(j)}} \left\{\left(\frac{k_1}{T_1}\alpha^2 + k_2 k_3 \delta\right)\{d_1 d_3 e^{-2i\omega\tau_i^{(j)}}(2iw + b_1) + (2iw + d_2)\left[(2iw - a_1)(2iw + b_1) + a_2 b_3 N e^{-2i\omega\tau_i^{(j)}}\right]\}\right\}}{(2i\omega + k_2T_2)\left\{a_2 b_3 N (2iw + d_2) + (2iw + b_1)\left[(2iw - a_1)(2iw + d_2) e^{2i\omega\tau_i^{(j)}} + d_1 d_3\right]\right\} - k_1 k_4 b_3 N (2iw + d_2)},$$

$$E_2^{(1)} = \frac{\frac{k_1}{T_1}\alpha\overline{\alpha}\left(e^{-i\omega\tau_i^{(j)}} + e^{i\omega\tau_i^{(j)}}\right) + k_2 k_3 \left(\overline{\delta} e^{-i\omega\tau_i^{(j)}} + \delta e^{i\omega\tau_i^{(j)}}\right) b_1 d_2 k_4}{d_2 b_3 N k_1 k_4 + k_2 T_2 (a_1 b_1 d_2 - a_2 d_2 b_3 N - b_1 d_1 d_3)} - \frac{\left(e^{-i\omega\tau_i^{(j)}} + e^{i\omega\tau_i^{(j)}}\right)\left[\frac{-a_1}{M} b_1 d_2 k_2 T_2 + b_3 d_2 (a_2 k_2 T_2 - k_1 k_4)\right]}{d_2 b_3 N k_1 k_4 + k_2 T_2 (a_1 b_1 d_2 - a_2 d_2 b_3 N - b_1 d_1 d_3)},$$

$$E_2^{(2)} = \frac{\left(e^{-i\omega\tau_i^{(j)}} + e^{i\omega\tau_i^{(j)}}\right)\left[\frac{a_1}{M} d_2 b_3 N k_2 T_2 + b_3 k_2 T_2 (d_1 d_3 - a_1 d_2)\right]}{d_2 b_3 N k_1 k_4 + k_2 T_2 (a_1 b_1 d_2 - a_2 d_2 b_3 N - b_1 d_1 d_3)} + \frac{\left[\frac{k_1}{T_1}\alpha\overline{\alpha}\left(e^{-i\omega\tau_i^{(j)}} + e^{i\omega\tau_i^{(j)}}\right) + k_2 k_3 \left(\overline{\delta} e^{-i\omega\tau_i^{(j)}} + \delta e^{i\omega\tau_i^{(j)}}\right)\right] d_2 k_4 b_3 N}{d_2 b_3 N k_1 k_4 + k_2 T_2 (a_1 b_1 d_2 - a_2 d_2 b_3 N - b_1 d_1 d_3)},$$

$$E_2^{(3)} = 0,$$

$$E_2^{(4)} = \frac{\left(e^{-i\omega\tau_i^{(j)}} + e^{i\omega\tau_i^{(j)}}\right)\left[b_1 d_1 k_2 T_2 \frac{a_1}{M} + b_3 d_1 (k_1 k_4 - a_2 k_2 T_2)\right]}{d_2 b_3 N k_1 k_4 + k_2 T_2 (a_1 b_1 d_2 - a_2 d_2 b_3 N - b_1 d_1 d_3)} + \frac{\left[\frac{k_1}{T_1}\alpha\overline{\alpha}\left(e^{-i\omega\tau_i^{(j)}} + e^{i\omega\tau_i^{(j)}}\right) + k_2 k_3 \left(\overline{\delta} e^{-i\omega\tau_i^{(j)}} + \delta e^{i\omega\tau_i^{(j)}}\right)\right] b_1 d_1 k_4}{d_2 b_3 N k_1 k_4 + k_2 T_2 (a_1 b_1 d_2 - a_2 d_2 b_3 N - b_1 d_1 d_3)},$$

$$E_2^{(5)} = -\frac{\left[\frac{k_1}{T_1}\alpha\overline{\alpha}\left(e^{-i\omega\tau_i^{(j)}} + e^{i\omega\tau_i^{(j)}}\right) + k_2 k_3 \left(\overline{\delta} e^{-i\omega\tau_i^{(j)}} + \delta e^{i\omega\tau_i^{(j)}}\right)\right](b_1 d_3 d_1 - d_2 a_1 b_1 + b_3 d_2 a_2 N)}{d_2 b_3 N k_1 k_4 + k_2 T_2 (a_1 b_1 d_2 - a_2 d_2 b_3 N - b_1 d_1 d_3)}$$

$$-\frac{\left(e^{-i\omega\tau_i^{(j)}} + e^{i\omega\tau_i^{(j)}}\right)\left[\frac{a_1}{M} k_1 d_2 b_3 N - b_3 k_1 (a_1 d_2 - d_1 d_3)\right]}{d_2 b_3 N k_1 k_4 + k_2 T_2 (a_1 b_1 d_2 - a_2 d_2 b_3 N - b_1 d_1 d_3)},$$

where $E_1 = \left(E_1^{(1)}, E_1^{(2)}, E_1^{(3)}, E_1^{(4)}, E_1^{(5)}\right)^T$, $E_2 = \left(E_2^{(1)}, E_2^{(2)}, E_2^{(3)}, E_2^{(4)}, E_2^{(5)}\right)^T$.

Substitute E_1 and E_2 into $W_{20}(\theta)$ and $W_{11}(\theta)$ respectively, and substitute $W_{20}(\theta)$ and $W_{11}(\theta)$ into g_{21}, then all of g_{20}, g_{11}, g_{02} and g_{21} can be obtained from the system (3-1).

Denote $c_1(0) = \dfrac{i}{2\omega\tau_0}\left(g_{20}g_{11} - 2\left|g_{11}\right|^2 - \dfrac{\left|g_{02}\right|^2}{3}\right) + \dfrac{g_{21}}{2}$, then we have the parameters below which determine the properties of Hopf bifurcation

$$\mu_{i2}^{(j)} = -\frac{\operatorname{Re}\{c_1(0)\}}{\operatorname{Re}\{\lambda'\}},\ \beta_{i2}^{(j)} = 2\operatorname{Re}\{c_1(0)\},\ \tau_{i2}^{(j)} = -\frac{\operatorname{Im}\{c_1(0)\} + \mu_2 \operatorname{Im}\left\{\lambda'\left(\tau_i^{(j)}\right)\right\}}{\omega\tau_i^{(j)}}.$$

According to positiveness or negativeness of the parameters above and the reference [21], we obtain the properties of Hopf bifurcation stated by the following theorem.

Theorem 4.1. If $\mu_{i2}^{(j)} > 0(\mu_{i2}^{(j)} < 0)$ $(i = 0, 1, 2,\ j = 0, 1, 2, \ldots, m)$ then the periodic solutions of Hopf bifurcation are supercritical (subcritical). If $\beta_{i2}^{(j)} < 0(\beta_{i2}^{(j)} > 0)$ $(i = 0, 1, 2,\ j = 0, 1, 2, \ldots, m)$, then the periodic solutions of Hopf bifurcation are asymptotically stable (unstable). If $\tau_{i2}^{(j)} > 0(\tau_{i2}^{(j)} < 0)$ $(i = 0, 1, 2,\ j = 0, 1, 2, \ldots, m)$, then the period of bifurcating period solutions increases (decreases).

4.1. Simulation

In this section, we will simulate the state of the delayed system (3-1) to illustrate the effectiveness of the Theorem 3.2 and Theorem 4.1. First of all, parameters are selected as following

$$\begin{cases} a_1 = 0.09,\ a_2 = 0.077,\ b_1 = 0.08,\ b_2 = 5,\ b_3 = 0.16,\ c_1 = 0.2,\ c_2 = 0.5,\ c_3 = 0.4, \\ d_1 = 0.01,\ d_2 = 0.06,\ d_3 = 0.08,\ M = 1.85, N = 0.6,\ k_1 = 0.68,\ k_2 = 1.4,\ k_3 = 0.49, \\ k_4 = 0.09,\ T_1 = 1.95,\ T_2 = 2.7, \end{cases}$$

and initial values have been chosen to be

$$x(0) = 0.016,\ y(0) = 0.11,\ z(0) = 0.07,\ u(0) = 0.37,\ w(0) = 0.002.$$

Then we obtain Figure 4 which shows that the system (1-1) is unstable when $\tau = 0$, and Figure 5 which shows that the system (3-1) is chaotic when $\tau = 0.2$.

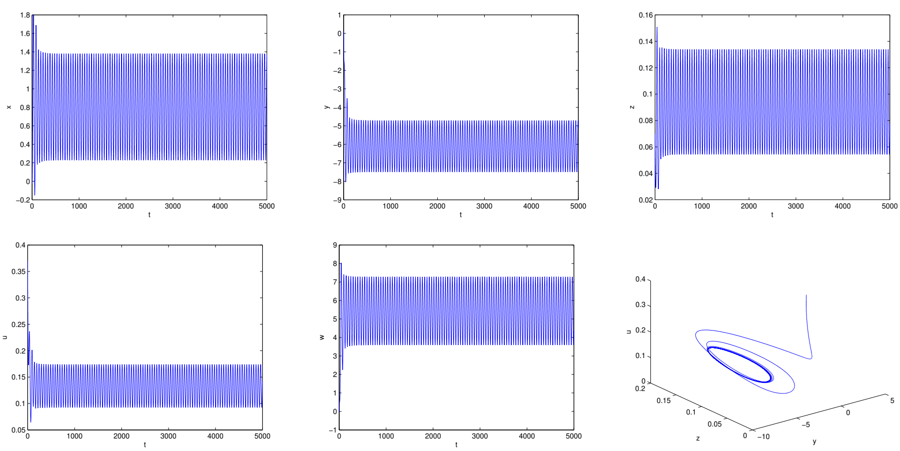

Fig 4. The equilibrium point S_0 is unstable when $\tau = 0$.

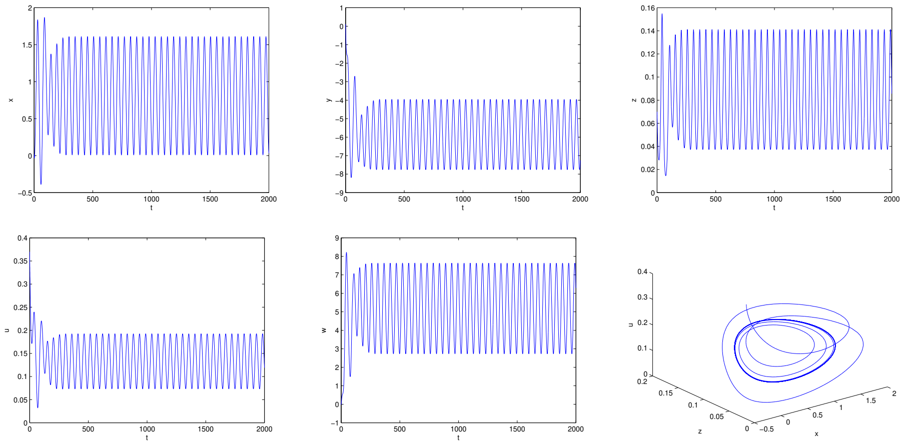

Fig 5. The equilibrium point S_0 is unstable when $\tau = 0.2$.

For the case that there exists only one root of the equation (28), we choose the parameters (18) under which the system (1-1) without time delay is stable as is shown in Figure 1. Through simple computations, we attain the following values

$$\omega = 0.1214 \quad \text{and} \quad \tau^{(n)} = 3.2974 + 25.8780n, n = 0, 1, 2, \ldots. \tag{52}$$

It is shown in the Theorem 3.2 that the equilibrium point S_0 is asymptotically stable when $\tau \in [0, 3.2974)$ and unstable when $\tau \in (3.2974, +\infty)$, moreover, Hopf bifurcation appears when $\tau = 3.2974$.

Now we set $\tau = 2.5 \in [0, 3.2974)$, then we attain Figure 6 which shows that system(3-1) is stable and the equilibrium pointS_0 is asymptotically stable.

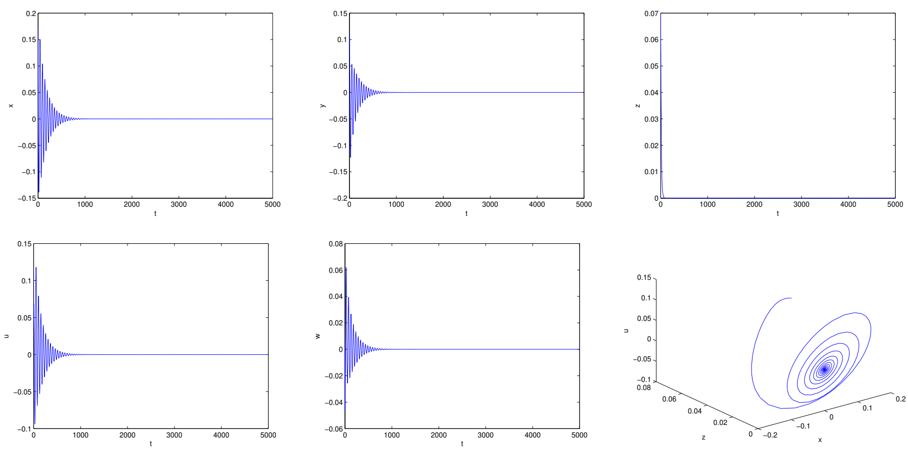

Fig 6. The equilibrium point S_0 is stable when $\tau = 2.5 \in [0, 3.2974)$.

Set $\tau = 8 \in (3.2974, +\infty)$, numerical simulation is given in Figure 7 which shows that the equilibrium point S_0 is unstable.

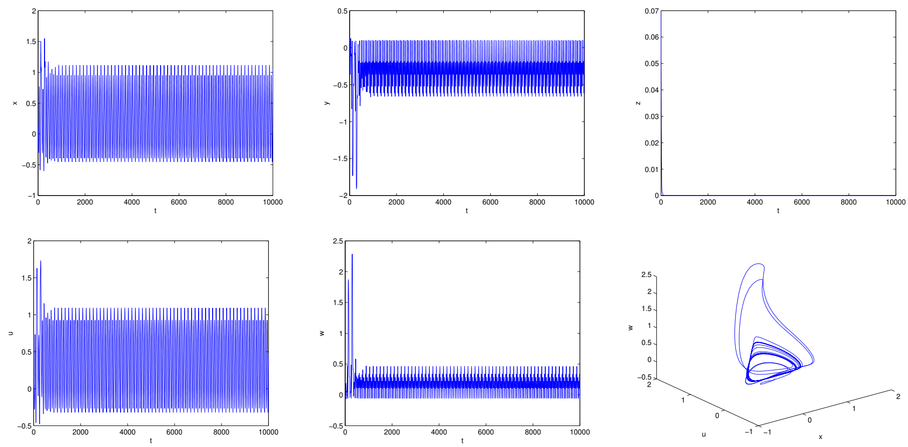

Fig 7. The equilibrium point S_0 is unstable when $\tau = 8 \in (3.2974, +\infty)$.

When $\tau = \tau_0 = 3.2974$, the parameters which determine the properties of Hopf bifurcation can be obtained as

$$\mu \approx -5361.8<0,\ \beta \approx 26.5680>0,\ \tau \approx -105.1107<0, \tag{53}$$

which together with Theorem 4.1 show that the periodic solutions of Hopf bifurcation are subcritical, unstable and decreases periodically. As we set $\tau = \tau_0 = 3.2974$, then we have Figure 8 which shows that the states of the system shock periodically near the equilibrium point S_0. Obviously Hopf bifurcation appears.

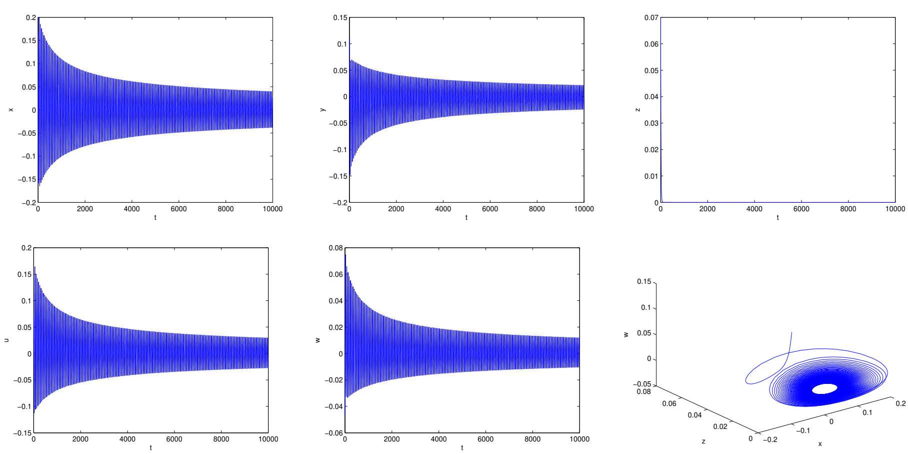

Fig 8. Hopf bifurcation appears when $\tau = 3.2974$.

5. CONCLUSION

In this paper, we consider the five-dimensional energy demand-supply system which is the high-dimensional ordinary differential equations and applied widely in reality. First of all, we analyze the stability and obtain the conditions of stability or instability for all of the equilibrium points. Numerical simulations have then been given in order to illustrate the effectiveness of all of the cases for the related theorems. Secondly, the phenomenon of time delay has been introduced in the energy model which results in the construction of the delayed system. The stability of the equilibrium points of the delayed system has been analyzed completely by the stability switching technique, which tells us the stability and instability intervals for the delay. This may be conducted in the actual process of the energy demand and supply. Especially, for the case that Hopf bifurcation appears, the explicit formulae of the parameters have been shown to illustrate the properties of the periodic solutions. Finally, the numerical simulations have been given to show the effectiveness of the main results. The related future work may be focused on the form of the controller to stabilize the applied high-dimensional ordinary differential equations.

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