SYMMETRIC IMPLICATIONAL RESTRICTION METHOD OF FUZZY INFERENCE

Yiming Tang, Wenbin Wu, Youcheng Zhang, Witold Pedrycz, Fuji Ren and Jun Liu

The symmetric implicational method is revealed from a different perspective based upon the restriction theory, which results in a novel fuzzy inference scheme called the symmetric implicational restriction method. Initially, the SIR-principles are put forward, which constitute optimized versions of the triple I restriction inference mechanism. Next, the existential requirements of basic solutions are given. The supremum (or infimum) of its basic solutions is achieved from some properties of fuzzy implications. The conditions are obtained for the supremum to become the maximum (or the infimum to be the minimum). Lastly, four concrete examples are provided, and it is shown that the new method is better than the triple I restriction method, because the former is able to let the inference more compact, and lead to more and superior particular inference schemes.

Keywords: fuzzy inference, fuzzy entropy, compositional rule of inference, continuity

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1. INTRODUCTION

Fuzzy inference [8, 11, 14] is a process of inference from an imprecise set of premises to a possible imprecise conclusion. In the human mind, the process of inference is often approximate. For example, from the rule "if a tomato is red, then it is ripe" and the premise "it is very red", one can immediately conclude that "the tomato is very ripe". The essential problems of fuzzy inference are FMP (fuzzy modus ponens) and FMT (fuzzy modus tollens) which assume the following forms:

$$\text{FMP: for a rule } A \to B \text{ and premise } A^*, \quad \text{calculate } B^*, \tag{1}$$

$$\text{FMT: for a rule } A \to B \text{ and premise } B^*, \quad \text{calculate } A^*, \tag{2}$$

where $A, A^* \in F(U)$ and $B, B^* \in F(V)$ ($F(U), F(V)$ respectively represent the sets of all fuzzy subsets of universes U , V).

Focusing on these problems, the CRI (compositional rule of inference) method presented by Zadeh has become the most widely known strategy to acquire the solutions

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for (1) and (2) (see [5, 15, 33]). To make better such approach, Wang [30] proposed the triple I method.

The solution to the triple I method was the smallest $B^* \in F(V)$ (or the largest $A^* \in F(U)$) making

$$(A(u) \to B(v)) \to (A^*(u) \to B^*(v)) \tag{3}$$

take its maximum for any $u \in U, v \in V$, in which $\to$ denoted a fuzzy implication on [0,1]. Moreover it was extended to the α-triple I method, and corresponding ideal solution was the smallest $B^* \in F(V)$ (or the largest $A^* \in F(U)$) letting

$$(A(u) \to B(v)) \to (A^*(u) \to B^*(v)) \geq \alpha \tag{4}$$

hold ($u \in U$, $v \in V$, $\alpha \in (0, 1]$). Later, for (4), starting from the opposite perspective based on the restriction theory, Song et al. put forward the triple I restriction method [19], which focused on the following formula ($\alpha \in (0, 1]$):

$$(A(u) \to B(v)) \to (A^*(u) \to B^*(v)) < \alpha, \tag{5}$$

whose optimal solution was the largest $B^* \in F(V)$ (or the smallest $A^* \in F(U)$) such that (5) holds ($u \in U$, $v \in V$). The triple I restriction method has provided essential theoretical basis to achieve index analyses of fuzzy controller [18].

Peng and Song et al. established the triple I restriction method, the restriction method and the reverse triple I method (see [18, 19, 20]). Liu and Wang [10] gave a unified form of the triple I restriction methods. Pei [23] presented the unified α-triple I method employing R-implications. Dai et al. analyzed the robustness of the triple I method and fully implicational restriction method [3]. Luo et al. presented a triple I method based upon interval-valued fuzzy sets, and revealed its robustness [13]. To sum up, the triple I method had many interesting and useful points (such as continuity, robustness, logical foundation, see [12, 22, 23] for details). But, it was revealed that it was not ideal from the angle of some fuzzy system, due to that it had weak response ability and only generated a few useful fuzzy systems (see [6, 7, 9]).

In response to this problem, we reformulated the triple I method. In fact, the middle implication in (3) reflected the "if-then" relation, while the first and third ones could be thought of as the implication connective in logic system. In view of such idea, the triple I method was developed to the symmetric implicational method in [27]. The symmetric implicational method tries to go in quest of the smallest $B^* \in F(V)$ (or the largest $A^* \in F(U)$) letting that

$$(A(u) \to_1 B(v)) \to_2 (A^*(u) \to_1 B^*(v)), \tag{6}$$

is maximized ($u \in U, v \in V$), in which $\to_1$ and $\to_2$ can be distinct.

What is more, we established more comprehensive α-symmetric implicational strategy, which came from ($\alpha \in [0, 1]$)

$$(A(u) \to_1 B(v)) \to_2 (A^*(u) \to_1 B^*(v)) \geq \alpha. \tag{7}$$

It was validated that the symmetric implicational method produced a sound inference strategy (see [27, 28]).

Inspired by the idea of the triple I restriction method, the key formula (7) of the α-symmetric implicational mechanism can be analyzed from the opposite perspective based upon the restriction theory, i. e., ($\alpha \in (0,1]$)

$$(A(u) \to_1 B(v)) \to_2 (A^*(u) \to_1 B^*(v)) < \alpha. \tag{8}$$

The fuzzy inference scheme constructed from (8) is said to be the symmetric implicational restriction method.

The objective of this paper is to systematically investigate the symmetric implicational restriction method. The originality of this work lies in three aspects. First, the symmetric implicational framework is enriched from the angle of restriction theory. Second, the SIR-principles for FMP and FMT are brought forward. Lastly, a uniform mode of the ideal solutions of the novel method is built, which is distilled from some key properties of fuzzy implications.

2. PRELIMINARIES

We briefly recall some prerequisites to make the paper self-contained.

Definition 2.1. (Klement et al. [15]) A mapping $T : [0,1]^2 \to [0,1]$ is called a t-norm if T is commutative, associative, increasing with the property $T(1,x) = x$ ($x \in [0,1]$).

Definition 2.2. (Baczyński and Jayaram [1]) A mapping $[0,1]^2 \to [0,1]$ I is said to be a fuzzy implication on $[0,1]$ when I has the following properties:

(C1) $I(0,0) = 1,\ I(1,1) = 1,\ I(1,0) = 0.$

(C2) $I(a,b)$ is decreasing for a,

(C3) $I(a,b)$ is increasing for b,

$I(a,b)$ can be denoted as $a \to b$ ($a, b \in [0,1]$).

In the light of Definition 2.2,

(C4) $I(0,a) = I(a,1) = 1$ ($a \in [0,1]$)

holds for any fuzzy implication I (noting $I(0,1) = 1$ evidently holds).

Definition 2.3. (Novák et al.[17]) Suppose that T, I are two $[0,1]^2 \to [0,1]$ mappings. If the residual condition holds:

$$T(a,b) \le c \iff b \le I(a,c) \quad (a,b,c \in [0,1]), \tag{9}$$

then (T,I) is called a residual couple.

Definition 2.4. (Monserrat et al. [16]) A fuzzy implication $I : [0,1]^2 \to [0,1]$ is called an R-implication when there is a t-norm T making the following formula hold:

$$I(a,b) = \sup\{x \in [0,1] |\ T(a,x) \le b\}, \quad a, b \in [0,1]. \tag{10}$$

Lemma 2.5. (Fodor and Roubens [4], Wang and Fu [31]) If I is an R-implication with left-continuous t-norm T, then I satisfies:

(C5) $a \leq b \iff I(a, b) = 1$,

(C6) $I(1, a) = a$,

(C7) $I(a, I(b, c)) = I(b, I(a, c))$,

(C8) $I(T(a, b), c) = I(a, I(b, c))$,

(C9) $I(\sup_{x \in X} x,\ a) = \inf_{x \in X} I(x, a)$,

(C10) $I(a, \inf_{x \in X} x) = \inf_{x \in X} I(a, x)$,

(C11) $a \leq I(b, c) \iff b \leq I(a, c)$,

in which $a, b, c, x \in [0, 1]$ and $X \subset [0, 1],\ X \neq \varnothing$.

Since (C4) implies

$$\{x \in [0, 1] \mid I(a, x) = 1\} \neq \varnothing \ (a \in [0, 1]),$$

one has Proposition 2.6 from [27].

Proposition 2.6. (Tang and Yang [27]) Suppose that I is a fuzzy implication satisfying

(C12) $I(a, b)$ is right-continuous w.r.t. b,

then the mapping T: $[0, 1]^2 \to [0, 1]$ expressed as

$$T(a, b) = \inf\{x \in [0, 1] \mid b \leq I(a, x)\}, \quad a, b \in [0, 1]$$

is residual to I, and (10) holds.

Definition 2.7. (Wang and Zhou [32]) Suppose that $F(Z)$ is the set of all fuzzy subsets on a non-empty set Z, if

$$A \leq_F B \iff A(z_0) \leq B(z_0),\ z_0 \in Z,\ A, B \in F(Z),$$

then $\leq_F$ is said to be a partial order relation on $F(Z)$.

Lemma 2.8. (Wang and Zhou [32]) $\langle F(Z),\ \leq_F \rangle$ is a complete lattice.

Example 2.9. (Klement et al. [15]) The following operations are all fuzzy implications which satisfy (C12) (where $a, b \in [0, 1]$, and x' represents $1 - x$).

$$I_{LK}(a, b) = \begin{cases} 1, & a \leq b, \\ a' + b, & a > b, \end{cases}$$

$$I_{FD}(a, b) = \begin{cases} 1, & a \leq b, \\ a' \vee b, & a > b, \end{cases} \ ([21]),$$

$$I_{GG}(a, b) = \begin{cases} 1, & a \leq b, \\ b/a, & a > b, \end{cases}$$

$$I_{GD}(a,b) = \begin{cases} 1, & a \le b, \\ b, & a > b, \end{cases}$$
$$I_{RC}(a,b) = a' + a \times b,$$
$$I_{GR}(a,b) = \begin{cases} 1, & a \le b, \\ 0, & a > b, \end{cases}$$
$$I_{KD}(a,b) = a' \vee b,$$
$$I_{YG}(a,b) = b^a \quad (I_{YG}(0,0) = 1),$$
$$I_{EP}(a,b) = \begin{cases} 1, & a \le b, \\ (2b - ab)/(a + b - ab), & a > b, \end{cases} \text{ ([29]).}$$

Furthermore, the mappings residual to I_{LK}, I_{FD}, I_{GG}, I_{GD}, I_{RC}, I_{GR}, I_{KD}, I_{YG} , I_{EP} are:

$$T_{LK}(a,b) = \begin{cases} a + b - 1, & a + b > 1, \\ 0, & a + b \le 1, \end{cases}$$
$$T_{FD}(a,b) = \begin{cases} a \wedge b, & a + b > 1, \\ 0, & a + b \le 1, \end{cases}$$
$$T_{GG}(a,b) = a \times b,$$
$$T_{GD}(a,b) = a \wedge b,$$
$$T_{RC}(a,b) = \begin{cases} (a + b - 1)/a, & a + b > 1, \\ 0, & a + b \le 1, \end{cases}$$
$$T_{GR}(a,b) = \begin{cases} a, & b > 0, \\ 0, & b = 0, \end{cases}$$
$$T_{KD}(a,b) = \begin{cases} b, & a + b > 1, \\ 0, & a + b \le 1, \end{cases}$$
$$T_{YG}(a,b) = \begin{cases} \sqrt[a]{b}, & a > 0, \\ 0, & a = 0, \end{cases}$$
$$T_{EP}(a,b) = ab/[2 - (a + b - ab)].$$

3. THE SYMMETRIC IMPLICATIONAL RESTRICTION METHOD OF FMP

Turning in the direction of the FMP problem (1), from the angle of the symmetric implicational idea and restriction mode, the following principle is provided:

SIR-Inference Principle for FMP: The conclusion B^* (in $\langle F(V), \le_F \rangle$) of FMP (1) is the largest fuzzy set which makes (8) hold.

Such principle improves the triple I restriction principle for FMP in [10, 19], due to the fact that the symmetric implicational mechanism is superior over the triple I method.

Definition 3.1. Let $A, A^* \in F(U)$, $B \in F(V)$, if B^* (in $\langle F(V), \le_F \rangle$) makes (8) hold for any $u \in U, v \in V$, then B^* is called a symmetric restriction solution of FMP.

Definition 3.2. Suppose that $A, A^* \in F(U)$, $B \in F(V)$, and that the non-empty set $\mathbb{G}$ is the set of all symmetric restriction solutions of FMP, and finally that D^* (in $\langle F(V), \leq_F\rangle$) is the supremum of $\mathbb{G}$. Then D^* is called a SIR-SupP-solution. And, if D^* is the maximum of $\mathbb{G}$, then D^* is also called a SIR-MaxP-solution.

Theorem 3.3. Let $A, A^* \in F(U)$, $B \in F(V)$, $\alpha \in (0, 1]$, and $\to_1$, $\to_2$ are two fuzzy implications. Then there exists $B^* \in F(V)$ as a symmetric restriction solution of FMP iff

$$(A(u) \to_1 B(v)) \to_2 (A^*(u) \to_1 0) < \alpha \tag{11}$$

holds for any $u \in U, v \in V$.

P r o o f. On the one hand, if (11) holds, then we take

$$B^* \equiv 0,$$

obviously B^* satisfies (8) and hence B^* is a symmetric restriction solution of FMP.

On the other hand, if there exists $B^* \in F(V)$ which is a symmetric restriction solution of FMP, then B^* satisfies (8), and since $\to_1$ satisfies (C3) we have that

$$A^*(u) \to_1 B^*(v) \geq A^*(u) \to_1 0,$$

and

$$\begin{aligned} \alpha &> (A(u) \to_1 B(v)) \to_2 (A^*(u) \to_1 B^*(v)) \\ &\geq (A(u) \to_1 B(v)) \to_2 (A^*(u) \to_1 0), \end{aligned}$$

i.e. (11) holds. □

In a similar way Proposition 3.4 can be obtained.

Proposition 3.4. If C_1 is a symmetric restriction solution of FMP, and $C_2 \leq_F C_1$(in which $C_1, C_2 \in \langle F(V), \leq_F\rangle$). Then C_2 is a symmetric restriction solution of FMP.

Remark 3.5. Suppose that (11) holds and C_1^* is any symmetric restriction solution of FMP. By virtue of Proposition 3.4, each fuzzy set C_2^* satisfying

$$C_2^* \leq_F C_1^*,$$

is a symmetric restriction solution of FMP (where $C_1^*, C_2^* \in \langle F(V), \leq_F\rangle$). As a result, it can be found that there are lots of symmetric restriction solutions of FMP, which incorporate

$$C_3^*(v) \equiv 0 \ (v \in V).$$

C_3^* is a special case, due to that (8) holds from start to finish regardless of what $A \to_1 B$ and A^* are employed. Consequently, when the ideal symmetric restriction solution of FMP exists, it is better to be the supremum (or the largest one).

Theorem 3.6. If $\to_1, \to_2$ are fuzzy implications satisfying (C12), and T_1, T_2 are the functions residual to $\to_1, \to_2$, and (11) holds. Then the SIR-SupP-solution is as follow:

$$B^*(v) = \inf_{u\in U}\{T_1(A^*(u), T_2(A(u) \to_1 B(v), \alpha))\}, \quad v \in V. \tag{12}$$

Proof. Let

$$G_1 = \{v \in V \mid B^*(v) = 0\}, G_2 = \{v \in V \mid B^*(v) > 0\}.$$

Suppose that $C \in F(V)$, and that $C(v) = 0$ for $v \in G_1$, $C(v) < B^*(v)$ for $v \in G_2$. We verify that C is a symmetric restriction solution of FMP, that is,

$$(A(u) \to_1 B(v)) \to_2 (A^*(u) \to_1 C(v)) < \alpha \tag{13}$$

holds for any $u \in U, v \in V$.

If $v \in G_1$, then we get from (11) that $C(v) = 0$ satisfies (13) for any $u \in U$.

If $v \in G_2$, then we have from (12) and $C(v) < B^*(v)$ that

$$C(v) < T_1(A^*(u), T_2(A(u) \to_1 B(v), \alpha)) \tag{14}$$

holds for any $u \in U$. Here the proof is carried out by contradiction. Assume that (13) does not hold. Then there exist $u_0 \in U$, $v_0 \in V$ making

$$(A(u_0) \to_1 B(v_0)) \to_2 (A^*(u_0) \to_1 C(v_0)) \geq \alpha$$

hold (evidently $v_0 \in G_2$). Hence one has from the residual condition (9) that

$$T_2(A(u_0) \to_1 B(v_0), \alpha) \leq A^*(u_0) \to_1 C(v_0),$$

and

$$T_1(A^*(u_0), T_2(A(u_0) \to_1 B(v_0), \alpha)) \leq C(v_0),$$

which contradicts (14). Thus (13) holds for any $u \in U, v \in V$. Therefore, C is a symmetric restriction solution of FMP.

Moreover, we show that B^* determined by (12) is the supremum of all symmetric restriction solutions of FMP. Suppose that

$$D(v_0) > B^*(v_0).$$

We verify that D is not a symmetric restriction solution of FMP. Actually, one has from (12) that there exists $u_0 \in U$ letting

$$D(v_0) > T_1(A^*(u_0), T_2(A(u_0) \to_1 B(v_0), \alpha))$$

hold. It follows from (9) that

$$T_2(A(u_0) \to_1 B(v_0), \alpha) \leq A^*(u_0) \to_1 D(v_0),$$

and

$$\alpha \leq (A(u_0) \to_1 B(v_0)) \to_2 (A^*(u_0) \to_1 D(v_0)).$$

So D is not a symmetric restriction solution of FMP.

Summarizing above, B^* expressed by (12) is the SIR-SupP-solution. □

Proposition 3.7. Suppose that $\to_2$ is a fuzzy implication which satisfies (C12), and that T_2 is the mapping residual to $\to_2$, and that (11) holds, then the SIR-SupP-solution is as follows ($v \in V$):

(i) If $\to_1$ takes I_{LK}, then the SIR-SupP-solution is

$$B^*(v) = \inf_{u \in U}\{A^*(u) + T_2(A(u) \to_{LK} B(v), \alpha) - 1\}.$$

(ii) If $\to_1$ takes I_{GD}, then the SIR-SupP-solution is

$$B^*(v) = \inf_{u \in U}\{A^*(u) \wedge T_2(A(u) \to_{GD} B(v), \alpha)\}.$$

(iii) If $\to_1$ takes I_{GG}, then the SIR-SupP-solution is

$$B^*(v) = \inf_{u \in U}\{A^*(u) \times T_2(A(u) \to_{GG} B(v), \alpha)\}.$$

(iv) If $\to_1$ takes I_{FD}, then the SIR-SupP-solution is

$$B^*(v) = \inf_{u \in U}\{A^*(u) \wedge T_2(A(u) \to_{FD} B(v), \alpha)\}.$$

(v) If $\to_1$ takes I_{RC}, then the SIR-SupP-solution is

$$B^*(v) = \inf_{u \in U}\{[A^*(u) + T_2(A(u) \to_{RC} B(v), \alpha) - 1]/A^*(u)\}.$$

(vi) If $\to_1$ takes I_{GR}, then the SIR-SupP-solution is

$$B^*(v) = \inf_{u \in U}\{A^*(u)\}.$$

(vii) If $\to_1$ takes I_{KD}, then the SIR-SupP-solution is

$$B^*(v) = \inf_{u \in U}\{T_2(A(u) \to_{KD} B(v), \alpha)\}.$$

(viii) If $\to_1$ takes I_{YG}, then the SIR-SupP-solution is

$$B^*(v) = \inf_{u \in U}\{T_2(A(u) \to_{YG} B(v), \alpha)^{\frac{1}{A^*(u)}}\}.$$

(ix) If $\to_1$ takes I_{EP}, then the SIR-SupP-solution is $B^*(v) = \inf_{u \in U}\{(A^*(u) \times T_2(A(u) \to_{EP} B(v), \alpha))/(2 - (A^*(u) + T_2(A(u) \to_{EP} B(v), \alpha) - A^*(u) \times T_2(A(u) \to_{EP} B(v), \alpha)))\}$.

P r o o f. If $\rightarrow_1 \in \{I_{LK}, I_{GD}, I_{GG}, I_{FD}, I_{RC}, I_{GR}, I_{KD}, I_{YG}, I_{EP}\}$ and $\rightarrow_2$ is a fuzzy implication which satisfies (C12), then it follows from Theorem 3.6, we have that the SIR-SupP-solution is

$$B^*(v) = \inf_{u \in U} \{T_1(A^*(u), T_2(A(u) \rightarrow_1 B(v), \alpha))\}, \quad v \in V,$$

where T_1, T_2 are the functions residual to $\rightarrow_1, \rightarrow_2$.

Then we need to provide the specific expression of B^*. We show the case of I_{LK} as an example, the other cases can be verified in the same manner.

Let $\rightarrow_1$ take I_{LK}. Note that T_{LK} is the mapping residual to I_{LK}.

It follows from (11) that we have

$$(A(u) \rightarrow_1 B(v)) \rightarrow_2 (A^*(u) \rightarrow_1 0) < \alpha,$$

which implies that

$$A^*(u) > 0$$

and

$$(A(u) \rightarrow_{LK} B(v)) \rightarrow_2 (1 - A^*(u)) < \alpha \; (u \in U, v \in V).$$

Following that,

$$\alpha \leq (A(u) \rightarrow_{LK} B(v)) \rightarrow_2 (1 - A^*(u))$$

does not hold. Then we have from the residual condition (9) that

$$T_2(A(u) \rightarrow_{LK} B(v), \alpha) \leq 1 - A^*(u)$$

does not hold. So one has that

$$T_2(A(u) \rightarrow_{LK} B(v), \alpha) > 1 - A^*(u).$$

Then

$$A^*(u) + T_2(A(u) \rightarrow_{LK} B(v), \alpha) > 1.$$

Finally we obtain

$$\begin{aligned} B^*(v) &= \inf_{u \in U} \{T_{LK}(A^*(u), T_2(A(u) \rightarrow_{LK} B(v), \alpha))\} \\ &= \inf_{u \in U} \{A^*(u) + T_2(A(u) \rightarrow_{LK} B(v), \alpha) - 1\}, \quad v \in V. \end{aligned}$$

□

Theorem 3.8. Under the same condition as Theorem 3.6, the condition which the SIR-SupP-solution B^* becomes the SIR-MaxP-solution, is shown in Table 1 (for any $u \in U$,$v \in V$), where $\Psi(u, v) = \frac{A^*(u) \times T_2(A(u) \rightarrow_{EP} B(v), \alpha)}{2 - (A^*(u) + T_2(A(u) \rightarrow_{EP} B(v), \alpha) - A^*(u) \times T_2(A(u) \rightarrow_{EP} B(v), \alpha))}$.

P r o o f. If the SIR-SupP-solution B^* is a symmetric restriction solution of FMP, then it is evident that B^* is the SIR-MaxP-solution (which B^* lets (8) hold for any $u \in U, v \in V$). The situations of I_{LK}, I_{EP} are proved through examples.

$\to_1$	The SIR-SupP-solution B^*	The condition which B^* is the SIR-MaxP-solution
I_{LK}	$\inf_{u\in U}\{A^*(u)+T_2(A(u)\to_{LK} B(v),\alpha)-1\}$	$A^*(u)+T_2(A(u)\to_{LK} B(v),\alpha) >$ $\inf_{u\in U}\{A^*(u)+T_2(A(u)\to_{LK} B(v),\alpha)\}$
I_{GD}	$\inf_{u\in U}\{A^*(u)\wedge T_2(A(u)\to_{GD} B(v),\alpha)\}$	$A^*(u)\wedge T_2(A(u)\to_{GD} B(v),\alpha) > B^*(v)$
I_{GG}	$\inf_{u\in U}\{A^*(u)\times T_2(A(u)\to_{GG} B(v),\alpha)\}$	$A^*(u)\times T_2(A(u)\to_{GG} B(v),\alpha) > B^*(v)$
I_{FD}	$\inf_{u\in U}\{A^*(u)\wedge T_2(A(u)\to_{FD} B(v),\alpha)\}$	$A^*(u)\wedge T_2(A(u)\to_{FD} B(v),\alpha) > B^*(v)$
I_{RC}	$\inf_{u\in U}\{[A^*(u)+T_2(A(u)\to_{RC} B(v),\alpha)-1]/A^*(u)\}$	$(A^*(u)+T_2(A(u)\to_{RC} B(v),\alpha)-1)/A^*(u) > B^*(v)$
I_{GR}	$\inf_{u\in U}\{A^*(u)\}$	$A^*(u) > B^*(v)$
I_{KD}	$\inf_{u\in U}\{T_2(A(u)\to_{KD} B(v),\alpha)\}$	$T_2(A(u)\to_{KD} B(v),\alpha) > B^*(v)$
I_{YG}	$\inf_{u\in U}\{T_2(A(u)\to_{YG} B(v),\alpha)^{\frac{1}{A^*(u)}}\}$	$T_2(A(u)\to_{YG} B(v),\alpha)^{\frac{1}{A^*(u)}} > B^*(v)$
I_{EP}	$\inf_{u\in U}\{\Psi(u,v)\}$	$\Psi(u,v) > B^*(v)$

Tab. 1. Some Conclusions of the Symmetric Implicational Restriction Inference Algorithm of FMP.

(i) Let $\to_1$ employ I_{LK}. We get from Theorem 3.6 that the SIR-MaxP-solution is

$$B^*(v) = \inf_{u\in U}\{A^*(u)+T_2(A(u)\to_{LK} B(v),\alpha)-1\}.$$

In the light of the condition given in (i), we have $(u\in U, v\in V)$

$$\begin{aligned} B^*(v) &= \inf_{u\in U}\{A^*(u)+T_2(A(u)\to_{LK} B(v),\alpha)-1\} \\ &= \inf_{u\in U}\{A^*(u)+T_2(A(u)\to_{LK} B(v),\alpha)\}-1 \\ &< A^*(u)+T_2(A(u)\to_{LK} B(v),\alpha)-1. \end{aligned}$$

Note that

$$T_2(A(u)\to_{LK} B(v),\alpha)-1\le 0,$$

then

$$A^*(u)\ge A^*(u)+T_2(A(u)\to_{LK} B(v),\alpha)-1 > B^*(v)$$

holds. So

$$T_2(A(u)\to_{LK} B(v),\alpha)\le 1-A^*(u)+B^*(v)$$

does not hold, and thus we have from (9) that

$$\alpha\le (A(u)\to_{LK} B(v))\to_2 (1-A^*(u)+B^*(v))$$

does not hold, then

$$\alpha > (A(u)\to_{LK} B(v))\to_2 (1-A^*(u)+B^*(v)).$$

Consequently, one has $(u\in U, v\in V)$

$$\begin{aligned} &(A(u)\to_1 B(v))\to_2 (A^*(u)\to_1 B^*(v)) \\ &= (A(u)\to_1 B(v))\to_2 (1-A^*(u)+B^*(v)) \\ &< \alpha. \end{aligned}$$

So B^* lets (8) hold for any $u \in U$,$v \in V$, and thus it is a symmetric restriction solution of FMP. As a result, the SIR-SupP-solution B^* becomes the SIR-MaxP-solution.

(ii) Let $\rightarrow_1$ take I_{EP}. We get the SIR-SupP-solution B^* from Theorem 3.6. According to the condition provided in (ix), we have $(u \in U, v \in V)$

$$T_2(A(u) \rightarrow_{EP} B(v), \alpha) > \frac{2B^*(v) - A^*(u) \times B^*(v)}{A^*(u) + B^*(v) - A^*(u) \times B^*(v)}.$$

Thus

$$T_2(A(u) \rightarrow_{EP} B(v), \alpha) \leq \frac{2B^*(v) - A^*(u) \times B^*(v)}{A^*(u) + B^*(v) - A^*(u) \times B^*(v)}$$

does not hold, so

$$\alpha \leq (A(u) \rightarrow_{EP} B(v)) \rightarrow_2 \left(\frac{2B^*(v) - A^*(u) \times B^*(v)}{A^*(u) + B^*(v) - A^*(u) \times B^*(v)} \right)$$

does not hold, one has

$$\alpha > (A(u) \rightarrow_{EP} B(v)) \rightarrow_2 \left(\frac{2B^*(v) - A^*(u) \times B^*(v)}{A^*(u) + B^*(v) - A^*(u) \times B^*(v)} \right).$$

From the condition provided in (ix), it is easy to find

$$0 < \frac{T_2(A(u) \rightarrow_{EP} B(v), \alpha)}{2 - (A^*(u) + T_2(A(u) \rightarrow_{EP} B(v), \alpha) - A^*(u) \times T_2(A(u) \rightarrow_{EP} B(v), \alpha))} < 1,$$

it follows that $A^*(u) > B^*(v)$.

Summarizing above, we obtain $(u \in U, v \in V)$

$$\begin{aligned} &(A(u) \rightarrow_1 B(v)) \rightarrow_2 (A^*(u) \rightarrow_1 B^*(v)) \\ &= (A(u) \rightarrow_1 B(v)) \rightarrow_2 \left(\frac{2B^*(v) - A^*(u) \times B^*(v)}{A^*(u) + B^*(v) - A^*(u) \times B^*(v)} \right) \\ &< \alpha. \end{aligned}$$

So B^* makes (8) hold for any $u \in U$,$v \in V$, and thus the SIR-SupP-solution B^* becomes the SIR-MaxP-solution. □

Example 3.9. Assume that (11) holds.

(i) If $\rightarrow_1$ takes I_{GD}, and $\rightarrow_2$ takes I_{GG}, then the SIR-SupP-solution is

$$B^*(v) = \inf_{u \in U} \left\{ A^*(u) \wedge ((A(u) \rightarrow_{GD} B(v)) \times \alpha) \right\}.$$

(ii) If $\rightarrow_1$ takes I_{LK}, and $\rightarrow_2$ takes I_{GG}, then the SIR-SupP-solution is

$$B^*(v) = \inf_{u \in U} \left\{ A^*(u) + ((A(u) \rightarrow_{LK} B(v)) \times \alpha) - 1 \right\}.$$

P r o o f. (i) Suppose that $\rightarrow_1$ takes I_{GD}, and $\rightarrow_2$ takes I_{GG}. Note that T_{GD} is the function residual to I_{GD}, and T_{GG} is the function residual to I_{GG}. Thus the SIR-SupP-solution is

$$\begin{aligned} B^*(v) &= \inf_{u\in U}\{T_{GD}(A^*(u), T_{GG}(A(u)\rightarrow_1 B(v),\alpha))\} \\ &= \inf_{u\in U}\{T_{GD}(A^*(u),(A(u)\rightarrow_1 B(v))\times\alpha))\} \\ &= \inf_{u\in U}\{A^*(u)\wedge((A(u)\rightarrow_{GD} B(v))\times\alpha)\}. \end{aligned}$$

(ii) Suppose that $\rightarrow_1$ takes I_{LK}, and that $\rightarrow_2$ takes I_{GG}. Note that T_{LK} is the function residual to I_{LK}, and that T_{GG} is the function residual to I_{GG}. By (11), one has that

$$A^*(u)>0,\ A^*(u)\rightarrow_1 0 = 1-A^*(u),$$
$$1-A^*(u)<\alpha\times(A(u)\rightarrow_1 B(v)).$$

We further obtain that $A^*(u)+(A(u)\rightarrow_1 B(v))\times\alpha>1$ holds for any $u\in U, v\in V$. So the SIR-SupP-solution is

$$\begin{aligned} B^*(v) &= \inf_{u\in U}\{T_{LK}(A^*(u), T_{GG}(A(u)\rightarrow_1 B(v),\alpha))\} \\ &= \inf_{u\in U}\{T_{LK}(A^*(u),(A(u)\rightarrow_1 B(v))\times\alpha)\} \\ &= \inf_{u\in U}\{A^*(u)+((A(u)\rightarrow_{LK} B(v))\times\alpha)-1\}. \end{aligned}$$

□

Example 3.10. Let $U=V=[0,1]$, $A(u)=(u+3)/4$, $B(v)=(1+3v)/4$, $A^*(u)=(2-u)/2$, $\alpha=1/2$, in which $u\in U, v\in V$. Suppose that $\rightarrow_1 = I_{GD}$, $\rightarrow_2 = I_{GG}$ in the symmetric restriction method of FMP. Here we show the computing process of the SIR-SupP-solution.

To begin with, we have

$$A(u)\rightarrow_1 B(v) = \begin{cases} 1 & \frac{u+3}{4}\le\frac{1+3v}{4} \\ \frac{1+3v}{4} & \frac{u+3}{4}>\frac{1+3v}{4} \end{cases} = \begin{cases} 1 & u\le 3v-2 \\ \frac{1+3v}{4} & u>3v-2 \end{cases}.$$

It is easy to verify that (11) holds.

Furthermore, we get from Example 3.9 that SIR-SupP-solution is as follows ($v\in V$):

$$\begin{aligned} B^*(v) &= \inf_{u\in U}\{A^*(u)\wedge((A(u)\rightarrow_{GD} B(v))\times\alpha)\} \\ &= \inf_{u\in[0,1]}\left\{\frac{2-u}{2}\wedge\frac{1}{2}\,\middle|\, u\le 3v-2\right\}\wedge \\ &\quad \inf_{u\in[0,1]}\left\{\frac{2-u}{2}\wedge\left(\frac{1+3v}{4}\times\frac{1}{2}\right)\,\middle|\, u>3v-2\right\}. \end{aligned}$$

It can be divided into three cases.

(i) Suppose $v = 1$, then

$$1 \in \{u \in [0,1] | u \leq 3v - 2\},$$

$$\{u \in [0,1] | u > 3v - 2\} = \varnothing.$$

Since $\frac{2-u}{2}$ is decreasing w.r.t. u, we have

$$B^*(v) = \frac{1}{2} \wedge (inf \varnothing) = \frac{1}{2} \wedge 1 = \frac{1}{2}.$$

(ii) Suppose $1 > v \geq \frac{2}{3}$, then

$$1 \in \{u \in [0,1] | u > 3v - 2\}.$$

Since $\frac{2-u}{2}$ is decreasing w.r.t. u, we have

$$\begin{aligned} B^*(v) &= \left(\frac{2-(3v-2)}{2} \wedge \frac{1}{2}\right) \wedge \left(\frac{2-1}{2} \wedge \frac{1+3v}{8}\right) \\ &= \frac{1}{2} \wedge \left(\frac{1}{2} \wedge \frac{1+3v}{8}\right) = \frac{1+3v}{8}. \end{aligned}$$

(iii) Suppose $\frac{2}{3} > v \geq 0$, then

$$\{u \in [0,1] | u \leq 3v - 2\} = \varnothing,$$

$$1 \in \{u \in [0,1] | u > 3v - 2\}.$$

Thus we have

$$\begin{aligned} B^*(v) &= (inf \varnothing) \wedge \left(\frac{2-1}{2} \wedge \frac{1+3v}{8}\right) \\ &= 1 \wedge \left(\frac{1}{2} \wedge \frac{1+3v}{8}\right) = \frac{1+3v}{8}. \end{aligned}$$

To sum up, we have that the SIR-SupP-solution is as follows:

$$B^*(v) = \frac{1+3v}{8}.$$

Example 3.11. Let U, V, A, B, A^*, α take the same values in Example 3.10. Assume that $\rightarrow_1 = I_{GG}$, $\rightarrow_2 = I_{GG}$ in the symmetric restriction method of FMP, which degenerates into the triple I restriction method for FMP using I_{GG}.

By doing a similar calculation, we obtain that the SIR-SupP-solution can be obtained as follows:

$$B^*(v) = \frac{1+3v}{16}.$$

Remark 3.12. On the one hand, aiming at the same U, V, A, B, A^*, α in Example 3.10 and Example 3.11, since

$$0 \leq v \leq 1 \Rightarrow \frac{1+3v}{8} > \frac{1+3v}{16},$$

the SIR-SupP-solution (derived from the symmetric implicational restriction method) in Example 3.10 is bigger than the optimal solution from the triple I restriction method in Example 3.11. According to the idea of the SIR-principle for FMP (which finds the largest one such that (8) holds), the symmetric implicational restriction method for FMP in Example 3.10 can make the inference more compact, then it is better than the triple I restriction method for FMP in Example 3.11. On the other hand, focusing on the scope of $\{I_{LK}, I_{GD}, I_{GG}, I_{FD}, I_{RC}, I_{GR}, I_{KD}, I_{YG}, I_{EP}\}$, from the symmetric implicational restriction method we can get

$$9 * 9 = 81$$

kinds of particular fuzzy inference strategies for FMP, in which $\rightarrow_1, \rightarrow_2$ respectively take such 9 implications. However, from the triple I restriction inference method, it is obvious that there are only 9 kinds of particular strategies for FMP. To sum up, the symmetric implicational restriction method can give more and better specific strategies than the triple I restriction method, and then it is superior over the latter.

4. THE SYMMETRIC IMPLICATIONAL RESTRICTION METHOD OF FMT

Facing up to the FMT problem (2), from the viewpoint of the symmetric implicational idea and restriction mode, we can provide the following principle:

SIR-Inference Principle for FMT: The conclusion A^* (in $\langle F(U), \leq_F \rangle$) of FMT (2) is the smallest fuzzy set which makes (8) hold.

Such principle improves the triple I restriction inference principle for FMT in [10, 19], since the symmetric implicational mechanism is better than the triple I method.

Definition 4.1. Let $A \in F(U)$, $B, B^* \in F(V)$, if A^* (in $\langle F(U), \leq_F \rangle$) makes (8) hold for any $u \in U, v \in V$, then A^* is called a symmetric restriction solution of FMT.

Definition 4.2. Suppose that $A \in F(U)$, $B, B^* \in F(V)$, and that the non-empty set $\mathbb{F}$ is the set of all symmetric restriction solutions of FMT, and finally that C^* (in $\langle F(U), \leq_F \rangle$) is the infimum of $\mathbb{F}$. Then C^* is called a SIR-InfT-solution. And, if C^* is the minimum of $\mathbb{F}$, then C^* is also called a SIR-MinT-solution.

Theorem 4.3. Suppose that $\rightarrow_1, \rightarrow_2$ are two fuzzy implications, $A \in F(U)$, $B, B^* \in F(V)$, $\alpha \in (0, 1]$. Then there exists an $A^* \in F(U)$ as a symmetric restriction solution of FMT iff the following inequality holds for any $u \in U, v \in V$:

$$(A(u) \rightarrow_1 B(v)) \rightarrow_2 (1 \rightarrow_1 B^*(v)) < \alpha. \tag{15}$$

P r o o f. If (15) holds, then we take

$$A^* \equiv 1,$$

obviously A^* satisfies (8) and hence A^* is a symmetric restriction solution of FMT.

Moreover, if there exists $A^* \in F(U)$ which is a symmetric restriction solution of FMT, then A^* satisfies (8). Because $\rightarrow_1, \rightarrow_2$ satisfy (C2) and (C3), one has that

$$A^*(u) \rightarrow_1 B^*(v) \geq 1 \rightarrow_1 B^*(v),$$

and

$$\begin{aligned}\alpha &> (A(u) \rightarrow_1 B(v)) \rightarrow_2 (A^*(u) \rightarrow_1 B^*(v)) \\ &\geq (A(u) \rightarrow_1 B(v)) \rightarrow_2 (1 \rightarrow_1 B^*(v)),\end{aligned}$$

i. e. (15) holds. □

Similar to Theorem 4.3, we can get Proposition 4.4.

Proposition 4.4. Suppose that C_1 is a symmetric restriction solution of FMT, and that $C_1 \leq_F C_2$ (in which $C_1, C_2 \in \langle F(U), \leq_F \rangle$).Then C_2 is a symmetric restriction solution of FMT.

Remark 4.5. Suppose that (15) holds and C_1^* is any symmetric restriction solution of FMT. In the light of Proposition 4.4, each fuzzy set C_2^* which is larger than C_1^*, is a symmetric restriction solution of FMT (in which $C_1^*, C_2^* \in \langle F(U), \leq_F \rangle$)). Consequently, there are lots of symmetric restriction solutions of FMT, which incorporate

$$C_3^*(u) \equiv 1 \ (u \in U).$$

C_3^* is a particular solution, since (8) holds from start to finish regardless of what $A \rightarrow_1 B$ and B^* are employed. Therefore, if the optimal symmetric restriction solution of FMT exists, then it should be the infimum (or the smallest one).

Theorem 4.6. If the fuzzy implication $\rightarrow_1$ satisfies (C11), and the fuzzy implication $\rightarrow_2$ satisfies (C12), and T is the function residual to $\rightarrow_2$, and (15) holds. Then the SIR-InfT-solution can be expressed as follows:

$$A^*(u) = \sup_{v \in V}\{T(A(u) \rightarrow_1 B(v),\ \alpha) \rightarrow_1 B^*(v)\}, \quad u \in U. \tag{16}$$

P r o o f. Because the fuzzy implication $\rightarrow_2$ satisfies (C12), the residual condition (9) holds for $\rightarrow_2$. Let

$$\begin{aligned}H_1 &= \{u \in U | A^*(u) = 1\}, \\ H_2 &= \{u \in U | A^*(u) < 1\}.\end{aligned}$$

Assume that $C \in F(U)$, and that $C(u) = 1$ for $u \in H_1$, $C(u) > A^*(u)$ for $u \in H_2$. We shall verify that C is a symmetric restriction solution of FMT, that is, the following formula holds for any $u \in U, v \in V$:

$$(A(u) \rightarrow_1 B(v)) \rightarrow_2 (C(u) \rightarrow_1 B^*(v)) < \alpha. \tag{17}$$

If $u \in H_1$, then we have from (15) that $C(u) = 1$ satisfies (17) for any $v \in V$.

If $u \in H_2$, then one has from (16) and $C(u) > A^*(u)$ that

$$C(u) > T(A(u) \to_1 B(v), \alpha) \to_1 B^*(v) \tag{18}$$

holds for any $v \in V$. We use the proof by contradiction. Assume that (17) does not hold. Then there exist $u_0 \in U$ and $v_0 \in V$ making

$$(A(u_0) \to_1 B(v_0)) \to_2 (C(u_0) \to_1 B^*(v_0)) \geq \alpha$$

hold (evidently $u_0 \in H_2$). So we have from (9) that

$$T(A(u_0) \to_1 B(v_0), \alpha) \leq C(u_0) \to_1 B^*(v_0),$$

and since (C11) holds for $\to_1$ we get

$$C(u_0) \leq T(A(u_0) \to_1 B(v_0), \alpha) \to_1 B^*(v_0),$$

which contradicts (18). So (15) holds for any $u \in U, v \in V$. Consequently, C is a symmetric restriction solution of FMT.

Furthermore, we prove that A^* expressed by (16) is the infimum of symmetric restriction solution of FMT. Assume that

$$D(u_0) < A^*(u_0).$$

We prove that D is not a symmetric restriction solution of FMT. In fact, we have from (16) that there exists $v_0 \in V$ such that

$$D(u_0) < T(A(u_0) \to_1 B(v_0)\,, \alpha) \to_1 B^*(v_0)$$

holds. Due to that (C11) holds for $\to_1$, it follows that

$$T(A(u_0) \to_1 B(v_0), \alpha) \leq D(u_0) \to_1 B^*(v_0),$$

and the residual condition (9) holds for $\to_2$, so we have

$$\alpha \leq (A(u_0) \to_1 B(v_0)) \to_2 (D(u_0) \to_1 B^*(v_0))$$

Thus, D is not a symmetric restriction solution of FMT.

As a result, A^* expressed by (16) is the infimum of symmetric restriction solution of FMT, i. e., the SIR-InfT-solution. □

Proposition 4.7. If $\to_2 \in \{I_{LK}, I_{GD}, I_{GG}, I_{FD}, I_{RC}, I_{GR}, I_{KD}, I_{YG}, I_{EP}\}$, and the fuzzy implication $\to_1$ satisfies (C11) and (15) holds, then the SIR-InfT-solution is as follows ($u \in U$):

(i) If $\to_2$ takes I_{LK}, then

$$A^*(u) = \sup_{v \in V}\{((A(u) \to_1 B(v)) + \alpha - 1) \to_1 B^*(v)\}.$$

(ii) If $\rightarrow_2$ takes I_{GD}, then

$$A^*(u) = \sup_{v\in V}\{((A(u) \rightarrow_1 B(v)) \wedge \alpha) \rightarrow_1 B^*(v)\}.$$

(iii) If $\rightarrow_2$ takes I_{GG}, then

$$A^*(u) = \sup_{v\in V}\{((A(u) \rightarrow_1 B(v)) \times \alpha) \rightarrow_1 B^*(v)\}.$$

(iv) If $\rightarrow_2$ takes I_{FD}, then

$$A^*(u) = \sup_{v\in V}\{((A(u) \rightarrow_1 B(v)) \wedge \alpha) \rightarrow_1 B^*(v)\}..$$

(v) If $\rightarrow_2$ takes I_{RC}, then $A^*(u) = \sup_{v\in V}\{[((A(u) \rightarrow_1 B(v)) + \alpha - 1)/(A(u) \rightarrow_1 B(v))] \rightarrow_1 B^*(v)\}$.

(vi) If $\rightarrow_2$ takes I_{GR}, then

$$A^*(u) = \sup_{v\in V}\{(A(u) \rightarrow_1 B(v)) \rightarrow_1 B^*(v)\}.$$

(vii) If $\rightarrow_2$ takes I_{KD}, then

$$A^*(u) = \sup_{v\in V}\{\alpha \rightarrow_1 B^*(v)\}.$$

(viii) If $\rightarrow_2$ takes I_{YG}, then

$$A^*(u) = \sup_{v\in V}\{\alpha^{\frac{1}{A(u)\rightarrow_1 B(v)}} \rightarrow_1 B^*(v)\}.$$

(ix) If $\rightarrow_2$ takes I_{EP}, then $A^*(u) = \sup_{v\in V}\{((A(u) \rightarrow_1 B(v)) \times \alpha)/(2 - (A(u) \rightarrow_1 B(v)) - \alpha + (A(u) \rightarrow_1 B(v)) \times \alpha) \rightarrow_1 B^*(v)\}$.

Proof. If $\rightarrow_2 \in \{I_{LK}, I_{GD}, I_{GG}, I_{FD}, I_{RC}, I_{GR}, I_{KD}, I_{YG}\}$, then it follows form Theorem 4.6 that the SIR-InfT-solution is

$$A^*(u) = \sup_{v\in V}\{T(A(u) \rightarrow_1 B(v), \alpha) \rightarrow_1 B^*(v)\}, \quad u \in U.$$

We prove the case of I_{LK} as an example, the rest can be proved in the same way.

Suppose that $\rightarrow_2$ takes I_{LK}. It follows from (15) that $(u \in U, v \in V)$

$$1 > B^*(v),$$

$\to_2$	The SIR-InfT-solution A^*	The condition which A^* is the SIR-MinT-solution
I_{LK}	$\sup_{v\in V}\{((A(u)\to_1 B(v))+\alpha-1)\to_1 B^*(v)\}$	$((A(u)\to_1 B(v))+\alpha-1)\to_1 B^*(v)<A^*$
I_{GD}	$\sup_{v\in V}\{((A(u)\to_1 B(v))\wedge\alpha)\to_1 B^*(v)\}$	$((A(u)\to_1 B(v))\wedge\alpha)\to_1 B^*(v)<A^*$
I_{GG}	$\sup_{v\in V}\{((A(u)\to_1 B(v))\times\alpha)\to_1 B^*(v)\}$	$((A(u)\to_1 B(v))\times\alpha)\to_1 B^*(v)<A^*$
I_{FD}	$\sup_{v\in V}\{((A(u)\to_1 B(v))\wedge\alpha)\to_1 B^*(v)\}$	$((A(u)\to_1 B(v))\wedge\alpha)\to_1 B^*(v)<A^*$
I_{RC}	$\sup_{v\in V}\{\frac{(A(u)\to_1 B(v))+\alpha-1}{A(u)\to_1 B(v)}\to_1 B^*(v)\}$	$\frac{(A(u)\to_1 B(v))+\alpha-1}{A(u)\to_1 B(v)}\to_1 B^*(v)<A^*$
I_{GR}	$\sup_{v\in V}\{(A(u)\to_1 B(v))\to_1 B^*(v)\}$	$(A(u)\to_1 B(v))\to_1 B^*(v)<A^*$
I_{KD}	$\sup_{v\in V}\{\alpha\to_1 B^*(v)\}$	$\alpha\to_1 B^*(v)<A^*$
I_{YG}	$\sup_{v\in V}\{\alpha^{\frac{1}{A(u)\to_1 B(v)}}\to_1 B^*(v)\}$	$\alpha^{\frac{1}{A(u)\to_1 B(v)}}\to_1 B^*(v)<A^*$
I_{EP}	$\sup_{v\in V}\{\Phi(u,v)\to_1 B^*(v)\}$	$\Phi(u,v)\to_1 B^*(v)<A^*$

Tab. 2. Some Conclusions of the Symmetric Implicational Restriction Inference Algorithm of FMT.

$$1-(A(u)\to_1 B(v))+(1\to_1 B^*(v))<\alpha.$$

We further obtain that $(u\in U, v\in V)$

$$(A(u)\to_1 B(v))+\alpha>1.$$

Thus, we obtain $(u\in U)$

$$\begin{aligned}A^*(u)&=\sup_{v\in V}\{T_{LK}(A(u)\to_1 B(v),\alpha)\to_1 B^*(v)\}\\&=\sup_{v\in V}\{((A(u)\to_1 B(v))+\alpha-1)\to_1 B^*(v)\}.\end{aligned}$$

□

Theorem 4.8. Under the same condition as Theorem 4.6, the condition which the SIR-InfT-solution A^* becomes the SIR-MinT-solution, is shown in Table 2 (for any $u\in U$,$v\in V$), where $\Phi(u,v)=\frac{(A(u)\to_1 B(v))\times\alpha}{2-(A(u)\to_1 B(v))-\alpha+(A(u)\to_1 B(v))\times\alpha}$.

Proof. If the SIR-InfT-solution A^* is a symmetric restriction solution of FMT, then A^* is the minimum of symmetric restriction solutions of FMT, i. e., the SIR-MinT-solution (which A^* should make (8) hold for any $u\in U, v\in V$). We prove the cases of I_{LK}, I_{GD} as examples.

(i) Let $\to_2$ take I_{LK}. It follows from Proposition 4.7 that the SIR-InfT-solution is $(u\in U)$

$$A^*(u)=\sup_{v\in V}\{((A(u)\to_1 B(v))+\alpha-1)\to_1 B^*(v)\}.$$

From the condition given in (i), we have $(u\in U, v\in V)$

$$A^*(u)>((A(u)\to_1 B(v))+\alpha-1)\to_1 B^*(v).$$

From another angle,

$$A^*(u)\le((A(u)\to_1 B(v))+\alpha-1)\to_1 B^*(v)$$

does not hold. Note that $\to_1$ satisfies (C11), then one has that

$$(A(u) \to_1 B(v)) + \alpha - 1 \leq A^*(u) \to_1 B^*(v)$$

does not hold. In other words,

$$(A(u) \to_1 B(v)) + \alpha - 1 > A^*(u) \to_1 B^*(v) \tag{19}$$

holds $(u \in U, v \in V)$, and hence

$$A(u) \to_1 B(v) > A^*(u) \to_1 B^*(v). \tag{20}$$

To sum up, we obtain from (19) and (20) that $(u \in U, v \in V)$

$$\begin{aligned}&(A(u) \to_1 B(v)) \to_2 (A^*(u) \to_1 B^*(v))\\&= 1 - (A(u) \to_1 B(v)) + (A^*(u) \to_1 B^*(v))\\&< \alpha.\end{aligned}$$

So A^* lets (8) hold for any $u \in U$,$v \in V$, and thus it is a symmetric restriction solution of FMP. Consequently, the SIR-InfT-solution A^* becomes the SIR-MinT-solution.

(ii) Let $\to_2$ take I_{GD}. We have from Proposition 4.7 that the SIR-InfT-solution is $(u \in U)$

$$A^*(u) = \sup_{v \in V}\{((A(u) \to_1 B(v)) \wedge \alpha) \to_1 B^*(v)\}.$$

From the condition given in (ii), we get $(u \in U, v \in V)$

$$A^*(u) > ((A(u) \to_1 B(v)) \wedge \alpha) \to_1 B^*(v).$$

Then

$$A^*(u) \leq ((A(u) \to_1 B(v)) \wedge \alpha) \to_1 B^*(v)$$

does not hold. Because $\to_1$ satisfies (C11), one has that

$$(A(u) \to_1 B(v)) \wedge \alpha \leq A^*(u) \to_1 B^*(v)$$

does not hold. In other words, we have $(u \in U, v \in V)$

$$(A(u) \to_1 B(v)) \wedge \alpha > A^*(u) \to_1 B^*(v) \tag{21}$$

and thus

$$A(u) \to_1 B(v) > A^*(u) \to_1 B^*(v),\ \alpha > A^*(u) \to_1 B^*(v). \tag{22}$$

To sum up, we get from (21) and (22) that $(u \in U, v \in V)$

$$\begin{aligned}&(A(u) \to_1 B(v)) \to_2 (A^*(u) \to_1 B^*(v))\\&= A^*(u) \to_1 B^*(v) < \alpha.\end{aligned}$$

Thus A^* makes (8) hold for any $u \in U$,$v \in V$, and hence it is a symmetric restriction solution of FMT. As a result, the SIR-InfT-solution A^* becomes the SIR-MinT-solution.
□

Example 4.9. Suppose that (15) holds.

(i) If $\to_1$ takes I_{GD}, and $\to_2$ takes I_{GG}, then the SIR-InfT-solution is

$$A^*(u) = \sup_{v\in V} \{B^*(v)\}.$$

(ii) If $\to_1$ takes I_{LK}, and $\to_2$ takes I_{GG}, then the SIR-InfT-solution is

$$A^*(u) = \sup_{v\in V} \{1 - (A^*(u) \to_{LK} B^*(v)) \times \alpha + B^*(v)\}.$$

Proof. (i) Suppose that $\to_1$ takes I_{GD}, and that $\to_2$ takes I_{GG}. Note that T_{GG} is the function residual to I_{GG}. By (15), we have that

$$\frac{B^*(v)}{A(u) \to_{GD} B(v)} < \alpha.$$

Thus one has from Theorem 4.6 that the SIR-InfT-solution is as follows:

$$\begin{aligned} A^*(u) &= \sup_{v\in V} \{T_{GG}(A(u) \to_1 B(v), \alpha) \to_1 B^*(v)\} \\ &= \sup_{v\in V} \{((A(u) \to_{GD} B(v)) \times \alpha) \to_{GD} B^*(v)\} \\ &= \sup_{v\in V} \{B^*(v)\}. \end{aligned}$$

(ii) Suppose that $\to_1$ takes I_{LK}, and that $\to_2$ takes I_{GG}. By (15), one has that

$$\frac{B^*(v)}{A(u) \to_{LK} B(v)} < \alpha.$$

One further gets that the SIR-InfT-solution is

$$\begin{aligned} A^*(u) &= \sup_{v\in V} \{T_{GG}(A(u) \to_1 B(v), \alpha) \to_1 B^*(v)\} \\ &= \sup_{v\in V} \{((A(u) \to_{LK} B(v)) \times \alpha) \to_{LK} B^*(v)\} \\ &= \sup_{v\in V} \{1 - (A(u) \to_{LK} B(v)) \times \alpha + B^*(v)\}. \end{aligned}$$

□

Example 4.10. Suppose that $U = V = [0,1]$, $A(u) = (3-u)/4$, $B(v) = (2+v)/4$, $B^*(v) = (1-v)/4$, $\alpha = 3/4$, in which $u \in U, v \in V$. Assume that $\to_1 = I_{LK}$, $\to_2 = I_{GG}$ in the symmetric restriction method of FMT. Here we show the computing process of the SIR-InfT-solution.

To begin with, we have

$$\begin{aligned} &A(u) \to_1 B(v) = I_{LK}(A(u), B(v)) \\ &= \begin{cases} 1 & \frac{3-u}{4} \le \frac{2+v}{4} \\ 1 - \frac{3-u}{4} + \frac{2+v}{4} & \frac{3-u}{4} > \frac{2+v}{4} \end{cases} = \begin{cases} 1 & u+v \ge 1 \\ \frac{u+v+3}{4} & u+v < 1 \end{cases}. \end{aligned}$$

Here (15) evidently holds.

Moreover, it is easy to verify that

$$(A(u) \to_1 B(v)) \times \alpha > B^*(v).$$

Then it follows from Proposition 4.7 that the SIR-InfT-solution is as follows ($v \in V$):

$$\begin{aligned}
A^*(u) &= \sup_{v \in V}\{((A(u) \to_1 B(v)) \times \alpha) \to_1 B^*(v)\} \\
&= \sup_{v\in[0,1]} \left\{1 - ((A(u) \to_1 B(v)) \times \alpha) + B^*(v) \;\middle|\; u+v<1\right\} \vee \\
&\quad \sup_{v\in[0,1]} \left\{1 - ((A(u) \to_1 B(v)) \times \alpha) + B^*(v) \;\middle|\; u+v\geq 1\right\} \\
&= \sup_{v\in[0,1]} \left\{1 - \frac{3\times(3+u+v)}{4\times 4} + \frac{1-v}{4} \;\middle|\; u+v<1\right\} \vee \\
&\quad \sup_{v\in[0,1]} \left\{1 - \frac{3}{4} + \frac{1-v}{4} \;\middle|\; u+v\geq 1\right\} \\
&= \sup_{v\in[0,1]} \left\{\frac{11-3u-7v}{16} \;\middle|\; u+v<1\right\} \vee \\
&\quad \sup_{v\in[0,1]} \left\{\frac{2-v}{4} \;\middle|\; u+v\geq 1\right\}.
\end{aligned}$$

It can be divided into two situations.

(i) Suppose $u = 1$, then $\{v \in [0,1] \mid u+v<1\} = \varnothing$, and

$$0 \in \{v \in [0,1] \mid u+v \geq 1\}.$$

Since $\frac{2-v}{4}$ is decreasing w.r.t. v, we get

$$A^*(u) = \sup \varnothing \vee \frac{2}{4} = 0 \vee \frac{2}{4} = \frac{1}{2}.$$

(ii) Suppose $1 > u \geq 0$, then

$$0 \in \{v \in [0,1] \mid u+v<1\}.$$

Because $\frac{11-3u-7v}{16}$, $\frac{2-v}{4}$ are decreasing w.r.t. v, we have

$$A^*(u) = \frac{11-3u}{16} \vee \frac{2-(1-u)}{4} = \frac{11-3u}{16}.$$

To sum up, we obtain that SIR-InfT-solution is:

$$A^*(u) = \frac{11-3u}{16}.$$

Example 4.11. Let U, V, A, B, B^*, α employ the same values in Example 4.10. Assume that $\rightarrow_1 = I_{LK}$, $\rightarrow_2 = I_{LK}$ in the symmetric restriction method of FMT, which degenerates into the triple I restriction method for FMT using I_{LK}.

A similar calculation is performed, then we achieve that the SIR-InfT-solution is as follows:

$$A^*(u) = \frac{3-u}{4}.$$

Remark 4.12. For one thing, aiming at the same U, V, A, B, B^*, α in Example 4.10 and Example 4.11, because

$$u = 1 \Rightarrow \frac{11-3u}{16} = \frac{3-u}{4},$$

$$0 \leq u < 1 \Rightarrow \frac{11-3u}{16} < \frac{3-u}{4},$$

the SIR-InfT-solution from the symmetric implicational restriction method in Example 4.10 is smaller than the optimal solution from the triple I restriction method in Example 4.11. From the angle of SIR-principle for FMT (which tries to find the smallest one such that (8) holds), the symmetric implicational restriction method for FMT in Example 4.10 lets the inference closer, then it is more reasonable than the triple I restriction method for FMT in Example 4.11. For another, in the environment of $\{I_{LK}, I_{GD}, I_{GG}, I_{FD}, I_{RC}, I_{GR}, I_{KD}, I_{YG}, I_{EP}\}$, the symmetric implicational restriction method can lead to $9 * 9 = 81$ kinds of particular fuzzy inference strategies for FMT, while the triple I restriction inference method only can result in 9 kinds. In summary, the symmetric implicational restriction method can generate more and better specific strategies than the triple I restriction method, and thus it performs better than the latter.

The symmetric implicational restriction method is different from the symmetric implicational method to a large extent. Taking into consideration that the α-symmetric implicational method is a generalization of the basic symmetric implicational method, here we show the major differences between the α-symmetric implicational method and the symmetric implicational restriction method. First, the fundamental idea is different. The idea of the α-symmetric implicational method is derived from (4). However, the idea of the symmetric implicational restriction method comes from (5). Second, the existing condition of basic solution is disparate. For this point, the symmetric implicational restriction method needs (11) for FMP and (15) for FMT. But there are no such harsh conditions for the symmetric implicational method. Third, the optimal solution is different. The SIR-SupP-solution of the symmetric implicational restriction method is expressed as (12) for FMP (noting that the condition (11) demands to be satisfied in advance) and the SIR-InfT-solution is expressed as (16) for FMT (noting that the condition (15) needs to be satisfied ahead of time). But the optimal solution of the symmetric implicational method is different (see Theorem 5.2, Theorem 5.4, Theorem 5.5 in [27]), in which the conditions (11) and (15) are not demanded. Note that the idea of proving process is also disparate. Lastly, the symmetric implicational restriction method needs special condition which lets the supremum to become the maximum (or the infimum to be the minimum). But such condition is not required for the symmetric implicational method.

It is important to emphasize that fuzzy reasoning is unlikely to solve all problems in a method. In fact, each method often has its own unique advantages. Aiming at the symmetric implicational restriction method and the symmetric implicational method, it is difficult to say which one is better, because both approaches have different advantages. At the same time, it is not possible to replace one directly with the other.

As for the symmetric implicational method, its idea is basically in the same line as the CRI method and the triple I method. Of course, compared with the triple I method, the symmetric implicational method forms a more optimal inference strategy from the dual perspectives of logical system and inference model. Moreover, the symmetric implicational method also has a solid logical basis. To be specific, Dai [2] established a predicate formal representation of the solution for the symmetric implicational method on account of a formal logic system, which has offered a sound logic foundation for the symmetric implicational method.

As for the symmetric implicational restriction method, its idea comes from the reverse thinking of the symmetric implicational method. The special advantages of the symmetric implicational restriction method are further explained here. In practical application, when we investigate the universal approximation problem for a fuzzy system constructed by the symmetric implicational method, the phenomenon of rule explosion often appears. To alleviate such tough problem, the highlight of restriction idea is that the number of elements in the fuzzy rule base can be greatly reduced in the given precision when designing fuzzy rules and corresponding fuzzy system [18, 19]. This is also the key advantage of the symmetric implicational restriction method. In addition, the research results of the symmetric implicational restriction method can further improve and enrich the theory system of fuzzy inference, which is able to provide the necessary theoretical basis for the study of performance indexes of new fuzzy controllers.

5. CONCLUSIONS

The symmetric implicational restriction method is put forward and studied with emphasis on the following points.

(i) We propose the SIR-principles, which form optimized versions of the triple I restriction principles.

(ii) We give the existential requirement of basic solutions of the symmetric implicational restriction method.

(iii) We achieve the supremum (or infimum) of the symmetric implicational restriction solutions for FMP (or FMT) from some properties of fuzzy implications. We show conditions for the supremum to be the maximum (or, the infimum to be the minimum).

(iv) We provide four concrete computing examples. We draw the conclusion that the symmetric implicational restriction method is superior over the triple I restriction method, because the former is able to make the inference more compact from the angle of SIR-principles (which find the optimal fuzzy set such that (8) holds), and lead to more and better particular inference schemes.

In future studies it is worth focusing on the symmetric implicational restriction method being considered from the perspective of fuzzy system. Moreover, one may extend it to granular fuzzy inference strategy by fusing the idea of granular computing (see [24, 25, 26]).

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Yiming Tang, Key Laboratory of Knowledge Engineering with Big Data, Ministry of Education, School of Computer and Information, Hefei University of Technology, Hefei, 230601, P. R. China; Anhui Province Key Laboratory of Affective Computing and Advanced Intelligent Machine, School of Computer and Information, Hefei University of Technology, Hefei, 230601, P. R. China; and Department of Electrical and Computer Engineering, University of Alberta, Edmonton, AB T6R 2V4. Canada.
e-mail: tym608@163.com

Wenbin Wu, School of Computer and Information, Hefei University of Technology, Hefei, 230601. P. R. China.
e-mail: wwb9221121@163.com

Youcheng Zhang, School of Computer and Information, Hefei University of Technology, Hefei, 230601. P. R. China.
e-mail: zyggood@qq.com

Witold Pedrycz, Department of Electrical and Computer Engineering, University of Alberta, Edmonton, AB T6R 2V4, Canada; and Department of Electrical and Computer Engineering, Faculty of Engineering, King Abdulaziz University, Jeddah, 21589, Saudi Arabia; and Systems Research Institute, Polish Academy of Sciences, Warsaw 01-447. Poland.
e-mail: wpedrycz@ualberta.ca

Fuji Ren, School of Computer and Information, Hefei University of Technology, Hefei, 230601. P. R. China.
e-mail: ren@is.tokushima-u.ac.jp

Jun Liu, School of Computer and Information, Hefei University of Technology, Hefei, 230601. P. R. China.
e-mail: liujun@hfut.edu.cn