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CHARACTERIZATION OF ADMISSIBLE LINEAR ESTIMATORS UNDER EXTENDED BALANCED LOSS FUNCTION

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In this paper, we study the admissibility of linear estimator of regression coefficient in linear model under the extended balanced loss function (EBLF). The sufficient and necessary condition for linear estimators to be admissible are obtained respectively in homogeneous and non-homogeneous classes. Furthermore, we show that admissible linear estimator under the EBLF is a convex combination of the admissible linear estimator under the sum of square residuals and quadratic loss function.

Keywords: admissibility, extended balanced loss function, linear admissible estimator

Classification: 62C15, 62F10, 62J05

1. INTRODUCTION

The following notations and operations on a matrix L are used throughout this paper. We will write $L \in \mathrm{R}^{p\times n}$, if L is a $p\times n$ real matrix, $L \in \mathrm{R}^p$ if L is a $p\times 1$ real matrix and $p\times p$ identity matrix written as I_p. For a matrix L, the symbols L', L^-, L^+, $tr(L), C(L)$ and $\lambda(L)$ stand for respectively the transpose, g-inverse, Moore–Penrose inverse, trace, column space(range) and eigenvalues of L. For symmetric matrices L and M, $L \geq M$ and $L > M$ represent the nonnegative definite (n.n.d) and positive definite (p.d) of matrix $L - M$. We shall write $Ly \sim \beta$ to denote that Ly is admissible for β, also $Ly \nsim \beta$ to denote that Ly is not admissible for β. Consider the linear model

$$y = X\beta + \varepsilon, \quad E(\varepsilon) = 0, \quad E(\varepsilon\varepsilon') = \sigma^2 I_n \tag{1}$$

where y is an $n \times 1$ vector of observations on the dependent variable, X is an $n \times p$ model matrix of observations on the p regressors and X has full of rank p, $\beta \in \mathrm{R}^p$ and $\sigma^2 > 0$ are unknown parameters , ε is an $n \times 1$ vector of disturbances.

Let β_* denote any estimator of β, then the quadratic loss function which reflects the goodness of fit of the model is

$$(y - X\beta_*)'(y - X\beta_*), \tag{2}$$

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where $X\beta_*$ is the predictor for y. Similarly, the precision of estimator of β is measured by the weighted loss function

$$(\beta_* - \beta)'X'X(\beta_* - \beta). \tag{3}$$

Generally, either of the criterion of (2) or (3) is used to judge the performance of any estimator. By combining the (2) and (3), Zellner [27] proposed a new criterion called balanced loss function (ZBLF), which focuses on estimates around true parameter and goodness of fit of model

$$ZBLF(\beta_*, \beta) = t\,(y - X\beta_*)^{'}\,(y - X\beta_*) + (1-t)(\beta_* - \beta)'X'X(\beta_* - \beta). \tag{4}$$

ZBLF has received considerable attention in the literature under different setups, see for example, Xu and Wu [25], Cao [7], Cao and He [8].

Further, Shalabh [21] has defined a target function for the purpose of simultaneous prediction of actual and average values of y as

$$T = ty + (1-t)E\,(y)\,, \tag{5}$$

where t is a scalar between 0 and 1.

The following predictive loss function arises when we use the predictor $X\beta_*$ for simultaneous prediction of actual and average values of y through the target function (5):

$$\begin{aligned} PLF(\beta_*, \beta, t) &= (X\beta_* - T)'(X\beta_* - T) \\ &= t^2\,(y - X\beta_*)^{'}\,(y - X\beta_*) + (1-t)^2(\beta_* - \beta)'X'X(\beta_* - \beta) \\ &\quad + 2t(1-t)(X\beta_* - y)'X(\beta_* - \beta). \end{aligned} \tag{6}$$

Predictive loss function not only incorporates the ZBLF as its particular case but also measures the correlation between the goodness of fit of model and concentration of estimates around the true parameter.

Appereciating the popularity of the ZBLF, Shalabh et al. [22] proposed the EBLF as follows:

$$\begin{aligned} EBLF(\beta_*, \beta, t_1, t_2) &= t_1\,(y - X\beta_*)^{'}\,(y - X\beta_*) + t_2(\beta_* - \beta)'X'X(\beta_* - \beta) \\ &\quad + (1 - t_1 - t_2)\,(X\beta_* - y)'X(\beta_* - \beta), \end{aligned} \tag{7}$$

where $t_1$and t_2 are the scalars lying between 0 and 1 characterizing the loss functions. EBLF encompass the following particular cases:

1. When $t_1 = 1$ and $t_2 = 0$, the EBLF in Equation (7) reduces to the (2) which is the criterion of goodness of fit of model .

2. If we put $t_1 = 0$ and $t_2 = 1$, the EBLF in Equation (7) reduces to the weighted quadratic loss function of (3) which is the criterion of precision of estimation.

3. If $t_1 = 0$ and $t_2 = 0$, the EBLF in Equation (7) reduces to the component$(X\beta_* - y)'X(\beta_* - \beta)$ which is the criterion of interaction or covariation between the precision of estimation and goodness of fit.

4. When we take $t_1 = t$ and $t_2 = 1 - t$, where $0 < t < 1$, we obtain the ZBLF in Equation (4).

Then, from Equations (7), Shalabh et al. [22] observed that

$$EBLF(\beta_*, \beta, t_1, t_2) = t_1 \varepsilon' \varepsilon - (1 + t_1 - t_2)\, \varepsilon' X(\beta_* - \beta) + (\beta_* - \beta)' X' X(\beta_* - \beta), \quad (8)$$

and the risk of any estimator β_* is the expected value of the EBLF which is calculated by

$$R(\beta_*, \beta, t_1, t_2) = E(EBLF(\beta_*, \beta, t_1, t_2)). \quad (9)$$

The main advantage of the EBLF is that it is more flexible in comparison to (4) and (6). The ZBLF in Equation (4) takes care only either the precision of estimation or the goodness of fit whereas the EBLF in Equation (7) extends it by considering the interaction or covariation between the precision of estimation and goodness of fit. Ignoring this covariation in the formulation of loss function may lead to wrong statistical inferences. The weights assigned in Equation (6) for precision of estimation, goodness of fit and their covariation depends only on one factor t assigned to one of the criterion only. The EBLF in Equation (7) provides a more flexible option of choosing different weights for precision of estimation, goodness of fit and covariation terms and, in the presence of covariation, it will obviously lead to improved statistical inferences (see Chaturvedi and Shalabh [6]).

There are some studies on EBLF by some authors. For example; Özbay and Kaçıranlar [18] discussed the performance of the adaptive optimal estimator of Farebrother (1975) under the EBLF. Kaçıranlar and Dawoud [15] introduced and derived the optimal EBLF estimators and predictors and discussed their performances.

The problem of admissibility of linear estimators was investigated first by Cohen [4]. Ten years later, an exhaustive study of the problem was given by Rao [19] for the more general model. Characterization of admissible linear estimators has received considerable attention in the literature under different models or loss function. Some important examples are given as follows: Baksalary and Markiewicz [1,2,3], Klonecki and Zontek [14], Stepniak [20], Hoffmann [13], Markiewicz [17], Yu Lu and Zhong Shi [26], Groß and Markiewicz [12]. Recently, Synowka Bejenka and Zontek [24] examined on admissibility of linear estimators in models with finitely generated parameter space. Stepniak [23] studied on admissible invariant estimators in a linear model. As far as we know, characterization of admissible linear estimators is not studied under EBLF, and this will be discussed in this article.

Throughout this paper, linear estimators are, respectively, from the following non-homogeneous and homogeneous classes

$$\mathcal{L} = \left\{ Ly + a : L \in \mathrm{R}^{p \times n}, a \in \mathrm{R}^p \right\}, \quad (10)$$

and

$$\mathcal{L}^h = \left\{ Ly : L \in \mathrm{R}^{p \times n} \right\}. \quad (11)$$

In this paper, Section 2 we give our main results, namely the sufficient and necessary conditions for linear estimators to be admissible and some corollaries. Section 3 contains some definitions and lemmas playings important roles in this paper. Proofs of the main results are given in Section 4 and details of the proofs are given in Appendix, Conclusions are assigned in Section 5.

2. THE MAIN RESULTS

In this section, the sufficient and necessary condition for linear estimators to be admissible are obtained respectively in $\mathcal{L}$ and $\mathcal{L}^h$ classes. In addition, admissible linear estimator under the EBLF is shown by the convex combination of linear admissible estimators under the sum of square residuals and quadratic loss function.

2.1. Explicit characterization

It is known that EBLF is more sensitive than the quadratic loss and reduces to the quadratic loss when $t_1 = 0$ and $t_2 = 1$. Thus, there exists some relationship between the EBLF and the quadratic loss. It is presented in Theorem 2.1 which plays a crucial role in proving the main results.

Theorem 2.1. Given linear model

$$\begin{cases} z = X'X\beta + u \\ E(u) = 0,\ E(uu') = \sigma^2 X'X, \end{cases} \tag{12}$$

where $\beta \in \mathrm{R}^p$ and $\sigma^2 > 0$ are unknown and u is a p- dimensional error vector. Let $C = (1-w)\, I_p$, $\tilde{\mathcal{L}} = \left\{\tilde{L}z : \tilde{L} \in \mathrm{R}^{p\times p}\right\}$, $\mathcal{L}_0^h = \{L_0 X'y : L_0 \in \mathrm{R}^{p\times p}\}$, the loss function

$$LF_B(\beta_*, C\beta) = (\beta_* - C\beta)' B (\beta_* - C\beta),$$

$\tilde{L} = L_0 - w\,(X'X)^{-1}$, and $C\beta$ is linearly estimable in model (12). Then, under model (12) and loss $LF_B(\beta_*, C\beta)$, $\tilde{L}z \overset{\tilde{\mathcal{L}}}{\sim} C\beta$ holds if and only if $L_0 X'y \overset{\mathcal{L}_0^h}{\sim} \beta$ holds under model (1) and the EBLF, where $w = (1 + t_1 - t_2)/2$ and $0 \le w \le 1$.

The following characterization can be derived from the above results.

Theorem 2.2. Under model (1) and the EBLF, $Ly \overset{\mathcal{L}^h}{\sim} \beta$ holds if and only if L can be written as

$$L = \left[w\,(X'X)^{-1} + (1-w)\,(X'X)^{-1/2}\, D\,(X'X)^{-1/2}\right] X',$$

where $w = (1 + t_1 - t_2)/2$, $0 \le w \le 1$ and $D = (X'X)^{1/2}\, LX\,(X'X)^{-1/2}$ is symmetric and $\lambda(D) \subset [0, 1]$.

According to the Theorem 2.2, we can obtain the following corollary.

Corollary 2.3. In linear model (1), if A is the set of admissible linear estimator of β in $\mathcal{L}^h$ under the $LF_I(\beta_*, \beta) = (\beta_* - \beta)'(\beta_* - \beta)$, and let A^* is the set of admissible linear estimator of β under the EBLF. Then, A^* can be demonstrated as follows

$$\begin{aligned} A^* &= \left\{w\hat{\beta} + (1-w)\tilde{\beta}_{mse} : \tilde{\beta}_{mse} \in A\right\} \\ &= \left\{\hat{\beta} - (1-w)\,(X'X)^{-1/2}\, D\,(X'X)^{-1/2}\, X'y : \tilde{\beta}_{mse} \in A\right\}, \end{aligned}$$

where $\hat{\beta}$ denotes least square estimator which is the admissible estimator under loss (2) and $\tilde{\beta}_{mse}$ is the admissible estimator under the loss $LF_I(\beta_*, \beta)$.

Theorem 2.4. Under model (1) and the EBLF, $Ly+a \overset{\mathcal{L}}{\sim} \beta$ holds if and only if $Ly \overset{\mathcal{L}^h}{\sim} \beta$ and $a \in C\left(LX - I_p\right)$ holds simultaneously.

2.2. Implicit characterization

Theorem 2.2 and Theorem 2.4 provide the explicit form of L and a which is necessary and sufficient for $Ly + a \overset{\mathcal{L}}{\sim} \beta$. From the following theorem, the implicit characterizations of admissible linear estimators $Ly+a$ by giving necessary and sufficient algebraic conditions on L and a can be deduced.

Theorem 2.5. Under model (1) and the EBLF, $Ly + a \overset{\mathcal{L}}{\sim} \beta$ holds if and only if

i. $XL = L'X'$,

ii. $\lambda(XL) \subset [0, 1]$,

iii. $a \in C\left(LX - I_p\right)$.

3. PRELIMINARIES

Lemma 3.1. Let y be an $n \times 1$ random vector with $E(y) = \mu$ and $Cov(y) = V$ and let W be an $n \times n$ non- stochastic matrix. Then the identity

$$E(y'Wy) = tr(WV) + \mu' W\mu,$$

holds true. Note that the above result holds for any not necessarily symmetric square matix W.

Definition 3.2. Let $\mathcal{L}$ be a class of estimators. For arbitrary $d\left(y\right) \notin \mathcal{L}$, if there exists an estimator $d_1\left(y\right) \in \mathcal{L}$ such that $d_1\left(y\right)$ is better than $d\left(y\right)$, then $\mathcal{L}$ is called a complete class (Lehmann and Casella [16]) .

Thus from this definition, it is known that we only need to search admissible estimators in the complete class.

Lemma 3.3. Let $\mathcal{L}_0^h = \{L_0X'y : L_0 \in \mathrm{R}^{p\times p}\}$. Then $\mathcal{L}_0^h$ is a complete class of $\mathcal{L}^h$ in Equation (11).

Proof. If $\beta_* = Ly \in \mathcal{L}^h$, using some properties of trace and Lemma 3.1, from Equation (7) and (9), we get

$$\begin{aligned} R\left(Ly;\ \beta,\ \sigma^2\right) &= E\left[t_1\left(y - XLy\right)^{'}\left(y - XLy\right) + t_2\left(Ly - \beta\right)^{'} X'X\left(Ly - \beta\right)\right] \\ &+ \left[(1 - t_1 - t_2)E\left(XLy - y\right)^{'} X\left(Ly - \beta\right)\right] \\ &= \sigma^2 tr\left[t_1\left(I_n - XL\right)^{'}\left(I_n - XL\right) + t_2\left(L'X'XL\right)\right] \\ &+ \sigma^2 tr\left(1 - t_1 - t_2\right)\left(\left(XL - I_n\right)^{'} XL\right) \\ &+ \beta'X'\left(I_n - XL\right)^{'}\left(I_n - XL\right)X\beta. \end{aligned} \tag{13}$$

On the other hand, if $Ly \in \mathcal{L}^h$, then, $LP_X y = L_0 X' y \in \mathcal{L}_0^h$, where $P_X = X\left(X'X\right)^{-1} X'$ and $L_0 = LX(X'X)^{-1}$. So ,

$$R\left(Ly;\ \beta,\ \sigma^2\right) - R\left(LP_X y;\ \beta,\ \sigma^2\right) = \sigma^2 tr\left[L'X'XL\left(I_n - P_X\right)\right] \geq 0,$$

(see Graybill [10], p.307). Moreover, the equality holds if and only if $L = LP_X$. The proof is given in detail in the Appendix 1. □

The following Lemma 3.4 is essentially due to Rao [19] as extended by Chen et al. [5].

Lemma 3.4. Let $\mathcal{L}^* = \left\{Lz : L \in \mathrm{R}^{k\times n}\right\}$ and

$$\begin{cases} z = X_0\beta + u\,; \\ E(u) = 0,\ E(uu') = \sigma^2 V, \end{cases} \tag{14}$$

where, $n \times p$ matrix X_0 and $V > 0$ are known, $\beta \in \mathrm{R}^p$ and $\sigma^2 > 0$ are unknown. If C is a $k \times p$ matrix and $C\beta$ is estimable, then $Lz \overset{\mathcal{L}^*}{\sim} C\beta$ holds under loss function

$$LF_I(\beta_*, C\beta) = (\beta_* - C\beta)'(\beta_* - C\beta),$$

if and only if the following statements are valid:

(a) $L = LX_0T^-X_0'V^{-1}$

(b) $LX_0T^-X_0'L' \leq LX_0T^-C'$,

where LX_0T^-C' is symmetric, $T = X_0'V^{-1}X_0$ and T^- is g - inverse of T.

We can see that if $k = p$, the set $\mathcal{L}^*$ turns into $\mathcal{L}^h$, and if $n = k = p$, the set $\mathcal{L}^*$ turns into $\tilde{\mathcal{L}}$.

Lemma 3.5. (Dong and Wu [9]) In the linear model (12), the sufficient and necessary condition for $\tilde{L}z \in \tilde{\mathcal{L}}$ to be an admissible estimator of $C\beta$ with respect to the $LF_B(\beta_*, C\beta)$ is unrelated to B only if $B > 0$.

4. PROOFS OF THE MAIN RESULTS

Proof of Theorem 2.1: Let $L_0X'y \in \mathcal{L}_0^h$. We get from Equation (13)

$$\begin{aligned} R\left(L_0X'y\,;\ \beta,\ \sigma^2\right) &= \sigma^2 tr(X\tilde{L}'B\tilde{L}X') + \sigma^2 tr\left(t_1 I_n - w^2 X\left(X'X\right)^{-1} X'\right) \\ &\quad + \beta'\left[\tilde{L}X'X - C\right]' B\left[\tilde{L}X'X - C\right]\beta, \end{aligned} \tag{15}$$

where $w = (1 + t_1 - t_2)/2$, $0 \leq w \leq 1$, $B = X'X$, $C = (1 - w)\, I_p$ and $\tilde{L} = L_0 - w\left(X'X\right)^{-1}$.

(The proof of Equation (15) is given in detail in the Appendix 2.)

Note that

$$ELF_B(\tilde{L}z, C\beta) = \sigma^2 tr(X\tilde{L}'B\tilde{L}X') + \beta'\left[\left(\tilde{L}X'X - C\right)' B\left(\tilde{L}X'X - C\right)\right]\beta. \tag{16}$$

If $L_0X'y \overset{\mathcal{L}_0^h}{\sim} \beta$, then for an arbitrary $M_0X'y \in \mathcal{L}_0^h$, we have

$$R\left(L_0X'y;\ \beta,\ \sigma^2\right) \leq R\left(M_0X'y;\ \beta,\ \sigma^2\right), \tag{17}$$

for all $\beta \in R^p$ and $\sigma^2 > 0$, where $\tilde{M} = M_0 - w\left(X'X\right)^{-1}$. We can see from (15) and (16),

$$R\left(L_0X'y;\ \beta,\ \sigma^2\right) - R\left(M_0X'y;\ \beta,\ \sigma^2\right) = ELF_B(\tilde{L}z, C\beta) - ELF_B(\tilde{M}z, C\beta).$$

Then,we get that (17) holds if and only if for any $\tilde{M}z \in \tilde{\mathcal{L}}$

$$ELF_B(\tilde{L}z, C\beta) \leq ELF_B(\tilde{M}z, C\beta), \tag{18}$$

holds for all $\beta \in \mathrm{R}^p$ and $\sigma^2 > 0$, where $\tilde{M} = M_0 - w\left(X'X\right)^{-1}$. Thus, from the definition of admissibility, (17) holds if and only if $\tilde{L}z \overset{\tilde{\mathcal{L}}}{\sim} C\beta$ in model (12). So, proof of Theorem 2.1 is completed. □

The following characterization can be derived from the above results.

Proof of Theorem 2.2. In this proof, first of all, we demonstrate that $\tilde{L}$ which is given in Theorem 2.1. Then, using Lemma 3.3 and relationship between the $\tilde{L}, L_0$ and L, we will get the desired result.

Let $k = n = p$. If $X_0 = V = X'X$, model (14) reduces to the model (12) and the set $\mathcal{L}^*$ turns into $\tilde{\mathcal{L}}$. Then, using Lemma 3.4, Lemma 3.5 and Theorem 2.1, under the model (12) when $B = X'X$, we get the necessary and sufficient condition for the $\tilde{L}z \overset{\tilde{\mathcal{L}}}{\sim} C\beta$ under

$$LF_{X'X}(\beta_*, C\beta) = (\beta_* - C\beta)'X'X(\beta_* - C\beta),$$

are as follows:

(a) $\tilde{L} = \tilde{L}X'X\left(X'X\left(X'X\right)^{-1}X'X\right)^{-}X'X\left(X'X\right)^{-1} = \tilde{L}$

(b) $\tilde{L}X'X\tilde{L}' \leq (1-w)\tilde{L},$ (19)

where $(1-w)\tilde{L}$ is symmetric and $T = V = X'X$.

If we take $D = (1-w)^{-1}\left(X'X\right)^{1/2}\tilde{L}\left(X'X\right)^{1/2}$, using (19), we get

$$\tilde{L}X'X\tilde{L}' \leq (1-w)\tilde{L} \Leftrightarrow DD' \leq D,$$

where

$$\begin{aligned} DD' &= (1-w)^{-2}\left(X'X\right)^{1/2}\tilde{L}X'X\tilde{L}'\left(X'X\right)^{1/2} \\ &\leq (1-w)^{-1}\left(X'X\right)^{1/2}\tilde{L}\left(X'X\right)^{1/2} = D, \end{aligned}$$

then $D' = D$ and $\lambda(D) \subset [0,1]$. So, from $D = (1-w)^{-1}\left(X'X\right)^{1/2}\tilde{L}\left(X'X\right)^{1/2}$ we can obtain the $\tilde{L} = (1-w)\left(X'X\right)^{-1/2}D\left(X'X\right)^{-1/2}$. Since (15) and Theorem 2.1, we know

$$L_0 = w\left(X'X\right)^{-1} + \tilde{L} \ \textit{and} \ \tilde{L}z \overset{\tilde{\mathcal{L}}}{\sim} C\beta \Leftrightarrow L_0X'y \overset{\mathcal{L}_0^h}{\sim} \beta.$$

Then, under the model (1) and the EBLF, $L_0X'y \overset{\mathcal{L}_0^h}{\sim} \beta$ holds if and only if

$$L_0 = w\left(X'X\right)^{-1} + (1-w)\left(X'X\right)^{-1/2} D \left(X'X\right)^{-1/2}. \tag{20}$$

As $Ly \in \mathcal{L}^h$ and $L_0X'y \in \mathcal{L}_0^h$, then, $L_0X'y = Ly \in \mathcal{L}^h$ and using Lemma 3.3, we can obtain the desired result that $Ly \overset{\mathcal{L}^h}{\sim} \beta$ under the EBLF holds if and only if L can be written as

$$L = \left[w\left(X'X\right)^{-1} + (1-w)\left(X'X\right)^{-1/2} D \left(X'X\right)^{-1/2}\right] X', \tag{21}$$

so, the proof of Theorem 2.2 is completed. □

According to the Theorem 2.2, we can obtain the following corollary.

Proof of Corollary 2.3. If we consider the special case of Lemma 3.4 when $k = p$, $X_0 = X$, $V = I_n$ and $C = I_p$, then , we get the sufficient and necessary condition for $Lz \overset{\mathcal{L}^h}{\sim} \beta$ under the loss function $LF_I(\beta_*, \beta)$ are as follows:

a) $L = LX\left(X'X\right)^{-1} X' = LP_X,$ (22)

b) $LP_xL' \leq LX\left(X'X\right)^{-1},$ (23)

where $LX\left(X'X\right)^{-1}$ is symmetric and $T = X'X$.

If we take $D = \left(X'X\right)^{1/2} LX\left(X'X\right)^{-1/2}$ and using (23), we get

$$LP_xL' \leq LX\left(X'X\right)^{-1} \Leftrightarrow DD' \leq D,$$

where $D' = D$ and $\lambda(D) \subset [0,1]$. From (22), $L = LP_X$, we get

$$L = LX\left(X'X\right)^{-1} X' = \left(X'X\right)^{-1/2} D \left(X'X\right)^{-1/2} X'.$$

For model (1), let

$$\begin{aligned} A &= \left\{Ly : L = LP_X \text{ and } LP_XL' \leq LX\left(X'X\right)^{-1}\right\} \\ &= \left\{\left(X'X\right)^{-1/2} D \left(X'X\right)^{-1/2} X'y : \ D' = D \text{ and } 0 \leq D \leq I_p\right\}. \end{aligned} \tag{24}$$

If $\tilde{\beta}_{mse} \in A$, then

$$\tilde{\beta}_{mse} = Ly = \left(X'X\right)^{-1/2} D \left(X'X\right)^{-1/2} X'y.$$

This is same as the results in Groß [11] when $a = 0$ (see p.221).

As A^* is the set of admissible linear estimators of β in $\mathcal{L}^h$ under the EBLF, from (22), we have

$$\begin{aligned} A^* &= \left\{ \begin{array}{l} Ly = \left[w\left(X'X\right)^{-1} + (1-w)\left(X'X\right)^{-1/2} D \left(X'X\right)^{-1/2}\right] X'y \\ \text{where } D = D' \text{ and } 0 \leq D \leq I_p \end{array} \right\} \\ &= \left\{w\hat{\beta} + (1-w)\tilde{\beta}_{mse} : \tilde{\beta}_{mse} \in A\right\}, \end{aligned} \tag{25}$$

for any $\beta^*_{EBLF} \in A^*$ and $L \in \mathcal{L}^h$. Then we get

$$\beta^*_{EBLF} = Ly = w\hat{\beta} + (1-w)\tilde{\beta}_{mse}.$$

We have the following conclusion from the above statements:
The expression of A^* shows that the admissible linear estimator under the EBLF is a convex combination of the admissible linear estimators under the unweighted squared error loss (2) and (3).

Since $D = D'$ and $0 \le D \le I_p$, then $D_1 = I_p - D$ also symmetric and $0 \le D_1 \le I_p$. So,

$$\begin{aligned} A &= \left\{ (X'X)^{-1/2} D \, (X'X)^{-1/2} X'y : \, D = D' \text{ and } 0 \le D \le I_p \right\} \\ &= \left\{ \hat{\beta} - (X'X)^{-1/2} D_1 \, (X'X)^{-1/2} X'y : \, D_1 = D_1' \text{ and } 0 \le D_1 \le I_p \right\}. \end{aligned} \tag{26}$$

Combining (25) and (26), A^* can be rewritten as follows

$$\begin{aligned} A^* &= w\hat{\beta} + (1-w)\tilde{\beta} \\ &= w\hat{\beta} + (1-w)\left(\hat{\beta} - (X'X)^{-1/2} D_1 \, (X'X)^{-1/2} X'y\right) \\ &= \hat{\beta} - (1-w)\tilde{\beta}_{mse}. \end{aligned} \tag{27}$$

P r o o f o f T h e o r e m 2.4. Sufficiency.
If a linear estimator $\beta_* = Ly + a \in \mathcal{L}$, then, using Equation (7) and (9), we get the risk of its with respect to the EBLF.

$$\begin{aligned} R\left(Ly + a;\ \beta,\ \sigma^2\right) &= t_1 E\left[(y - X(Ly+a))' \, (y - X(Ly+a))\right] \\ &\quad + t_2 E\left[((Ly+a) - \beta)' \, X'X \, ((Ly+a) - \beta)\right] \\ &\quad + (1 - t_1 - t_2)\left[E\left(X(Ly+a) - y\right)' X \left((Ly+a) - \beta\right)\right] \\ &= \sigma^2 tr\left[t_1 \left(I_n - XL\right)' \left(I_n - XL\right) + t_2 \left(L'X'XL\right)\right] \\ &\quad + (1 - t_1 - t_2)\sigma^2 tr\left(\left(XL - I_n\right)' XL\right) \\ &\quad + \left[\left(LX - I_p\right)\beta + a\right]' X'X \left[\left(LX - I_p\right)\beta + a\right] \end{aligned} \tag{28}$$

(The proof of Equation (28) is given in detail in the Appendix 3).

As $a \in C\left(LX - I_p\right)$, there exists $a_0 \in R^p$ such that $a = \left(LX - I_p\right) a_0$. Hence, using (28) we get

$$R\left(Ly + a;\ \beta,\ \sigma^2\right) = R\left(Ly;\ \beta + a_0,\ \sigma^2\right). \tag{29}$$

Suppose $Ly + a \not\sim \beta$, then there exists an estimator $Qy + q$ such that

$$R\left(Qy + q;\ \beta,\ \sigma^2\right) \le R\left(Ly + a;\ \beta,\ \sigma^2\right), \tag{30}$$

holds for all $\beta \in R^p$ and $\sigma^2 > 0$.

Let $\beta = -a_0$ in (30), then by (28) and (29)

$$\begin{aligned} &[q-(QX-I_p)\,a_0]'\,X'X\,[q-(QX-I_p)\,a_0] \\ &=\lim_{\sigma^2\to 0^+} R\left(Qy+q\,;\ -a_0,\sigma^2\right) \le \lim_{\sigma^2\to 0^+} R\left(Ly+a\,;\ \beta,\ \sigma^2\right) \\ &=\lim_{\sigma^2\to 0^+} R\left(Ly\,;\ 0,\sigma^2\right). \end{aligned} \tag{31}$$

Since $X'X > 0$ and the right side of inequality (31) is approaching zero. Therefore,

$$q = (QX - I_p)\,a_0. \tag{32}$$

From (29), (30) and (32),

$$\begin{aligned} R\left(Qy\,;\ \beta+a_0,\ \sigma^2\right) &= R\left(Qy+a_0\,;\ \beta,\ \sigma^2\right) \\ &\le R\left(Ly+a_0\,;\ \beta,\ \sigma^2\right) = R\left(Ly\,;\ \beta+a_0,\ \sigma^2\right), \end{aligned}$$

and we get

$$R\left(Qy\,;\ \beta_0,\ \sigma^2\right) \le R\left(Ly\,;\ \beta_0,\ \sigma^2\right),$$

for all $\beta_0 \in R^p$ and $\sigma^2 > 0$, where $\beta_0 = \beta + a_0$. Which means that Qy is better than Ly, this is contradicting $Ly \sim \beta$. Hence, the sufficiency is proved.

Necessity. Firstly, it is proved that $a \in C\,(LX - I_p)$ when $Ly + a \sim \beta$.
Denote S as the orthogonal matrix of projection onto $C\left\{(X'X)^{1/2}\,(LX-I_p)\right\}$ and $s = (X'X)^{-1/2}\,S\,(X'X)^{1/2}\,a$. Let $\eta = (X'X)^{1/2}\,(LX-I_p)$, using the knowledge of projection matrix in Groß [11] (see pp.242-243) , we get

$$\begin{aligned} S &= \eta(\eta'\eta)^+\eta' \\ &= (X'X)^{1/2}\,(LX-I_p)\,[(LX-I_p)'\,(X'X)\,(LX-I_p)]^+\,(LX-I_p)'\,(X'X)^{1/2}, \end{aligned} \tag{33}$$

and

$$s = (LX-I_p)\,[(LX-I_p)'\,(X'X)\,(LX-I_p)]^+\,(LX-I_p)'\,(X'X)\,a. \tag{34}$$

So, $s \in C\,(LX - I_p)$ holds. Using Moore–Penrose properties such as $\eta'\eta\,(\eta'\eta)^+\,\eta' = \eta'$, we have

$$\begin{aligned} &R\left(Ly+a\,;\ \beta,\ \sigma^2\right) - R\left(Ly+s\,;\ \beta,\ \sigma^2\right) \\ &= a'X'Xa - s'X'Xs = a'\,(X'X)^{1/2}\,(I_p - S)\,(X'X)^{1/2}\,a \ge 0. \end{aligned}$$

From which we get

$$R\left(Ly+a\,;\ \beta,\ \sigma^2\right) \ge R\left(Ly+s\,;\ \beta,\ \sigma^2\right), \tag{35}$$

for all $\beta \in R^p$ and $\sigma^2 > 0$. The inequality holds if and only if $(X'X)^{1/2}a = S(X'X)^{1/2}a$ holds, which is $a = (X'X)^{-1/2}S(X'X)^{1/2}a = s$. Inequality (35) and the admissibility of $Ly + a$ together result in $a = s$. Thus $a \in C\,(LX - I_p)$ is right.

Next, we prove that $Ly \sim \beta$, when $Ly + a \sim \beta$ holds.
If $Ly \not\sim \beta$, there exists an estimator Ky such that

$$R\left(Ky;\ \beta,\ \sigma^2\right) \leq R\left(Ly;\ \beta,\ \sigma^2\right), \tag{36}$$

is right for all (β, σ^2) with the strict inequality holding at some point (β_1, σ_1^2). As $a \in C\left(LX - I_p\right)$, there exists $a_0 \in R^p$ such that $a = \left(LX - I_p\right) a_0$.

So, by (36),

$$\begin{aligned} R\left(Ly + a;\ \beta,\ \sigma^2\right) &= R\left(Ly;\ \beta + a_0,\ \sigma^2\right) \\ &\geq R\left(Ky;\ \beta + a_0,\ \sigma^2\right) \\ &= R\left(Ky + (KX - I_p)a_0;\ \beta,\ \sigma^2\right), \end{aligned}$$

holds for all (β, σ^2) and the strict inequality is true at point $\left(\beta_1 - a_0,\ \sigma_1^2\right)$, which means that $Ky + (KX - I_p)a_0$ is better than $Ly + a$, this is contradictory to the admissibility of $Ly + a$. Hence necessity is proved. Therefore, the proof of Theorem 2.4 is completed. □

P r o o f o f T h e o r e m 2.5. Using (21) in Theorem 2.2, we have

$$XL = wX\left(X'X\right)^{-1}X' + (1-w)X\left(X'X\right)^{-1/2}D\left(X'X\right)^{-1/2}X'. \tag{37}$$

If we set

$$H = X\left(X'X\right)^{-1}X', G = X\left(X'X\right)^{-1/2}D\left(X'X\right)^{-1/2}X',$$

since D and H is symmetric, then $XL = wH + (1-w)G$ is symmetric, which means condition (i) satisfied. The non-zero eigenvalues of G coincide with the nonzero eigenvalues of $D\left(X'X\right)^{-1/2}X'X\left(X'X\right)^{-1/2} = D$, where $\lambda(D) \subset [0,1]$. This shows that, all eigenvalues of the matrix G lie in [0,1], which denoted by $\lambda(G) \subset [0,1]$ (see Groß [11] Theorem A.38).

Let H and G be two symmetric $n \times n$ matrices satisfying $GH = HG$. Then there exists an $n \times n$ orthogonal matrix U such that

$$H = U\Lambda U' \text{ and } G = U\Gamma U',$$

where Λ and Γ are two $n \times n$ diagonal matrices (see also Groß [11] Theorem A.47). Then, XL can be written as

$$XL = U(w\Lambda + (1-w)\Gamma)U'.$$

Since H is idempotent, H has only eigenvalues 0 or 1 and $\lambda(G) \subset [0,1]$ means that diagonal element $\gamma_i, i = 1, \ldots, p$ lies in the interval $[0,1]$. Then, we get

$$\lambda(w\Lambda + (1-w)\Gamma) \subset [0,1] \Leftrightarrow \lambda(XL) \subset [0,1],$$

where $0 \leq w \leq 1$ (see Graybill [10], p.43). So, condition (ii) also satisfied.

Using results of Theorem 2.4, we have $a \in C\left(LX - I_p\right)$. Thus, the proof of Theorem 2.4 is completed. □

Appendix 1: Proof of Lemma 3.3

Note that, $\beta_* = Ly \in \mathcal{L}^h$, using some properties of trace and Lemma 3.1, from Equation (7) and Equation (9), we get

$$\begin{aligned} R\left(Ly;\beta,\sigma^2\right) &= \sigma^2 tr\left[t_1\left(I_n - XL\right)^{'}\left(I_n - XL\right) + t_2\left(L'X'XL\right)\right] \\ &+ \sigma^2 tr\left(1 - t_1 - t_2\right)\left(\left(XL - I_n\right)^{'} XL\right) \\ &+ \beta' X'\left(I_n - XL\right)^{'}\left(I_n - XL\right) X\beta. \end{aligned}$$

On the other hand, if $Ly \in \mathcal{L}^h$, then $LP_X y = L_0 X'y \in \mathcal{L}_0^h$, where $P_X = X\left(X'X\right)^{-1} X'$and $L_0 = LX(X'X)^{-1}$. So ,

$$\begin{aligned} R\left(Ly;\beta,\sigma^2\right) - R\left(LP_X y;\beta,\sigma^2\right) &= \sigma^2 tr\left[t_1\left(XLL'X'\right) + t_2\left(XLL'X'\right)\right. \\ &- t_1\left(XLP_X L'X'\right) - t_2\left(XLP_X L'X'\right) \\ &\left.+ \left(1 - t_1 - t_2\right)\left(L'X'XL - XLP_X L'X'\right)\right] \\ &= \sigma^2 tr\left[t_1 L'X'XL\left(I_n - P_X\right) + t_2 L'X'XL\left(I_n - P_X\right)\right. \\ &\left.+ \left(1 - t_1 - t_2\right) L'X'XL\left(I_n - P_X\right)\right] \\ &= \sigma^2 tr\left[L'X'XL\left(I_n - P_X\right)\right] \geq 0. \end{aligned}$$

Moreover, the equality holds if and only if $L = LP_X$.

Appendix 2: The proof of Equations (15)**.**

Firstly, let $L_0 X'y \in \mathcal{L}_0^h$ we get Equation (15) using (13):

$$\begin{aligned} R\left(L_0 X'y;\ \beta,\ \sigma^2\right) &= \sigma^2 tr\left[t_1\left(I_n - XL_0X'\right)^{'}\left(I_n - XL_0X'\right) + t_2\left(XL_0'X'XL_0X'\right)\right. \\ &\left.+ \left(1 - t_1 - t_2\right)\left(\left(XL_0X' - I_p\right)^{'} XL_0X'\right)\right] \\ &+ \beta' X'\left(I_n - XL_0X'\right)^{'}\left(I_n - XL_0X'\right) X\beta \\ &= \sigma^2 tr\left[XL_0'X'XLX' - \left(1 + t_1 - t_2\right) XL_0X'\right] + \sigma^2 tr\left(t_1 I_n\right) \\ &+ \beta'\left(L_0X'X - I_p\right)^{'} X'X\left(L_0X'X - I_p\right)\beta. \end{aligned}$$

Since $0 \leq t_1, t_2 \leq 1$, we can find such a w denoted by $w = \left(1 + t_1 - t_2\right)/2$, where $0 \leq w \leq 1$. Then above equation can be rewritten as follows:

$$\begin{aligned} R\left(L_0 X'y;\ \beta,\ \sigma^2\right) &= \sigma^2 tr\left[\left(L_0X' - w\left(X'X\right)^{-1} X'\right)^{'} X'X\left(L_0X' - w\left(X'X\right)^{-1} X'\right)\right] \\ &+ \sigma^2 tr\left(t_1 I_n - w^2 X\left(X'X\right)^{-1} X'\right) \\ &+ \beta'\left(L_0X'X - I_p\right)^{'} X'X\left(L_0X'X - I_p\right)\beta. \end{aligned}$$

Let

$$B = X'X, C = (1-w)\, I_p, \tilde{L} = L_0 - w\,(X'X)^{-1}.$$

Then, above equation equal to

$$R\left(L_0X'y\,;\ \beta,\ \sigma^2\right) = \sigma^2 trX\tilde{L}'B\tilde{L}X' + \sigma^2 tr\left(t_1 I_n - w^2 X\,(X'X)^{-1} X'\right)$$
$$+\,\beta'\left[\tilde{L}X'X - C\right]' B\left[\tilde{L}X'X - C\right]\beta.$$

Appendix 3: The proof of Equation (28)**.**

$$\begin{aligned}
R\left(Ly+a;\beta,\sigma^2\right) &= t_1\Big[\sigma^2 tr\,(I_n - XL)'\,(I_n - XL) + \beta'X'\,(I_n - XL)'\,(I_n - XL)\,X\beta \\
&-a'X'(I_n - XL)X\beta - \beta'X'\,(I_n - XL)'\,Xa + a'X'Xa\Big] \\
&+ t_2\left[\sigma^2 tr(L'X'XL) + \beta'X'L'X'XLX\beta + \beta'X'L'X'X(a-\beta)\right. \\
&\left.+(a-\beta)'X'XLX\beta + (a-\beta)'XX(a-\beta)\right] \\
&+ (1-t_1-t_2)\Big[\sigma^2 tr\,(XL - I_n)'\,XL + \beta'X'\,(I_n - XL)'\,XLX\beta \\
&+a'X'XLX\beta + \beta'X'\,(XL - I_n)'\,X(a-\beta) + a'X'X(a-\beta)\Big] \\
&= \sigma^2 tr\Big[t_1\,(I_n - XL)'\,(I_n - XL) + t_2\,(L'X'XL) \\
&+ (1-t_1-t_2)\left((XL - I_n)'\,XL\right)\Big] \\
&+ \left[(LX - I_p)\,\beta + a\right]'\,X'X\left[(LX - I_p)\,\beta + a\right].
\end{aligned}$$

5. CONCLUSIONS

We investigate the explict and implict characterization of admissible linear estimator with respect to the EBLF. The relationship between the EBLF and the quadratic loss is presented in Theorem 2.1. For the admissibility of an estimator of form Ly under the EBLF, the structure of L is given in Theorem 2.2. We also show that admissible linear estimator under the EBLF is a convex combination of the admissible linear estimator under the sum of square residuals and quadratic loss function. The relationship between the admissibility of homogeneous and heterogeneous estimators is given in Theorem 2.4. Furthermore, the alternative theorem for the admissibility of heterogeneous estimators is given as Theorem 2.5.

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