KYBERNETIKA — VOLUME *57* (2021), NUMBER 4, PAGES 594–612

STRONG X-ROBUSTNESS OF INTERVAL MAX-MIN MATRICES

Helena Myšková and Ján Plavka

In max-min algebra the standard pair of operations plus and times is replaced by the pair of operations maximum and minimum, respectively. A max-min matrix A is called strongly robust if the orbit $x, A\otimes x, A^2\otimes x, \ldots$ reaches the greatest eigenvector with any starting vector. We study a special type of the strong robustness called the strong $\boldsymbol{X}$-robustness, the case that a starting vector is limited by a lower bound vector and an upper bound vector. The equivalent condition for the strong $\boldsymbol{X}$-robustness is introduced and efficient algorithms for verifying the strong $\boldsymbol{X}$-robustness is described. The strong $\boldsymbol{X}$-robustness of a max-min matrix is extended to interval vectors $\boldsymbol{X}$ and interval matrices $\boldsymbol{A}$ using for-all–exists quantification of their interval and matrix entries. A complete characterization of AE/EA strong $\boldsymbol{X}$-robustness of interval circulant matrices is presented.

Keywords: max-min algebra, interval matrix, strong robustness, AE(EA) robustness

Classification: 15A18, 15A80, 93C55

1. INTRODUCTION

This paper is concerned with a problem of max-min algebra, which is one of the subareas of tropical mathematics. In a wider algebraic context, tropical mathematics (also known as idempotent mathematics) can be viewed as a mathematical theory developed over idempotent semi-rings. Note that the operation of taking maximum of two numbers is the simplest and the most useful example of an idempotent addition.

Idempotent semi-rings can be used in a range of practical problems related to scheduling and optimization. There are several monographs [9, 10, 12] and collections of papers on tropical mathematics and its applications. Tropical algebra plays a crucial role in the study of dynamic systems with discrete events associated with optimization problems, such as project planning or management, in which the target function depends on maximum and minimum operations.

The development of idempotent semi-rings was motivated by multi-machine interaction processes. In these processes we have n machines which work in stages. In the algebraic model of their interactive work, the entries of a vector $x^{(k)} \in \mathbb{B}(n)$ represent the state of machines after some stage k, and the entries of a matrix $A \in \mathbb{B}(n,n)$ encode the influence of machines on each other. If we assume that A does not change from

DOI: 10.14736/kyb-2021-4-0594

stage to stage then the orbit x, $A \otimes x, \dots A^k \otimes x$ represents the evolution of such process. Regarding the orbits, one wishes to know the set of starting vectors from which a given objective can be achieved. One of objectives in is to achieve the greatest eigenvector. Matrices for which the greatest eigenvector is achieved starting at each starting vector are called strongly robust, see [21]. In general,the set of starting vectors from which an eigenvector can be achieved, contains the set of all eigenvectors, but it can be also as big as the whole space. Matrices with this property are called weakly robust, see [3]. Several types of the robustness in other extremal algebra, so-called max-plus algebra, have been studied in [16, 22].

In practice, the values of starting vector are not exact numbers and usually they are rather contained in some intervals. Considering matrices and vectors with interval entries is therefore of practical importance, see [5, 14, 21]. Problems in which the input data depends on the parameter can be considered as a predecessor of interval calculations [19].

In the present paper, we consider a special type of the strong robustness called the strong $\boldsymbol{X}$-robustness, where $\boldsymbol{X}$ is a given interval vector. Matrices for which the greatest eigenvector is achieved starting at each starting vector from $\boldsymbol{X}$ are called strongly $\boldsymbol{X}$-robust. In Theorem 3.3, we give the equivalent conditions for the strong $\boldsymbol{X}$-robustness. The strong $\boldsymbol{X}$-robustness is extended to interval vectors $\boldsymbol{X}$ using for-all and exists-quantification of their interval entries. We characterize the strong $\boldsymbol{X}^{EA}$-robustness and $\boldsymbol{X}^{AE}$-robustness in Theorems 3.6 and 3.7.

Section 4 is devoted to the special type of matrices, so-called circulant matrices. We give necessary and sufficient conditions to the possible, universal, AE and EA strong $\boldsymbol{X}$-robustness of interval circulant matrices in Theorems 4.17, 4.18, 4.20, 4.21 and 4.23. Finally, the AE/EA strong $\boldsymbol{X}^{AE}$-robustness and the AE/EA strong $\boldsymbol{X}^{EA}$-robustness are characterized in Theorem 4.26 and Theorem 4.27.

The concepts of $\boldsymbol{X}^{AE/EA}$ robustness and AE/EA robustness of interval circulant matrices were studied in [17] and [18].

2. BACKGROUND OF THE PROBLEM

2.1. Preliminaries

The max-min algebra $\mathcal{B}$ is the triple $(\mathcal{I}, \oplus, \otimes)$, where $(\mathcal{I}, \leq)$ is a bounded linearly ordered set with binary operations *maximum* and *minimum*, denoted by $\oplus$ and $\otimes$, respectively. The least element in $\mathcal{I}$ will be denoted by O, the greatest one by I.

By $\mathbb{N}$ we denote the set of all natural numbers and we use the notation $\mathbb{N}_0$ for the set $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. The greatest common divisor of a set S, $S \subseteq \mathbb{N}$ is denoted by $\gcd S$ and the least common multiple by $\operatorname{lcm} S$. For a, $b \in \mathbb{N}$, the symbol $a \mid b$ means that the number a is a divisor of the number b. For a given natural number $n \in \mathbb{N}$, we use the notations $N = \{1, 2, \dots, n\}$ and $N_0 = \{0, 1, \dots, n-1\}$.

For any $n \in \mathbb{N}$ and $\mathcal{I}(n, n)$ denotes the set of all square matrices of order n, $\mathcal{I}(n)$ denotes the set of all n-dimensional column vectors over $\mathcal{B}$. For a given $\alpha \in \mathcal{I}$ a constant vector is denoted by $\alpha^* = (\alpha, \dots, \alpha)^T$. The matrix operations over $\mathcal{B}$ are defined formally in the same manner (with respect to $\oplus$, $\otimes$) as matrix operations over any field. The rth power of a matrix A is denoted by A^r.

For $A \in \mathcal{I}(n,n)$ and $C \in \mathcal{I}(n,n)$ we write $A \leq C$ ($A < C$) if $a_{ij} \leq c_{ij}$ ($a_{ij} < c_{ij}$) holds true for all i, $j \in N$.

By digraph we understand a pair $\mathcal{G} = (V_{\mathcal{G}}, E_{\mathcal{G}})$, where $V_{\mathcal{G}}$ is a non-empty finite set, called the node set, and $E_{\mathcal{G}}$, $E_{\mathcal{G}} \subseteq V_{\mathcal{G}} \times V_{\mathcal{G}}$ is called the arc set. A digraph $\mathcal{G}'$ is a subdigraph of digraph $\mathcal{G}$, if $V_{\mathcal{G}'} \subseteq V_{\mathcal{G}}$ and $E_{\mathcal{G}'} \subseteq E_{\mathcal{G}}$. A path in $\mathcal{G}$ is the sequence of nodes $\mathcal{P} = (v_0, v_1, \ldots, v_l)$ such that $(v_{k-1}, v_k) \in E_{\mathcal{G}}$ for all $k = 1, 2, \ldots, l$. The number $l \geq 0$ is called the length of $\mathcal{P}$. If $v_0 = v_l$, then $\mathcal{P}$ is a cycle of length l. A cycle is elementary if all nodes except the terminal node are distinct. A digraph is called strongly connected if any two distinct nodes of $\mathcal{G}$ are contained in a common cycle.

By a *strongly connected component* of a digraph $\mathcal{G} = (V_{\mathcal{G}}, E_{\mathcal{G}})$ we mean a subdigraph $\mathcal{K} = (V_{\mathcal{K}}, E_{\mathcal{K}})$, where the node set $V_{\mathcal{K}} \subseteq V_{\mathcal{G}}$ is such that any two distinct nodes $i, j \in V_{\mathcal{K}}$ are contained in a common cycle, $E_{\mathcal{K}} = E_{\mathcal{G}} \cap (V_{\mathcal{K}} \times V_{\mathcal{K}})$ and $V_{\mathcal{K}}$ is the maximal subset with this property. A strongly connected component $\mathcal{K}$ of a digraph is called non-trivial, if there is a cycle of positive length in $\mathcal{K}$. For any non-trivial strongly connected component $\mathcal{K}$, the *period* of $\mathcal{K}$ is defined as

$$\operatorname{per}\mathcal{K} = \gcd\{\,\ell(c);\ c \text{ is a cycle in } \mathcal{K},\ \ell(c) > 0\,\}.$$

If $\mathcal{K}$ is trivial, then we define $\operatorname{per}\mathcal{K} = 1$. By $\operatorname{SCC}^*\mathcal{G}$ we denote the set of all non-trivial strongly connected components of $\mathcal{G}$. The set of all strongly connected components of $\mathcal{G}$ is denoted by $\operatorname{SCC}\mathcal{G}$. The period of $\mathcal{G}$ is defined as follows:

$$\operatorname{per}\mathcal{G} = \operatorname{lcm}\{\operatorname{per}\mathcal{K};\ \mathcal{K} \in \operatorname{SCC}^*\mathcal{G}\}.$$

For a given matrix $A \in \mathcal{I}(n,n)$ and a number $h \in \mathcal{I}$, the *threshold digraph* $\mathcal{G}(A,h)$ is the digraph with the node set $V_{\mathcal{G}(A,h)} = N$ and the edge set $E_{\mathcal{G}(A,h)} = \{(i,j) \in N \times N;\ a_{ij} \geq h\}$.

The following lemma describes the relation between matrices and corresponding threshold digraphs.

Lemma 2.1. (Molnárová et al. [13]) Let A, $C \in \mathcal{I}(n,n)$ and h, h_1, $h_2 \in \mathcal{I}$.

(i) If $A \leq C$ then $\mathcal{G}(A,h) \subseteq \mathcal{G}(C,h)$,

(ii) if $h_1 < h_2$ then $\mathcal{G}(A,h_2) \subseteq \mathcal{G}(A,h_1)$.

For a given matrix A and a given vector x, define a *threshold matrix* $A_{(h)}$ and *threshold vector* $x_{(h)}$, respectively, as follows:

$$(a_{(h)})_{ij} = \begin{cases} I & \text{if } a_{ij} \geq h, \\ O & \text{otherwise.} \end{cases} \qquad (x_{(h)})_i = \begin{cases} I & \text{if } x_j \geq h, \\ O & \text{otherwise.} \end{cases}$$

For a given matrix $A \in \mathcal{I}(n,n)$, denote $H(A) = \{a_{ij};\ i,j \in N\}$.

Lemma 2.2. (Semančíková [24]) The decomposition of a matrix A over $\mathcal{I}$ to its threshold matrices has the following properties:

(i) $A = \bigoplus_{h \in H}(h \otimes A_{(h)})$ for any set H such that $H(A) \subseteq H \subseteq \mathcal{I}$.

(ii) For any two $\otimes$-compatible matrices A and B, $(A \otimes B)_{(h)} = A_{(h)} \otimes B_{(h)}$ for any $h \in \mathcal{I}$. Hence $A \otimes B = \bigoplus_{h \in H}(h \otimes A_{(h)} \otimes B_{(h)})$ for any set H such that $H(A) \cup H(B) \subseteq H \subseteq \mathcal{I}$.

2.2. Periodicity of matrices and orbits

Let $\lambda \in \mathcal{I}$. A matrix $A \in \mathcal{I}(n,n)$ is *ultimately* λ*-periodic* if there are natural numbers p and R such that the equality

$$A^{k+p} = \lambda \otimes A^k$$

holds for each $k \geq R$. The smallest natural number p with the above property is called the period of A, denoted by $\operatorname{per}(A,\lambda)$. The smallest R with the above property is called the *defect* of A, denoted by $\operatorname{def}(A,\lambda)$. It is well known that in max-min algebra, $\operatorname{def}(A) \leq (n-1)^2 + 1$.

In case $\lambda = I$ we us the notation $\operatorname{per} A$ instead of $\operatorname{per}(A,I)$.

Definition 2.3. For any $A \in \mathcal{I}(n,n)$ and $x \in \mathcal{I}(n)$ the orbit of A generated by x is the vector sequence $\mathcal{O}(A,x) = (x(r); r \in \mathbb{N}_0)$ whose initial vector is $x(0) = x$ and successive members are defined by the formula $x(r+1) = A \otimes x(r)$. The ith coordinate of $x(r)$ is denoted by $x_i(r)$. The ith coordinate orbit is the sequence $\mathcal{O}_i(A,x) = (x_i(r);\ r \in \mathbb{N}_0)$.

Both operations in max-min algebra are idempotent, so no new numbers are created in the process of generating of matrix powers and an orbit. Hence a power sequence, an orbit $\mathcal{O}(A,x)$ and a coordinate orbit $\mathcal{O}_i(A,x)$ are always ultimately periodic sequences. Their periods will be called the *period* of A, the *orbit period* and the *coordinate-orbit* period of $\mathcal{O}(A,x)$, in notation $\operatorname{per}(A)$, $\operatorname{per}(A,x)$ and $\operatorname{per}(A,x,i)$, respectively. Similarly, we denote the defects by $\operatorname{def}(A,x)$ and $\operatorname{def}(A,x,i)$.

Theorem 2.4. (Gavalec [7], Semančíková [24]) Let $A \in \mathcal{I}(n,n)$ and $x \in \mathcal{I}(n)$. Then

(i) $\operatorname{per}(A) = \operatorname*{lcm}_{x \in \mathcal{I}(n)} \operatorname{per}(A,x)$;

(ii) $\operatorname{per}(A,x) = \operatorname*{lcm}_{i \in N} \operatorname{per}(A,x,i)$.

Corollary 2.5. Let $A \in \mathcal{I}(n,n)$ and $x \in \mathcal{I}(n)$. Then $\operatorname{per}(A,x) \,|\, \operatorname{per}(A)$ for each $x \in \mathcal{I}(n)$.

Denote by $\mathcal{G}(A,x,h)$ the digraph with the same vertex set and edge set as $\mathcal{G}(A,h)$, but with evaluated vertices as follows.

Denote by $\mathcal{O}^{\text{per}}(A,x)$ and $\mathcal{O}_i^{\text{per}}(A,x)$ the *periodic part* of $\mathcal{O}(A,x)$ and $\mathcal{O}_i(A,x)$, respectively, i. e., $\mathcal{O}^{\text{per}}(A,x) = (x(r);\ r > \operatorname{def}(A,x))$ and $\mathcal{O}_i^{\text{per}}(A,x) = (x_i(r);\ r > \operatorname{def}(A,x,i))$. By (O) and (I) we understand the infinite sequences of the same elements O and I, respectively.

2.3. Robustness of matrices

For a given matrix $A \in \mathcal{I}(n,n)$, define an eigenvalue and an eigenvector as a number $\lambda \in \mathcal{I}$ and a vector $x \in \mathcal{I}(n)$ which satisfy the equality

$$A \otimes x = \lambda \otimes x.$$

An *eigenspace* $V(A,\lambda)$ is defined as the set of all eigenvectors of A corresponding to eigenvalue λ, i. e.,

$$V(A,\lambda) = \{x \in \mathcal{I}(n);\ A \otimes x = \lambda \otimes x\}.$$

An *attraction set* is the set defined as follows:

$$\mathrm{attr}(A,\lambda) = \{x \in \mathcal{I}(n);\ O(A,x) \cap V(A,\lambda) \neq \emptyset\}.$$

For $\lambda = I$ we will use notations $V(A)$ and $\mathrm{attr}(A)$ instead of $V(A,\lambda)$ and $\mathrm{attr}(A,\lambda)$, respectively.

Let $A = (a_{ij}) \in \mathcal{I}(n,n)$, $\lambda \in \mathcal{I}$ be given. Define the greatest eigenvector $x^{\oplus}(A,\lambda)$ corresponding to a matrix A and λ as

$$x^{\oplus}(A,\lambda) = \bigoplus_{x \in V(A,\lambda)} x.$$

For $\lambda = I$ let us denote use abbreviation $x^{\oplus}(A)$ instead of $x^{\oplus}(A,I)$.
Denote

$$r_j(A) = \bigoplus_{i \in N} a_{ij}, \quad c(A) = \bigotimes_{i \in N} r_j(A).$$

In [25] it was stated for a more general algebraic structure (distributive lattice) that the greatest eigenvector $x^{\oplus}(A,\lambda)$ exists for every matrix A and every $\lambda \in B$ whereby its entries are given by the efficient formula presented in the next theorem.

Theorem 2.6. (Yi-Jia Tan [26]) Let $B(\oplus,\otimes)$ be a distributive lattice. Let $A \in B(n,n)$ and $\lambda \in B$ be given. Then

$$x_j^{\oplus}(A,\lambda) = \begin{cases} I, & \text{if } \lambda \oplus r_j(A) \leq \lambda \otimes (A^n \otimes I^*)_j, \\ \lambda \otimes (A^n \otimes I^*)_j, & \text{otherwise.} \end{cases} \tag{1}$$

Corollary 2.7. Let $A \in \mathcal{I}(n,n)$ be given. Then $x^{\oplus}(A) = A^n \otimes I^*$.

Computing the greatest eigenvector lying in the given interval $\boldsymbol{X}$ has been studied in [23].

Denote by $\mathrm{attr}^*(A,\lambda)$ the set

$$\mathrm{attr}^*(A,\lambda) = \{x \in \mathcal{I}(n); x^{\oplus}(A,\lambda) \in O(A,x)\}.$$

It is easy to see that $x^{\oplus}(A) \geq c^*(A) = (c(A),\dots,c(A))^T$ holds true and $x^{\oplus}(A)$ can not be reached with a vector $x \in \mathcal{I}(n)$ such that $x < c^*(A)$.

Definition 2.8. Let $A \in \mathcal{I}(n,n)$ be given. Then A is called

1. robust if $\mathrm{attr}(A) = \mathcal{I}(n)$,
2. strongly robust if $\mathrm{attr}^*(A) = \mathcal{I}(n) \setminus \{x \in \mathcal{I}(n);\ x < c^*(A)\}$.

Theorem 2.9. (Plavka and Szabó [20]) Let $A \in \mathcal{I}(n,n)$, $\lambda \in \mathcal{I}$ be given. Then A is robust if and only if $\mathrm{per}(A,\lambda) = 1$.

Theorem 2.10. (Plavka and Szabó [20]) Let $A \in \mathcal{I}(n,n)$ be given. Then A is strongly robust if and only if $x^{\oplus}(A,\lambda) = c^*(A)$ and $\mathcal{G}(A,c(A))$ is a strongly connected digraph with period equal to 1.

The concepts of the robustness and strong robustness have been studied in [1, 2, 20]. Equivalent conditions and efficient algorithms for interval case have been presented in [13, 21].

2.4. $\boldsymbol{X}$-robustness

In this part, we will formulate the necessary and sufficient conditions for (strong) $\boldsymbol{X}$-robustness. Similarly as in [5, 13, 15, 17], we define an interval vector $\boldsymbol{X}$ as follows:

$$\boldsymbol{X} = [\underline{x}, \overline{x}] = \{ x \in \mathcal{I}(n); \underline{x} \leq x \leq \overline{x} \},$$

where $\underline{x}, \overline{x} \in \mathcal{I}(n)$, $\underline{x} \leq \overline{x}$.

Definition 2.11. Let $A \in \mathcal{I}(n,n)$ and $\boldsymbol{X}$ be given. Then A is called $\boldsymbol{X}$-robust if $\boldsymbol{X} \subseteq \operatorname{attr}(A)$.

The following lemma follows directly from Definition 3.1.

Lemma 2.12. Let $A, \boldsymbol{X}$ be given. The following assertions are equivalent

1. A is $\boldsymbol{X}$-robust,
2. $(\forall x \in \boldsymbol{X})(\exists k \in \mathbb{N})[A^k \otimes x \in V(A) \cap \boldsymbol{X}]$,
3. $(\forall x \in \boldsymbol{X})[\operatorname{per}(A, x) = 1]$.

Let $\boldsymbol{X} = [\underline{x}, \overline{x}] \subset \mathcal{I}(n)$ be an interval vector. For a given index $i \in N$ denote the vector

$$x^{(i)} = (\underline{x}_1, \ldots, \underline{x}_{i-1}, \overline{x}_i, \underline{x}_{i+1}, \ldots, \underline{x}_n)^T.$$

Lemma 2.13. (Myšková and Plavka [15]) Let $\boldsymbol{X}$ be given and $x \in \boldsymbol{X}$. Then

$$x = \bigoplus_{i=1}^{n} x_i \otimes x^{(i)}.$$

Theorem 2.14. (Myšková and Plavka [15]) Let $A \in \mathcal{I}(n,n)$ and $\boldsymbol{X}$ be given. A matrix A is $\boldsymbol{X}$-robust if and only if $\operatorname{per}(A, x^{(j)}) = 1$ for each $j \in N$.

The decision whether a given matrix A is $\boldsymbol{X}$-robust using Theorem 2.14 requires $O(n^4 \log n)$ arithmetic operations.

3. THE TYPES OF STRONG ROBUSTNESS WITH BOUNDED VECTOR

3.1. Strong $\boldsymbol{X}$-robustness

Definition 3.1. Let $A \in \mathcal{I}(n,n)$ and $\boldsymbol{X}$ be given. Then A is called strongly $\boldsymbol{X}$-robust if $\boldsymbol{X} \subseteq \operatorname{attr}^*(A)$.

It is easy to see that for a given matrix A and a given interval vector $\boldsymbol{X}$, the strong $\boldsymbol{X}$-robustness of A implies its $\boldsymbol{X}$-robustness. It can be expected that the converse implication does not apply. In this part, we give the necessary and sufficient conditions for the strong $\boldsymbol{X}$-robustness.

Lemma 3.2. Let $A, \boldsymbol{X}$ be given. A matrix A is strongly $\boldsymbol{X}$-robust if and only if $A^{n^2} \otimes x = x^{\oplus}(A)$ for each $x \in \boldsymbol{X}$.

P r o o f. The assertion trivially follows from the above definitions and the fact that $\operatorname{def}(A) \leq (n-1)^2+1 \leq n^2$. Therefore, the existence of $k \in \mathbb{N}$ such that $A^k \otimes x = x^{\oplus}(A)$ we can replace with the condition $A^{n^2} \otimes x = x^{\oplus}(A)$. $\square$

Theorem 3.3. Let $A, \boldsymbol{X}$ be given. Then A is strongly $\boldsymbol{X}$-robust if and only if $\underline{x} \in \operatorname{attr}^*(A)$.

P r o o f. Let $x \in \boldsymbol{X}$ be arbitrary. Then

$$x^{\oplus}(A) = A^{n^2} \otimes \underline{x} \leq A^{n^2} \otimes x \leq A^{n^2} \otimes \overline{x} \leq A^{n^2} \otimes I^* = x^{\oplus}(A),$$

where the last equality follows from Corollary 2.7. We have $A^{n^2} \otimes x = x^{\oplus}(A)$ for each $x \in \boldsymbol{X}$. By Lemma 3.2 a matrix A is strongly $\boldsymbol{X}$-robust. $\square$

Theorem 3.4. Let A and $\boldsymbol{X}$ be given. If A is strongly $\boldsymbol{X}$-robust then $\max_{i \in N} \underline{x}_i \geq \max_{k \in N} x_k^{\oplus}(A)$.

P r o o f. Suppose that the inequality $\underline{x}_j < \max_{k \in N} x_k^{\oplus}(A) = x_t^{\oplus}(A)$ holds for each $j \in N$. Then we have

$$(A^{n^2} \otimes \underline{x})_t = \bigoplus_{k \in N} a_{tk}^{n^2} \otimes \underline{x}_k \leq \bigoplus_{k \in N} \underline{x}_k < x_t^{\oplus}(A).$$

Since $A^{n^2} \otimes \underline{x} \neq x^{\oplus}(A)$, by Theorem 3.3 the matrix A is not strongly $\boldsymbol{X}$-robust. $\square$

3.2. Strong $\boldsymbol{X}^{AE}/\boldsymbol{X}^{EA}$-robustness

In this part, we define the strong $\boldsymbol{X}^{AE}$ and $\boldsymbol{X}^{EA}$-robustness and give equivalent conditions for them. Similarly as in [11, 17, 18], each element of $\boldsymbol{X}$ can be associated either with the universal, or with the existential quantifier. Then we can split the interval vector as $\boldsymbol{X} = \boldsymbol{X}^{\forall} \oplus \boldsymbol{X}^{\exists}$, where $\boldsymbol{X}^{\forall}$ is the interval vector comprising universally quantified coefficients and $\boldsymbol{X}^{\exists}$ concerns existentially quantified coefficients.

Denote by $N^{\forall}$ ($N^{\exists}$) the set of indices corresponding with universal (existential) quantifier in $\boldsymbol{X}$, respectively. In the other words, $\underline{x}_i^{\exists} = \overline{x}_i^{\exists} = O$ for each $i \in N^{\forall}$ and $\underline{x}_i^{\forall} = \overline{x}_i^{\forall} = O$ for each $i \in N^{\exists}$.

Definition 3.5. A matrix A is called

- *strongly $\boldsymbol{X}^{EA}$-robust* if

$$(\exists x^{\exists} \in \boldsymbol{X}^{\exists})(\forall x^{\forall} \in \boldsymbol{X}^{\forall})\ x^{\exists} \oplus x^{\forall} \in \operatorname{attr}^*(A),$$

- *strongly $\boldsymbol{X}^{AE}$-robust* if

$$(\forall x^{\forall} \in \boldsymbol{X}^{\forall})(\exists x^{\exists} \in \boldsymbol{X}^{\exists})\ x^{\exists} \oplus x^{\forall} \in \operatorname{attr}^*(A).$$

Theorem 3.6. Let $A, \boldsymbol{X}$ be given. Then A is strongly $\boldsymbol{X}^{EA}$-robust if and only if $\overline{x}^{\exists} \oplus \underline{x}^{\forall} \in \operatorname{attr}^*(A)$.

Proof. Suppose that $\overline{x}^{\exists} \oplus \underline{x}^{\forall} \in \operatorname{attr}^*(A)$, i.e., $A^{n^2} \otimes (\overline{x}^{\exists} \oplus \underline{x}^{\forall}) = x^{\oplus}(A)$. Let $x^{\forall} \in \boldsymbol{X}^{\forall}$ be arbitrary. We obtain

$$x^{\oplus}(A) = A^{n^2} \otimes (\overline{x}^{\exists} \oplus \underline{x}^{\forall}) \leq A^{n^2} \otimes (\overline{x}^{\exists} \oplus x^{\forall}) \leq A^{n^2} \otimes I^* = x^{\oplus}(A).$$

Therefore there exists $x^{\exists} \in \boldsymbol{X}^{\exists}$, namely $x^{\exists} = \overline{x}^{\exists}$ such that $A^{n^2} \otimes (\overline{x}^{\exists} \oplus x^{\forall}) = x^{\oplus}(A)$ for each $x^{\forall} \in \boldsymbol{X}^{\forall}$. Hence A is strongly $\boldsymbol{X}$-robust.

For the converse implication suppose that $A^{n^2} \otimes (\overline{x}^{\exists} \oplus \underline{x}^{\forall}) \neq x^{\oplus}(A)$. Then there exists $i \in N$ such that $[A^{n^2} \otimes (\overline{x}^{\exists} \oplus \underline{x}^{\forall})]_i < x_i^{\oplus}(A)$. Let $x^{\exists} \in \boldsymbol{X}^{\exists}$ be arbitrary, but fixed. We have

$$[A^{n^2} \otimes (x^{\exists} \oplus \underline{x}^{\forall})]_i \leq [A^{n^2} \otimes (\overline{x}^{\exists} \oplus \underline{x}^{\forall})]_i < x_i^{\oplus}(A).$$

Since for each $x^{\exists} \in \boldsymbol{X}^{\exists}$ there exists $x^{\forall} \in \boldsymbol{X}^{\forall}$, namely $x^{\forall} = \underline{x}^{\forall}$ such that $x^{\exists} \oplus x^{\forall} \notin \operatorname{attr}^*(A)$, a matrix A is not $\boldsymbol{X}^{EA}$-robust. □

Theorem 3.7. Let $A, \boldsymbol{X}$ be given. Then A is strongly $\boldsymbol{X}^{EA}$-robust if and only if A is strongly $\boldsymbol{X}^{AE}$-robust.

Proof. From Definition 3.5 it follows that the strong $\boldsymbol{X}^{EA}$-robustness implies the strong $\boldsymbol{X}^{AE}$-robustness.

For the converse implication suppose that A is not strongly $\boldsymbol{X}^{EA}$-robust. According to Theorem 3.6, $\overline{x}^{\exists} \oplus \underline{x}^{\forall} \notin \operatorname{attr}^*(A)$, i.e., there exists $i \in N$ such that $[A^{n^2} \otimes (\overline{x}^{\exists} \oplus \underline{x}^{\forall})]_i < x_i^{\oplus}(A)$.

Let $x^{\exists} \in \boldsymbol{X}^{\exists}$ be arbitrary, but fixed. We obtain

$$[A^{n^2} \otimes (x^{\exists} \oplus \underline{x}^{\forall})]_i \leq [A^{n^2} \otimes (\overline{x}^{\exists} \oplus \underline{x}^{\forall})]_i < x_i^{\oplus}(A).$$

Therefore, there exists $x^{\forall} \in \boldsymbol{X}^{\forall}$, namely $x^{\forall} = \underline{x}^{\forall}$, such that for each $x^{\exists} \in \boldsymbol{X}^{\exists}$ we have $x^{\exists} \oplus \underline{x}^{\forall} \notin \operatorname{attr}^*(A)$. Hence A is not strongly $\boldsymbol{X}^{AE}$-robust. □

4. STRONG ROBUSTNESS OF INTERVAL CIRCULANT MATRICES

In this section we shall deal with the special class of matrices, the circulant matrices. The notions of a possible, universal, AE and EA $\boldsymbol{X}^{EA}/\boldsymbol{X}^{AE}$ strong robustness of an interval circulant matrix are defined and the polynomial algorithms for checking of them are introduced in this section.

Definition 4.1. A matrix $A \in \mathcal{I}(n,n)$ is called circulant, if it is of the form

$$A = \begin{pmatrix} a_0 & a_1 & a_2 & \dots & a_{n-2} & a_{n-1} \\ a_{n-1} & a_0 & a_1 & \dots & a_{n-3} & a_{n-2} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ a_1 & a_2 & a_3 & \dots & a_{n-1} & a_0 \end{pmatrix}.$$

We denote a circulant matrix A by abbreviation $A = \mathcal{C}(a_0, \ldots, a_{n-1})$.

For each $k \in N_0$ the entries a_k create the kth stripe. In the associated digraph $\mathcal{G}(A)$ the arcs corresponding to the kth stripe are of the form $(i, i+k)$ for $i \in N$, where all the numbers here are considered modulo n. Denote by $E^{(k)}$ the set of all arcs (i,j) in $\mathcal{G}(A)$ corresponding to the kth stripe. The arcs of $E^{(k)}$ fall into a set of disjoint cycles, all with the same length equal to $\dfrac{n}{\gcd(n,k)}$.

Realize that for a circulant matrix A we have $H(A) = \{a_i;\ i \in N_0\}$. Denote $m(A) = \max H(A)$ and $J(A) = \{i \in N_0 : a_i = m(A)\}$.

If a vector x is also given, denote $H(A,x) = H(A) \cup \{x_j;\ j \in N\}$.

Lemma 4.2. (Molnárová et al. [13]) Let $A \in \mathcal{I}(n,n)$ be circulant matrix. The following assertions hold true:

(i) For each $h \in H(A)$ the threshold digraph $\mathcal{G}(A,h)$ is either strongly connected or all strongly connected components are nontrivial.

(ii) If k, $l \in N$ are not lying in the same strongly connected component in $\mathcal{G}(A,h)$ then there is no edge from k to l.

The following theorem gives the formula for the computation the period of a circulant matrix.

Theorem 4.3. (Gavalec [6]) Let $A = \mathcal{C}(a_0, \ldots, a_{n-1})$ be a circulant matrix and $J(A) = \{i_0, i_1, \ldots, i_{k-1}\}$. Then

$$\mathrm{per}(A) = \gcd\left(\frac{n}{\gcd(n,i_0)}, \frac{i_0 - i_1}{\gcd(n,i_0,i_1)}, \frac{i_0 - i_2}{\gcd(n,i_0,i_1,i_2)}, \ldots, \frac{i_0 - i_{k-1}}{\gcd(n,i_0,\ldots,,i_{k-1})}\right).$$

According to [6] we can compute per(A) is $O(n)$ time.

Corollary 4.4. Let A be a circulant matrix. Then $\mathrm{per}(A) \le n$.

Definition 4.5. A path in $\mathcal{G}(A,x,h)$ with terminal node i is called an *orbit path* if $(x_{(h)})_i = I$.

Theorem 4.6. (Cechlárová [4]) For $A \in \mathcal{I}(n,n)$, $x \in \mathcal{I}(n)$, $h \in \mathcal{I}$, $r \in \mathbb{N}$ and $i,j \in N$,

(i) $(A^r)_{ij} \ge h$ if and only if there is a path in $\mathcal{G}(A,h)$ from i to j of length r,

(ii) $\mathcal{O}_i(A,x)(r) \ge h$ if and only if there is an orbit path in $\mathcal{G}(A,x,h)$ of length r starting at i.

Denote by $\mathcal{G}(A,x)[i]$ the strongly connected component of $\mathcal{G}(A,x)$ containing node i.

Lemma 4.7. (Myšková and Plavka [15]) Let $A \in \mathcal{I}(n,n)$ be a binary circulant matrix and $x \in \mathcal{I}(n)$ be a binary vector and $i \in N$. Then $\mathcal{O}_i^{\mathrm{per}}(A,x) \ne (O)$ if and only if $A \ne O$ and there exists $j \in V_{\mathcal{G}(A,x)[i]}$ such that $x_j = I$.

Remark 4.8. Lemma 4.7 does not hold for a general binary matrix. If $x_j = O$ for each $j \in V_{\mathcal{G}(A,x)[i]}$ and there exists an edge from $\mathcal{G}(A,x)[i]$ to $\mathcal{K} \in \mathrm{SCC}^* \mathcal{G}(A,x)$ such that $x_l = I$ for some $l \in V_{\mathcal{K}}$ then $\mathcal{O}_i^{\mathrm{per}}(A,x) \neq (O)$.

For a given $h \in \mathcal{I}$ we shall use the following notations:

$$\mathrm{SCC}_0^* \mathcal{G}(A,x,h) = \{\mathcal{K} \in \mathrm{SCC}^* \mathcal{G}(A,h);\ (x_{(h)})_i = O \text{ for each } i \in V_{\mathcal{K}}\},$$
$$\mathrm{SCC}_1^* \mathcal{G}(A,x,h) = \{\mathcal{K} \in \mathrm{SCC}^* \mathcal{G}(A,h);\ (x_{(h)})_i = I \text{ for some } i \in V_{\mathcal{K}}\}.$$

Definition 4.9. Let A be a circulant matrix and $h \in \mathcal{I}$. We say that a strongly connected component $\mathcal{K} \in \mathrm{SCC}_1^* \mathcal{G}(A,x,h)$ *has the property* (P), if for each $i \in V_{\mathcal{K}}$ there exist $r \in \mathbb{N}_0$ such that for each $k > r$ there exists an orbit path from i in $\mathcal{K}$ of length k.

Theorem 4.10. (Myšková and Plavka [15]) Let $A^C = \mathcal{C}(a_0, a_1, \ldots, a_{n-1})$ be a circulant matrix, $A^C \neq O$. Then $\mathrm{per}(A,x) = 1$ if and only if each strongly connected component $\mathcal{K} \in \mathrm{SCC}_1^* \mathcal{G}(A,x,h)$ has the property (P) for each $h \in H(A,x)$.

Lemma 4.11. (Myšková and Plavka [15]) Let $A, C \in \mathcal{I}(n,n)$, $x \in \mathcal{I}(n)$ and $\mathcal{K}_1 \in \mathrm{SCC}_1^* \mathcal{G}(A,x,h_1)$, $\mathcal{K}_2 \in \mathrm{SCC}_1^* \mathcal{G}(C,x,h_2)$ be such that $\mathcal{K}_1 \subseteq \mathcal{K}_2$, $h_1 \geq h_2$ and $\mathcal{K}_1$ has the property (P). Then $\mathcal{K}_2$ has the property (P).

According to Corollary 2.7, for a circulant matrix $A = \mathcal{C}(a_0, a_i, \ldots, a_{n-1})$ we obtain

$$x^{\oplus}(A) = m(A)^* \in \mathcal{I}(n). \tag{2}$$

Theorem 4.12. Let $A = \mathcal{C}(a_0, a_i, \ldots, a_{n-1})$ and $x \in \mathcal{I}(n)$ be given. Then $x \in \mathrm{attr}^*(A)$ if and only if for each $\mathcal{K} \in \mathrm{SCC}^* \mathcal{G}(A, m(A))$ the following assertions hold:

- $\mathcal{K} \in \mathrm{SCC}_1^* \mathcal{G}(A,x,m(A))$;
- $\mathcal{K}$ has property (P).

Proof. Denote $h(A,x,i) = \max\{h \in H(A,x);\ \mathcal{O}_i^{\mathrm{per}}(A_{(h)}, x_{(h)}) \neq (O)\}$. According to Lemma 2.2 we have

$$x_i(r) = (A^r \otimes x)_i = \bigoplus_{h \leq h(A,x,i),\, h \in H(A,x)} h \otimes (A^r_{(h)} \otimes x_{(h)})_i$$

for each $r > \max_{h \in H(A,x)} \mathrm{def}(A_{(h)}, x_{(h)}, i)$, or equivalently, $\mathcal{O}_i^{\mathrm{per}}(A,x) = \bigoplus_{h \leq h(A,x,i)} h \otimes \mathcal{O}_i^{\mathrm{per}}(A_{(h)}, x_{(h)})$. A vector x belongs to $\mathrm{attr}^*(A)$ if and only if $A^{n^2} \otimes x = m^*(A) \in \mathcal{I}(n)$. This means that for each $i \in N$ it has to be $\mathcal{O}_i^{\mathrm{per}}(A_{(m(A))}, x_{(m(A))}) = (I)$.

Suppose that there exists $\mathcal{K} \in \mathrm{SCC}^* \mathcal{G}(A, m(A))$ such that either

$$\mathcal{K} \in \mathrm{SCC}_0^* \mathcal{G}(A_{m(A)}, x_{(m(A))}) \text{ or } \mathcal{K} \text{ does not have property (P).}$$

In the first case, by Lemma 4.7, we obtain $\mathcal{O}_i^{\mathrm{per}}(A_{(m(A))}, x_{(m(A))}) = (O)$ for each $i \in V_{\mathcal{K}}$. Then $x \notin \mathrm{attr}^*(A)$.

If $\mathcal{K}$ does not have property (P) then according to Theorem 4.10 we have $\mathrm{per}(A,x) \neq 1$ which implies $x \notin \mathrm{attr}(A)$ and consequently $x \notin \mathrm{attr}^*(A)$.

For the converse implication suppose that for each $\mathcal{K} \in \mathrm{SCC}^* \mathcal{G}(A, m(A))$ we have $\mathcal{K} \in \mathrm{SCC}_1^* \mathcal{G}(A,x,m(A))$ and $\mathcal{K}$ has property (P). Consequently, $\mathcal{O}_i^{\mathrm{per}}(A_{(m(A))}, x_{(m(A))}) = (I)$ for each $i \in N$ which implies $A^{n^2} \otimes x = x^{\oplus}(A)$. Hence $x \in \mathrm{attr}^*(A)$. $\square$

4.1. Possible and universal strong $\boldsymbol{X}$-robustness of interval circulant matrices

Definition 4.13. An interval circulant matrix $\boldsymbol{A}^C$ is the set of all circulant matrices $A \in \boldsymbol{A}$ where

$$\boldsymbol{A} = \begin{pmatrix} \boldsymbol{a}_0 & \boldsymbol{a}_1 & \boldsymbol{a}_2 & \dots & \boldsymbol{a}_{n-2} & \boldsymbol{a}_{n-1} \\ \boldsymbol{a}_{n-1} & \boldsymbol{a}_0 & \boldsymbol{a}_1 & \dots & \boldsymbol{a}_{n-3} & \boldsymbol{a}_{n-2} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ \boldsymbol{a}_1 & \boldsymbol{a}_2 & \boldsymbol{a}_3 & \dots & \boldsymbol{a}_{n-1} & \boldsymbol{a}_0 \end{pmatrix},$$

and $\mathbf{a}_i = [\underline{a}_i, \overline{a}_i]$ for each $i \in N_0$. We denote an interval circulant matrix $\boldsymbol{A}^C$ by abbreviation $\boldsymbol{A}^C = \mathcal{C}(\mathbf{a}_0, \dots, \mathbf{a}_{n-1})$.

There are matrices in $\boldsymbol{A}$ that are not circulant, so $\boldsymbol{A} \neq \boldsymbol{A}^C$. On the other hand $\underline{A}$, $\overline{A} \in \boldsymbol{A}^C$, therefore the set $\boldsymbol{A}^C$ is always non-empty.

Definition 4.14. An interval circulant matrix $\boldsymbol{A}^C = \mathcal{C}(\mathbf{a}_0, \dots, \mathbf{a}_{n-1})$ is

- possibly strongly $\boldsymbol{X}$-robust, if there exists $A \in \boldsymbol{A}^C$ such that A is strongly $\boldsymbol{X}$-robust,
- universally strongly $\boldsymbol{X}$-robust, if any matrix $A \in \boldsymbol{A}^C$ is strongly $\boldsymbol{X}$-robust.

To give a necessary and sufficient condition for the possible strong $\boldsymbol{X}$-robustness let us define the circulant matrix $A^* = \mathcal{C}(a_0^*, a_1^*, \dots, a_{n-1}^*)$ as follows:

$$a_i^* = \min\{m(\underline{A}), \overline{a}_i\} \tag{3}$$

for each $i \in N_0$.

Lemma 4.15. (Molnárová et al. [13]) Let $A \in \boldsymbol{A}^C$ be arbitrary and let A^* be defined by (3). Then $J(A) \subseteq J(A^*)$.

Lemma 4.16. (Myšková and Plavka [15]) Let $x \in \mathcal{I}(n)$. Then there exists a matrix $A \in \boldsymbol{A}^C$ such that $\mathrm{per}(A, x) = 1$ if and only if $\mathrm{per}(A^*, x) = 1$.

Theorem 4.17. An interval circulant matrix $\boldsymbol{A}^C = \mathcal{C}(\mathbf{a}_0, \dots, \mathbf{a}_{n-1})$ is possibly strongly $\boldsymbol{X}$-robust if and only if the matrix A^* defined by (3) is strongly $\boldsymbol{X}$-robust.

Proof. Suppose that A^* is not strongly $\boldsymbol{X}$-robust. According to Theorem 3.3, we have $\underline{x} \notin \mathrm{attr}^*(A^*)$. By Theorem 4.12 there exists $\mathcal{K} \in \mathrm{SCC}^* \,\mathcal{G}(A^*, m(A^*))$ such that either $\mathcal{K} \in \mathrm{SCC}_0^* \,\mathcal{G}(A^*, \underline{x}, m(A^*))$ or $\mathcal{K}$ does not have property (P). Let $A \in \boldsymbol{A}^C$ be arbitrary, but fixed. It follows from Lemma 4.15 that $\mathcal{G}(A, m(A) \subseteq \mathcal{G}(A^*, m(A^*))$. We shall distinguish two possibilities.

(i) If $\mathcal{K} \in \mathrm{SCC}_0^* \,\mathcal{G}(A^*, \underline{x}, m(A^*))$ then for each $\mathcal{K}_1 \in \mathrm{SCC}^* \,\mathcal{G}(A, (m(A)))$, $\mathcal{K}_1 \subseteq \mathcal{K}$ we have $\mathcal{K}_1 \in \mathrm{SCC}_0^* \,\mathcal{G}(A, \underline{x}, m(A))$. According to Theorem 4.12 we obtain $\underline{x} \notin \mathrm{attr}^*(A)$ and by Theorem 3.3 the matrix A is not strongly $\boldsymbol{X}$-robust.

(ii) Let $\mathcal{K} \in \mathrm{SCC}_1^* \mathcal{G}(A^*, \underline{x}, m(A^*))$ be such that $\mathcal{K}$ does not have property (P) and let $\mathcal{K}_1 \in \mathrm{SCC}^* \mathcal{G}(A, m(A))$, $\mathcal{K}_1 \subseteq \mathcal{K}$. If $\mathcal{K}_1 \in \mathrm{SCC}_1^* \mathcal{G}(A, \underline{x}, m(A))$ then, according to Lemma 4.11, $\mathcal{K}_1$ does not have property (P). Another possibility is that $\mathcal{K}_1 \in \mathrm{SCC}_0^* \mathcal{G}(A, \underline{x}, m(A))$.

In both cases, by Theorem 4.12, we obtain $\underline{x} \notin \mathrm{attr}^*(A)$. According to Theorem 3.3 the matrix A is not strongly $\boldsymbol{X}$-robust.

The converse implication trivially follows. □

We will proceed with the universal strong $\boldsymbol{X}$-robustness. For this purpose, for a given number $k \in N_0$ denote by $A^{(k)}$ the circulant matrix

$$A^{(k)} = \mathcal{C}(\underline{a}_0, \underline{a}_1, \ldots, \underline{a}_{k-1}, \overline{a}_k, \underline{a}_{k+1}, \ldots, \underline{a}_{n-1}).$$

Theorem 4.18. An interval circulant matrix $\boldsymbol{A}^C = \mathcal{C}(\mathbf{a}_0, \ldots, \mathbf{a}_{n-1})$ is universally strongly $\boldsymbol{X}$-robust if and only if for each $k \in N_0$ the matrix $A^{(k)}$ is strongly $\boldsymbol{X}$-robust.

P r o o f. Suppose that $\boldsymbol{A}^C$ is not universally strongly $\boldsymbol{X}$-robust. We shall prove that there exists $l \in N_0$ such that the matrix $A^{(l)}$ is not strongly $\boldsymbol{X}$-robust.

Let $A \in \boldsymbol{A}^C$ be such that A is not strongly $\boldsymbol{X}$-robust. According to Lemma 3.3, we have $\underline{x} \notin \mathrm{attr}^*(A)$. By Theorem 4.12 there exists $\mathcal{K} \in \mathrm{SCC}^* \mathcal{G}(A, m(A))$ such that either $\mathcal{K} \in \mathrm{SCC}_0^* \mathcal{G}(A, \underline{x}, m(A))$ or $\mathcal{K}$ does not have property (P).

Let $l \in J(A)$. For the matrix $A^{(l)}$ we have $m(A^{(l)}) = \overline{a}_l \geq a_l = m(A)$ and $\mathcal{G}(A^{(l)}, m(A^{(l)})) \subseteq \mathcal{G}(A, m(A))$. We shall distinguish two possibilities.

(i) If $\mathcal{K} \in \mathrm{SCC}_0^* \mathcal{G}(A, \underline{x}, m(A))$ then for each $\mathcal{K}_1 \in \mathrm{SCC}^* \mathcal{G}(A^{(l)}, \overline{a}_l)$, $\mathcal{K}_1 \subseteq \mathcal{K}$ we have $\mathcal{K}_1 \in \mathrm{SCC}_0^* \mathcal{G}(A^{(l)}, \underline{x}, \overline{a}_l)$. According to Theorem 4.12 we obtain $\underline{x} \notin \mathrm{attr}^*(A^{(l)})$ and by Theorem 3.3 the matrix $A^{(l)}$ is not strongly $\boldsymbol{X}$-robust.

(ii) Let $\mathcal{K} \in \mathrm{SCC}_1^* \mathcal{G}(A, \underline{x}, m(A))$, be such that $\mathcal{K}$ does not have property (P). Let $\mathcal{K}_1 \in \mathrm{SCC}^* \mathcal{G}(A^{(l)}, \overline{a}_l)$, $\mathcal{K}_1 \subseteq \mathcal{K}$ be arbitrary. If $\mathcal{K}_1 \in \mathrm{SCC}_1^* \mathcal{G}(A^{(l)}, \underline{x}, \overline{a}_l)$, then according to Lemma 4.11, $\mathcal{K}_1$ does not have property (P). The second possibility is that $\mathcal{K}_1 \in \mathrm{SCC}_0^* \mathcal{G}(A^{(l)}, \underline{x}, \overline{a}_l)$. In both cases, according to Theorem 4.12, we obtain $\underline{x} \notin \mathrm{attr}^*(A^{(l)})$ and by Theorem 3.3 the matrix $A^{(l)}$ is not strongly $\boldsymbol{X}$-robust.

The converse implication is trivial. □

4.2. AE/EA strong $\boldsymbol{X}$-robustness

Similarly as an interval vector, an interval matrix can be split as $\boldsymbol{A}^C = \boldsymbol{A}^\forall \oplus \boldsymbol{A}^\exists$, where $\boldsymbol{A}^\forall$ is the interval matrix comprising universally quantified entries and $\boldsymbol{A}^\exists$ concerns existentially quantified entries. Thereafter we denote by $N_0^\exists$ and $N_0^\forall$ the corresponding sets of indices. In other words, $\underline{a}_i^\exists = \overline{a}_i^\exists = O$ for each $i \in N_0^\forall$ and $\underline{a}_i^\forall = \overline{a}_i^\forall = O$ for each $i \in N_0^\exists$. We define the other two types of the strong $\boldsymbol{X}$-robustness of interval circulant matrices.

Definition 4.19. An interval circulant matrix $\boldsymbol{A}^C = \mathcal{C}(\mathbf{a}_0, \dots, \mathbf{a}_{n-1})$ is called

(i) *EA strongly $\boldsymbol{X}$-robust* if there exists $A^\exists \in \boldsymbol{A}^\exists$ such that for each $A^\forall \in \boldsymbol{A}^\forall$ the matrix $A^\exists \oplus A^\forall$ is strongly $\boldsymbol{X}$-robust;

(ii) *AE strongly $\boldsymbol{X}$-robust* if for each $A^\forall \in \boldsymbol{A}^\forall$ there exists $A^\exists \in \boldsymbol{A}^\exists$ such that the matrix $A^\exists \oplus A^\forall$ is strongly $\boldsymbol{X}$-robust.

Theorem 4.20. An interval circulant matrix $\boldsymbol{A}^C = \mathcal{C}(\mathbf{a}_0, \dots, \mathbf{a}_{n-1})$ is AE strongly $\boldsymbol{X}$-robust if and only if for each $j \in N_0^\forall \cup \{n\}$ there exists $A^\exists \in \boldsymbol{A}^\exists$ such that $A^\exists \oplus A^{\forall(j)}$ is strongly $\boldsymbol{X}$-robust.

Proof. Suppose that $\boldsymbol{A}$ is not AE strongly $\boldsymbol{X}$-robust, i.e., there exists $A^\forall \in \boldsymbol{A}^\forall$ such that for each $A^\exists \in \boldsymbol{A}^\exists$, the matrix $A^\exists \oplus A^\forall$ is not strongly $\boldsymbol{X}$-robust. We shall prove that there exists $j \in N_0^\forall \cup \{n\}$ such that for each $A^\exists \in \boldsymbol{A}^\exists$, the matrix $A^\exists \oplus A^{\forall(j)}$ is not strongly $\boldsymbol{X}$-robust.

The assumption is equivalent to that the interval circulant matrix $\mathbf{B} = \boldsymbol{A}^\exists \oplus [A^\forall, A^\forall]$ is not possibly strongly $\boldsymbol{X}$-robust. This means that the matrix B^* defined

$$b_i^* = \begin{cases} \min\{m(\underline{A}^\exists \oplus A^\forall), \overline{a}_i^\exists\} & \text{for } i \in N_0^\exists; \\ a_i^\forall & \text{for } i \in N_0^\forall, \end{cases} \tag{4}$$

is not strongly $\boldsymbol{X}$-robust. It is easy to see that $m(B^*) = m(\underline{A}^\exists \oplus A^\forall)$. According to Lemma 3.3 we have $\underline{x} \notin \operatorname{attr}^*(B^*)$. Hence, by Theorem 4.12, there exists $\mathcal{K} \in \operatorname{SCC}^* \mathcal{G}(B^*, m(B^*))$ such that $\mathcal{K} \in \operatorname{SCC}_0^* \mathcal{G}(B^*, \underline{x}, m(B^*))$ or $\mathcal{K} \in \operatorname{SCC}_1^* \mathcal{G}(B^*, \underline{x}, m(B^*))$ is such that $\mathcal{K}$ does not have property (P). We shall distinguish two possibilities for which we shall refer as Case 1 and Case 2.

Case 1. If $m(B^*) = m(\underline{A}^\exists)$, then the interval circulant matrix $\mathbf{C} = \boldsymbol{A}^\exists \oplus [A^{\forall(n)}, A^{\forall(n)}]$ is possibly strongly $\boldsymbol{X}$-robust if and only if the matrix

$$c_i^* = \begin{cases} \min\{m(\underline{A}), \overline{a}_i^\exists\} & \text{for } i \in N_0^\exists; \\ \underline{a}_i^\forall & \text{for } i \in N_0^\forall. \end{cases} \tag{5}$$

obtained by (3) is $\boldsymbol{X}$-robust. The equality $m(B^*) = m(\underline{A}^\exists)$ implies $b_i^* = c_i^*$ for each $i \in N_0^\exists$. Since $m(B^*) = m(C^*)$ and $C^* \leq B^*$, there exists $\mathcal{K}_1 \in \operatorname{SCC}^* \mathcal{G}(C^*, m(C^*))$, $\mathcal{K}_1 \subseteq \mathcal{K}$ such that $\mathcal{K}_1 \in \operatorname{SCC}_0^* \mathcal{G}(C^*, \underline{x}, m(C^*))$ or $\mathcal{K}_1$ does note have property (P). Since $\underline{x} \notin \operatorname{attr}^*(C^*)$, C^* is not strongly $\boldsymbol{X}$-robust. According to Theorem 4.17, $\mathbf{C}$ is not possibly strongly $\boldsymbol{X}$-robust, which means that $\boldsymbol{A}$ is not AE strongly $\boldsymbol{X}$-robust.

Case 2. If $m(B^*) = m(A^\forall)$, then $a_j^\forall = m(B^*)$ for some $j \in N_0^\forall$. Let us define the interval circulant matrix $\mathbf{D} = \boldsymbol{A}^\exists \oplus [A^{\forall(j)}, A^{\forall(j)}]$. By (3), we obtain the matrix D^*

$$d_i^* = \begin{cases} \min\{\overline{a}_i^\exists, \overline{a}_j^\forall\} & \text{for } i \in N_0^\exists; \\ a_i^{\forall(j)} & \text{for } i \in N_0^\forall. \end{cases} \tag{6}$$

Since $m(B^*) \leq m(D^*) = \overline{a}_j^\forall$, we obtain $\{i \in N_0^\exists;\, d_i^* = m(D^*)\} \subseteq \{i \in N_0^\exists;\, b_i^* = m(B^*)\}$. The inequalities $d_i^* \leq b_i^*$ for $i \neq j$, $i \in N_0^\forall$ and $d_j^* = m(D^*)$ imply that $\{i \in$

$N_0^\forall;\ d_i^* = m(D^*)\} \subseteq \{i \in N_0^\forall;\ b_i^* = m(B^*)\}$. We obtain $\mathcal{G}(D^*, m(D^*)) \subseteq \mathcal{G}(B^*, m(B^*))$. Similarly as in Case 1 the matrix D^* is not strongly $\boldsymbol{X}$-robust, so there is no matrix $A^\exists \in \boldsymbol{A}^\exists$ such that $A^\exists \oplus A^{\forall(j)}$ is strongly $\boldsymbol{X}$-robust.

The converse implication trivially holds. □

4.3. EA strong $\boldsymbol{X}$-robustness

To give the necessary and sufficient condition for the EA strong $\boldsymbol{X}$-robustness let us denote

$$m(\boldsymbol{A}) = m(\underline{A}^\exists \oplus \overline{A}^\forall), \quad S(\boldsymbol{A}^\exists) = \{A^\exists \in \boldsymbol{A}^\exists; m(A^\exists \oplus A^{\forall(n)}) \geq m(\boldsymbol{A})\}.$$

and define the matrix $\tilde{A}^\exists \in \boldsymbol{A}^\exists$ as follows:

$$\tilde{a}_i^\exists = \min\{\overline{a}_i^\exists, m(\boldsymbol{A})\} \quad \text{for each } i \in N_0^\exists. \tag{7}$$

Theorem 4.21. Let $\boldsymbol{A}^C = \mathcal{C}(\mathbf{a}_0, \dots, \mathbf{a}_{n-1})$ be such that $m(\overline{A}^\exists \oplus A^{\forall(n)}) < m(\boldsymbol{A})$. Then $\boldsymbol{A}$ is EA strongly $\boldsymbol{X}$-robust if and only if min $\_i \in N\underline{x}_i \geq m(\boldsymbol{A})$.

Proof. The assumption $m(\overline{A}^\exists \oplus A^{\forall(n)}) < m(\boldsymbol{A})$ is equivalent to that $m(\boldsymbol{A}) = m(\overline{A}^\forall)$.

Suppose that $\min_{i\in N} \underline{x}_i < m(\boldsymbol{A})$. Then there exist $k \in N$ and $j \in N_0^\forall$ be such that $\underline{x}_k = \min_{i\in N} \underline{x}_i < m(\overline{A}^\forall) = \overline{a}_j^\forall$. Let $A^\exists \in \boldsymbol{A}^\exists$ be arbitrary but fixed. For the matrix $A^\exists \oplus A^{\forall(j)}$ we have $J(A^\exists \oplus A^{\forall(j)}) = \{j\}$. Let $\mathcal{K} \in \mathrm{SCC}^* \,\mathcal{G}(A^\exists \oplus A^{\forall(j)}, \overline{a}_j^\forall)$ be such that $k \in V_\mathcal{K}$. If $j = 0$ then $\mathcal{K}$ consists of a cycle of length zero, so $\mathcal{K} \in \mathrm{SCC}_0^* \,\mathcal{G}(A^\exists \oplus A^{\forall(j)}, \underline{x}, \overline{a}_j^\forall)$. If $j \neq 0$, then $\mathcal{K}$ consists of a cycle of length $l(c) = \frac{n}{\gcd(n,j)}$. Since $\mathcal{K} \in \mathrm{SCC}_1^* \,\mathcal{G}(A^\exists \oplus A^{\forall(j)}, \underline{x}, \overline{a}_j^\forall)$, there are no orbit paths of lengths $t \cdot l(c)$, $t \in \mathbb{N}$ starting at point k in $\mathcal{K}$. This means that $\mathcal{K}$ does not have property (P).

In both cases, according to Theorem 4.12, the vector $\underline{x} \notin \mathrm{attr}^*(A^\exists \oplus A^{\forall(j)})$, so $A^\exists \oplus A^{\forall(j)}$ is not strongly $\boldsymbol{X}$-robust. Hence $\boldsymbol{A}$ is not EA strongly $\boldsymbol{X}$-robust.

For the converse implication suppose that $\min_{i\in N} \underline{x}_i \geq m(\boldsymbol{A})$. Then for each $A^\exists \in \boldsymbol{A}^\exists$, for each $A^\forall \in \boldsymbol{A}^\forall$ and for each $i \in N$ we have $\underline{x}_i \geq m(A^\exists \oplus A^\forall)$. Then for each $\mathcal{K} \in \mathrm{SCC}^* \,\mathcal{G}(A^\exists \oplus A^\forall, m(A^\exists \oplus A^\forall))$ we have $\mathcal{K} \in \mathrm{SCC}_1^* \,\mathcal{G}(A^\exists \oplus A^\forall, \underline{x}, m(A^\exists \oplus A^\forall))$ and each path in $\mathcal{K}$ is an orbit path. Since there are the paths of length r for each $r \in \mathbb{N}$, $\mathcal{K}$ has property (P). Hence $\underline{x} \in \mathrm{attr}^*(A^\exists \oplus A^\forall)$, so $\boldsymbol{A}^C$ is EA strongly $\boldsymbol{X}$-robust. □

Lemma 4.22. Let an interval circulant matrix $\boldsymbol{A}^C = \mathcal{C}(\mathbf{a}_0, \dots, \mathbf{a}_{n-1})$ be such that $m(\overline{A}^\exists \oplus A^{\forall(n)}) \geq m(\boldsymbol{A})$. Then $J(A^\exists \oplus A^{\forall(n)}) \subseteq J(\tilde{A}^\exists \oplus A^{\forall(n)})$ for each $A^\exists \in S(\boldsymbol{A}^\exists)$.

Proof. The inequality $m(\overline{A}^\exists \oplus A^{\forall(n)}) \geq m(\boldsymbol{A})$ implies $S(\boldsymbol{A}^\exists) \neq \emptyset$. It is easy to see that $m(\tilde{A}^\exists \oplus A^{\forall(n)}) = m(\boldsymbol{A})$ and $J(\tilde{A}^\exists \oplus A^{\forall(n)}) = \{i \in N_0^\exists;\ \overline{a}_i^\exists \geq m(\boldsymbol{A})\} \cup \{i \in N_0^\forall; \underline{a}_i^\forall = \overline{a}_i^\forall = m(\boldsymbol{A})\}$. We shall distinguish two possibilities.

Case 1. If $A^\exists \in S(\boldsymbol{A}^\exists)$ is such that there exists $r \in N_0^\exists$, $r \in J(A^\exists \oplus A^{\forall(n)})$, then

$$\overline{a}_r^\exists \geq a_r^\exists \geq m(\boldsymbol{A}).$$

Hence $r \in J(\tilde{A}^\exists \oplus A^{\forall(n)})$, which implies $J(A^\exists \oplus A^{\forall(n)}) \subseteq J(\tilde{A}^\exists \oplus A^{\forall(n)}))$.

Case 2. If $A^\exists \in \boldsymbol{A}^\exists$ is such that $J(A^\exists \oplus A^{\forall(n)}) = \{i \in N_0^\forall; \underline{a}_i^\forall = m(\boldsymbol{A})\}$ then $J(A^\exists \oplus A^{\forall(n)}) = J(\tilde{A}^\exists \oplus A^{\forall(n)})$. □

Theorem 4.23. Let $\boldsymbol{A}^C = \mathcal{C}(\mathbf{a}_0, \ldots, \mathbf{a}_{n-1})$ be such that $m(\overline{A}^{\exists} \oplus A^{\forall(n)}) \geq m(\boldsymbol{A})$. Then $\boldsymbol{A}$ is EA strongly $\boldsymbol{X}$-robust if and only if the matrix $\tilde{A}^{\exists} \oplus A^{\forall(n)}$ is strongly $\boldsymbol{X}$-robust or $\min_{i\in N} \underline{x}_i \geq m(\boldsymbol{A})$.

Proof. Suppose that the matrix $\tilde{A}^{\exists}\oplus A^{\forall(n)}$ is strongly $\boldsymbol{X}$-robust or $\min_{i\in N} \underline{x}_i \geq m(\boldsymbol{A})$.

In the first case, according to Lemma 3.3, we have $\underline{x} \in \operatorname{attr}^*(\tilde{A}^{\exists}\oplus A^{\forall(n)})$. By Theorem 4.12 for each $\mathcal{K} \in \mathrm{SCC}^* \,\mathcal{G}(\tilde{A}^{\exists} \oplus A^{\forall(n)}, m(\boldsymbol{A}))$ we have $\mathcal{K} \in \mathrm{SCC}^*_1 \,\mathcal{G}(\tilde{A}^{\exists} \oplus A^{\forall(n)}, \underline{x}, m(\boldsymbol{A}))$ and $\mathcal{K}$ has property (P). Let $A^{\forall} \in \boldsymbol{A}^{\forall}$ be arbitrary. Since $\mathcal{G}(\tilde{A}^{\exists} \oplus A^{\forall(n)}, m(\boldsymbol{A})) \subseteq \mathcal{G}(\tilde{A}^{\exists}\oplus A^{\forall}, m(\boldsymbol{A}))$, for each $\mathcal{K}_1 \in \mathrm{SCC}^* \,\mathcal{G}(\tilde{A}^{\exists}\oplus A^{\forall}, m(\boldsymbol{A}))$ there exists $\mathcal{K} \in \mathrm{SCC}^* \,\mathcal{G}(\tilde{A}^{\exists}\oplus A^{\forall(n)}, m(\boldsymbol{A}))$ such that $\mathcal{K} \subseteq \mathcal{K}_1$. Since $\mathcal{K} \in \mathrm{SCC}^*_1 \,\mathcal{G}(\tilde{A}^{\exists} \oplus A^{\forall(n)}, \underline{x}, m(\boldsymbol{A}))$ and $\mathcal{K}$ has property (P), $\mathcal{K}_1 \in \mathrm{SCC}^*_1 \,\mathcal{G}(\tilde{A}^{\exists} \oplus A^{\forall}, \underline{x}, m(\boldsymbol{A}))$ and $\mathcal{K}$ has property (P). We obtain $\underline{x} \in \operatorname{attr}^*(\tilde{A}^{\exists} \oplus A^{\forall})$ and consequently $\tilde{A}^{\exists} \oplus A^{\forall}$ is strongly $\boldsymbol{X}$-robust. Hence $\boldsymbol{A}$ is EA strongly $\boldsymbol{X}$-robust.

In the second case suppose that $\min_{i\in N} \underline{x}_i \geq m(\boldsymbol{A})$. From $m(\underline{A}^{\exists} \oplus A^{\forall(n)}) \leq m(\underline{A}^{\exists} \oplus \overline{A}^{\forall}) = m(\boldsymbol{A})$ we obtain $\underline{x}_k \geq m(\underline{A}^{\exists} \oplus A^{\forall})$ for each $A^{\forall} \in \boldsymbol{A}^{\forall}$ and for each $k \in N$. This is equivalent to $(\underline{x}_k)_{(h)} = I$ for each $k \in N$, where $h = m(\underline{A}^{\exists} \oplus A^{\forall})$. Then for each $\mathcal{K} \in \mathrm{SCC}^* \,\mathcal{G}(\underline{A}^{\exists} \oplus A^{\forall}, m(\underline{A}^{\exists} \oplus A^{\forall}))$ we have $\mathcal{K} \in \mathrm{SCC}^*_1 \,\mathcal{G}(\underline{A}^{\exists} \oplus A^{\forall}, \underline{x}, m(\underline{A}^{\exists} \oplus A^{\forall}))$ and each path in $\mathcal{K}$ is an orbit path. Since there are the paths of length r for each $r \in \mathbb{N}$, $\mathcal{K}$ has property (P). We obtain $\underline{x} \in \operatorname{attr}^*(\underline{A}^{\exists} \oplus A^{\forall})$ and consequently $\underline{A}^{\exists} \oplus A^{\forall}$ is strongly $\boldsymbol{X}$-robust. Since there exists $A^{\exists} \in \boldsymbol{A}^{\exists}$, namely $A^{\exists} = \underline{A}^{\exists}$, such that $A^{\exists} \oplus A^{\forall}$ is strongly $\boldsymbol{X}$-robust for each $A^{\forall} \in \boldsymbol{A}^{\forall}$, an interval circulant matrix $\boldsymbol{A}^C$ is EA strongly $\boldsymbol{X}$-robust.

For the converse implication suppose that $\tilde{A}^{\exists} \oplus A^{\forall(n)}$ is not strongly $\boldsymbol{X}$-robust and $\min_{i\in N} \underline{x}_i < m(\boldsymbol{A})$. By Lemma 3.3 we have $\underline{x} \notin \operatorname{attr}^*(\tilde{A}^{\exists} \oplus A^{\forall(n)})$. Hence, according to Theorem 4.12, there exists $\mathcal{K} \in \mathrm{SCC}^* \,\mathcal{G}(\tilde{A}^{\exists} \oplus A^{\forall(n)})$ such that $\mathcal{K} \in \mathrm{SCC}^*_0 \,\mathcal{G}(\tilde{A}^{\exists} \oplus A^{\forall(n)}, \underline{x}, m(\tilde{A}^{\exists} \oplus A^{\forall(n)}))$ or $\mathcal{K}$ does not have property (P). Let $A^{\exists} \in \boldsymbol{A}^{\exists}$ be arbitrary. We shall distinguish two possibilities.

(i) If $A^{\exists} \in S(\boldsymbol{A}^{\exists})$, then according to Lemma 4.22 we have

$$\mathcal{G}(A^{\exists} \oplus A^{\forall(n)}, m(A^{\exists} \oplus A^{\forall(n)})) \subseteq \mathcal{G}(\tilde{A}^{\exists} \oplus A^{\forall(n)}, m(\tilde{A}^{\exists} \oplus A^{\forall(n)})).$$

Hence there exists $\mathcal{K}_1 \in \mathrm{SCC}^* \,\mathcal{G}(A^{\exists} \oplus A^{\forall(n)})$, $\mathcal{K}_1 \subseteq \mathcal{K}$ such that $\mathcal{K}_1 \in \mathrm{SCC}^*_0 \,\mathcal{G}(A^{\exists} \oplus A^{\forall(n)}, \underline{x}, m(A^{\exists} \oplus A^{\forall(n)}))$ or $\mathcal{K}_1$ does not have property (P).

It means that for each $A^{\exists} \in S(\boldsymbol{A}^{\exists})$ there exists $A^{\forall} \in \boldsymbol{A}^{\forall}$, namely $A^{\forall} = A^{\forall(n)}$ such that $A^{\exists} \oplus A^{\forall(n)}$ is not strongly $\boldsymbol{X}$-robust.

(ii) If $A^{\exists} \notin S(\boldsymbol{A}^{\exists})$, then the inequality $\min_{i\in N} \underline{x}_i < m(\boldsymbol{A})$ implies that there exist $k \in N$ and $j \in N_0^{\forall}$ be such that $\underline{x}_k = \min_{i\in N} \underline{x}_i < m(\overline{A}^{\forall}) = \overline{a}_j^{\forall}$. For the matrix $A^{\exists} \oplus A^{\forall(j)}$ we have $J(A^{\exists} \oplus A^{\forall(j)}) = \{j\}$. Similarly as in the proof of Theorem 4.21, for the strongly connected component $\mathcal{K} \in \mathrm{SCC}^* \,\mathcal{G}(A^{\exists} \oplus A^{\forall(j)}, \overline{a}_j^{\forall})$ containg node k we have either $\mathcal{K} \in \mathrm{SCC}^*_0 \,\mathcal{G}(A^{\exists} \oplus A^{\forall(j)}, \underline{x}, \overline{a}_j^{\forall})$ or $\mathcal{K}$ does not have property (P). Hence $A^{\exists} \oplus A^{\forall(j)}$ is not strongly $\boldsymbol{X}$-robust.

In both cases, there exists $A^{\forall} \in \boldsymbol{A}^{\forall}$ such that $A^{\exists} \oplus A^{\forall}$ is not strongly $\boldsymbol{X}$-robust. Therefore, $\boldsymbol{A}^C$ is not EA strongly $\boldsymbol{X}$-robust. $\square$

Example 4.24. Let us have

$$\boldsymbol{A} = ([3,5],[4,7],[6,8],[5,7],[4,8],[5,7]), \boldsymbol{X} = ([7,9],[8,10],[8,9],[7,8],[4,6],[5,8])^\top$$

and $N_0^\exists = \{0,1,2,3\},\ N_0^\forall = \{4,5\}$.

Decide whether the given interval circulant matrix is strongly EA $\boldsymbol{X}$-robust and strongly AE $\boldsymbol{X}$-robust.

Solution. We have

$$\boldsymbol{A}^\exists = \boldsymbol{A}^\exists([3,5],[4,7],[6,8],[5,7],[0,0],[0,0])$$

and

$$\boldsymbol{A}^\forall = \boldsymbol{A}^\forall([0,0],[0,0],0,0],[0,0],[4,8],[5,7]).$$

First, we check the strong EA $\boldsymbol{X}$-robustness. Since $m(\overline{A}^\exists \oplus A^{\forall(n)}) = m(\boldsymbol{A}) = 8$ and $\min_{i\in N} \underline{x}_i = 4 < m(\boldsymbol{A})$, according to Theorem 4.23 we have to decide about the strong $\boldsymbol{X}$-robustness of $\tilde{A}^\exists \oplus A^{\forall(n)}$.

According to (7) we have $\tilde{A}^\exists = (5,7,8,7,0,0)$ and $\tilde{A}^\exists \oplus A^{\forall(6)} = (5,7,8,7,4,5)$. We obtain

$$\mathcal{O}(\tilde{A}^\exists \oplus A^{\forall(6)}, \underline{x}) = \big((7,8,8,7,4,5)^T, (8,7,7,7,7,8)^T, (7,7,7,8,8,7)^T,$$

$$(7,8,8,7,7,7)^T, (,7,7,7,7,8)^T, \dots\big)\,.$$

We can see that $\mathrm{per}(A^{*\exists} \oplus A^{(6)}, \underline{x}) = 3$, so the matrix $A^{*\exists} \oplus A^{(6)}$ is not strongly $\boldsymbol{X}$-robust. Hence the given interval circulant matrix is not EA strongly $\boldsymbol{X}$-robust.

To check the strong AE $\boldsymbol{X}$-robustness, we check the possible strong $\boldsymbol{X}$-robustness od interval matrices $\boldsymbol{A}^\exists \oplus [A^{(j)}, A^{(j)}]$ for $j = 4,\ 5,\ 6$.

For $j = 1$, we obtain $\boldsymbol{A}^\exists \oplus [A^{(4)}, A^{(4)}] = \mathcal{C}([3,5],[4,7],[6,8],[5,7],[8,8],[5,5])$. According to Theorem 4.17, we have to check the strong $\boldsymbol{X}$-robustness of the matrix $A^{*(4)} = \mathcal{C}(5,7,8,7,8,5)$ computed by (3). We have

$$\mathcal{O}(A^{*(4)}, \underline{x}) = \big((7,8,8,7,4,5)^T, (8,7,7,8,8,8)^T, (8,8,8,8,8,8)^T, (8,8,8,8,8,8)^T, \dots\big)$$

Since $x^\oplus(A^{*(4)}) = (8,8,8,8,8,8)^T \in \mathcal{O}(A^{*(4)}, \underline{x})$, the matrix $A^{*(4)}$ is strongly $\boldsymbol{X}$-robust, which means that $\boldsymbol{A}^\exists \oplus [A^{(4)}, A^{(4)}]$ is possibly strongly $\boldsymbol{X}$-robust.

For $j = 5$, we have $\boldsymbol{A}^\exists \oplus [A^{(5)}, A^{(5)}] = \mathcal{C}([3,5],[4,7],[6,8],[5,7],[4,4],[7,7])$. We check the strong $\boldsymbol{X}$-robustness of the matrix $A^{*(5)} = \mathcal{C}(5,7,7,7,4,7)$ computed by (3). We have

$$\mathcal{O}(A^{*(5)}, \underline{x}) = \big((7,8,8,7,4,5)^T, (7,7,7,7,7,7)^T, (7,7,7,7,7,7)^T, \dots\big)$$

which implies that the matrix $A^{*(5)}$ is strongly $\boldsymbol{X}$-robust because of $x^\oplus(A^{*(5)}) = (7,7,7,7,7,7)^T \in \mathcal{O}(A^{*(5)}, \underline{x})$. Hence $\boldsymbol{A}^\exists \oplus [A^{(5)}, A^{(5)}]$ is possibly strongly $\boldsymbol{X}$-robust.

At last, $\boldsymbol{A}^\exists \oplus [A^{(6)}, A^{(6)}] = \mathcal{C}([3,5],[4,7],[6,8],[5,7],[4,4],[5,5])$ is possibly strongly $\boldsymbol{X}$-robust because of for the matrix $A^{*(6)} = (5,6,6,6,4,5)$ we obtain

$$\mathcal{O}(A^{*(6)}, \underline{x}) = \big((7,8,8,7,4,5)^T, (6,6,6,6,6,6)^T, (6,6,6,6,6,6)^T, \dots\big)$$

so $x^{\oplus}(A^{*(6)}) = (6,6,6,6,6,6)^T \in \mathcal{O}(A^{*(6)}, \underline{x})$.

Since interval matrices $\boldsymbol{A}^{\exists} \oplus [A^{(4)}, A^{(4)}]$, $\boldsymbol{A}^{\exists} \oplus [A^{(5)}, A^{(5)}]$ and $\boldsymbol{A}^{\exists} \oplus [A^{(6)}, A^{(6)}]$ are possibly strongly $\boldsymbol{X}$-robust, the given interval circulant matrix is AE strongly $\boldsymbol{X}$-robust. $\surd$

4.4. AE/EA strong $\boldsymbol{X}^{AE}/\boldsymbol{X}^{EA}$-robustness

Definition 4.25. Let an interval circulant matrix $\boldsymbol{A}^C = \mathcal{C}(\mathbf{a}_0, \ldots, \mathbf{a}_{n-1})$ and an interval vector X be given. An interval circulant matrix $\boldsymbol{A}^C$ is called

(i) EA strongly $\boldsymbol{X}^{AE}$-robust ($\boldsymbol{X}^{EA}$-robust) if there exists $A^{\exists} \in \boldsymbol{A}^{\exists}$ such that for each $A^{\forall} \in \boldsymbol{A}^{\forall}$ the matrix $A^{\exists} \oplus A^{\forall}$ is strongly $\boldsymbol{X}^{AE}$-robust ($\boldsymbol{X}^{EA}$-robust);

(ii) AE strongly $\boldsymbol{X}^{AE}$-robust ($\boldsymbol{X}^{EA}$-robust) if for each $A^{\forall} \in \boldsymbol{A}^{\forall}$ there exists $A^{\exists} \in \boldsymbol{A}^{\exists}$ such that the matrix $A^{\exists} \oplus A^{\forall}$ is $\boldsymbol{X}^{AE}$-robust ($\boldsymbol{X}^{EA}$-robust).

Theorem 4.26. An interval circulant matrix $\boldsymbol{A}^C = \mathcal{C}(\mathbf{a}_0, \ldots, \mathbf{a}_{n-1})$ is AE strongly $\boldsymbol{X}^{AE}$-robust ($\boldsymbol{X}^{EA}$-robust) if and only if for each $j \in N_0^{\forall} \cup \{n\}$ there exists $A^{\exists} \in \boldsymbol{A}^{\exists}$ such that $A^{\exists} \oplus A^{\forall(j)}$ is strongly $\boldsymbol{X}^{AE}$-robust ($\boldsymbol{X}^{EA}$-robust), respectively.

Proof. The proof is similar to the proof of Theorem 4.20. $\square$

Theorem 4.27. Let $\boldsymbol{A}^C = \mathcal{C}(\mathbf{a}_0, \ldots, \mathbf{a}_{n-1})$ and $\boldsymbol{X}$ be given. Then $\boldsymbol{A}$ is EA strongly $\boldsymbol{X}^{AE}$-robust ($\boldsymbol{X}^{EA}$-robust) if and only if the matrix $\tilde{A}^{\exists} \oplus A^{\forall(n)}$ is strongly $\boldsymbol{X}^{AE}$-robust ($\boldsymbol{X}^{EA}$-robust), respectively, or $\min\{\min_{i \in N^{\exists}} \underline{x}_i^{\forall}, \min_{i \in N^{\exists}} \overline{x}_i^{\exists}\} \geq m(\boldsymbol{A})$.

Proof. The proof is similar to the proofs of Theorem 4.21 and Theorem 4.23. $\square$

5. CONCLUSION

In the paper, we dealt with max-min matrices for which the greatest eigenvector is achieved by orbit starting at each vector from a given interval vector $\boldsymbol{X}$. Such matrices are called strongly $\boldsymbol{X}$-robust matrices. We extended the studying of the strong $\boldsymbol{X}$-robustness by defining the strong $\boldsymbol{X}^{EA}$-robustness and $\boldsymbol{X}^{AE}$-robustness of a given matrix A. These new types were created using the quantification of interval entries of vector $\boldsymbol{X}$ by universal or existential quantifier.

Next we focused on a special type of matrices, circulant matrices and their interval version. By quantifying its interval elements, we obtained other types of the strong $\boldsymbol{X}$-robustness, AE strong $\boldsymbol{X}$-robustness and EA strong $\boldsymbol{X}$-robustness. All presented types of the strong $\boldsymbol{X}$- robustness can be checked in polynomial time. Our goal for the future is to extend the strong $\boldsymbol{X}$-robustness to general interval matrices, which, to our knowledge, has not yet been done.

(Received February 9, 2021)

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Helena Myšková, Department of Mathematics and Theoretical Informatics, Technical University, Němcovej 32, 04200 Košice. Slovak Republic.
e-mail: helena.myskova@tuke.sk

Ján Plavka, Department of Mathematics and Theoretical Informatics, Technical University, Němcovej 32, 04200 Košice. Slovak Republic.
e-mail: jan.plavka@tuke.sk