ON THE CONSTRUCTIONS OF T-NORMS AND T-CONORMS ON SOME SPECIAL CLASSES OF BOUNDED LATTICES

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Recently, the topic related to the construction of triangular norms and triangular conorms on bounded lattices using ordinal sums has been extensively studied. In this paper, we introduce a new ordinal sum construction of triangular norms and triangular conorms on an appropriate bounded lattice. Also, we give some illustrative examples for clarity. Then, we show that a new construction method can be generalized by induction to a modified ordinal sum for triangular norms and triangular conorms on an appropriate bounded lattice, respectively. And we provide some illustrative examples.

1. INTRODUCTION

Triangular norms (briefly t-norms) and triangular conorms (briefly t-conorms) were introduced by Schweizer and Sklar [28] in the study of probabilistic metric spaces as a special kind of associative functions defined on the unit interval $[0, 1]$. Although the t-norms and t-conorms were strictly defined on the unit interval $[0, 1]$, they were mostly studied on bounded lattices.

The notion of ordinal sum of semigroups in Cliffords sense [9] was further developed by Mostert and Shields [22] and later used for introducing new t-norms and conorms on the unit interval $[0, 1]$, see [20]. Note that there is a minor difference in ordinal sum construction for triangular norms (based on min operator) with those for triangular conorms (based on max operator). Since Goguen' s [17] generalization of the classical fuzzy sets (with membership values from $[0, 1]$) to L-fuzzy sets (with membership values from a bounded lattice L), there is a growing interest in t-norms and t-conorms on bounded lattices, in particular in ordinal sum constructions. Saminger [27] focused on ordinal sums of t-norms acting on some particular bounded lattice which is not necessarily a chain or an ordinal sum of lattices. Also, she provided necessary and sufficient conditions for an ordinal sum operation yielding again a t-norm on some bounded lattice whereas the operation is determined by an arbitrary selection of subintervals as carriers for arbitrary summand t-norms. Medina [21] presented several necessary and sufficient conditions for ensuring whether an ordinal sum on a bounded lattice of arbitrary t-norms is a t-norm.

Recently, ordinal sums of t-norms and t-conorms on bounded lattices have been studied intensively. In 2015, a modification of ordinal sums of t-norms and t-conorms resulting to a t-norms and t-conorms on an arbitrary bounded lattice was shown by Ertuğrul, Karaçal, Mesiar [16]. Further modifications were proposed by Çaylı [11, 13], Aşıcı, Mesiar [3, 4], Ouyang, Zhang, Baets [23] and Dan, Hu, Qiao [14]. In 2020, a new ordinal sum construction of t-norms and t-conorms on bounded lattices based on interior and closure operators was proposed by Dvořák, Holčapek [15]. Also, the proposed method generalized several known constructions and provided a simple tool to introduce new classes of t-norms and t-conorms.

In this paper, we introduce a new ordinal sum construction of t-norms and t-conorms on an arbitrary bounded lattice satisfying some constraints for a fixed element $a \in L \setminus \{0, 1\}$, by using the existence of t-norms on the sublattice $[0, a]$ and of t-conorms on the sublattice $[a, 1]$, respectively.

This paper is organized as follows. In Section 2, some basic notions dealing with bounded lattices and t-norms (t-conorms) on them are recalled. In Section 3, we introduce a new ordinal sum construction of t-norms and t-conorms on an arbitrary bounded lattice with a fixed element $a \in L \setminus \{0, 1\}$ based on the existence of a t-norm V acting on $[0, a]$ and a t-conorm W on $[a, 1]$, respectively, where some additional conditions on its $a \in L \setminus \{0, 1\}$ are required. We should add these conditions to satisfy commutativity of t-norms and t-conorms on bounded lattices (see the cases 1.3. and 3.2.1.). In addition, the role of these conditions is stressed by providing some illustrative examples and then our constructions yield a t-norm and t-conorm on a bounded lattice in particular cases. Then, we provide some examples to illustrate that our new construction approaches presented in this paper are different from the approaches proposed by Ertuğrul, Karaçal, Mesiar [16] and Çaylı [13]. In Section 4, we present our modified ordinal sum construction in its full generality. And we provide some illustrative examples. Finally, some concluding remarks are added.

2. PRELIMINARIES

A lattice [1, 5, 8, 10, 12, 18, 19] is a partially ordered set $(L, \leq)$ in which each two element subset $\{x, y\}$ has an infimum, denoted as $x \wedge y$, and a supremum, denoted as $x \vee y$. A bounded lattice $(L, \leq, 0, 1)$ is a lattice that has the bottom and top elements written as 0 and 1, respectively.

Given a bounded lattice $(L, \leq, 0, 1)$ and $a, b \in L$, if a and b are incomparable, in this case, we use the notation $a \parallel b$. We denote the set of elements which are incomparable with a by I_a. So $I_a = \{x \in L \mid x \parallel a\}$.

Given a bounded lattice $(L, \leq, 0, 1)$ and $a, b \in L$, $a \leq b$, a subinterval $[a, b]$ of L is defined as

$[a, b] = \{x \in L \mid a \leq x \leq b\}$
Similarly, $[a, b) = \{x \in L \mid a \leq x < b\}$, $(a, b] = \{x \in L \mid a < x \leq b\}$ and $(a, b) = \{x \in L \mid a < x < b\}$.

Definition 2.1. (Aşıcı and Karaçal [6], Saminger [27]) Let $(L, \leq, 0, 1)$ be a bounded lattice. A *triangular norm* T (t-norm) is a binary operation on L which is commutative, associative, increasing with respect to both variables and it satisfies $T(x, 1) = x$ for all $x \in L$.

Definition 2.2. (Aşıcı [2], Saminger [27]) Let $(L, \leq, 0, 1)$ be a bounded lattice. A *triangular conorm* S (t-conorm) is a binary operation on L which is commutative, associative, increasing with respect to both variables and it satisfies $S(x, 0) = x$ for all $x \in L$.

Extremal t-norms $T_\wedge$ and T_W on an arbitrary bounded lattice L are defined as follows, respectively:
$$T_\wedge(x, y) = x \wedge y$$
$$T_W(x, y) = \begin{cases} x \wedge y & \text{if } 1 \in \{x, y\} , \\ 0 & \text{otherwise.} \end{cases}$$
Similarly, the t-conorms $S_\vee$ and S_W on L are defined as follows, respectively:
$$S_\vee(x, y) = x \vee y$$
$$S_W(x, y) = \begin{cases} x \vee y & \text{if } 0 \in \{x, y\} , \\ 1 & \text{otherwise.} \end{cases}$$

3. A NEW METHOD FOR ORDINAL SUM CONSTRUCTION OF T-NORMS AND T-CONORMS ON BOUNDED LATTICES

In this section, we give some illustrative examples to discuss the literature. Then, we list the construction approaches for t-norms and t-conorms presented by [16] and [13]. Also, we construct ordinal sums of t-norms and t-conorms on an arbitrary bounded lattice L in Theorem 3.6 and Theorem 3.19, respectively, where $a \in L \setminus \{0, 1\}$, V is t-norm on $[0, a]$ and W is t-conorm on $[a, 1]$, respectively. Also, we give some illustrative examples for clarity.

The following definition of an ordinal sum of t-norms defined on subintervals of a bounded lattice $(L, \leq, 0, 1)$ has been extracted from [27], which generalizes the methods given in [20] on subintervals of $[0, 1]$.

Definition 3.1. (Saminger [27]) Let $(L, \leq, 0, 1)$ be a bounded lattice and fix some subinterval $[a, b]$ of L. Let V be a t-norm on $[a, b]$. Then $T^\sim : L^2 \to L$ defined by

$$T^\sim(x, y) = \begin{cases} V(x, y) & \text{if } (x, y) \in [a, b]^2 , \\ x \wedge y & \text{otherwise} \end{cases} \tag{1}$$

is an ordinal sum $(< a, b, V >)$ of V on L.

Definition 3.2. (Saminger [27]) Let $(L, \leq, 0, 1)$ be a bounded lattice and fix some subinterval $[a, b]$ of L. Let W be a t-conorm on $[a, b]$. Then $S^\sim : L^2 \to L$ defined by

$$S^\sim(x, y) = \begin{cases} W(x, y) & \text{if } (x, y) \in [a, b]^2 , \\ x \vee y & \text{otherwise} \end{cases} \tag{2}$$

is an ordinal sum $(< a, b, W >)$ of W on L.

However, the operation $T^\sim$ (resp. $S^\sim$) given by Formula (1) (resp. Formula (2)) need not be a t-norm (resp. t-conorm), in general. Observe that condition ensuring that $T^\sim$ (resp. $S^\sim$) given by (1) ((2)) is a t-norm (t-conorm) on L are given in Saminger's paper [27]. If L is a chain, then this $T^\sim$ ($S^\sim$) is a t-norm (t-conorm) for any $[a,b] \subseteq L$. Now, we will give an example only for t-norms because of the example can be obtained for t-conorms from duality.

Example 3.3. Consider the lattice $(L_1 = \{0_{L_1}, t, a, r, 1_{L_1}\}, \leq, 0_{L_1}, 1_{L_1})$ given in Figure 1 and define the t-norm $V : [0_{L_1}, a]^2 \to [0_{L_1}, a]$ as follows.

$$V(x,y) = \begin{cases} x \wedge y & \text{if } a \in \{x,y\} \ , \\ 0_{L_1} & \text{otherwise.} \end{cases}$$

Then, using Formula (1), the operation $T^\sim$ on L_1 given by (1) defined by Table 1 is not a t-norm.
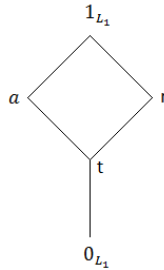
Fig. 1. The lattice L_1.

$T^\sim$	0_{L_1}	t	a	r	1_{L_1}
0_{L_1}	0_{L_1}	0_{L_1}	0_{L_1}	0_{L_1}	0_{L_1}
t	0_{L_1}	0_{L_1}	t	t	t
a	0_{L_1}	t	a	t	a
r	0_{L_1}	t	t	r	r
1_{L_1}	0_{L_1}	d	a	b	1_{L_1}

Tab. 1. The operation $T^\sim$ on L_1.

Indeed, the operation $T^\sim$ does not satisfy associativity, because $T^\sim(a, T^\sim(r,t)) = T^\sim(a,t) = t \neq 0_{L_1} = T^\sim(t,t) = T^\sim(T^\sim(a,r),t)$. We obtain that $T^\sim$ is not a t-norm on L_1.

Theorem 3.4. (Ertuğrul et al. [16]) Let $(L, \leq, 0, 1)$ be a bounded lattice and $a \in L \setminus \{0,1\}$. If V_T is a t-norm on $[a,1]$ and W_S is a t-conorm on $[0,a]$, then the functions

$T^* : L^2 \to L$ and $S^* : L^2 \to L$ are a t-norm and a t-conorm on L, respectively, where

$$T^*(x,y) = \begin{cases} x \wedge y & \text{if } x = 1 \text{ or } y = 1 , \\ V_T(x,y) & \text{if } x, y \in [a,1) , \\ x \wedge y \wedge a & \text{otherwise.} \end{cases}$$

$$S^*(x,y) = \begin{cases} x \vee y & \text{if } x = 0 \text{ or } y = 0 , \\ W_S(x,y) & \text{if } x, y \in (0,a] , \\ x \vee y \vee a & \text{otherwise.} \end{cases}$$

Theorem 3.5. (Çaylı [13]) Let $(L, \leq, 0, 1)$ be a bounded lattice and $a \in L \setminus \{0,1\}$. If V_T is a t-norm on $[a,1]$ and W_S is a t-conorm on $[0,a]$, then the functions $T^{**} : L^2 \to L$ and $S^{**} : L^2 \to L$ are a t-norm and a t-conorm on L, respectively, where

$$T^{**}(x,y) = \begin{cases} V_T(x,y) & \text{if } (x,y) \in [a,1)^2 , \\ x \wedge y & 1 \in \{x,y\}, \\ 0 & \text{otherwise.} \end{cases}$$

$$S^{**}(x,y) = \begin{cases} W_S(x,y) & \text{if } (x,y) \in (0,a]^2 , \\ x \vee y & 0 \in \{x,y\}, \\ 1 & \text{otherwise.} \end{cases}$$

Now, we introduce a new ordinal sum construction of t-norms on an arbitrary bounded lattice L with some properties related to fixed element $a \in L \setminus \{0,1\}$.

Theorem 3.6. Let $(L, \leq, 0, 1)$ be a bounded lattice and $a \in L \setminus \{0,1\}$. If $x \parallel y$ for all $x \in I_a$ and $y \in (0,a]$, and $x < y$ for all $x \in I_a$ and $y \in (a,1]$, then the function $T : L^2 \to L$ defined as follows is a t-norm on L, where V is a t-norm on $[0,a]$.

$$T(x,y) = \begin{cases} V(x,y) & \text{if } (x,y) \in [0,a]^2 , \\ y & \text{if } (x,y) \in (a,1] \times I_a , \\ x & \text{if } (x,y) \in I_a \times (a,1] , \\ 0 & \text{if } (x,y) \in [0,a] \times I_a \cup I_a \times [0,a] , \\ x \wedge y & \text{otherwise.} \end{cases}$$

P r o o f. It is easy to see that $T(x,y) = T(y,x)$ for all $x,y \in L$. So, commutativity of T is hold. Also, we have $T(x,1) = x \wedge 1 = x$ for all $x \in L$. So, the fact that $1 \in L$ is a neutral element of T.

i) Monotonicity: We prove that if $x \leq y$, then $T(x,z) \leq T(y,z)$ for all $z \in L$. If $\{x,y,z\} \cap \{0,1\} \neq \emptyset$ the monotonicity follows from the boundary condition of T. So we will verify them for $\{x,y,z\} \in L \setminus \{0,1\}$. If $(x,z) \in [0,a] \times I_a \cup I_a \times [0,a]$, then it must be $T(x,z) = 0$. So, $T(x,z) = 0 \leq T(y,z)$ for all $y \in L$. The proof can be split into all possible cases.

1. $x \in (0, a]$

   1.1 $y \in (0, a]$

      1.1.1. $z \in (0, a]$
      $$T(x, z) = V(x, z) \leq V(y, z) = T(y, z)$$

      1.1.2. $z \in (a, 1)$
      $$T(x, z) = x \leq y = T(y, z)$$

   1.2. $y \in (a, 1)$

      1.2.1. $z \in (0, a]$
      $$T(x, z) = V(x, z) \leq z = y \wedge z = T(y, z)$$

      1.2.2. $z \in (a, 1)$
      $$T(x, z) = x \wedge z \leq y \wedge z = T(y, z)$$

   1.3. $y \in I_a$
      Since $x \in (0, a]$ and $y \in I_a$, then it holds $x \parallel y$. So, it can not be the case $y \in I_a$.

2. $x \in (a, 1)$
   Then, it must be the case that $y \in (a, 1]$.

   2.1 $y \in (a, 1)$

      2.1.1. $z \in (0, a]$ or $z \in I_a$
      $$T(x, z) = z = T(y, z)$$

      2.1.2. $z \in (a, 1)$
      $$T(x, z) = x \wedge z \leq y \wedge z = T(y, z)$$

3. $x \in I_a$
   Then, it must be the case that $y \in I_a$ or $y \in (a, 1]$.

   3.1 $y \in I_a$

      3.1.1. $z \in (a, 1)$
      $$T(x, z) = x \leq y = T(y, z)$$

      3.1.2. $z \in I_a$
      $$T(x, z) = x \wedge z \leq y \wedge z = T(y, z)$$

   3.2. $y \in (a, 1)$

      3.2.1. $z \in (a, 1)$
      It can be obtained from the constraint of Theorem 3.6 that $x < z$ for $x \in I_a$ and $z \in (a, 1)$.
      $$T(x, z) = x < y \wedge z = T(y, z)$$

      3.2.2. $z \in I_a$
      $$T(x, z) = x \wedge z \leq z = T(y, z)$$

ii) Associativity: We need to prove that $T(x, T(y, z)) = T(T(x, y), z)$ for all $x, y, z \in L$. If at least one of x, y, z in L is 0 or 1, then it is obvious. So we will verify them for $\{x, y, z\} \in L \setminus \{0, 1\}$. Also, if $(x, z) \in (0, a] \times I_a \cup I_a \times (0, a]$ and $(x, y) \in (0, a] \times I_a \cup I_a \times (0, a]$, then the equality holds because of both sides are zero. So, the proof is split into all possible cases.

1. $x \in (0, a]$

    1.1 $y \in (0, a]$

        1.1.1. $z \in (0, a]$

$$T(x, T(y, z)) = T(x, V(y, z)) = V(x, V(y, z)) = V(V(x, y), z) = T(T(x, y), z)$$

        1.1.2. $z \in (a, 1)$

$$T(x, T(y, z)) = T(x, y) = V(x, y) = T(V(x, y), z) = T(T(x, y), z)$$

    1.2. $y \in (a, 1)$

        1.2.1. $z \in (0, a]$

$$T(x, T(y, z)) = T(x, z) = T(T(x, y), z)$$

        1.2.2. $z \in (a, 1)$

$$T(x, T(y, z)) = T(x, y \wedge z) = x = T(x, z) = T(T(x, y), z)$$

2. $x \in (a, 1)$

    2.1 $y \in (0, a]$

        2.1.1. $z \in (0, a]$

$$T(x, T(y, z)) = T(x, V(y, z)) = V(y, z) = T(y, z) = T(T(x, y), z)$$

        2.1.2. $z \in (a, 1)$

$$T(x, T(y, z)) = T(x, y) = y = T(y, z) = T(T(x, y), z)$$

        2.1.3. $z \in I_a$

$$T(x, T(y, z)) = T(x, 0) = 0 = T(y, z) = T(T(x, y), z)$$

    2.2. $y \in (a, 1)$

        2.2.1. $z \in (0, a]$ or $z \in I_a$

$$T(x, T(y, z)) = T(x, z) = z = T(x \wedge y, z) = T(T(x, y), z)$$

        2.2.2. $z \in (a, 1)$

$$T(x, T(y, z)) = T(x, y \wedge z) = x \wedge y \wedge z = T(x \wedge y, z) = T(T(x, y), z)$$

    2.3. $y \in I_a$

2.3.1. $z \in (0, a]$

$$T(x, T(y, z)) = T(x, 0) = 0 = T(y, z) = T(T(x, y), z)$$

2.3.2. $z \in (a, 1)$

$$T(x, T(y, z)) = T(x, y) = y = T(y, z) = T(T(x, y), z)$$

2.3.3. $z \in I_a$
Let $y \wedge z \in I_a$ or $y \wedge z = 0$.

$$T(x, T(y, z)) = T(x, y \wedge z) = y \wedge z = T(y, z) = T(T(x, y), z)$$

3. $x \in I_a$

3.1. $y \in (a, 1)$
3.1.1. $z \in (a, 1)$

$$T(x, T(y, z)) = T(x, y \wedge z) = x = T(x, z) = T(T(x, y), z)$$

3.1.2. $z \in I_a$

$$T(x, T(y, z)) = T(x, z) = T(T(x, y), z)$$

3.2. $y \in I_a$
3.2.1. $z \in (0, a]$
Let $x \wedge y \in I_a$ or $x \wedge y = 0$.

$$T(x, T(y, z)) = T(x, 0) = 0 = T(x \wedge y, z) = T(T(x, y), z)$$

3.2.2. $z \in (a, 1)$
Let $x \wedge y \in I_a$ or $x \wedge y = 0$.

$$T(x, T(y, z)) = T(x, y) = x \wedge y = T(x \wedge y, z) = T(T(x, y), z)$$

3.2.3. $z \in I_a$
Let $y \wedge z \in I_a$ and $(x \wedge y \in I_a$ or $x \wedge y = 0)$.

$$T(x, T(y, z)) = T(x, y \wedge z) = x \wedge y \wedge z = T(x \wedge y, z) = T(T(x, y), z)$$

Let $y \wedge z = 0$ and $(x \wedge y = 0$ or $x \wedge y \in I_a)$.

$$T(x, T(y, z)) = T(x, y \wedge z) = x \wedge y \wedge z = T(x \wedge y, z) = T(T(x, y), z)$$

So, we have the fact that T is a t-norm on L. □

Remark 3.7. The t-norm T introduced in Theorem 3.6 can be alternatively described as follows:

$$T(x, y) = \begin{cases} V(x, y) & \text{if } (x, y) \in [0, a]^2 , \\ y & \text{if } (x, y) \in (a, 1] \times I_a \cup [a, 1] \times [0, a] , \\ x & \text{if } (x, y) \in I_a \times (a, 1] \cup [0, a] \times [a, 1] , \\ 0 & \text{if } (x, y) \in [0, a] \times I_a \cup I_a \times [0, a] , \\ x \wedge y & \text{if } (x, y) \in [a, 1]^2 \cup I_a \times I_a. \end{cases}$$

Corollary 3.8. If we take $V = T_{\wedge}$ on $[0, a]$ given in Theorem 3.6, then we obtain the following t-norm on L.

$$T(x, y) = \begin{cases} y & \text{if } (x, y) \in (a, 1] \times I_a \ , \\ x & \text{if } (x, y) \in I_a \times (a, 1] \ , \\ 0 & \text{if } (x, y) \in [0, a] \times I_a \cup I_a \times [0, a] \ , \\ x \wedge y & \text{otherwise.} \end{cases}$$

Example 3.9. Consider the infinite lattice $(L_2 = \{0_{L_2}, m, a, r_1, r_2, \cdots_1, n, \cdots_2, k_{-2}, k_{-1}, k_0, k_1, k_2, \cdots_3, 1_{L_2}\}, \leq, 0_{L_2}, 1_{L_2})$ in Figure 2 satisfies the constraints of Theorem 3.6 (for element $a \in L_2$). Consider the t-norm $V : [0_{L_2}, a]^2 \to [0_{L_2}, a]$ as follows:

$$V(x, y) = \begin{cases} x \wedge y & \text{if } a \in \{x, y\} \ , \\ 0_{L_2} & \text{otherwise.} \end{cases}$$
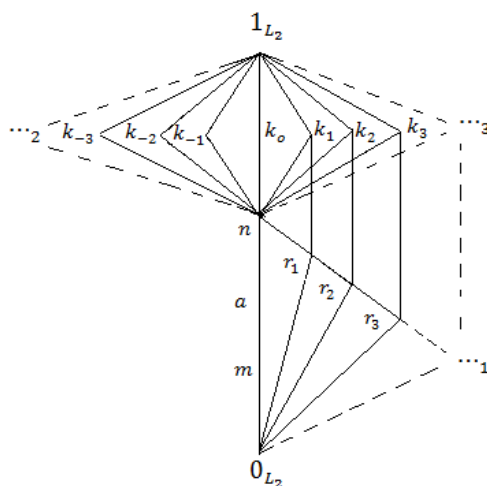
Fig. 2. The lattice L_2.

Then, the function T on L_2 defined by Table 2 is a t-norm.

Remark 3.10. Let $(L, \leq, 0, 1)$ be a bounded lattice and $a \in L \setminus \{0, 1\}$. In Theorem 3.6, observe that the condition for all $x \in I_a$ and $y \in (0, a]$ it holds $x \parallel y$ can not be omitted, in general. The following example illustrates the fact that the function $T : L^2 \to L$ defined by Theorem 3.6 is not a t-norm.

Example 3.11. The lattice $(L_3 = \{0_{L_3}, m, k, a, n, p, q, s, 1_{L_3}\}, \leq, 0_{L_3}, 1_{L_3})$ in Figure 3 does not satisfy (for $a \in L_3$) one of the constraints of Theorem 3.6. That is, there is

T	0_{L_2}	m	a	r_1	r_2	$\cdots_1$	n	$\cdots_2$	k_{-1}	k_0	k_1	$\cdots_3$	1_{L_2}
0_{L_2}	0_{L_2}	0_{L_2}	0_{L_2}	0_{L_2}	0_{L_2}	0_{L_2}	0_{L_2}	0_{L_2}	0_{L_2}	0_{L_2}	0_{L_2}	0_{L_2}	0_{L_2}
m	0_{L_2}	0_{L_2}	m	0_{L_2}	0_{L_2}	0_{L_2}	m	m	m	m	m	m	m
a	0_{L_2}	m	a	0_{L_2}	0_{L_2}	0_{L_2}	a	a	a	a	a	a	a
r_1	0_{L_2}	0_{L_2}	0_{L_2}	r_1	r_2	$\cdots_1$	r_1	r_1	r_1	r_1	r_1	r_1	r_1
r_2	0_{L_2}	0_{L_2}	0_{L_2}	r_2	r_2	$\cdots_1$	r_2	r_2	r_2	r_2	r_2	r_2	r_2
$\cdots_1$	0_{L_2}	0_{L_2}	0_{L_2}	$\cdots_1$	$\cdots_1$	$\cdots_1$	$\cdots_1$	$\cdots_1$	$\cdots_1$	$\cdots_1$	$\cdots_1$	$\cdots_1$	$\cdots_1$
n	0_{L_2}	m	a	r_1	r_2	$\cdots_1$	n	n	n	n	n	n	n
$\cdots_2$	0_{L_2}	m	a	r_1	r_2	$\cdots_1$	n	$\cdots_2$	n	n	n	n	$\cdots_2$
k_{-1}	0_{L_2}	m	a	r_1	r_2	$\cdots_1$	n	n	k_{-1}	n	n	n	k_{-1}
k_0	0_{L_2}	m	a	r_1	r_2	$\cdots_1$	n	n	n	k_0	n	n	k_0
k_1	0_{L_2}	m	a	r_1	r_2	$\cdots_1$	n	n	n	n	k_1	n	k_1
$\cdots_3$	0_{L_2}	m	a	r_1	r_2	$\cdots_1$	n	n	n	n	n	$\cdots_3$	$\cdots_3$
1_{L_2}	0_{L_2}	m	a	r_1	r_2	$\cdots_1$	n	$\cdots_2$	k_{-1}	k_0	k_1	$\cdots_3$	1_{L_2}

Tab. 2. The t-norm T on L_2.

the element $m \in L_3$ such that $m < k$ for $k \in I_a$ and $m \in (0_{L_3}, a)$. Consider the t-norm $V : [0_{L_3}, a]^2 \to [0_{L_3}, a]$, $V(x, y) = x \wedge y$.

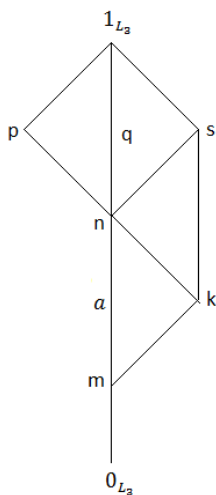Then, the function T on L_3 defined by Table 3 is not a t-norm. Indeed, it does not satisfy monotonicity. Clearly, $m < k$ and $T(m, m) = m \nleq 0_{L_3} = T(k, m)$.

T	0_{L_3}	m	k	a	n	p	q	s	1_{L_3}
0_{L_3}	0_{L_3}	0_{L_3}	0_{L_3}	0_{L_3}	0_{L_3}	0_{L_3}	0_{L_3}	0_{L_3}	0_{L_3}
m	0_{L_3}	m	0_{L_3}	m	m	m	m	m	m
k	0_{L_3}	0_{L_3}	k	0_{L_3}	k	k	k	k	k
a	0_{L_3}	m	0_{L_3}	a	a	a	a	a	a
n	0_{L_3}	m	k	a	n	n	n	n	n
p	0_{L_3}	m	k	a	n	p	n	n	p
q	0_{L_3}	m	k	a	n	n	q	n	q
s	0_{L_3}	m	k	a	n	n	n	s	s
1_{L_3}	0_{L_3}	m	k	a	n	p	q	s	1_{L_3}

Tab. 3. The function T on L_3.

Remark 3.12. Let $(L, \leq, 0, 1)$ be a bounded lattice and $a \in L \setminus \{0, 1\}$. In Theorem 3.6, observe that the condition for all $x \in I_a$ and $y \in (a, 1]$ it holds $x < y$ can not be omitted, in general. The following example illustrates the fact that the function $T : L^2 \to L$ defined by Theorem 3.6 is not a t-norm.

Example 3.13. The lattice $(L_4 = \{0_{L_4}, m, k, a, n, p, q, s, 1_{L_4}\}, \leq, 0_{L_4}, 1_{L_4})$ in Figure 4 does not satisfy one of the constraints of Theorem 3.6. Namely, there is the element

Fig. 3. The lattice L_3.

$n \in L_4$ such that $n \parallel k$ for $k \in I_a$ and $n \in (a, 1_{L_4})$. Consider the t-norm $V : [0_{L_4}, a]^2 \to [0_{L_4}, a]$ as follows:

$$V(x, y) = \begin{cases} x \wedge y & \text{if } a \in \{x, y\} \, , \\ 0_{L_4} & \text{otherwise.} \end{cases}$$

Then, the function T on L_4 defined by Table 4 is not a t-norm. Again, it does not satisfy monotonicity. Clearly $k < s$ and $T(k, n) = k \not\leq n = T(s, n)$.

T	0_{L_4}	m	k	a	n	p	q	s	1_{L_4}
0_{L_4}	0_{L_4}	0_{L_4}	0_{L_4}	0_{L_4}	0_{L_4}	0_{L_4}	0_{L_4}	0_{L_4}	0_{L_4}
m	0_{L_4}	0_{L_4}	0_{L_4}	m	m	m	m	m	m
k	0_{L_4}	0_{L_4}	k	0_{L_4}	k	k	k	k	k
a	0_{L_4}	m	0_{L_4}	a	a	a	a	a	a
n	0_{L_4}	m	k	a	n	n	n	n	n
p	0_{L_4}	m	k	a	n	p	n	n	p
q	0_{L_4}	m	k	a	n	n	q	n	q
s	0_{L_4}	m	k	a	n	n	n	s	s
1_{L_4}	0_{L_4}	m	k	a	n	p	q	s	1_{L_4}

Tab. 4. The function T on L_4.

Remark 3.14. The t-norm defined in Theorem 3.6 satisfies the t-norm defined in Definition 3.1 since we have constraints on the bounded lattice. Let's examine the examples below to explain the differences between the two t-norms.
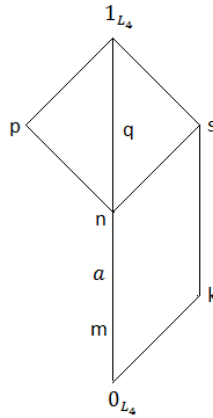
Fig. 4. The lattice L_4.

Example 3.15. Consider the bounded lattice $(L_3 = \{0_{L_3}, m, k, a, n, p, q, s, 1_{L_3}\}, \leq, 0_{L_3}, 1_{L_3})$ in Figure 3 and consider the t-norm $V : [0_{L_3}, a]^2 \to [0_{L_3}, a]$, $V(x, y) = x \wedge y$. According to the Definition 3.1, the function $T^\sim$ defined by Table 5 is a t-norm on L_3.

$T^\sim$	0_{L_3}	m	k	a	n	p	q	s	1_{L_3}
0_{L_3}	0_{L_3}	0_{L_3}	0_{L_3}	0_{L_3}	0_{L_3}	0_{L_3}	0_{L_3}	0_{L_3}	0_{L_3}
m	0_{L_3}	m	m	m	m	m	m	m	m
k	0_{L_3}	m	k	m	k	k	k	k	k
a	0_{L_3}	m	m	a	a	a	a	a	a
n	0_{L_3}	m	k	a	n	n	n	n	n
p	0_{L_3}	m	k	a	n	p	n	n	p
q	0_{L_3}	m	k	a	n	n	q	n	q
s	0_{L_3}	m	k	a	n	n	n	s	s
1_{L_3}	0_{L_3}	m	k	a	n	p	q	s	1_{L_3}

Tab. 5. The t-norm $T^\sim$ on L_3.

On the other side, we proved that the function T defined by Table 3 in Example 3.11 is not a t-norm on L_3.

Example 3.16. Consider the bounded lattice $(L_4 = \{0_{L_4}, m, k, a, n, p, q, s, 1_{L_4}\}, \leq, 0_{L_4}, 1_{L_4})$ in Figure 4 and consider the t-norm $V : [0_{L_4}, a]^2 \to [0_{L_4}, a]$ as follows:

$$V(x, y) = \begin{cases} x \wedge y & \text{if } a \in \{x, y\}, \\ 0_{L_4} & \text{otherwise.} \end{cases}$$

According to the Definition 3.1, the function $T^\sim$ defined by Table 6 is a t-norm on L_4.

$T^{\sim}$	0_{L_4}	m	k	a	n	p	q	s	1_{L_4}
0_{L_4}	0_{L_4}	0_{L_4}	0_{L_4}	0_{L_4}	0_{L_4}	0_{L_4}	0_{L_4}	0_{L_4}	0_{L_4}
m	0_{L_4}	0_{L_4}	0_{L_4}	m	m	m	m	m	m
k	0_{L_4}	0_{L_4}	k	0_{L_4}	0_{L_4}	0_{L_4}	0_{L_4}	k	k
a	0_{L_4}	m	0_{L_4}	a	a	a	a	a	a
n	0_{L_4}	m	0_{L_4}	a	n	n	n	n	n
p	0_{L_4}	m	0_{L_4}	a	n	p	n	n	p
q	0_{L_4}	m	0_{L_4}	a	n	n	q	n	q
s	0_{L_4}	m	k	a	n	n	n	s	s
1_{L_4}	0_{L_4}	m	k	a	n	p	q	s	1_{L_4}

Tab. 6. The t-norm $T^{\sim}$ on L_4.

On the other side, we proved that the function T defined by Table 4 in Example 3.13 is not a t-norm on L_4.

Remark 3.17. The t-norm defined in Theorem 3.6 is different from the t-norms defined in Theorem 3.4 and Theorem 3.5 proposal by [16] and [13], respectively. In general $T \neq T^*$ and $T \neq T^{**}$. We show these arguments by an example as follows.

Example 3.18. Consider the lattice $(L_5 = \{0_{L_5}, m, k, a, n, p, q, s, 1_{L_5}\}, \leq, 0_{L_5}, 1_{L_5})$ in Figure 5 and consider the t-norms $V_T : [a, 1_{L_5}]^2 \to [a, 1_{L_5}]$ and $V : [0_{L_5}, a]^2 \to [0_{L_5}, a]$ as follows:

$$V_T(x, y) = \begin{cases} x \wedge y & \text{if } 1_{L_5} \in \{x, y\}, \\ a & \text{otherwise.} \end{cases}$$

$$V(x, y) = \begin{cases} x \wedge y & \text{if } a \in \{x, y\}, \\ 0_{L_5} & \text{otherwise.} \end{cases}$$

Using the construction approaches in Theorem 3.6, Theorem 3.4 and Theorem 3.5, we define the t-norms T, T^* and T^{**} by Table 7, Table 8 and Table 9 respectively. Then,

- $T \neq T^*$ since $T(p, k) = k \neq 0_{L_5} = T^*(p, k)$.
- $T \neq T^{**}$ since $T(q, p) = n \neq a = T^{**}(q, p)$.

Next, we introduce a new ordinal sum construction of t-conorm on arbitrary bounded lattice L with some properties related to an element $a \in L \setminus \{0, 1\}$. We omit the proof of the next Theorem due to its similarity to the proof of Theorem 3.6.

Theorem 3.19. Let $(L, \leq, 0, 1)$ be a bounded lattice and $a \in L \setminus \{0, 1\}$. If $x \parallel y$ for all $x \in I_a$ and $y \in [a, 1)$, and $x > y$ for all $x \in I_a$ and $y \in [0, a)$, then the function $S : L^2 \to L$ defined as follows is a t-conorm on L, where W is a t-conorm on $[a, 1]^2$.
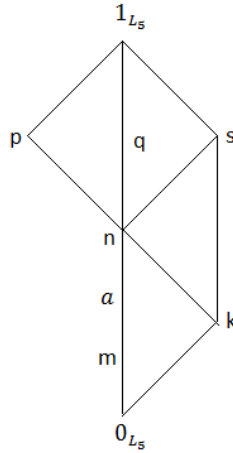
Fig. 5. The lattice L_5.

T	0_{L_5}	m	k	a	n	p	q	s	1_{L_5}
0_{L_5}	0_{L_5}	0_{L_5}	0_{L_5}	0_{L_5}	0_{L_5}	0_{L_5}	0_{L_5}	0_{L_5}	0_{L_5}
m	0_{L_5}	0_{L_5}	0_{L_5}	m	m	m	m	m	m
k	0_{L_5}	0_{L_5}	k	0_{L_5}	k	k	k	k	k
a	0_{L_5}	m	0_{L_5}	a	a	a	a	a	a
n	0_{L_5}	m	k	a	n	n	n	n	n
p	0_{L_5}	m	k	a	n	p	n	n	p
q	0_{L_5}	m	k	a	n	n	q	n	q
s	0_{L_5}	m	k	a	n	n	n	s	s
1_{L_5}	0_{L_5}	m	k	a	n	p	q	s	1_{L_5}

Tab. 7. The t-norm T on L_5.

$$S(x,y) = \begin{cases} W(x,y) & \text{if } (x,y) \in [a,1]^2 , \\ y & \text{if } (x,y) \in [0,a) \times I_a , \\ x & \text{if } (x,y) \in I_a \times [0,a) , \\ 1 & \text{if } (x,y) \in [a,1] \times I_a \cup I_a \times [a,1] , \\ x \vee y & \text{otherwise.} \end{cases}$$

The proof of this Theorem is the same as the related proof of Theorem 3.6 and therefore omitted. This argument is based on the the fact that exchanging, in original bounded lattice $\mathcal{L} = (L, \wedge, \vee, 0_L, 1_L)$, $\wedge$ and $\vee$, and 0_L and 1_L, i.e., considering $\overline{\mathcal{L}} = (L, \overline{\wedge}, \overline{\vee}, \overline{0_L}, \overline{1_L})$ with $\overline{\wedge} = \vee$, $\overline{\vee} = \wedge$, $\overline{0_L} = 1_L$, $\overline{1_L} = 0_L$, we obtain a dual lattice $\overline{\mathcal{L}}$, in

T^*	0_{L_5}	m	k	a	n	p	q	s	1_{L_5}
0_{L_5}	0_{L_5}	0_{L_5}	0_{L_5}	0_{L_5}	0_{L_5}	0_{L_5}	0_{L_5}	0_{L_5}	0_{L_5}
m	0_{L_5}	m	0_{L_5}	m	m	m	m	m	m
k	0_{L_5}	0_{L_5}	0_{L_5}	0_{L_5}	0_{L_5}	0_{L_5}	0_{L_5}	0_{L_5}	k
a	0_{L_5}	m	0_{L_5}	a	a	a	a	a	a
n	0_{L_5}	m	0_{L_5}	a	a	a	a	a	n
p	0_{L_5}	m	0_{L_5}	a	a	a	a	a	p
q	0_{L_5}	m	0_{L_5}	a	a	a	a	a	q
s	0_{L_5}	m	0_{L_5}	a	a	a	a	a	s
1_{L_5}	0_{L_5}	m	k	a	n	p	q	s	1_{L_5}

Tab. 8. The t-norm T^* on L_5.

T^{**}	0_{L_5}	m	k	a	n	p	q	s	1_{L_5}
0_{L_5}	0_{L_5}	0_{L_5}	0_{L_5}	0_{L_5}	0_{L_5}	0_{L_5}	0_{L_5}	0_{L_5}	0_{L_5}
m	0_{L_5}	0_{L_5}	0_{L_5}	0_{L_5}	0_{L_5}	0_{L_5}	0_{L_5}	0_{L_5}	m
k	0_{L_5}	0_{L_5}	0_{L_5}	0_{L_5}	0_{L_5}	0_{L_5}	0_{L_5}	0_{L_5}	k
a	0_{L_5}	0_{L_5}	0_{L_5}	a	a	a	a	a	a
n	0_{L_5}	0_{L_5}	0_{L_5}	a	a	a	a	a	n
p	0_{L_5}	0_{L_5}	0_{L_5}	a	a	a	a	a	p
q	0_{L_5}	0_{L_5}	0_{L_5}	a	a	a	a	a	q
s	0_{L_5}	0_{L_5}	0_{L_5}	a	a	a	a	a	s
1_{L_5}	0_{L_5}	m	k	a	n	p	q	s	1_{L_5}

Tab. 9. The t-norm T^{**} on L_5.

this duality, t-norms on $\mathcal{L}$ are linked to t-conorms on $\overline{\mathcal{L}}$. Also, we omitted the examples related to t-conorms from this duality.

Remark 3.20. The t-conorm S introduced in Theorem 3.19 can be described alternatively as follows:

$$S(x,y) = \begin{cases} W(x,y) & \text{if } (x,y) \in [a,1]^2 , \\ y & \text{if } (x,y) \in [0,a) \times I_a \cup [0,a] \times [a,1] , \\ x & \text{if } (x,y) \in I_a \times [0,a) \cup [a,1] \times [0,a] , \\ 1 & \text{if } (x,y) \in [a,1] \times I_a \cup I_a \times [a,1] , \\ x \vee y & \text{if } (x,y) \in [0,a]^2 \cup I_a \times I_a. \end{cases}$$

Corollary 3.21. If we take $W = S_\vee$ on $[a,1]^2$ given in Theorem 3.19, then we obtain

the following t-conorm on L.

$$S(x,y) = \begin{cases} y & \text{if } (x,y) \in [0,a) \times I_a , \\ x & \text{if } (x,y) \in I_a \times [0,a) , \\ 1 & \text{if } (x,y) \in [a,1] \times I_a \cup I_a \times [a,1] , \\ x \vee y & \text{otherwise.} \end{cases}$$

4. MODIFIED ORDINAL SUM CONSTRUCTION OF T-NORMS AND T-CONORMS ON BOUNDED LATTICES

In [13, 16], we know that new t-norms and t-conorms on bounded lattices can be obtained by means of recursion. In this section, based on the approaches of constructing t-norms and t-conorms proposed in Section 3, we introduce a new ordinal sum construction of t-norms and t-conorms on an arbitrary bounded lattice L by means of recursion.

First, we introduce a new ordinal sum construction of t-norms and t-conorms on an arbitrary bounded lattice L by means of recursion in Theorem 4.1 and Theorem 4.3.

Theorem 4.1. Let $(L, \leq, 0, 1)$ be a bounded lattice and $\{a_0, a_1, a_2, \cdots, a_n\}$ be a finite chain in L such that $0 = a_0 < a_1 < a_2 < \cdots < a_n = 1$. Let $x \parallel y$ for all $x \in I_{a_i}$ and $y \in (0, a_i]$, and $x < y$ for all $x \in I_{a_i}$ and $y \in (a_i, 1]$, and $V : [0, a_1]^2 \to [0, a_1]$ be a t-norm. Then, the function $T_n : L^2 \to L$ defined as follows is a t-norm, where $T_1 = V$ and for $i \in \{2, \cdots, n\}$, the function $T_i : [0, a_i]^2 \to [0, a_i]$ is given by

$$T_i(x,y) = \begin{cases} T_{i-1}(x,y) & \text{if } (x,y) \in [0, a_{i-1}]^2 , \\ y & \text{if } (x,y) \in (a_{i-1}, a_i] \times I_{a_{i-1}} , \\ x & \text{if } (x,y) \in I_{a_{i-1}} \times (a_{i-1}, a_i] , \\ 0 & \text{if } (x,y) \in [0, a_{i-1}] \times I_{a_{i-1}} \cup I_{a_{i-1}} \times [0, a_{i-1}] , \\ x \wedge y & \text{otherwise.} \end{cases} \tag{3}$$

Proof. The proof follows easily from Theorem 3.6 by induction and therefore it is omitted. □

It should be pointed out that if L is a chain then the Formula (3), can be reformulated into

$$T_i(x,y) = \begin{cases} T_{i-1}(x,y) & \text{if } (x,y) \in [0, a_{i-1}]^2 , \\ x \wedge y & \text{otherwise.} \end{cases}$$

Example 4.2. Consider the lattice $(L_6 = \{0_{L_6}, d, m, b, c, k, t, 1_{L_6}\}, \leq, 0_{L_6}, 1_{L_6})$ described in Figure 6 with the finite chain $0_{L_6} < d < m < k < t < 1_{L_6}$ in L_6 and define the t-norm $V : [0_{L_6}, d]^2 \to [0_{L_6}, d]$ by $V = T_\wedge$. By using Theorem 4.1, where $V = T_1$, t-norms $T_2 : [0_{L_6}, m]^2 \to [0_{L_6}, m]$, $T_3 : [0_{L_6}, k]^2 \to [0_{L_6}, k]$, $T_4 : [0_{L_6}, t]^2 \to [0_{L_6}, t]$, $T_5 : L_6^2 \to L_6$ are defined in Tables 10-13.
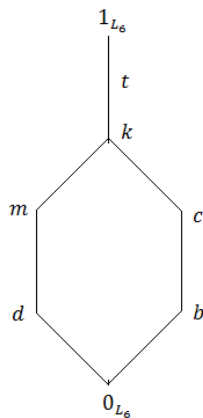
Fig. 6. The lattice L_6.

T_2	0_{L_6}	d	m
0_{L_6}	0_{L_6}	0_{L_6}	0_{L_6}
d	0_{L_6}	d	d
m	0_{L_6}	d	m

Tab. 10. The t-norm T_2 on L_6.

T_3	0_{L_6}	d	m	b	c	k
0_{L_6}	0_{L_6}	0_{L_6}	0_{L_6}	0_{L_6}	0_{L_6}	0_{L_6}
d	0_{L_6}	d	d	0_{L_6}	0_{L_6}	d
m	0_{L_6}	d	m	0_{L_6}	0_{L_6}	m
b	0_{L_6}	0_{L_6}	0_{L_6}	b	b	b
c	0_{L_6}	0_{L_6}	0_{L_6}	b	c	c
k	0_{L_6}	d	m	b	c	k

Tab. 11. The t-norm T_3 on L_6.

Theorem 4.3. Let $(L, \leq, 0, 1)$ be a bounded lattice and $\{a_0, a_1, a_2, \ldots, a_n\}$ be a finite chain in L such that $1 = a_0 > a_1 > a_2 > \cdots > a_n = 0$. Let $x \parallel y$ for all $x \in I_{a_i}$ and $y \in [a_i, 1)$, and $x > y$ for all $x \in I_{a_i}$ and $y \in [0, a_i)$, and $W : [a_1, 1]^2 \to [a_1, 1]$ be a t-conorm. Then, the function $S_n : L^2 \to L$ defined recursively as follows is a t-conorm,

T_4	0_{L_6}	d	m	b	c	k	t
0_{L_6}	0_{L_6}	0_{L_6}	0_{L_6}	0_{L_6}	0_{L_6}	0_{L_6}	0_{L_6}
d	0_{L_6}	d	d	0_{L_6}	0_{L_6}	d	d
m	0_{L_6}	d	m	0_{L_6}	0_{L_6}	m	m
b	0_{L_6}	0_{L_6}	0_{L_6}	b	b	b	b
c	0_{L_6}	0_{L_6}	0_{L_6}	b	c	c	c
k	0_{L_6}	d	m	b	c	k	k
t	0_{L_6}	d	m	b	c	k	t

Tab. 12. The t-norm T_4 on L_6.

T_5	0_{L_6}	d	m	b	c	k	t	1_{L_6}
0_{L_6}	0_{L_6}	0_{L_6}	0_{L_6}	0_{L_6}	0_{L_6}	0_{L_6}	0_{L_6}	0_{L_6}
d	0_{L_6}	d	d	0_{L_6}	0_{L_6}	d	d	d
m	0_{L_6}	d	m	0_{L_6}	0_{L_6}	m	m	m
b	0_{L_6}	0_{L_6}	0_{L_6}	b	b	b	b	b
c	0_{L_6}	0_{L_6}	0_{L_6}	b	c	c	c	c
k	0_{L_6}	d	m	b	c	k	k	k
t	0_{L_6}	d	m	b	c	k	t	t
1_{L_6}	0_{L_6}	d	m	b	c	k	t	1_{L_6}

Tab. 13. The t-norm T_5 on L_6.

where $S_1 = W$ and for $i \in \{2, \ldots, n\}$, the function $S_i : [a_i, 1]^2 \to [a_i, 1]$ is given by

$$S_i(x, y) = \begin{cases} S_{i-1}(x, y) & \text{if } (x, y) \in [a_{i-1}, 1]^2 , \\ y & \text{if } (x, y) \in [a_i, a_{i-1}) \times I_{a_{i-1}} , \\ x & \text{if } (x, y) \in I_{a_{i-1}} \times [a_i, a_{i-1}) , \\ 1 & \text{if } (x, y) \in [a_{i-1}, 1] \times I_{a_{i-1}} \cup I_{a_{i-1}} \times [a_{i-1}, 1] , \\ x \vee y & \text{otherwise.} \end{cases} \qquad (4)$$

It should be pointed out that if L is a chain then the Formula (4), can be reformulated into

$$S_i(x, y) = \begin{cases} S_{i-1}(x, y) & \text{if } (x, y) \in [a_{i-1}, 1]^2 , \\ x \vee y & \text{otherwise.} \end{cases}$$

We omitted the examples with related to t-conorms from duality.

5. CONCLUDING REMARKS

We have introduced a new construction method for building t-norms and t-conorms on an arbitrary bounded lattice with some constraints. Based on this ordinal sum

method, we have introduced a new class of t-norms T and t-conorms S on an arbitrary relevant bounded lattice, by using the existence of a t-norm V on a sublattice $[0, a]$ and a t-conorm W on a sublattice $[a, 1]$, respectively. In order to better understand the introduced t-norms T and t-conorms S, we have given some illustrative examples. Also, we have shown that our new construction methods for t-norms T and t-conorms S can be generalized by induction to a modified ordinal sum for t-norms and t-conorms on relevant bounded lattice, respectively. Again, we have given some illustrative examples. Our methods allow to construct of t-norms and t-conorms with the unitary subsets of $[0, 1]$ playing the role of its identity in the bounded lattices frequently considered in several branches of uncertainty modeling and information systems such as group decision making problems [26] and computing with words [24, 25], including set-valued fuzzy sets, hesitant fuzzy sets and typical hesitant fuzzy sets [7, 29].

ACKNOWLEDGEMENT

R E F E R E N C E S

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