KYBERNETIKA — VOLUME *57* (2021), NUMBER 2, PAGES 193–219

POLE PLACEMENT AND MIXED SENSITIVITY OF LTI MIMO SYSTEMS HAVING CONTROLLED OUTPUTS DIFFERENT FROM MEASUREMENTS

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Multi-Input Multi-Output (MIMO) Linear Time-Invariant (LTI) controllable and observable systems where the controller has access to some plant outputs but not others are considered. Analytical expressions of coprime factorizations of a given plant, a solution of the Diophantine equation and the two free parameters of a two-degrees of freedom (2DOF) controller based on observer stabilizing control are presented solving a pole placement problem, a mixed sensitivity criterion, and a reference tracking problem. These solutions are based on proposed stabilizing gains solving a pole placement problem by output feedback. The proposed gains simplify the coprime factorizations of the plant and the controller, and allow assigning a decoupled characteristic polynomial. The 2DOF stabilizing control is based on the Parameterization of All Stabilizing Controllers (PASC) where the free parameter in the feedback part of the controller solves the mixed sensitivity robust control problem of attenuation of a Low-Frequency (LF) additive disturbance at the input of the plant and of a High-Frequency (HF) additive disturbance at the measurement, while the free parameter in the reference part of the controller assures that the controlled output tracks the reference at LF such as step or sinusoidal inputs. With the proposed expressions, the mixed sensitivity problem is solved without using weighting functions, so the controller does not increase its order; and the infinite norm of the mixed sensitivity criterion, as well as the assignment of poles, is determined by a set of control parameters.

Keywords: stabilizing control, mixed sensitivity, pole placement, reference tracking, linear systems, robust control, 2DOF control configuration

Classification: 93C05, 93C35, 93D09, 93C73, 93C41

1. INTRODUCTION

In control and mechatronic systems in addition to achieving stability, it is desired to meet performance criteria even in the presence of uncertainties and disturbances. Robust control theory (see [20]) gives us tools to achieve these goals. This theory bases its criterion on the minimization of the $\mathcal{H}_\infty$ norm of the functions that relate inputs to outputs of bounded energy. Due to the relationships that exist between these functions, it is sometimes not possible to satisfy several criteria at the same time, so there are techniques such as loop shaping and mixed sensitivity (see [15] and [12], respectively),

DOI: 10.14736/kyb-2021-2-0193

for an output feedback control. For controlled outputs different from the measured outputs, a feedback configuration can be rearranged to meet the standard robust control framework by separating the controller, getting an augmented plant with two sets of inputs: the disturbance signals that are desired to be attenuated and the control inputs; and two sets of outputs: the measured outputs and the signals of interest. The $\mathcal{H}_\infty$ norm of the relationships between disturbance signals and the signals of interest is the criterion to be minimized.

It is well known that the pole-placement approach for MIMO systems is non-unique and can lead to unsatisfactory closed-loop performance. These problems are tackled adding additional control design requirements. Work in this direction is the one of [13] where a Linear Matrix Inequality (LMI) technique is described to perform robust pole placement by proportional-derivative feedback on Linear Time-Invariant (LTI) Multi-Input Multi-Output (MIMO) systems subject to polytopic or norm-bounded uncertainty. In the work of [14], a robust pole assignment is presented, consisting of making exact pole assignment finding optimal feedback gains by using recurrent neural networks, guaranteeing closed-loop stability for a disturbed or uncertain plant state matrix. Also, the possible undesired pole-zero cancellation of the mixed sensitivity designs proposed by [12] was examined in the work of [18], it is pointed out that pole-zero cancellation is dependent upon the choice of weighting functions and the particular construction of weighting function is given to prevent the phenomenon. Two techniques are compared in the work of [4] that prevent pole-zero cancellation of the Riccati-based mixed sensitivity approach. In this paper, the proposed approach is to solve the closed pole placement control problem using the static gains of the formulas of [16] for the Parameterization of All Stabilizing Controllers (PASC) and the remaining free parameters solve a mixed sensitivity problem.

The state-space solution for the standard control framework given by [3] consists of solving two algebraic Riccati equations in an iterated way to find feedback gains for the controller. With the LMI approach developed by [6], it is also possible to assign the closed-loop poles in a prescribed LMI region (see for example [1]). Both solutions require an augmented plant, which can include weighting functions to delimit frequency bandwidths, that allows to minimize the criterion and implying an increase in the controller order. An approach by soft computing techniques like Evolution Algorithms can be found in the work of [17] that presents pole assignment and mixed sensitivity criterion in a multi-objective function, following with an optimization procedure.

Different techniques, methods, and tools are used in the literature to obtain a controller that satisfies several specifications. This paper extends the work of [5], that has the specifications to track step input reference for controlled outputs different from feedback outputs, to place closed-loop real poles, and to solve a mixed sensitivity criterion. These specifications are set-up in a 2DOF control configuration since each parameter can be assigned for different specifications; in particular, the control scheme presented in the work of [2] that was used for input-output decoupling, which consists on separating the plant into two plants, one for the controlled output and other for feedback output. The controller is based on the PASC (see [19]) that consists of coprime factorizations of the plant, a solution of the Diophantine equation, and two free parameters. Analytical expressions of the PASC were obtained using the approach presented in the works of [7, 8, 9], where the expressions are in the frequency domain with elements of the state-space representation of the given square LTI plant, *i.e.*, the number of inputs equals the number of outputs. In those works, first, coprime factorizations of the plant are proposed and then the controller factorization is obtained algebraically by solving the correspondent Diophantine equation to finally propose the two free parameters that solve the mixed sensitivity criterion and the input reference tracking. Under this approach, different mixed sensitivity problems have been solved and one of its advantages is that it does not require the use of weighting functions that can increase the controller order.

In this paper, the improvements are to solve the problem of attenuating the most common disturbances in a feedback system, which are, the additive disturbances to the plant input at Low-Frequencies (LF) and disturbances in the measurement of the output in High-Frequencies (HF) by a mixed sensitivity criterion to tracking the input reference at LF, such as step or sinusoidal inputs, for controlled outputs different from feedback outputs, and to solve pole placement by assigning poles at specific locations in the complex plane to guarantee a satisfactory transient response. The analytical expressions of the coprime factorizations of the plant and the controller are obtained by using the formulas presented by [16] using proposed feedback gains that assign poles at pre-specified locations.

The present work is organized as follows. Section 2 reviews the control scheme and its PASC, the considered disturbances to establish a mixed sensitivity problem, and the considered class of systems with two change of basis given by [9] to get the feedback gains that will place the closed-loop poles, to finally presents the overall problem and the procedure to solve it. In Section 3 we give the proposed feedback gains and factorizations for the plant and the controller, elements of the PASC; and the expressions of the two free parameters for solving a mixed sensitivity problem and an input reference tracking problem. In Section 4 the results are applied to a two-cart system and a half-car active suspension system.

Notation.

$\mathcal{R}$ denotes the set of real numbers; $\mathcal{R}(s)$ the set of all real rational functions in the complex variable s with real coefficients; $\mathcal{RH}_\infty$ the set of stable proper rational functions in the complex variable s; I_m the identity matrix of dimensions $m \times m$; and $A_l := \lim_{s\to 0} A(s)$ and $A_h := \lim_{s\to\infty} A(s)$ the asymptotic approximations of a matrix $A(s) \in \Re(s)$, in LF and HF, respectively.

2. PRELIMINARIES AND PROBLEM STATEMENT

In this section, the assumptions for the control system are considered and the problem statement is presented.

Consider the feedback system shown in Figure 1, proposed by [2], where P is the given nominal plant, K is the two-degrees of freedom (2DOF) controller to be designed, W_3 and W_4 are stable strictly proper weighting functions, y_o is the output to be controlled, y_m is the measured output, y_d is the reference, d_i, d_m, and d_h are disturbances additive at the input, at the measurement and additive at the output of the plant, respectively; with, P and K satisfying the following assumptions, A1 Let $P \in \mathcal{R}(s)^{(p_o+m)\times m}$ with the following right-coprime factorization (r.c.f.),

$$P = \begin{bmatrix} P_o \\ P_m \end{bmatrix} = \begin{bmatrix} N_o \\ N_m \end{bmatrix} D^{-1} \tag{1}$$

where, $N_o \in \mathcal{RH}_\infty^{p_o\times m}$, $N_m \in \mathcal{RH}_\infty^{m\times m}$, and $D \in \mathcal{RH}_\infty^{m\times m}$ is a common right coprime denominator of P_o and P_m, and,

A2 Let $K \in \mathcal{R}(s)^{m\times(p_o+m)}$ with the following left-coprime factorization (l.c.f.) ,

$$K = \widetilde{D}_K^{-1} \begin{bmatrix} Q & \widetilde{N}_K \end{bmatrix} \tag{2}$$

where $\widetilde{D}_K \in \mathcal{RH}_\infty^{m\times m}$ is a common left coprime denominator of the feedback and reference controller parts, $Q \in \mathcal{RH}_\infty^{m\times p_o}$ is a free control parameter, and $\widetilde{N}_K \in \mathcal{RH}_\infty^{m\times m}$.

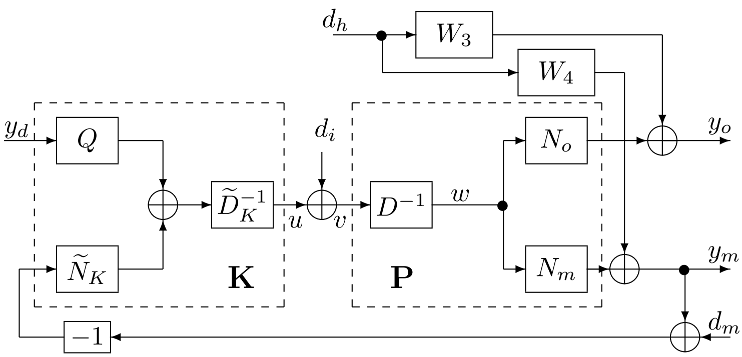

Fig. 1. Two-parameter control configuration.

Under assumption A1, the plant P_m is square, i. e., the signals y_m and u in Figure 1 have the same dimensions. Also, from Figure 1 the outputs y_o and y_m are described by,

$$y_o = N_o w + W_3 d_h \tag{3}$$

$$y_m = N_m w + W_4 d_h \tag{4}$$

where

$$w = \left(\widetilde{D}_K D + \widetilde{N}_K N_m\right)^{-1} \left(Q y_d + \widetilde{D}_K d_i - \widetilde{N}_K d_m - \widetilde{N}_K W_4 d_h\right). \tag{5}$$

As can be seen from equations (3) to (5), with coprime factorizations of P and K over $\mathcal{RH}_\infty$, the system is internally stable if the Diophantine equation,

$$\widetilde{D}_K D + \widetilde{N}_K N_m = I_m \tag{6}$$

is solved over $\mathcal{RH}_\infty$ for $\widetilde{D}_K$ and $\widetilde{N}_K$, which involves the given coprime factorizations only of P_m and not P_o. So, the role of K, in particular of the feedback part of K is to assure the stability of P_m through solving Eq. (6), and the overall system is stable since Q and N_o belong to $\mathcal{RH}_\infty$.

The next Theorem given by [19] gives the PASC for the feedback system shown in Figure 1.

Theorem 2.1. Let P be a given plant on the feedback configuration shown in Figure 1 satisfying assumption A1. Suppose (N_m, D), $(\widetilde{D}_m, \widetilde{N}_m)$, are any right and left-coprime factorizations of P_m over $\mathcal{RH}_\infty$, and that X, $Y \in \mathcal{RH}_\infty$ satisfy $XN_m + YD = I_m$. Then the two-parameter controller that stabilizes P satisfying assumption A2 is given by,

$$K = \widetilde{D}_K^{-1} \begin{bmatrix} Q & \widetilde{N}_K \end{bmatrix} \tag{7}$$

where

$$\widetilde{D}_K := Y - R\widetilde{N}_m \tag{8}$$

$$\widetilde{N}_K := X + R\widetilde{D}_m \tag{9}$$

with the free control parameters $Q \in \mathcal{RH}_\infty^{m\times p_o}$ and $R \in \mathcal{RH}_\infty^{m\times m}$ such that $\det\left(Y - R\widetilde{N}_m\right) \neq 0$.

Using the definitions given by equations (8) and (9), and the *r.c.f* and *l.c.f.* of $P_m = N_m D^{-1} = \widetilde{D}_m^{-1}\widetilde{N}_m$, the Diophantine equation given in Eq. (6), reduces to,

$$XN_m + YD = I_m. \tag{10}$$

The condition $det\left(Y - R\widetilde{N}_m\right) \neq 0$ of Theorem 2.1 is almost always satisfied for all s, and if P_m satisfies the parity interlacing property then a stable controller exists among the PASC (see [19]). The following is a standard assumption in robust control assuring, by Parseval's Lemma, that the energies of the outputs are bounded, when the $\mathcal{H}_\infty$ norm of the input-output relations are bounded.

A3 The energies of the disturbances d_i, d_m and d_h are considered to be bounded.

To attenuate the effects of d_i, d_m, and d_h over y_o, we reduce the infinity norm of the involved functions. Since W_3 is a stable low-pass filter, the HF external disturbance d_h does not affect y_o, also, since W_4 is a stable low-pass filter, from equations (3) to (6), d_h is attenuated over y_o, as $\|\widetilde{N}_K\|_\infty$ is minimized. Thus, from equations (3) to (6) and Theorem 2.1, to attenuate the effects of d_i, d_m, and d_h over y_o corresponds to minimize over R the mixed sensitive criterion,

$$J_1 = \left\| \begin{bmatrix} W_1 N_o \widetilde{D}_K \\ W_2 N_o \widetilde{N}_K \end{bmatrix} \right\|_\infty \tag{11}$$

where W_1 and W_2 are weighting functions that represent LF and HF bandwidths, respectively, assuming that d_i is more significant at LF and d_m and d_h are more significant at HF. So, in the same way, as in the works of [10] and [7], the criterion given in Eq. (11) is transformed into,

$$J_2 = \left\| \begin{bmatrix} N_o\widetilde{D}_{Kl} \\ N_o\widetilde{N}_{Kh} \end{bmatrix} \right\|_\infty \tag{12}$$

where $N_o\widetilde{D}_{Kl}$ is the approximation of $N_o\widetilde{D}_K$ at LF and $N_o\widetilde{N}_{Kh}$ is the approximation of $N_o\widetilde{N}_K$ at HF; i. e., $N_o\widetilde{D}_{Kl} = \lim_{s\to 0}(N_o\widetilde{D}_K)$ and $N_o\widetilde{N}_{Kh} = \lim_{s\to\infty}(N_o\widetilde{N}_K)$. The criterion given in Eq. (12) involves the simultaneous minimization of $N_o\widetilde{D}_{Kl}$ and $N_o\widetilde{N}_{Kh}$, this is, minimize over R,

$$J_3 = \|N_o\widetilde{D}_{Kl}\|_\infty + \mu\left(\|N_o\widetilde{D}_{Kl}\|_\infty - \|N_o\widetilde{N}_{Kh}\|_\infty\right) \tag{13}$$

where μ is a Lagrange multiplier; or equivalently, the criterion,

$$\min_R \quad \left\|N_o\widetilde{D}_{Kl}\right\|_\infty \tag{14a}$$

$$\text{subject to} \quad \left\|N_o\widetilde{D}_{Kl}\right\|_\infty = \left\|N_o\widetilde{N}_{Kh}\right\|_\infty. \tag{14b}$$

Note that in minimizing the function $\|N_o\widetilde{N}_{Kh}\|_\infty$, if $\|\widetilde{N}_{Kh}\|_\infty$ is the function that is most minimized, it implies to a certain extent that $\|N_m\widetilde{N}_{Kh}\|_\infty$, which correspond to an output multiplicative unstructured uncertainty model of P_m (see [20]), is also minimized at HF, where the unstructured uncertainties are more significant.

One of the advantages of using approximation is that it does not require the use of filters that can increase the controller dimension.

The following is the class of rectangular systems considered for the nominal plant. Consider that the state-space representation of the given nominal plant P satisfies,

A4 Let the state-space realization of $P \in \mathcal{R}(s)^{(p_o+m)\times m}$ be,

$$\begin{cases} \dot{x} = Fx + Gv \\ y = Hx \end{cases} \tag{15}$$

with $y = [y_o^T \ y_m^T]^T$. Allow $F \in \mathcal{R}^{n\times n}$, $G \in \mathcal{R}^{n\times m}$ and $H \in \mathcal{R}^{(p_o+m)\times n}$ be block partitioned as,

$$F = \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix}, \quad G = \begin{bmatrix} G_{11} \\ G_{21} \end{bmatrix}, \quad H = \begin{bmatrix} H_1 \\ H_2 \end{bmatrix} \tag{16}$$

being

$$H_1 = \begin{bmatrix} H_{11} & H_{12} \end{bmatrix} \quad \text{and} \quad H_2 = \begin{bmatrix} H_{21} & H_{22} \end{bmatrix} \tag{17}$$

where $n = 2m$ is even, $p_o \le m$, H_{11}, $H_{12} \in \mathcal{R}^{p_o\times m}$ and F_{11}, F_{12}, F_{21}, F_{22}, G_{11}, G_{21}, H_{21}, $H_{22} \in \mathcal{R}^{m\times m}$. Consequently, (F, G, H_1) and (F, G, H_2) are the state-space realizations for P_o and P_m, respectively, with P_m controllable and observable.

Remark 2.2. In this paper, the proposed factorizations of the plant P_m and the controller are based on the formulas of [16] and the change of basis proposed in the work of [9]. Although, the formulas of [16] hold for $p_m \neq m$, where p_m is the dimension of y_m, the change of basis of [9] becomes self-involved for $p_m \neq m$. The plant P_o can be non-square since this plant is involved only on the mixed sensitivity criterion. If $p_m > m$ then we can use a left-inverse H^+ such that $H^+H = I_{n\times n}$. Assumption A4 avoid using pseudo-inverse matrices that both loss uniqueness of the solution and closed-loop stability can be lost, so, an analysis is required to integrate the kernel parameters of H^+ into the design, as in [10] was done.

For a proper system, a proposed scheme presented in [8] for a 2DOF controller can be used to work only with the strictly proper part of the system.

The formulas presented in the work of [16] allows obtaining coprime factorizations for the given plant and a solution for the Diophantine equation given in Eq. (10) based on feedback gains. The following formulas are based on the state-space representation of the plant under assumption A4; i. e.,

$$\begin{aligned}
\begin{bmatrix} N_o \\ N_m \end{bmatrix} &= H\left(sI_n - F + G\mathcal{K}\right)^{-1} G && (18)\\
D &= I_m - \mathcal{K}\left(sI_n - F + G\mathcal{K}\right)^{-1} G && (19)\\
\widetilde{N}_m &= H_2\left(sI_n - F + LH_2\right)^{-1} G && (20)\\
\widetilde{D}_m &= I_m - H_2\left(sI_n - F + LH_2\right)^{-1} L && (21)\\
X &= \mathcal{K}\left(sI_n - F + LH_2\right)^{-1} L && (22)\\
Y &= I_m + \mathcal{K}\left(sI_n - F + LH_2\right)^{-1} G && (23)
\end{aligned}$$

with $\mathcal{K}$ and L such that the characteristic polynomials of $F - G\mathcal{K}$ and $F - LH_2$ are stable.

The works of [16] and [9] are based on the separation principle such that stability is preserved for the whole system when $\mathcal{K}$ and L ensure separately the stability of the characteristic polynomials of $F - G\mathcal{K}$ and $F - LH_2$. Also, as mentioned in [20], the PASC is an observer-based stabilizing controller.

The following transformations presented by [9] and updated in [8], are based on the state-space representation of P given in assumption A4, giving us a special structure that allows us to propose the feedback gains $\mathcal{K}$ and L.

Lemma 2.3. Consider P satisfying assumptions A1 and A4 with non-singular matrices G_{21} and H_{22}. Then, a change of basis $\xi = T_1 x$ where,

$$\begin{aligned}
T_1 &:= \begin{bmatrix} I_m & -G_{11}G_{21}^{-1} \\ V_1\Theta_1 & I_m \end{bmatrix}\\
T_1^{-1} &= \begin{bmatrix} \Delta_1^{-1} & \Delta_1^{-1}G_{11}G_{21}^{-1} \\ -V_1\Theta_1\Delta_1^{-1} & I_m - V_1\Theta_1\Delta_1^{-1}G_{11}G_{21}^{-1} \end{bmatrix}
\end{aligned} \qquad (24)$$

with $V_1 := \left(F_{12} - G_{11}G_{21}^{-1}F_{22}\right)^{-1}$, $\Theta_1 := F_{11} - G_{11}G_{21}^{-1}F_{21}$ and $\Delta_1 := I_m + G_{11}G_{21}^{-1}V_1\Theta_1$;

arrives at the following structure of the system realization in new coordinates,

$$A_k = \begin{bmatrix} 0 & A_{12k} \\ A_{21k} & A_{22k} \end{bmatrix}, \; B_k = \begin{bmatrix} 0 \\ B_m \end{bmatrix}, \; C_k = \begin{bmatrix} C_{11k} & C_{12k} \\ C_{21k} & C_{22k} \end{bmatrix}. \tag{25}$$

Consequently, $\left(A_k, B_k, \begin{bmatrix} C_{11k} & C_{12k} \end{bmatrix}\right)$ and $\left(A_k, B_k, \begin{bmatrix} C_{21k} & C_{22k} \end{bmatrix}\right)$ are state space realizations in new coordinates of P_o and P_m respectively. Also, a change of basis $\eta = T_2 x$ where,

$$\begin{aligned} T_2 &:= \begin{bmatrix} \Delta_2^{-1} & -\Delta_2^{-1}\Theta_2 V_2 \\ H_{22}^{-1} H_{21} \Delta_2^{-1} & I_m - H_{22}^{-1} H_{21} \Delta_2^{-1} \Theta_2 V_2 \end{bmatrix} \\ T_2^{-1} &= \begin{bmatrix} I_m & \Theta_2 V_2 \\ -H_{22}^{-1} H_{21} & I_m \end{bmatrix} \end{aligned} \tag{26}$$

with $V_2 := \left(F_{21} - F_{22} H_{22}^{-1} H_{21}\right)^{-1}$, $\Theta_2 := F_{11} - F_{12} H_{22}^{-1} H_{21}$ and $\Delta_2 := I_m + \Theta_2 V_2 H_{22}^{-1} H_{21}$; arrives at the following structure of the system realization in new coordinates,

$$A_o = \begin{bmatrix} 0 & A_{12o} \\ A_{21o} & A_{22o} \end{bmatrix}, \; B_o = \begin{bmatrix} B_{1o} \\ B_{2o} \end{bmatrix}, \; C_o = \begin{bmatrix} C_{11o} & C_{12o} \\ 0 & C_m \end{bmatrix}. \tag{27}$$

Consequently, $\left(A_o, B_o, \begin{bmatrix} C_{11o} & C_{12o} \end{bmatrix}\right)$ and $\left(A_o, B_o, \begin{bmatrix} 0 & C_m \end{bmatrix}\right)$ are state-space realizations in new coordinates of P_o and P_m respectively.

Remark 2.4. As discussed in [7] fully actuated Euler–Lagrange formulation, is a class of non-linear dynamic systems that has a linearized realization of the form $(A_k,\, B_k)$ given in Eq. (25). In this case, T_1 is not needed. Also, in Lemma 2.3 it is assumed without loss of generality that G_{21} and H_{22} are non-singular matrices, that is, let $\left(\widehat{F},\, \widehat{G},\, \widehat{H}\right)$ be a realization of P, where $\widehat{G}$ and $\begin{bmatrix} \widehat{H}_{21} & \widehat{H}_{22} \end{bmatrix}$ have m linearly independent columns and rows, respectively because the inputs of v and outputs of y_m are linearly independent. In case $\widehat{G}_{21}$ and $\widehat{H}_{22}$ are not invertible, then there are unimodular matrices U and V such that,

$$\begin{aligned} G &:= U\widehat{G} = \begin{bmatrix} 0 \\ G_{21} \end{bmatrix} \\ H &:= \widehat{H} V = \begin{bmatrix} C_{11o} & C_{12o} \\ 0 & H_{22} \end{bmatrix} \end{aligned} \tag{28}$$

with G_{21} and H_{22} being invertible; where U and V^{-1} play the role of transformations that are applied before T_1 and T_2. As a consequence B_m and C_m are invertible. Also, A_{12k} and A_{21o} are invertible because their rows are linearly independent; otherwise, the rank of the controllability and observability matrices would be less than n, contradicting that the realization of P_m is controllable and observable.

The tackled problem is.

Problem 1. To track the input reference to the desired output of P, y_o, in the configuration scheme of Figure 1, with transient response according to pre-specified poles and diminish additive disturbances at the plant input, d_i, at LF, and in the measurement, d_m, and additive disturbances at the plant output, d_h, at HF.

Steps to solve this problem:

- Propose feedback gains $\mathcal{K}$ and L based on the change of basis given by Lemma 2.3 to solve a pole placement problem.
- Use the feedback gains to obtain coprime factorizations of the plant and controller by the formulas given by [16].
- Obtain the mixed sensitivity criterion functions given in Eq. (14b) based on the factorizations and propose the first free parameter R to solve it.
- Propose the second free parameter Q that relates the input reference and the desired output, to solve the input reference tracking.

3. MIXED SENSITIVITY AND PARAMETRIZATION

The following Lemma presents feedback gains based on the state-space representation of the plant with free constant parameters such that we can assign closed-loop poles. Also, these gains simplify the closed-loop matrices $F-G\mathcal{K}$ and $F-LH_2$ into equations (18) to (23). With these gains, we develop the formulas of [16] to obtain analytical expressions of the coprime factorizations of the plant and controller.

Lemma 3.1. Consider P_m under the assumption A4, the change of basis T_1 and T_2 given in Lemma 2.3 and $0 < a_1, a_2, b_1, b_2, c_1, c_2 \in \mathcal{R}$ then,

$$\mathcal{K} = \bar{K}T_1 \tag{29}$$

$$L = T_2^{-1}\bar{L} \tag{30}$$

where

$$\bar{K} := B_m^{-1}\begin{bmatrix} A_{21k} + \frac{c_1}{a_1}A_{12k}^{-1} & A_{22k} + \frac{b_1}{a_1}I_m \end{bmatrix} \tag{31}$$

$$\bar{L} := \begin{bmatrix} \left(A_{12o} + \frac{c_2}{a_2}A_{21o}^{-1}\right)C_m^{-1} \\ \left(A_{22o} + \frac{b_2}{a_2}I_m\right)C_m^{-1} \end{bmatrix} \tag{32}$$

are state feedback gains such that the matrices $F - G\mathcal{K}$ and $F - LH_2$ have stable characteristic polynomials ϕ_1^m and ϕ_2^m, respectively, where $\phi_1 := s^2 + \frac{b_1}{a_1}s + \frac{c_1}{a_1}$ and $\phi_2 := s^2 + \frac{b_2}{a_2}s + \frac{c_2}{a_2}$.

Proof. From equations (25), (29) and (31) the characteristic polynomial of the matrix $F - G\mathcal{K}$ simplifies to,

$$\begin{aligned} sI_n - F + G\mathcal{K} &= sI_n - T_1^{-1}A_kT_1 + T_1^{-1}B_k\mathcal{K} \\ &= T_1^{-1}\left(sI_n - A_k + B_k\bar{K}\right)T_1 \\ &= T_1^{-1}\begin{bmatrix} sI_m & -A_{12k} \\ \frac{c_1}{a_1}A_{12k}^{-1} & \left(s+\frac{b_1}{a_1}\right)I_m \end{bmatrix}T_1. \end{aligned} \tag{33}$$

Applying the matrix decomposition formula [20, see p. 22–23]

$$\begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} = \begin{bmatrix} I & 0 \\ M_{21}M_{11}^{-1} & I \end{bmatrix}\begin{bmatrix} M_{11} & 0 \\ 0 & \Delta \end{bmatrix}\begin{bmatrix} I & M_{11}^{-1}M_{12} \\ 0 & I \end{bmatrix} \tag{34}$$

where $\Delta := M_{22} - M_{21}M_{11}^{-1}M_{12}$ is the Schur complement of M_{11}; then, its determinant is equal to $det(M_{11})det(\Delta)$. So, we have that,

$$\begin{aligned} det\left(sI_n - F + G\mathcal{K}\right) &= det\left(T_1^{-1}\right) det\left(\begin{bmatrix} sI_m & -A_{12k} \\ \frac{c_1}{a_1}A_{12k}^{-1} & \left(s+\frac{b_1}{a_1}\right)I_m \end{bmatrix}\right) det\left(T_1\right) \\ &= det\left(\begin{bmatrix} sI_m & -A_{12k} \\ \frac{c_1}{a_1}A_{12k}^{-1} & \left(s+\frac{b_1}{a_1}\right)I_m \end{bmatrix}\right) \\ &= det\left(sI_m\right) det\left(\left(s+\frac{b_1}{a_1}+\frac{c_1}{a_1}\frac{1}{s}\right)I_m\right) \\ &= \phi_1^m. \end{aligned} \tag{35}$$

Using Routh–Hurwitz criterion, with $a_1, b_1, c_1 > 0$ we get a stable polynomial. In the same way for $sI_n - F + LH_2$ using equations (27), (30) and (32), the characteristic polynomial of the matrix $F - LH_2$ is,

$$\begin{aligned} sI_n - F + LH_2 &= sI_n - T_2^{-1}A_oT_2 + L\begin{bmatrix} 0 & C_m \end{bmatrix} T_2 \\ &= T_2^{-1}\left(sI_n - A_o + \bar{L}\begin{bmatrix} 0 & C_m \end{bmatrix}\right) T_2 \\ &= T_2^{-1}\begin{bmatrix} sI_m & \frac{c_2}{a_2}A_{21o}^{-1} \\ -A_{21o} & \left(s+\frac{b_2}{a_2}\right)I_m \end{bmatrix} T_2. \end{aligned} \tag{36}$$

Then,

$$\begin{aligned} det\left(sI_n - F + LH_2\right) &= det\left(T_2^{-1}\right) det\left(\begin{bmatrix} sI_m & \frac{c_2}{a_2}A_{21o}^{-1} \\ -A_{21o} & \left(s+\frac{b_2}{a_2}\right)I_m \end{bmatrix}\right) det\left(T_2\right) \\ &= \phi_2^m \end{aligned} \tag{37}$$

with $a_2, b_2, c_2 > 0$, by the Routh–Hurwitz criterion, we get a stable polynomial. □

The following Lemma gives an *r.c.f.* for P, an l.c.f. for P_m and a solution to the Diophantine equation given in Eq. (10) based on the feedback gains proposed in Lemma 3.1.

Lemma 3.2. Consider K and a given nominal plant P under assumptions A1, A2 and A4 on the feedback configuration shown in Figure 1, the change of basis T_1 and T_2 given by Lemma 2.3 and the feedback gains given in Lemma 3.1 with $a_1, a_2, b_1, b_2, c_1, c_2 > 0$. Let $\phi_1 := s^2 + \frac{b_1}{a_1}s + \frac{c_1}{a_1}$ and $\phi_2 := s^2 + \frac{b_2}{a_2}s + \frac{c_2}{a_2}$. Then a right-coprime factorization of P over $\mathcal{RH}_\infty$ is,

$$\begin{bmatrix} N_o \\ N_m \end{bmatrix} = \frac{1}{\phi_1}\begin{bmatrix} (C_{12k}s + C_{11k}A_{12k})B_m \\ (C_{22k}s + C_{21k}A_{12k})B_m \end{bmatrix} \tag{38}$$

$$D = \frac{1}{\phi_1}B_m^{-1}(s^2I_m - A_{22k}s - A_{21k}A_{12k})B_m \tag{39}$$

a left-coprime factorization of P_m is,

$$\widetilde{N}_m = \frac{1}{\phi_2}C_m(B_{2o}s + A_{21o}B_{1o}) \tag{40}$$

$$\widetilde{D}_m = \frac{1}{\phi_2}C_m(s^2I_m - A_{22o}s - A_{21o}A_{12o})C_m^{-1} \tag{41}$$

and a solution to the Diophantine equation $XN_m + YD = I_m$ is,

$$X = \frac{1}{\phi_2}\begin{bmatrix}\widetilde{K}_1 & \widetilde{K}_2\end{bmatrix}\begin{bmatrix}\left[\left(A_{12o} + \frac{c_2}{a_2}A_{21o}^{-1}\right)s + \frac{b_2}{a_2}A_{12o} - \frac{c_2}{a_2}A_{21o}^{-1}A_{22o}\right]C_m^{-1} \\ \left[\left(A_{22o} + \frac{b_2}{a_2}I_m\right)s + A_{21o}A_{12o} + \frac{c_2}{a_2}I_m\right]C_m^{-1}\end{bmatrix} \tag{42}$$

$$Y = \frac{1}{\phi_2}\begin{bmatrix}\widetilde{K}_1 & \widetilde{K}_2\end{bmatrix}\begin{bmatrix}B_{1o}s + \frac{b_2}{a_2}B_{1o} - \frac{c_2}{a_2}A_{21o}^{-1}B_{2o} \\ B_{2o}s + A_{21o}B_{1o}\end{bmatrix} + I_m \tag{43}$$

where

$$\begin{bmatrix}\widetilde{K}_1 & \widetilde{K}_2\end{bmatrix} = \bar{K}T_1T_2^{-1} \tag{44}$$

being $\bar{K} = B_m^{-1}\begin{bmatrix}A_{21k} + \frac{c_1}{a_1}A_{12k}^{-1} & A_{22k} + \frac{b_1}{a_1}I_m\end{bmatrix}$.

P r o o f. From equations (33) and (36) given into the proof of Lemma 3.1, we have,

$$\begin{aligned}(sI_n - F + G\mathcal{K})^{-1} &= T_1^{-1}\begin{bmatrix}sI_m & -A_{12k} \\ \frac{c_1}{a_1}A_{12k}^{-1} & \left(s + \frac{b_1}{a_1}\right)I_m\end{bmatrix}^{-1}T_1 \\ (sI_n - F + LH_2)^{-1} &= T_2^{-1}\begin{bmatrix}sI_m & \frac{c_2}{a_2}A_{21o}^{-1} \\ -A_{21o} & \left(s + \frac{b_2}{a_2}\right)I_m\end{bmatrix}^{-1}T_2\end{aligned} \tag{45}$$

and,

$$(sI_n - F + G\mathcal{K})^{-1} = T_1^{-1}\frac{1}{\phi_1}\begin{bmatrix}\left(s + \frac{b_1}{a_1}\right)I_m & A_{12k} \\ -\frac{c_1}{a_1}A_{12k}^{-1} & sI_m\end{bmatrix}T_1 \tag{46}$$

$$(sI_n - F + LH_2)^{-1} = T_2^{-1}\frac{1}{\phi_2}\begin{bmatrix}\left(s + \frac{b_2}{a_2}\right)I_m & -\frac{c_2}{a_2}A_{21o}^{-1} \\ A_{21o} & sI_m\end{bmatrix}T_2 \tag{47}$$

that clearly satisfies $MM^{-1} = I$ with $M = sI_n - F + G\mathcal{K}$ or $M = sI_n - F + LH_2$. By using the formulas of the work of [16] given in equations (18) to (23) with T_1 given in Lemma 2.3 for P; and substituting Eq. (46) into Eq. (18) we have,

$$\begin{bmatrix}N_o \\ N_m\end{bmatrix} = \begin{bmatrix}C_{11k} & C_{12k} \\ C_{21k} & C_{22k}\end{bmatrix}\frac{1}{\phi_1}\begin{bmatrix}A_{12k}B_m \\ B_ms\end{bmatrix} \tag{48}$$

arriving at the result of Eq. (38). From T_1 given in Lemma 2.3 for P, $\mathcal{K}$ is given in Lemma 3.1, and substituting Eq. (46) into Eq. (19),

$$D = I_m - \frac{1}{\phi_1}B_m^{-1}\begin{bmatrix}A_{21k} + \frac{c_1}{a_1}A_{12k}^{-1} & A_{22k} + \frac{b_1}{a_1}I_m\end{bmatrix}\begin{bmatrix}A_{12k}B_m \\ B_ms\end{bmatrix} \tag{49}$$

reaching the result of Eq. (39). Through T_2 given in Lemma 2.3 for P_m, and substituting Eq. (47) into Eq. (20),

$$\widetilde{N}_m = \frac{1}{\phi_2}\begin{bmatrix}C_mA_{21o} & C_ms\end{bmatrix}\begin{bmatrix}B_{1o} \\ B_{2o}\end{bmatrix} \tag{50}$$

arriving at Eq. (40). By using T_2 given in Lemma 2.3 for P_m, L is given in Lemma 3.1, and substituting Eq. (47) into Eq. (21),

$$\widetilde{D}_m = I_m - \tfrac{1}{\phi_2}\begin{bmatrix} C_m A_{21o} & C_m s \end{bmatrix} \bar{L} \tag{51}$$

reaching the result of Eq. (41). Applying T_2 given in Lemma 2.3 for P_m, $\mathcal{K}$ and L given in Lemma 3.1, and substituting Eq. (47) into Eq. (22),

$$X = \mathcal{K} T_2^{-1} \tfrac{1}{\phi_2} \begin{bmatrix} \left(s + \frac{b_2}{a_2}\right) I_m & -\frac{c_2}{a_2} A_{21o}^{-1} \\ A_{21o} & s I_m \end{bmatrix} \bar{L} \tag{52}$$

arriving at Eq. (42). By using T_2 given in Lemma 2.3 for P_m, $\mathcal{K}$ and L are given in Lemma 3.1, and substituting Eq. (47) into Eq. (23),

$$Y = \mathcal{K} T_2^{-1} \tfrac{1}{\phi_2} \begin{bmatrix} \left(s + \frac{b_2}{a_2}\right) I_m & -\frac{c_2}{a_2} A_{21o}^{-1} \\ A_{21o} & s I_m \end{bmatrix} \begin{bmatrix} B_{1o} \\ B_{2o} \end{bmatrix} + I_m \tag{53}$$

reaching the result of Eq. (43). □

Due to the *l.c.f.* and *r.c.f.* of P_m that are given in the above Lemma, the poles of all the closed-loop transfer functions are determined by the roots of the polynomials ϕ_1 and ϕ_2. So, the pole placement control problem (part of Problem 1) is solved by assigning the desired poles into ϕ_1 and ϕ_2. Once, given in Lemma 3.2 *l.c.f.* and *r.c.f.* of P_m and a solution for the Diophantine equation (10), the stabilizing controller is given by Theorem 2.1. The following Theorem, presents a solution for Problem 1, giving explicit formulas for the free parameters of the controller. Of course, any other election of R and Q into $\mathcal{RH}_\infty$ is possible. The proposed ones simplify the mixed sensitivity problem and do not increase the order of the controller.

Theorem 3.3. Under assumptions A1 to A4, consider the parametrization of all stabilizing controllers given in Theorem 2.1 for the feedback configuration shown in Figure 1, with the coprime factorizations and the solution of the Diophantine equation given in Lemma 3.2, the change of basis T_1 and T_2 given by Lemma 2.3, and the criterion given in Eq. (12). Let $\begin{bmatrix} \widetilde{K}_1 & \widetilde{K}_2 \end{bmatrix} := \bar{K} T_1 T_2^{-1}$ where $\bar{K}$ is given by Eq. (31), $0 < r \in \mathcal{R}$, ω_h be a fixed frequency in the HF bandwidth of P and,

$$Y_c := \tfrac{b_2}{a_2} \widetilde{K}_1 B_{1o} - \tfrac{c_2}{a_2} \widetilde{K}_1 A_{21o}^{-1} B_{2o} + \widetilde{K}_2 A_{21o} B_{1o} + \tfrac{c_2}{a_2} I_m. \tag{54}$$

Then,

1. If $\|Y_c\|_\infty > \|Y_c - C_m A_{21o} B_{1o}\|_\infty$ and $\left\|N_o \widetilde{D}_{Kl}\right\|_\infty \leq \left\|N_o \widetilde{N}_{Kh}\right\|_\infty$ for some value of r as shown in Figure 2, where,

$$\|N_o \widetilde{D}_{Kl}\|_\infty = \tfrac{a_1 a_2}{c_1 c_2} \|C_{11k} A_{12k} B_m \left(Y_c - r C_m A_{21o} B_{1o}\right)\|_\infty \tag{55}$$

and

$$\|N_o \widetilde{N}_{Kh}\|_\infty = \tfrac{1}{\omega_h} r \|C_{12k} B_m\|_\infty \tag{56}$$

choose the free parameter of the stabilizing controller,

$$R = rI_{m\times m} \in \mathcal{RH}_\infty \tag{57}$$

with

$$r = \frac{d_{ku}}{n_{ku}+d_{kl}-d_{ku}} \tag{58}$$

being

$$\begin{aligned}
n_{ku} &:= \frac{1}{\omega_h}\alpha && (59)\\
d_{ku} &:= \frac{a_1a_2}{c_1c_2}\beta && (60)\\
d_{kl} &:= \frac{a_1a_2}{c_1c_2}\gamma && (61)\\
\alpha &:= \|C_{12k}B_m\|_\infty && (62)\\
\beta &:= \|C_{11k}A_{12k}B_mY_c\|_\infty && (63)\\
\gamma &:= \|C_{11k}A_{12k}B_m\left(Y_c - C_mA_{21o}B_{1o}\right)\|_\infty. && (64)
\end{aligned}$$

The value of R ensures that,

$$\|N_o\widetilde{D}_{Kl}\|_\infty = \|N_o\widetilde{N}_{Kh}\|_\infty = \frac{a_1a_2\alpha\beta}{a_1a_2\omega_h(\beta-\gamma)+c_1c_2\alpha}. \tag{65}$$

2. If $C_mA_{21o}B_{1o}$ is invertible, then choose

$$R = r\ Y_c(C_mA_{21o}B_{1o})^{-1} \in \mathcal{RH}_\infty \tag{66}$$

where

$$r = \frac{d_{ku}}{d_{ku}+n_{ku}} \tag{67}$$

being

$$\begin{aligned}
n_{ku} &:= \frac{1}{\omega_h}\alpha && (68)\\
d_{ku} &:= \frac{a_1a_2}{c_1c_2}\beta && (69)\\
\alpha &:= \|C_{12k}B_mY_c(C_mA_{21o}B_{1o})^{-1}\|_\infty && (70)\\
\beta &:= \|C_{11k}A_{12k}B_mY_c\|_\infty. && (71)
\end{aligned}$$

The value of R ensures that,

$$\|N_o\widetilde{D}_{Kl}\|_\infty = \|N_o\widetilde{N}_{Kh}\|_\infty = \frac{a_1a_2\alpha\beta}{a_1a_2\beta\omega_h+c_1c_2\alpha} \tag{72}$$

where the norms of the LF and HF approximations of the functions $\|N_o\widetilde{D}_K\|$ and $\|N_o\widetilde{N}_K\|$ are,

$$\|N_o\widetilde{D}_{Kl}\|_\infty = \left(\frac{a_1a_2}{c_1c_2}\right)\beta|1-r| \tag{73}$$

and

$$\|N_o\widetilde{N}_{Kh}\|_\infty = \frac{1}{\omega_h}r\alpha \tag{74}$$

respectively.

Also, let r_c the number of rows of $C_{11k} \in \mathcal{R}^{p_o \times m}$ linearly independent. Then, the free parameter Q guaranteeing that the r_c controlled outputs of y_o track r_c input references of y_d at stationary state with time response determined by the poles of $N_o Q$ is,

$$Q = \tfrac{c_1}{a_1} \left(A_{12k} B_m\right)^{-1} C_q \in \mathcal{RH}_\infty \tag{75}$$

being C_q such that

$$C_{11k} C_q = \begin{bmatrix} I_{r_c} & \mathbf{0}_{r_c \times (p_o - r_c)} \\ \mathbf{0}_{(p_o - r_c) \times r_c} & \mathbf{0}_{(p_o - r_c) \times (p_o - r_c)} \end{bmatrix}. \tag{76}$$

P r o o f. First, we take the approximations of the functions $N_o \widetilde{D}_K$ and $N_o \widetilde{N}_K$ in LF and HF bandwidths, respectively, with the factorizations given in Lemma 3.2 and assuming that $R \in \mathcal{R}^{m \times m}$.

With $\widetilde{D}_K = Y - R\widetilde{N}_m$ from Theorem 2.1, and N_m and Y are given by equations (38) and (43) from Lemma 3.2 we have,

$$N_{ol} = \tfrac{a_1}{c_1} C_{11k} A_{12k} B_m \tag{77}$$

$$\widetilde{D}_{Kl} = \tfrac{a_2}{c_2} \left(Y_c - R C_m A_{21o} B_{1o}\right) \tag{78}$$

with $C_{11k} A_{12k} \neq 0$ then,

$$N_o \widetilde{D}_{Kl} = \tfrac{a_1 a_2}{c_1 c_2} C_{11k} A_{12k} B_m \left(Y_c - R C_m A_{21o} B_{1o}\right). \tag{79}$$

With $\widetilde{N}_K = X + R\widetilde{D}_m$ from Theorem 2.1, and $\widetilde{D}_m$ and X are given by equations (41) and (42) from Lemma 3.2 we have,

$$\begin{aligned} N_{oh} &= \tfrac{1}{w_h} C_{12k} B_m \\ \widetilde{N}_{Kh} &= \tfrac{1}{w_h} X_c + R \end{aligned} \tag{80}$$

where

$$X_c = \widetilde{K}_1 \left(A_{12o} + \tfrac{c_2}{a_2} A_{21o}^{-1}\right) + \widetilde{K}_2 \left(A_{22o} + \tfrac{b_2}{a_2} I_m\right). \tag{81}$$

In Eq. (80) it is assumed that $\|R\|_\infty >> \|\frac{1}{w_h} X_c\|_\infty$ due to a high value of ω_h. So, we have

$$N_o \widetilde{N}_{Kh} = \tfrac{1}{w_h} C_{12k} B_m R. \tag{82}$$

Next, we proof 1 and 2,

1 Equations (55) and (56) follows substituting R from Eq. (57) into Eq. (79) and Eq. (82), respectively. Using R given by Eq. (57), the norms of the functions $N_o \widetilde{D}_{Kl}$ and $N_o \widetilde{N}_{Kh}$ given by equations (55) and (56), are described as shown in Figure 2, if for some value of r, $\|N_o \widetilde{D}_{Kl}\|_\infty \leq \|N_o \widetilde{N}_{Kh}\|_\infty$, ensuring the intersection of the lines. The value for r given in Eq. (58) occurs when both infinity norms are equal. Hence, using Eq. (58) into equations (55) and (56) follows the result of Eq. (65).

2 Equations (73) and (74) follow substituting R from Eq. (66) into Eq. (79) and Eq. (82), respectively. Using R given by Eq. (66) and assuming that the value of r varies between 0 and 1; the norms of the functions $N_o\widetilde{D}_{Kl}$ and $N_o\widetilde{N}_{Kh}$ given by equations (73) and (74) are described as shown in Figure 3. The value for r given in Eq. (67) occurs when both infinity norms are equal. Hence, using Eq. (67) into equations (73) and (74) follows the result of Eq. (72).

Finally, we proof the proposed Q in Eq. (75)

From equations (3) and (5), $y_o = N_o\left(\widetilde{D}_K D + \widetilde{N}_K N_m\right)^{-1} Q y_d$, that reduces to $y_o = N_o Q y_d$ applying the factorizations given by Lemma 3.2 that satisfies equations (6) and (10). Then, with the proposed Q in Eq. (75),

$$y_o = \tfrac{c_1}{a_1\phi_1}\left(C_{12k}s + C_{11k}A_{12k}\right)B_m\left(A_{12k}B_m\right)^{-1}C_q y_d \tag{83}$$

where $\phi_1 = s^2 + \frac{b_1}{a_1}s + \frac{c_1}{a_1}$ and the time response is determined by de poles of N_o chosen with a_1, b_1 and c_1. By approximating at LF bandwidth Eq. (83), we get $y_o = C_{11k}A_{12k}B_m\left(A_{12k}B_m\right)^{-1}C_q y_d$. Since A_{12k} and B_m are invertible (see Remark 2.4) we get $y_o = C_{11k}C_q y_d$ and with C_q satisfying equation (76)

$$y_o = \begin{bmatrix} I_{r_c} & \mathbf{0}_{r_c\times(p_o-r_c)} \\ \mathbf{0}_{(p_o-r_c)\times r_c} & \mathbf{0}_{(p_o-r_c)\times(p_o-r_c)} \end{bmatrix} y_d. \tag{84}$$

$\square$

If there are transmission zeros of P_o that could cause unwanted effects in the transient response, we can add to the free parameter Q extra poles with the term $\frac{z_i}{s+z_i}$, this is,

$$Q = \tfrac{c_1}{a_1}\left(A_{12k}B_m\right)^{-1}C_q\tfrac{z_i}{s+z_i} \tag{85}$$

where z_i for $i = 1,\ldots,m$ is the real part of the transmission zeros of P_o. This new term does not affect the reference tracking since the approximation at LF holds.

As ω_h is increased, from Eq. (65) or Eq. (72), $\|N_o\widetilde{D}_{Kl}\|_\infty$ and $\|N_o\widetilde{N}_{Kh}\|_\infty$ are minimized simultaneously, and criterion given by Eq. (12) is achieved. Parameters a_1, a_2, c_1 and c_2 can also be used; however, the free location of the closed-loop poles would be limited. Also, with this result, both functions will have the same norm and therefore criterion given by Eq. (12), and not vice versa where first the corresponding criterion norm is obtained, possibly reaching a different norm for each function.

The mixed sensitivity functions are obtained with the factorizations given in Lemma 3.2. They do not need to be a normalized factorization for solving the criterion as in [12] or require filters to delimit the bands to be minimized.

The following procedure is proposed to synthesize the controller

1. Verify that P satisfies assumptions $A1$, $A2$, and $A4$; and obtain the state-space representation of the given plant as in equations (25) and (27), using the transformations of Lemma 2.3.

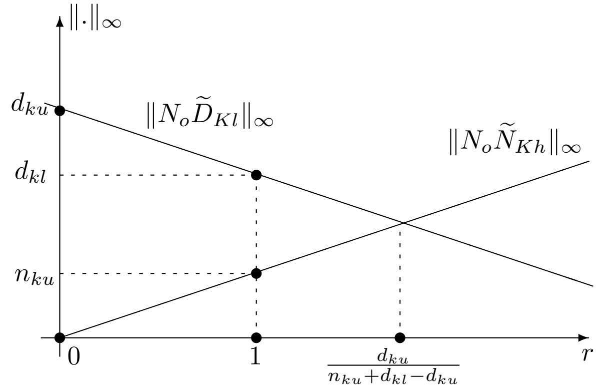

Fig. 2. Intersection function of case 1 of Theorem 3.3.

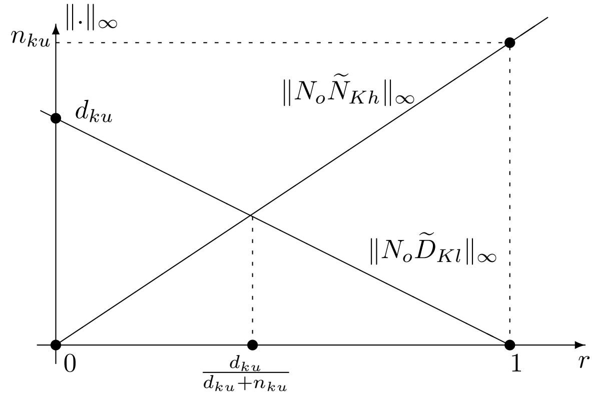

Fig. 3. Intersection function of case 2 of Theorem 3.3.

2. Select a performance level γ that meets the given closed-loop specifications and such that $\gamma_{opt} \leq J_2 \leq \gamma$, where J_2 is given by Eq. (12) and the minimal value of J_2, γ_{opt}, can be gotten by the work of [12].

3. Select the desired closed-loop poles using the parameters a_1, a_2, b_1, b_2, c_1 and c_2 and obtain the factorizations given in Lemma 3.2,

4. Select a free parameter R from Theorem 3.3 and use the parameter ω_h to minimize J_2 using equations (65) or (72). If the value of J_2 is not satisfactory, i. e. $J_2 > \gamma$ either return to step 3 selecting another value of a_1, a_2, b_1, b_2, c_1 and c_2, until $J_2 \leq \gamma$, or return to step 2 selecting another value of γ, until $J_2 \leq \gamma$. Otherwise, i. e., $J_2 \leq \gamma$, keep the value of $R \in \mathcal{RH}_\infty$, and,

5. Get the free parameter Q given in Eq. (75). Modify it according to Eq. (85) to cancel the unwanted dynamics of the zeros, if they exist.

4. EXAMPLE

In this section, we present three examples to show how to apply the presented results to solve Problem 1 by selecting the free constant parameters under assumptions A1-A4. In the first example, it is shown how to minimize the criterion norm using the parameter ω_h, once the poles are selected in closed-loop, and how the second free parameter can be readjusted when there are transmission zeros. The second example is a two-cart system, where two selections of poles are proposed to show how they affect the bandwidth for the closed-loop system, while in the third example, a controller is obtained for a half-car active suspension system. All examples are subject to external disturbances and have a square P_m, however, the two-cart system has a non-square P_o with more inputs than outputs.

Example 4.1. In this example P has unstable poles at 1, 2, 3 and 4; and P_o transmission zeros at -1.1232 and 13.8464 with a state space representation according to equation (16) in assumption A4, being,

$$F_{11} = \begin{bmatrix} -6 & -11 \\ 8 & 16 \end{bmatrix}, \quad F_{12} = \begin{bmatrix} 3 & -1 \\ -4 & 2 \end{bmatrix}, \quad F_{21} = \begin{bmatrix} 9 & 26 \\ 2 & 12 \end{bmatrix}, \quad F_{22} = \begin{bmatrix} -6 & 6 \\ -4 & 6 \end{bmatrix} \tag{86}$$

$$G_{11} = \begin{bmatrix} -1 & 2 \\ -3 & 4 \end{bmatrix}, \quad G_{21} = \begin{bmatrix} 5 & 6 \\ 7 & -8 \end{bmatrix} \tag{87}$$

$$H_{11} = \begin{bmatrix} 1 & -2 \\ -5 & -6 \end{bmatrix}, \quad H_{12} = \begin{bmatrix} 3 & -4 \\ 7 & 8 \end{bmatrix}, \quad H_{21} = \begin{bmatrix} -9 & 10 \\ 13 & -14 \end{bmatrix}, \quad H_{22} = \begin{bmatrix} 11 & 12 \\ 15 & 16 \end{bmatrix}. \tag{88}$$

First, we select a performance level $J_2 \leq 0.25$ and propose all closed-loop poles at -7. The characteristic closed-loop polynomial is $\phi_1^2 = (s^2+14s+49)^2$ with $a_1 = a_2 = 1$, $b_1 = b_2 = 14$ and $c_1 = c_2 = 49$ and then, the factorizations given in Lemma 3.2 are obtained.

Next, we obtain the expressions of R given in Theorem 3.3.
Since $\|Y_c\|_\infty < \|Y_c - C_m A_{21o} B_{1o}\|_\infty$ and $C_m A_{21o} B_{1o}$ is invertible for the proposed poles, we choose R given by equation (66).

Then we select the constant parameter ω_h to reduce the infinity norm of the criterion given in equation (72). From different values of ω_h according to Table 1, we choose $\omega_h = 180$, then $J_2 = 0.2070$. Using Parseval Lemma this means that d_i and d_m effects are reduced about 80% over the output y_o; measured by the 2-norm of d_i, d_m and y_o.

ω_h	$\left\|N_o\widetilde{D}_{Kl}\right\|_\infty = \left\|N_o\widetilde{N}_{Kh}\right\|_\infty$
100	0.3726
180	0.2070
300	0.1242

Tab. 1.

Finally, we use Q given by equation (75). Under T_1 the two rows of C_{11k} are linearly independent; then, we can track both references. The elements obtained for the two parameter controller are,

$$Q = \begin{bmatrix} 0.2996 & -0.1559 \\ 0.9103 & -0.3609 \end{bmatrix} \quad R = \begin{bmatrix} -0.4789 & -0.1566 \\ -0.4433 & -0.4526 \end{bmatrix} \tag{89}$$

$$\widetilde{D}_K = \frac{1}{\phi_1}\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} s^2 + \begin{bmatrix} 132.6 & -20.25 \\ 164.8 & -10.6 \end{bmatrix} s + \begin{bmatrix} 0.4609 & -87.35\times 10^{-3} \\ 0.5057 & -15.59\times 10^{-3} \end{bmatrix} \right) \tag{90}$$

$$\widetilde{N}_K = \frac{1}{\phi_1}\left(\begin{bmatrix} -0.4789 & -0.1566 \\ -0.4433 & -0.4526 \end{bmatrix} s^2 + \begin{bmatrix} 1.086 & -1.45 \\ 2.527 & -2.966 \end{bmatrix} s + \begin{bmatrix} 0.3044 & -1.761 \\ 2.341 & -4.27 \end{bmatrix} \right). \tag{91}$$

Figure 4 shows the maximum singular values of the function N_oQ that relates the input y_d to the output y_o and the mixed sensitivity functions $N_o\widetilde{D}_K$ and $N_o\widetilde{N}_K$ with their bandwidths in low and high frequency respectively; delimited by the norm value of 0.2070, which corresponds to $-13.68dB$. For this case the function N_oQ is over $0dB$ although closed-loop poles are not complex conjugate poles, this is due to the transmission zero at -1.1232. This will cause an overshoot in the response as a consequence.

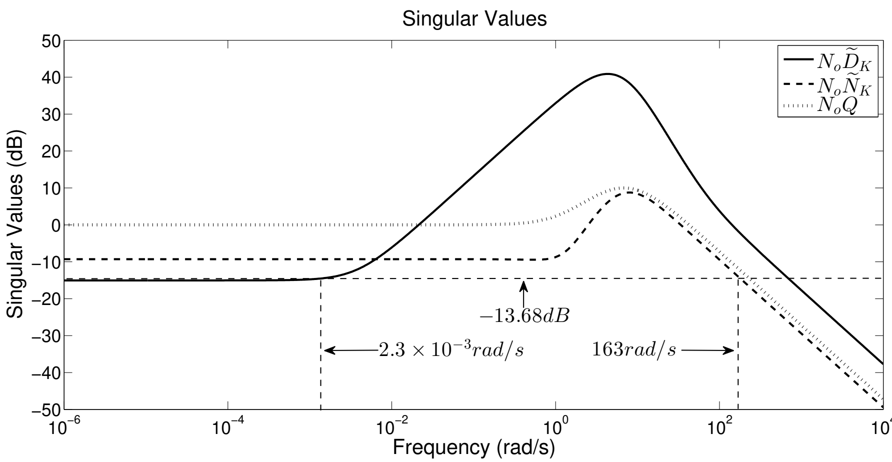

Fig. 4. Maximum singular values.

So, we use the modified Q given by equation (85), to cancel the unwanted effects of that transmission zero.

$$Q = \frac{1.1232}{s+1.1232} \begin{bmatrix} 0.2996 & -0.1559 \\ 0.9103 & -0.3609 \end{bmatrix}. \tag{92}$$

Note that, in this case, the new pole is closer to the origin, becoming a dominant pole.

Figure 5 shows the output response for the input $y_d = \begin{bmatrix} 3 & sin\ (0.1t) \end{bmatrix}^T$ under $d_i = sin\ (0.001t)$ for $t \geq 15\ s$ in low frequencies, and $d_m = sin\ (180t)$ for $t \geq 30\ s$ in high frequencies, with no overshoot for the step input and settling time according to the new dominant pole.

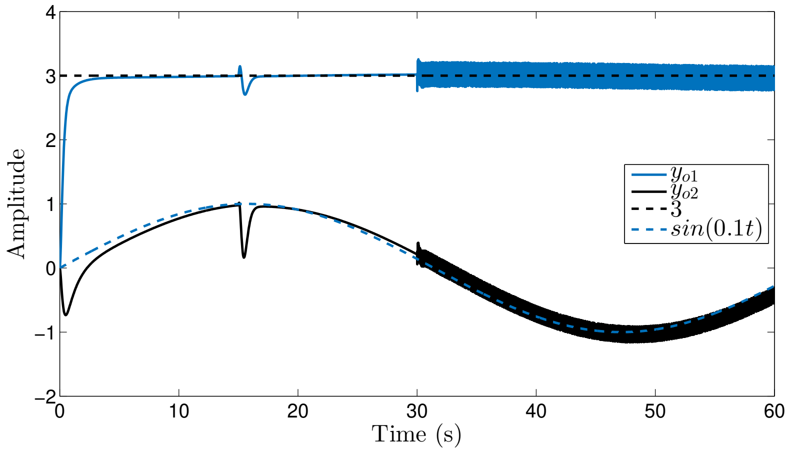

Fig. 5. Time response of the output y_o with the modified function N_oQ.

In this example we chose the high frequency bandwidth with ω_h. In the next example ω_h no longer determines the high frequency bandwidth since the criterion norm becomes 0 no matter which ω_h we choose.

Example 4.2. A state-space description of the two-cart system shown in Figure 6 is given by Eq. (15), with,

$$F = \begin{bmatrix} 0 & I_2 \\ F_{21} & F_{22} \end{bmatrix}, \quad G = \begin{bmatrix} 0 \\ G_{21} \end{bmatrix}, \quad H = \begin{bmatrix} H_{11} & 0 \\ H_{21} & H_{22} \end{bmatrix}. \tag{93}$$

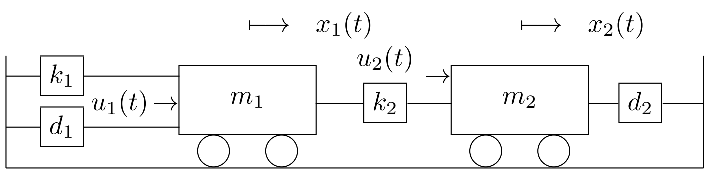

Fig. 6. Two-cart system.

where

$$F_{21} = \begin{bmatrix} \frac{-k_1}{m_1} - \frac{k_2}{m_1} & \frac{k_2}{m_1} \\ \frac{k_2}{m_2} & \frac{-k_2}{m_2} \end{bmatrix}, \qquad F_{22} = \begin{bmatrix} \frac{-d_1}{m_1} & 0 \\ 0 & -\frac{d_2}{m_2} \end{bmatrix} \qquad G_{21} = \begin{bmatrix} \frac{1}{m_1} & 0 \\ 0 & \frac{1}{m_2} \end{bmatrix} \tag{94}$$

$$H_{11} = \begin{bmatrix} 1 & 0 \end{bmatrix}, \qquad H_{21} = \begin{bmatrix} 3 & 6 \\ 7 & 8 \end{bmatrix}, \qquad H_{22} = \begin{bmatrix} 1 & 5 \\ 2 & 9 \end{bmatrix} \tag{95}$$

being m_1 and m_2 the masses of the carts, k_1 and k_2 the spring constants and d_1 and d_2 the damping coefficients with values shown in Table 2; $x(t) = [x_1(t)\ x_2(t)\ \dot{x}_1(t)\ \dot{x}_2(t)]^T$ the state vector of positions and velocities of each centre of masses. The transformations of Lemma 2.3 can be applied since G_{21} and H_{22} are non-singular matrices. Since P_m has only stable poles and does not have zeros, its realization is minimal, and P_m satisfies the parity interlacing property (see [19]). So, a stable compensator exists among the PASC.

Parameter	**Value**	**Unit**
m_1	1	Kg
m_2	1	Kg
k_1	0.01	N/m
k_2	0.01	N/m
d_1	1	N·s/m
d_2	1	N·s/m

Tab. 2. Parameters of the two-cart system.

In this example it is considered that only one position output tracks the input reference, and with $G_{11} = 0$, under transformation T_1, $C_{12k} = H_{12}$, then we can track the reference. Also $C_{12k} = 0$. This implies that N_o has no transmission zeros according to eq. (38) that could cause undesired effects on the output. Also, α from equations (62) and (70) becomes zero, then $\|N_o\widetilde{N}_{Kh}\|_\infty = 0$. Since there is no r in Eq. (55) such that $\|N_o\widetilde{D}_{Kl}\|_\infty \leq \|N_o\widetilde{N}_{Kh}\|_\infty = 0$, and $C_mA_{21o}B_{1o}$ is invertible, we choose R given by Eq. (66), then also $\|N_o\widetilde{D}_{Kl}\|_\infty = 0 = J_2$. In this case, ω_h no longer determines at which HF the norm corresponds.

Two cases are proposed, the first case, closed-loop poles with real part at -2.1 and the second case with real part at -4.9; both cases with damping ratio of 0.7. The characteristic closed-loop polynomials are; $\phi_1^2 = (s^2 + 4.2s + 9)^2$ with $a_1 = a_2 = 1$, $b_1 = b_2 = 4.2$ and $c_1 = c_2 = 9$, and $\phi_2^2 = (s^2 + 9.8s + 49)^2$ choosing $a_1 = a_2 = 1$, $b_1 = b_2 = 9.8$ and $c_1 = c_2 = 49$. The elements obtained for the 2DOF controller for the first case are,

$$Q = \begin{bmatrix} 9 \\ 0 \end{bmatrix}, \ R = \begin{bmatrix} -9.2203 & 7.4805 \\ 9.9214 & -4.2063 \end{bmatrix} \tag{96}$$

$$\widetilde{D}_K = \frac{1}{\phi_1}\left(I_2 s^2 + \begin{bmatrix} 1.659 & -21.22 \\ -1.509 & -4.351 \end{bmatrix} s\right) \tag{97}$$

$$\widetilde{N}_K = \frac{1}{\phi_1}\left(\begin{bmatrix} -9.22 & 7.48 \\ 9.921 & -4.206 \end{bmatrix} s^2 + \begin{bmatrix} -49.9 & 29.58 \\ 17.42 & -8.109 \end{bmatrix} s + \begin{bmatrix} -36 & 27 \\ 31.5 & -13.5 \end{bmatrix}\right) \tag{98}$$

and for the second case are,

$$Q = \begin{bmatrix} 49 \\ 0 \end{bmatrix}, \; R = \begin{bmatrix} 383.1975 & -116.2293 \\ 238.8403 & -88.3416 \end{bmatrix} \tag{99}$$

$$\widetilde{D}_K = \frac{1}{\phi_2}\left(I_2 s^2 + \begin{bmatrix} -132.1 & -869.9 \\ -62.16 & -380.5 \end{bmatrix} s \right) \tag{100}$$

$$\widetilde{N}_K = \frac{1}{\phi_2}\left(\begin{bmatrix} 383.2 & -116.2 \\ 238.8 & -88.34 \end{bmatrix} s^2 + \begin{bmatrix} -910.4 & 450.3 \\ 20.16 & -28.06 \end{bmatrix} s + \begin{bmatrix} -1067 & 800.3 \\ 933.7 & -400.2 \end{bmatrix} \right). \tag{101}$$

Figure 7 shows the maximum singular values of the mixed sensitivity functions $N_o\widetilde{D}_K$ and $N_o\widetilde{N}_K$ for both cases. Frequencies 9 rad/s and 50 rad/s corresponds to a magnitude of 0.2 for the function $N_o\widetilde{N}_K$ for the respective case, diminishing d_m about 80%.

The controller is implemented in the feedback configuration of Figure 1, where $d_h = 0$. Figure 8 shows the output for an input $y_d = 2$, under $d_i = sin\ (0.01t)$ for $t \geq 5\ s$ for both cases, and $d_m = sin\ (9t)$ for $t \geq 5\ s$ and $d_m = sin\ (50t)$ for $t \geq 5\ s$, for the respective case.

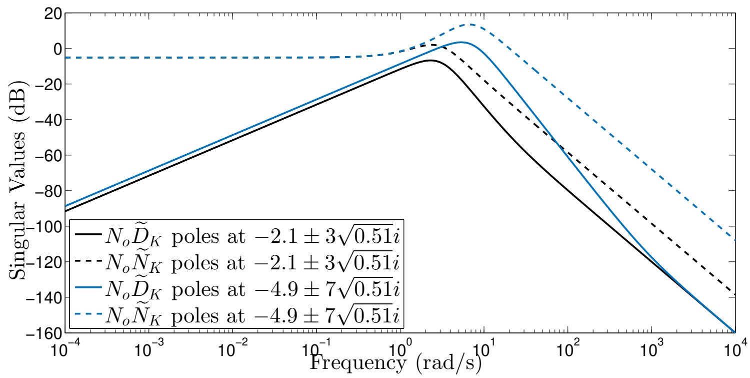

Fig. 7. Maximum singular values of $N_o\widetilde{D}_K$ and $N_o\widetilde{N}_K$.

Example 4.3. Consider the half-car active suspension system shown in Figure 9 where the unsprung masses were neglected, m and J are the sprung mass and the moment of inertia of the half-car vehicle body respectively, d_1 and d_2 are the distances of the front and rear suspension locations from the centre of mass of the vehicle body, k_1 and k_2 are the elasticity coefficients of the front and rear suspensions, and b_{s1} and b_{s2} are the damping coefficients of the front and rear suspensions, V_m and w_J are the vertical and angular velocities of the vehicle body at the centre of mass, F_{act2} and F_{act1} are the front and rear active forces produced by the actuators, $F_{masstransfer}$ is the mass transfer force due to braking or accelerating effects applied on the centre of mass, and V_{road2} and V_{road1} are the vertical ground front and rear velocities, respectively, as seen from the vehicle moving at speed V.

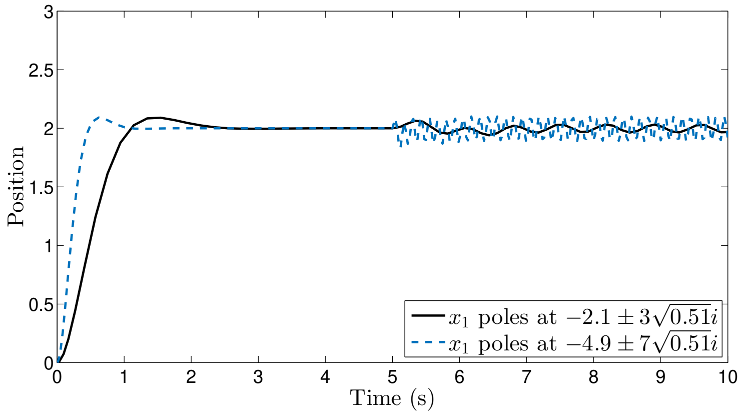

Fig. 8. Time response of the masses positions.

Let $x(t) = [x_{rel1}\ x_{rel2}\ Jw_J\ mV_m]^T$, where x_{rel1} and x_{rel2} are the relative positions of the front and rear suspensions respectively; $d = [V_{road1}\ F_{masstransfer}\ V_{road2}]^T$, where $F_{masstransfer}$ and $v = [F_{act1}\ F_{act2}]^T$, and V_{road1} and V_{road2} are measurable and non-mensurable external disturbance inputs, respectively. Then, an state-space description of the system is,

$$\begin{aligned} \dot{x} &= \begin{bmatrix} 0 & F_{12} \\ F_{21} & F_{22} \end{bmatrix} x + \begin{bmatrix} 0 \\ G_{21} \end{bmatrix} v + Ed_h \\ y &= \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} x \end{aligned} \tag{102}$$

with,

$$F_{12} = \begin{bmatrix} \frac{-1}{m} & \frac{1}{J}d_1 \\ \frac{-1}{m} & \frac{-1}{J}d_2 \end{bmatrix}, \quad F_{21} = \begin{bmatrix} k_1 & k_2 \\ -d_1k_1 & -d_2k_2 \end{bmatrix}, \quad F_{22} = \begin{bmatrix} \frac{-(b_{s2}+b_{s1})}{m} & \frac{-(d_2b_{s2}-d_1b_{s1})}{J} \\ \frac{-(d_2b_{s2}-d_1b_{s1})}{m} & \frac{-(d_2^2b_{s2}+d_1^2b_{s1})}{J} \end{bmatrix} \tag{103}$$

$$G_{21} = \begin{bmatrix} -1 & -1 \\ d_1 & -d_2 \end{bmatrix}, \quad E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ b_{s1} & 1 & b_{s2} \\ -d_1b_{s1} & 0 & d_2b_{s2} \end{bmatrix}, \quad H_{11} = I_2 \tag{104}$$

$$H_{12} = 0 \qquad H_{21} = \begin{bmatrix} 3 & 6 \\ 7 & 8 \end{bmatrix}, \qquad H_{22} = \begin{bmatrix} 1 & 5 \\ 2 & 9 \end{bmatrix} \tag{105}$$

and values that are shown in Table 3 given in [11]. As in example 4.2, G_{21} and H_{22} are non-singular matrices, and P_m satisfies the parity interlacing property since all the poles of P_m are stable and P_m does not have transmission zeros.

The system is represented by equations (3) and (4) in the scheme of Figure 1, where $W_3 = (F,\ E,\ [H_{11}\ H_{12}])$ and $W_4 = (F,\ E,\ [H_{21}\ H_{22}])$. Since W_3 and W_4 contain the

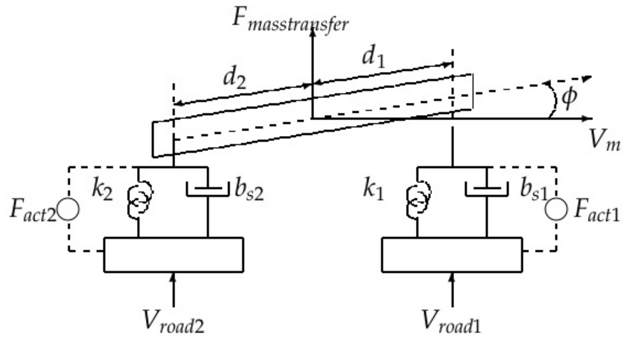

Fig. 9. Half-car active suspension system.

Parameter	**Value**	**Unit**
m	1794.4	Kg
J	3443.05	Kg
k_1	18615	N/m
k_2	66824.4	N/m
d_1	1.716	m
d_2	1.271	m
b_{s1}	1000	N·s/m
b_{s2}	1190	N·s/m

Tab. 3. Parameters of the half-car active suspension system.

dynamics of the plant in open-loop, and this is stable, it does not affect the stability of the closed-loop system. Also, W_3 and W_4 behave with low gain at HF and by minimizing $N_o\widetilde{N}_K$ at HF according to the criterion (12), the effects of d_h on y_o in HF will be minimized since $y_o = \left(W_3 - N_o\widetilde{N}_K P_{ymdh}\right) d_h$ according to Eq. (3).

In this example it is considered to regulate x_{rel1} y x_{rel2}; then, $H_{11} = I_2$ and $H_{12} = 0$ and under T_1 $C_{11k} = H_{11}$. As in the previous example, since $C_m A_{21o} B_{1o}$ is invertible, we choose R given by Eq. (66), then $\|N_o\widetilde{D}_{Kl}\|_\infty = \|N_o\widetilde{N}_{Kh}\|_\infty = J_2 = 0$, and ω_h no longer determines at which HF the norm corresponds. Poles are proposed in closed-loop with real part in -4.9 and damping ratio of 0.7, obtaining the polynomial $\phi_1^2 = (s^2 + 9.8s + 49)^2$, with $a_1 = a_2 = 1$, $b_1 = b_2 = 9.8$ ans $c_1 = c_2 = 49$. The elements obtained for the 2DOF controller are,

$$Q = 1 \times 10^4 \begin{bmatrix} 3.4829 & 0.2595 \\ 0.2585 & 4.7928 \end{bmatrix}, \quad R = 1 \times 10^6 \begin{bmatrix} -2.3862 & 1.3506 \\ -2.0357 & 1.1526 \end{bmatrix} \tag{106}$$

$$\widetilde{D}_K = \frac{1}{\phi_1} \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} s^2 + \begin{bmatrix} -6.994 \times 10^4 & 6.001 \times 10^5 \\ -6.434 \times 10^4 & 5.166 \times 10^5 \end{bmatrix} s \right) \tag{107}$$

$$\widetilde{N}_K = \frac{1}{\phi_1} \left(1 \times 10^6 \begin{bmatrix} -2.386 & 1.351 \\ -2.036 & 1.153 \end{bmatrix} s^2 + 1 \times 10^6 \begin{bmatrix} -2.632 & 1.495 \\ -2.241 & 1.274 \end{bmatrix} s \right.$$
$$\left. +1 \times 10^5 \begin{bmatrix} -7.092 & 5.478 \\ 8.57 & -3.492 \end{bmatrix} \right). \tag{108}$$

Figure 10 shows the maximum singular values of the functions $N_o\widetilde{D}_K$, $N_o\widetilde{N}_K$, N_oQ and $W_3 - N_o\widetilde{N}_K W_4$ which relate the inputs d_i, d_m, y_d and d_h to the output y_o, respectively. Figure 11 shows the output for an input reference $y_d = \begin{bmatrix} 1 & 1.25 \end{bmatrix}^T$, under $d_i = 0.5sin(0.1t)$, $t \geq 2$, $d_m = 0.5sin(500t)$, $t \geq 4$, $V_{road1} = 0.1sin(300t)$, $t \geq 6$, $F_{masstransfer} = 1$, $t \geq 8$ and $V_{road2} = 0.1sin(300t)$, $t \geq 10$.

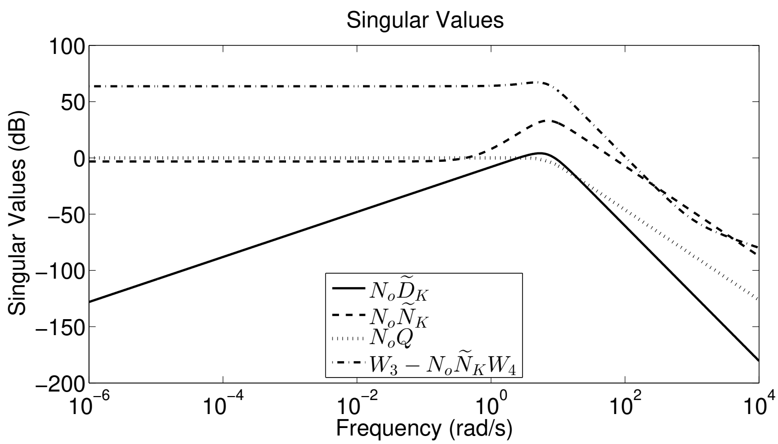

Fig. 10. Maximum singular values of $N_o\widetilde{D}_K$, $N_o\widetilde{N}_K$, N_oQ and $W_3 - N_o\widetilde{N}_K W_4$.

Although the assumptions for using $R = rI$ given in Eq. (57) that guarantees $\|N_o\widetilde{D}_{Kl}\|_\infty = \|N_o\widetilde{N}_{Kh}\|_\infty$ are not satisfied for the selected poles, with $r = 1$ we have $\|N_o\widetilde{D}_{Kl}\|_\infty = 3.6111 \times 10^{-4}$ and together with $\|N_o\widetilde{N}_{Kh}\|_\infty = 0$ that we already had, we can reduce the effects of d_i, d_h and d_m. Figure 12 shows the output for the same reference and disturbances previously given.

In the three examples, the system response is stable under the presence of disturbances of LF and HF, and smooth according to the selected pole assignment. Also, the objective of minimizing criterion given by Eq. (12) is achieved, that is, to attenuate $\|d_m\|_2$ and $\|d_h\|_2$ at HF, and $\|d_i\|_2$ at LF over $\|y_o\|_2$, achieving stability and robust performance.

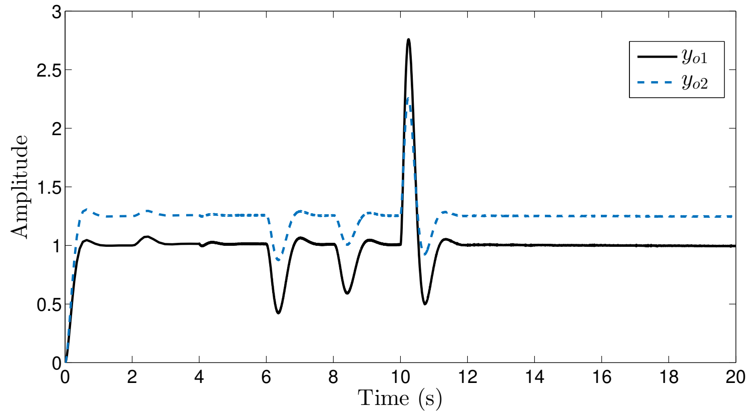

Fig. 11. Time response of $y_o(t)$.

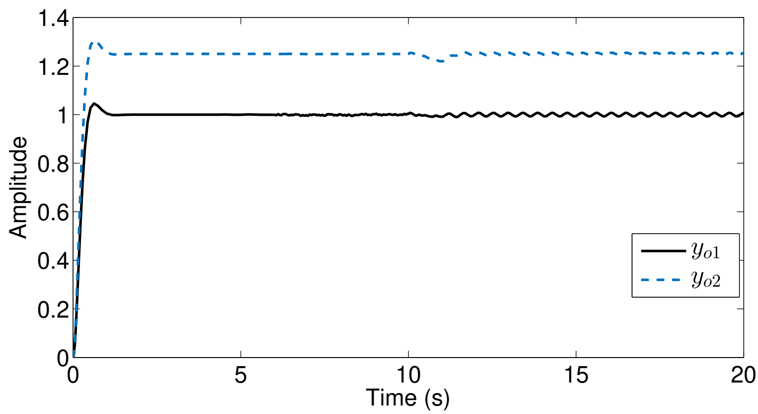

Fig. 12. Time response of $y_o(t)$ for $R = I$.

5. CONCLUSION

Analytical expressions of coprime factorizations of the given plant, a solution of the Diophantine equation and the two free parameters of the two-degrees of freedom stabilizing control are presented solving a pole placement problem, a mixed sensitivity criterion and tracking the input reference at LF. The 2DOF stabilizing control is based on the parameterization of all stabilizing controllers where the first free parameter solves a mixed sensitivity robust control problem, while the second free parameter assures that the controlled output tracks the input reference at LF.

With the proposed expressions, a pole placement problem and a mixed sensitivity problem, depends on a set of control parameters. The parameter that determines the infinite norm of the mixed sensitivity criterion delimits the high-frequency bandwidth. The transient response can be affected for the established poles if undesired transmission zeros are present and there is a correct input reference; nevertheless, the second free parameter can be modified to reduce this effect.

A stable and smooth response is obtained according to the selected pole assignment, despite the disturbances that are attenuated at the output of interest, to the extent established by the mixed sensitivity criterion. With the given analytical expressions, the controller has known dimension beforehand, are suitable for on-line implementation or adaptive control in future works.

ACKNOWLEDGEMENT

This work was supported by the National Council of Science and Technology (CONACYT).

(Received June 21, 2018)

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