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CARISTI'S FIXED POINT THEOREM IN PROBABILISTIC METRIC SPACES

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In this work, we define a partial order on probabilistic metric spaces and establish some new Caristi's fixed point theorems and Ekeland's variational principle for the class of (right) continuous and Archimedean t-norms. As an application, a partial answer to Kirk's problem in metric spaces is given.

Keywords: probabilistic metric space, Caristi's fixed point, Archimedean t-norm

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1. PRELIMINARIES

In [4], Caristi established a celebrated fixed point theorem which is very useful in the theory of nonlinear analysis. Since then, several generalizations of the theorem were obtained by many authors in generalized spaces, for instance, see [1, 2, 6, 12–15, 28]. Recall that this theorem states that any map $T : X \to X$ defined on a complete metric space (X, d) has a fixed point provided that there exists a lower semi-continuous map $\varphi : X \to [0, +\infty)$ such that

$$d(x, T(x)) \leq \varphi(x) - \varphi(T(x)), \qquad \text{for every} \qquad x \in X. \tag{1}$$

The notion of a probabilistic metric space as a field of probabilistic analysis was introduced in 1942 by K. Menger [23], which is a generalization of metric spaces. Since then, this theory has been developed in many directions, besides in its axiomatic growth and defining topological structure therein (e. g., fuzzy metric spaces in the sense of Michalek and Kramosil [22] and George and Veeramani [7]), fixed point theory was studied by several authors in these spaces (see [1, 6, 8, 11, 27]). The aim of the present paper is to prove a new Caristi's fixed point theorem in probabilistic metric spaces based on the contraction introduced in [1] and as a consequence deduce some related Caristi's type fixed point theorems in metric and probabilistic metric spaces. Also a partial answer to Krik's problem in ordinary metric space (X, d) is given. In the following, we recall some well-known definitions and results in the theory of probabilistic metric spaces.

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Recall (see [25]) that a t-norm is a binary operation $* : [0,1]^2 \to [0,1]$ which satisfies the following conditions:

(a) $x * y = y * x$,

(b) $x * (y * z) = (x * y) * z$,

(c) $a * 1 = a$ for all $a \in [0,1]$,

(d) $a * b \le c * d$ where $a \le c$ and $b \le d$ and $a, b, c, d \in [0,1]$.

Also, $*$ is said to be continuous (upper or lower semicontinuous, briefly, u.s.c. or l.s.c.) if it is a continuous (u.s.c. or l.s.c.) function in $[0,1]^2$. Since a t-norm $*$ is non-decreasing and commutative, the following proposition allows us to speak about left-continuous and right continuous t-norms instead of lower and upper semicontinuous t-norms, respectively.

Proposition 1.1. (Klement and Mesiar [18]) A t-norm $*$ is l.s.c. (u.s.c.) if and only if it is left-continuous (right continuous) in its first component, that is, if and only if for each $y \in [0,1]$ and for all non-decreasing (non-increasing) sequences $(x_n)_{n \ge 1}$ we have

$$\lim_{n \to +\infty} (x_n * y) = (\lim_{n \to +\infty} x_n * y).$$

For the class of all t-norms (which includes non-continuous t-norms) the only existing characterization is by the axioms (a)-(d) and among the important algebraic subclasses of t-norms, the following conceptions are more important. The t-norm $*$ is called Archimedean if for each $x, y \in (0,1)$, there is $n \in \mathbb{N}$ such that

$$x^n = \overbrace{x * x * \cdots * x}^{n \text{ times}} < y,$$

and the t-norm $*$ is said to be strictly monotone if $x * y < x * z$ whenever $x > 0$ and $y < z$, also a t-norm $*$ is called strict if it is continuous and strictly monotone [11, 18].

Remark 1.2. (Kolesárová [21], Klement and Mesiar [18], Klement and Mesiar [16]) Every left-continuous Archimedean t-norm is necessarily continuous.

The following representation theorem holds.

Theorem 1.3. (see Schweizer and Sklar [26], Klement and Mesiar [18], Abbasi and Mottaghi Golshan [1]) For a t-norm $* : [0,1]^2 \to [0,1]$ the following are equivalent:

(i) $*$ is a Archimedean t-norm,

(ii) $a * b \ge a \Rightarrow b = 1, \forall a \in (0,1]$.

Indeed, if $*$ be a continuous t-norm then above condition is equivalent to the following (see [16, 18, 20]).

(iii) There exists a continuous, strictly decreasing function $\vartheta : [0,1] \to [0,+\infty]$ such that $\vartheta(1) = 0$ (so-called additive generator), and such that for all $x, y \in [0,1]$,

$$\vartheta(x) + \vartheta(y) \in \operatorname{Rang}(\vartheta) \cup [\vartheta(0), +\infty],$$

$$x * y = \vartheta^{(-1)}(\vartheta(x) + \vartheta(y)),$$

where, $\vartheta^{(-1)}$ is the pseudo-inverse of ϑ, i. e.,

$$\vartheta^{(-1)}(x) = \sup\{y \in [a,b], \vartheta(y) > x\}.$$

Moreover, the function ϑ^{-1} is non-increasing and right-continuous.

(iv) There exists a continuous, strictly increasing function $\theta : [0,1] \to [0,1]$ such that $\theta(1) = 1$ (so-called multiplicative generator), and such that for all $x, y \in [0,1]$

$$\theta(x)\theta(y) \in \operatorname{Rang}(\theta) \cup [0, \theta(0)],$$

$$x * y = \theta^{(-1)}(\theta(x)\theta(y)),$$

where, $\theta^{(-1)}$ is the pseudo-inverse of θ, i. e.,

$$\theta^{(-1)}(x) = \sup\{y \in [0,1], \theta(y) < x\}.$$

Moreover, $*$ is a strict t-norm if and only if every continuous additive (multiplicative) generator ϑ (θ) of $*$ satisfies $\vartheta(1) = 0$ $(\theta(0) = 0)$ (see [16, 18]).

In general, additive generator and multiplicative generator are only assumed right-continuous at 0 (see [19, Definition 2.4] and [18]). Among the important examples of t-norm we mention the following. For every $x, y \in [0,1]$,

(i) The minimum t-norm is defined by $x *_\wedge y = \wedge\{x, y\}$,

(ii) The product t-norm is defined by $x *_p y = x \cdot y$,

(iii) The Schweizer-Sklar t-norms is defined by $x\mathfrak{L}_\lambda y = \vee(\{x^\lambda + y^\lambda - 1, 0\})^{1/\lambda}, \lambda > 0$.

(iv) The family $(T_\lambda^Y)_{\lambda\in(0,+\infty)}$ of Yager t-norms is given by

$$T_\lambda^Y(x,y) = \vee\left\{1 - ((1-x)^\lambda + (1-y)^\lambda)^{\frac{1}{\lambda}}, 0\right\}.$$

(v) The family $(T_\lambda^{SW})_{\lambda\in(-1,+\infty)}$ of Sugeno-Weber t-norm is given by

$$T_\lambda^{SW}(x,y) = \vee\left\{\frac{x + y - 1 + \lambda xy}{1+\lambda}, 0\right\}.$$

The Yager and Sugeno-Weber t-norms are increasing and $T_1^Y = T_0^{SW} = \mathfrak{L}_1$. Minimum t-norm has no any additive generator and $-\ln(x)$, $(1-x^\lambda)/\lambda$ and $(1-x)^\lambda$ are additive generators of (ii), (iii) and (iv), respectively.

In the standard notation, Δ^+ stands for the set of all non-decreasing functions $F : \mathbb{R} \to \mathbb{R}^+$ satisfying $\sup_{t\in\mathbb{R}} F(t) = 1$ and $F(0) = 0$. Since any function from Δ^+ is equal zero on $[-\infty, 0]$, we can consider the set Δ^+ consisting of non-decreasing functions F defined on $[0, +\infty]$ that satisfy $F(0) = 0$ and $F(+\infty) = 1$. The class Δ^+ plays important role in the probabilistic fixed point theorems.

Denote by ϵ_0 the specific distribution function defined by

$$\epsilon_0 = \begin{cases} 1 & t > 0, \\ 0, & t = 0. \end{cases}$$

Definition 1.4. (see Menger [23], Schweizer and Sklar [26], Hadzic and Pap [11])
A Menger probabilistic metric space (briefly in the sequel we use PM-space) is a triple $(X, F, *)$, where X is a non-empty set and F is a mapping from $X \times X$ into Δ^+. We denote the function $F(x, y)(\cdot)$ by $F_{x,y}(\cdot)$, for all $x, y \in X$. The function F is assumed to satisfy the following conditions:

(PM1) $F_{x,y}(t) = 0$ if $t = 0$,

(PM2) $F_{x,y}(t) = \epsilon_0$ if and only if $x = y$,

(PM3) $F_{x,y}(t) = F_{y,x}(t)$,

(PM4) $F_{x,y}(t + s) \geq F_{x,z}(t) * F_{y,z}(s)$,

for all $x, y, z \in X$ and $t, s > 0$.

Axiom (PM2) is equivalent to the following condition:

(PM2) $F_{x,y}(t) = 1$ for all $t > 0$ if and only if $x = y$.

Any fuzzy metric space $(X, M, *)$ is equivalent to a Menger space [22, Corollary of Theorem 1] in the sense that $M(x, y, t) = F_{x,y}(t)$, for all $x, y \in X, t > 0$. Note that in the most literature F is considered left-continuous function, but, in general by virtue of Corollary 1.8 it may be assumed that F is right-continuous (also, see the last paragraph below Definition 7 of [22]). Indeed, in this paper, in order to obtain the main results, like the same argument in [7, Axiom (GV1)], we will assume that $F_{x,y}(t) > 0, \forall t > 0, x, y \in X$.

Recall that if $(X, F, \star)$ be a PM-space and $*$ be a t-norm such that $a \star b \geq a * b$, for each $a, b \in [0, 1]$, then $(X, F, *)$ is a PM-space but the converse, in general, is not true.

The (ε, λ)-topology [25] in X is introduced by the family of neighbourhoods $\{U_{\varepsilon,\lambda}(x), \varepsilon > 0, \lambda \in (0, 1)\}_{x\in X}$ where $U_{\varepsilon,\lambda}(x)$ is defined by

$$U_{\varepsilon,\lambda}(x) = \{(x, y) \in X, F_{xy}(\varepsilon) > 1 - \lambda\},$$

according to this definition, a sequence $(x_n)_{n\geq1}$ in a PM-space is said to converge to a point x if for every $r \in (0, 1), t > 0$, there exists $n_0 \in \mathbb{N}$ (n_0 depends on r and t) such that $x_n \in U_{t,r}(x)$ whenever $n \geq n_0$. Notice that $x_n \to x$ if and only if, $F_{x,x_n} \to \epsilon_0$, i.e. for every $t > 0$, $\lim_n F_{x,x_n}(t) = 1$. If a t-norm $*$ is such that $\sup_{a<1} a * a = 1$, then X in the (ε, λ)-topology is a metrizable topological space.

Definition 1.5. (see George and Veeramani [7], Schweizer and Sklar [26], Gregori et al. [9])

(a) $(x_n)_{n\geq 1}$ is called a F-Cauchy sequence in $(X, F, *)$ if for any given $t > 0$ and $\epsilon \in (0, 1]$, there exists $n_0 = n_0(\epsilon, t) \in \mathbb{N}$ such that $F_{x_n,x_m}(t) > 1 - \epsilon$ whenever $n, m \geq n_0$.

(b) A PM-space in which every F-Cauchy sequence is convergent is called F-complete.

In general note that the convergence and so completeness of X is independent of the choice of t-norm $*$.

Remark 1.6. (Grabiec [8]) In a PM-space $(X, F, *)$, $F_{x,y}(\cdot)$ is non-decreasing for all $x, y \in X$.

It follows that at each point $t > 0$, the right limit $F^+_{x,y} = \lim_{s\to t^+} F_{x,y}(s)$ and the left limit $F^-_{x,y} = \lim_{s\to t^-} F_{x,y}(s)$ exist in $[0, 1]$ and the following two inequalities hold:

$$F^-_{x,y}(t) \leq F_{x,y}(t) \leq F^+_{x,y}(t).$$

Lemma 1.7. (see Lemma 6 and Corollary 7 of Grabiec [8]) If $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$. Then

$$F_{x,y}(t - \epsilon) \leq \liminf_{n\to\infty} F_{x_n,y_n}(t) \leq \limsup_{n\to\infty} F_{x_n,y_n}(t) \leq F_{x,y}(t + \epsilon),$$

for all $t > 0$ and $0 < \epsilon < t/2$.

Corollary 1.8. Let $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$. Then

(i) If $F_{x,y}(\cdot)$ is right continuous, for all $x, y \in X$, then $\limsup_{n\to\infty} F_{x_n,y_n}(t) \leq F_{x,y}(t)$, for all $t > 0$, that is, $F_{(\cdot,\cdot)}(t)$ is an upper semi-continuous function on $X \times X$, for all $t > 0$.

(ii) If $F_{x,y}(\cdot)$ is left continuous, for all $x, y \in X$, then $\liminf_{n\to\infty} F_{x_n,y_n}(t) \geq F_{x,y}(t)$, for all $t > 0$, that is, $F_{(\cdot,\cdot)}(t)$ is a lower semi-continuous function on $X \times X$, for all $t > 0$.

(iii) If $F_{x,y}(\cdot)$ is continuous, for all $x, y \in X$, then $\lim_{n\to\infty} F_{x_n,y_n}(t) = F_{x,y}(t)$, for all $t > 0$, that is, $F_{(\cdot,\cdot)}(t)$ is a continuous function on $X \times X$, for all $t > 0$ (see also [26]).

Lemma 1.9. If (X, F, T^{SW}_λ) be a PM-space for some $\lambda \in (-1, +\infty)$ then the triangular inequality (PM4) is equivalent to the following inequality

$$(1 + \lambda)F_{x,y}(t + s) + 1 \geq F_{x,z}(s) + F_{y,z}(t) + \lambda F_{x,z}(s)F_{y,z}(t), \tag{2}$$

for all $x, y, z \in X$ and $s, t > 0$.

Proof. It immediately follows from $T^{SW}_\lambda(a, b) \leq c \Leftrightarrow a + b + \lambda ab \leq (1 + \lambda)c + 1$, for all $a, b, c \in [0, 1]$. □

2. MAIN RESULTS

In this section we extend the results in [1] by a new proof based on partially ordered set and Zermelo-Bourbaki-Kneser fixed point theorem for the class of (right) continuous and Archimedean t-norms in PM-space.

A partially ordered set is a pair $(X, \preccurlyeq)$, where X is a nonempty set and $\preccurlyeq$ is a relation in X which is reflexive ($x \preccurlyeq x$ for all $x \in X$), antisymmetric (for $x, y \in X, x \preccurlyeq y$ and $y \preccurlyeq x$ imply $x = y$) and transitive (for $x, y, z \in X, x \preccurlyeq y$ and $y \preccurlyeq z$ imply $x \preccurlyeq z$). A nonempty subset C of X is said to be a chain if every pair of elements is comparable, i. e. for given $x, y \in C$, either $x \preccurlyeq y$ or $y \preccurlyeq x$. A mapping $T : X \to X$ is said to be progressive (regressive) if $x \preccurlyeq T(x)$ ($T(x) \preccurlyeq x$), for all $x \in X$, respectively.

The Zermelo fixed point is a general principle and is one of the first fixed point theorems using the order structure on X. Many important fixed point theorems can be derived from it.

Theorem 2.1. (Zermelo, Bourbaki, and Kneser [29], see also Zeidler [28]) Let $(X, \preccurlyeq)$ be a partially ordered set, in which every chain has a supremum. If T is a progressive, then T has a fixed point. Also the theorem remains true if "supremum" is replaced by "infimum" and "progressive" by "regressive".

Lemma 2.2. Let $(X, F, *)$ be a PM-space endowed with an Archimedean t-norm $*$ and φ be a non-trivial (i. e. $\exists x \in X$ such that $\varphi(x) \neq 0$) fuzzy set on X. Define the binary relation $\preccurlyeq$ on X as follows, for every $x \in X$, if $\varphi(x) = 0$ then $x \preccurlyeq y$ if and only if $x = y$ and if $\varphi(x) \neq 0$ then

$$x \preccurlyeq y \Longleftrightarrow \varphi(y) * F_{x,y}(t) \geq \varphi(x), \qquad \forall t > 0. \tag{3}$$

Then $(X, \preccurlyeq)$ is the partial order set induced by $\preccurlyeq$.

Proof. Since if $\varphi(x) \neq 0$ then by (3) $\varphi(y) \neq 0$, the relation $\preccurlyeq$ is well defined. The reflexivity of $\preccurlyeq$ follows immediately. Suppose that for given $x, y \in X$, $x \preccurlyeq y$ and $y \preccurlyeq x$. The cases $\varphi(x) = 0$ or $\varphi(y) = 0$ are trivial, if not, by (3) we have

$$\varphi(y) * F_{x,y}(t) \geq \varphi(x) \qquad \text{and} \qquad \varphi(x) * F_{x,y}(t) \geq \varphi(y), \qquad \forall t > 0.$$

So, for all $t > 0$ we have $\varphi(y) * F_{x,y}(t) \geq \varphi(x) \geq \varphi(x) * F_{x,y}(t) \geq \varphi(y)$. Archimedean condition implies that $F_{x,y}(t) = 1$, for all $t > 0$, hence, axiom (PM2) implies that $x = y$, thus, $\preccurlyeq$ is antisymmetric. Suppose that for given $x, y, z \in X$, $x \preccurlyeq y$ and $y \preccurlyeq z$. We may assume that $\varphi(z) \neq 0$. It follows from (3) that

$$\varphi(y) * F_{x,y}(t) \geq \varphi(x) \quad \text{and} \quad \varphi(z) * F_{z,y}(t) \geq \varphi(y), \qquad \forall t > 0,$$

so,

$$\varphi(z) * F_{x,z}(t) \geq \varphi(z) * F_{x,y}(t/2) * F_{y,z}(t/2) \ \geq \varphi(y) * F_{x,y}(t/2) \geq \varphi(x), \qquad \forall t \geq 0.$$

Thus, $x \preccurlyeq z$. This proves transitivity. □

Theorem 2.3. (Caristi's fixed point theorem in complete PM-space) Let $(X, F, \star)$ be a complete PM-space for arbitrary t-norm $\star$, T be a mapping from X into itself and $*$ be a t-norm such that $* \leq \star$. If one of the following holds:

(i) Suppose that $*$ is a right continuous and Archimedean t-norm, F is right continuous and that φ is a non-trivial and u.s.c. fuzzy set of X. Moreover, assume that

$$\varphi(Tx) * F_{x,Tx}(t) \geq \varphi(x), \qquad \forall x \in X, t > 0, \tag{4}$$

(ii) Suppose that $*$ is a continuous and Archimedean t-norm, F is left continuous and that φ is a l.s.c. fuzzy set of X and $\inf_{x\in X} \varphi(x) > 0$. Moreover, assume that

$$\varphi(x) * F_{x,Tx}(t) \geq \varphi(Tx), \qquad \forall x \in X, t > 0, \tag{5}$$

then T has a fixed point in X.

Proof. (i) Define the partial order $\preccurlyeq$ on X as in Lemma 2.2, then any chain has a supremum element. Indeed, let $C = (x_\alpha)_{\alpha\in\Omega}$ be a chain and $\eta = \sup_{x\in C} \varphi(x)$. The case $\eta = 0$ is trivial, so suppose that $\eta > 0$. Let $(\alpha_n)_{n\geq 1}$ be elements from Ω such that

$$\varphi(x_{\alpha_n}) \nearrow \eta. \tag{6}$$

From $0 < \varphi(x_{\alpha_1}) \leq \varphi(x_{\alpha_2}) \leq \cdots$, it follows that $x_{\alpha_1} \preccurlyeq x_{\alpha_2} \preccurlyeq \cdots$. Hence, the following inequality holds:

$$\varphi(x_{\alpha_m}) * F_{x_{\alpha_n},x_{\alpha_m}}(t) \geq \varphi(x_{\alpha_n}), \qquad \forall t > 0, m \geq n \in \mathbb{N}. \tag{7}$$

Contrary to the assumptions suppose that $(x_{\alpha_n})_{n\geq 1}$ is not a Cauchy sequence, so there exist $0 < \epsilon < 1$ and $t > 0$ such that for all $n \in \mathbb{N}$ there exists $m \geq n$ such that

$$F_{x_{\alpha_n},x_{\alpha_m}}(t) \leq (1 - \epsilon). \tag{8}$$

By (6) for each $0 < \epsilon' < 1$ there is $N \in \mathbb{N}$ such that $\varphi(x_{\alpha_n}) \geq \eta(1 - \epsilon')$, for all $n \geq N$. From (7) one can conclude that $\eta*(1-\epsilon) \geq \eta(1-\epsilon')$, for all $0 < \epsilon' < 1$, which contradicts the Archimedean condition, so $(x_{\alpha_n})_{n\geq 1}$ is a Cauchy sequence. By the completeness of $(X, F, \star)$, $(x_{\alpha_n})_{n\geq 1}$ converges to some $u \in X$ and taking limsup from both sides of (7) we have

$$\varphi(x_{\alpha_n}) \leq \limsup_{m\to+\infty}(\varphi(x_{\alpha_m}) * F_{x_{\alpha_n},x_{\alpha_m}}(t)) \leq \varphi(u) * F_{x_{\alpha_n},u}(t), \qquad \forall t > 0. \tag{9}$$

This shows that $x_{\alpha_n} \preccurlyeq u$, for all $n \in \mathbb{N}$, which means that u is a upper bound for $(x_{\alpha_n})_{n\geq 1}$. In order to see that u is also a upper bound for C, let $\beta \in \Omega$ be such that $x_{\alpha_n} \preccurlyeq x_\beta$, for all $n \geq 1$. Then for each $n \in \mathbb{N}$, we have

$$\varphi(x_\beta) * F_{x_{\alpha_n},x_\beta}(t) \geq \varphi(x_{\alpha_n}), \qquad \forall t > 0. \tag{10}$$

Hence

$$\varphi(x_{\alpha_n}) \leq \varphi(x_\beta), \qquad \forall n \geq 1,$$

which implies,

$$\varphi(x_\beta) = \sup\{\varphi(x_\alpha), \alpha \in \Omega\} = \lim_{n\to+\infty} \varphi(x_{\alpha_n}) = \eta.$$

Thus from (10) and the Archimedean condition we get $\lim_{n\to+\infty} x_{\alpha_n} = x_\beta$, which implies $x_\beta = u$. Also, condition (4) implies that T is progressive, so, Theorem 2.1 implies that T has a fixed point in X.

(ii) The proof is a bit different from part (i), with the same symbols from the previous part, let $\eta = \inf_{x\in C} \varphi(x) > 0$ and $(\alpha_n)_{n\geq 1}$ be elements from Ω such that $\varphi(x_{\alpha_n}) \searrow \eta$. From $\varphi(x_{\alpha_1}) \geq \varphi(x_{\alpha_2}) \geq \cdots$, it follows that $x_{\alpha_1} \succcurlyeq x_{\alpha_2} \succcurlyeq \cdots$. Hence, instead of (7) the following inequality holds:

$$\varphi(x_{\alpha_n}) * F_{x_{\alpha_m}, x_{\alpha_m}}(t) \geq \varphi(x_{\alpha_m}), \qquad \text{for all} \qquad t > 0, m \geq n \in \mathbb{N}. \tag{11}$$

For each $0 < \epsilon' < 1$ there is $N \in \mathbb{N}$ such that $\varphi(x_{\alpha_n}) \leq \eta(1-\epsilon')$, for all $n \geq N$. Like as part (i) from (8) and (11) it is concluded that $\eta(1-\epsilon') * (1-\epsilon) \geq \eta$, for all $0 < \epsilon' < 1$, contrary to the Archimedean condition on t-norm. Hence (x_{α_n}) converges to some $u \in X$, take $\liminf$ in (11), it follows that $x_{\alpha_n} \succcurlyeq u, n \in \mathbb{N}$, which means that u is a lower bound for $(x_{\alpha_n})_{n\geq 1}$, Similarly to (i) one can prove that u is a lower bound for net $C = (x_\alpha)_{\alpha\in\Omega}$, also condition (5) implies that T is regressive, so, Theorem 2.1 implies that T has a fixed point in X. □

Note that in Theorem 2.3-(i) (2.3-(ii)), the mapping φ need only be assumed to be u.s.c. (l.s.c.) relative to sequences $x_n \to x$ for which $\varphi(x_n) \leq \varphi(x_{n+1}), n \geq 1$, $(\varphi(x_{n+1}) \leq \varphi(x_n), n \geq 1)$.

The next example comes from [3] and shows that conditions of Theorem 2.3-(i) are satisfied.

Example 2.4. Let $X = [0,1] \cup \{2,3,\ldots\}$. For the metric, let $F_{x,y}(0) = 0$ and

$$F_{x,y}(t) = \frac{\min\{x,y\}+1}{\max\{x,y\}+1}, \qquad \forall x,y \in X, t > 0.$$

In [9, Corollary 23, Proposition 14-(iv)] it is shown that $([0,+\infty), F, *_p)$ is a complete PM-space and its topology is the usual topology of $\mathbb{R}$ restricted to $[0,+\infty)$, so X is a closed subset of it and $(X, F, *_p)$ is a complete PM-space. Let T and φ on X defined by

$$T(x) = \begin{cases} x - \frac{x^2}{2}, & x \in [0,1], \\ x - 1, & x = 2,3,\ldots \end{cases} \qquad \text{and} \quad \varphi(x) = e^{-x}, \qquad x \in X.$$

Since $T(x) \leq x$, for all $x \in X$, we have $F_{x,Tx}(t) = \frac{Tx+1}{x+1}, x \in X, t > 0$. Indeed, it is easy to check that $f(x) = e^{-x}(x+1)$ is a decreasing function on $[0,+\infty)$, thus, we get $e^{-T(x)}(T(x)+1) \geq e^{-x}(x+1)$, for all $x \in X$, which prove that condition (4) holds.

Using Lemma 1.9 and the same proof in [1, Corollary 3.3 and 3.4], the following corollaries are obtained without the Archimedean condition on t-norm.

Corollary 2.5. Let $(X, F, *)$ be a complete PM-space and let T be a mapping from X into itself. Also, let $*$ be a right continuous t-norm such that $a * b \geq \mathfrak{L}_1(a, b)$ for all $a, b \in [0, 1]$ and let φ be a u.s.c. upper bounded function from X into $[-\infty, +\infty]$. Additively, suppose that $\exists x \in X$ such that $\varphi(x) \neq -\infty$ and

$$F_{x,Tx}(t) \geq 1 + \varphi(x) - \varphi(Tx), \tag{12}$$

holds for all $x \in X$ and $t > 0$. Then T has a fixed point in X.

Corollary 2.6. Let $(X, F, *)$ be a complete PM-space and let T be a mapping from X into itself. Also assume that $*$ be a right continuous t-norm such that $a * b \geq T_\lambda^Y(a, b)$, for all $a, b \in [0, 1]$ and for some $\lambda > -1$. Let φ be a u.s.c. bounded function from X into $[0, +\infty)$. Moreover suppose that

$$\varphi(Tx) F_{x,Tx}(t) \geq \varphi(x), \tag{13}$$

for all $x \in X$ and $t > 0$. Then T has a fixed point in X.

The following theorem is obtained by the characterization of t-norms. This theorem is comparable with a problem raised by Kirk [4, 5] (see also [2, 12, 13]) in ordinary metric space, for improving Caristi's fixed point theorem. This indicates what condition can be given for a function $\eta : [0, +\infty) \to [0, +\infty)$ such that Caristi's fixed point theorem remains true for a function $\eta \circ d$ instead of d (see also [14]). More precisely, if (1) is substituted by

$$\eta(d(x, T(x)) \leq \varphi(x) - \varphi(T(x)), \qquad \forall x \in X. \tag{14}$$

Theorem 2.7. Let $(X, F, \wedge)$ be a complete PM-space and let T be a mapping from X into itself, if one of the following holds,

(i) Let F be right continuous, $\vartheta : [0, 1] \to [0, +\infty]$ be a strictly decreasing right continuous (or equivalently u.s.c.) function with $\vartheta(1) = 0$ such that

$$\vartheta(x) + \vartheta(y) \in \mathrm{Rang}(\vartheta) \cup [\vartheta(0^+), +\infty].$$

φ is a function from X into $[0, +\infty]$ such that $\mathrm{Rang}(\varphi) \subseteq \mathrm{Rang}(\vartheta)$, $\vartheta^{(-1)} \circ \varphi$ is u.s.c and $\exists x \in X$ such that $\varphi(x) \neq +\infty$. Assume that

$$\vartheta(F_{x,Tx}(t)) \leq \varphi(x) - \varphi(Tx), \qquad \forall x \in X, t > 0, \tag{15}$$

(ii) Let F be left continuous, $\vartheta : [0, 1] \to [0, +\infty]$ be a strictly decreasing continuous function with $\vartheta(1) = 0$ such that

$$\vartheta(x) + \vartheta(y) \in \mathrm{Rang}(\vartheta) \cup [\vartheta(0^+), +\infty],$$

and φ is a bounded function from X into $[0, +\infty)$ such that $\mathrm{Rang}(\varphi) \subseteq \mathrm{Rang}(\vartheta)$ and $\vartheta^{(-1)} \circ \varphi$ is l.s.c. and $\inf_{x \in X} \theta^{(-1)} \circ \varphi(x) > 0$. Assume that

$$\vartheta(F_{x,Tx}(t)) \leq \varphi(Tx) - \varphi(x), \qquad \forall x \in X, t > 0, \tag{16}$$

(iii) Let F be right continuous, $*$ is a right continuous and Archimedean t-norm and that φ is a non-trivial fuzzy set of X and $\theta : [0,1] \to [0,1]$ is a strictly increasing bijection such that $\theta^{(-1)} \circ \varphi$ is u.s.c. Assume that

$$\varphi(Tx) * \theta(F_{x,Tx}(t)) \geq \varphi(x), \qquad \forall x \in X, t > 0, \tag{17}$$

(iv) Let F be left continuous, $*$ is a continuous and Archimedean t-norm and that φ is a fuzzy set of X and $\theta : [0,1] \to [0,1]$ is a strictly increasing bijection such that $\theta^{(-1)} \circ \varphi$ is l.s.c. and $\inf_{x \in X} \theta^{(-1)} \circ \varphi(x) > 0$. Assume that

$$\varphi(x) * \theta(F_{x,Tx}(t)) \geq \varphi(Tx), \qquad \forall x \in X, t > 0,$$

then T has a fixed point in X.

P r o o f.

(i) Set $\varphi' = \vartheta^{(-1)} \circ \varphi$, so, φ' is a non-trivial and u.s.c. fuzzy set of X. Define the function $*_\vartheta : [0,1]^2 \to [0,1]$ by

$$a *_\vartheta b = \vartheta^{(-1)}(\vartheta(a) + \vartheta(b)), \qquad \forall a, b \in [0,1].$$

Theorem 3.23 and Proposition 3.29 of [18] imply that $*_\vartheta$ is an Archimedean t-norm, also since $*_\vartheta \leq \wedge$, $(X, F, *_\vartheta)$ is a fuzzy metric space. indeed, ϑ is a right continuous function, so $*_\vartheta$ is too (see [18, Proposition 3.27]). Since $\vartheta^{(-1)}$ is a decreasing function, from (15) we find

$$\vartheta^{(-1)}(\varphi(Tx) + \vartheta(F_{x,Tx}(t))) \geq \vartheta^{(-1)}(\varphi(x)), \qquad \forall x \in X, t > 0.$$

From $\mathrm{Rang}(\varphi) \subseteq \mathrm{Rang}(\vartheta)$ and [18, Remark 3.4-(v)] we have $\vartheta \circ \varphi' = \vartheta \circ (\vartheta^{(-1)} \circ \varphi) = \varphi$ and so the preceding inequality leads to

$$\vartheta^{(-1)}(\vartheta(\varphi'(Tx)) + \vartheta(F_{x,Tx}(t))) \geq \varphi'(x), \qquad \forall x \in X, t > 0,$$

or

$$\varphi'(Tx) *_\vartheta F_{x,Tx}(t) \geq \varphi'(x), \qquad \forall x \in X, t > 0,$$

thus, Theorem 2.3-(i) concludes that T has a fixed point in X.

(ii) It follows from Theorem 2.3-(ii) and similar to the previous one.

(iii) Let $\theta^{(-1)}$ is the pseudo-inverse of θ and set $\varphi' = \theta^{(-1)} \circ \varphi$. Define the function $*_\theta : [0,1]^2 \to [0,1]$ by

$$a *_\theta b = \theta^{(-1)}(\theta(a) * \theta(b)), \qquad \forall a, b \in [0,1].$$

t-norm $*_\theta$ is right continuous and Archimedean (see Proposition 2.28-(iii) and Remark 2.30-(ii) of [18]), also since $*_\vartheta \leq \wedge$, $(X, F, *_\vartheta)$ is a fuzzy metric space. Equation (17) implies that

$$\theta^{(-1)}(\theta(\varphi'(Tx)) * \theta(F_{x,Tx}(t))) \geq \varphi'(x), \qquad \forall x \in X, t > 0,$$

or

$$\varphi'(Tx) *_\theta F_{x,Tx}(t) \geq \varphi'(x), \qquad \forall x \in X, t > 0.$$

Thus, Theorem 2.3-(i) concludes that T has a fixed point in X.

(iv) It follows from Theorem 2.3-(ii) and similar to the previous one.

□

Remark 2.8. (i) In Theorem 2.7 it is enough to consider fuzzy metric space $(X, F, *)$ such that Archimedean t-norm $*_\vartheta$ induced by ϑ satisfies in $*_\vartheta \leq *$. Also, more information about comparison of t-norms may be found in [17, 18, 20].

(ii) Note that composition of two u.s.c. (or l.s.c.) functions is not necessarily u.s.c. (or l.s.c.), but if one of the following hold then $\vartheta^{-1} \circ \varphi$ is u.s.c., see also [24] for further results.

(a) $\vartheta(t) = (1-t)^p$ or $1 - t^p, p > 0$ and φ is a l.s.c.

(b) ϑ is increasing and u.s.c. and φ is an u.s.c. function.

and if one of the following hold then $\vartheta^{-1} \circ \varphi$ is l.s.c.

(c) $\vartheta(t) = (1-t)^p$ or $1 - t^p, p > 0$ and φ is a u.s.c.

(d) ϑ is decreasing and l.s.c. and φ is a l.s.c. function.

The following metric fixed point Corollary is deduced from Theorem 2.7 and Remark 2.8. This Corollary provides a partial answer to Kirk's problem in ordinary metric space.

Corollary 2.9. Let (X, d) be a complete metric space and let T be a mapping from X into itself. If one of the following holds,

(i) Let ϕ be a l.s.c. function from X into $[0, +\infty]$ and $\eta : [0, +\infty] \to [0, +\infty]$ be a strictly increasing l.s.c. function such that $\eta(0) = 0$, $*_{\eta(-\ln(s))} \leq *_p$ and $\mathrm{Rang}(\varphi) \subseteq \mathrm{Rang}(\eta)$. Moreover, assume that (14) holds.

(ii) Let $d < 1$ and ϕ be a l.s.c. function from X into $[0, +\infty]$ and $\eta : [0, 1] \to [0, +\infty]$ be a strictly increasing l.s.c. function such that $\eta(0) = 0$, $*_{\eta(1-s)} \leq *_{\mathfrak{L}_1}$ and $\mathrm{Rang}(\varphi) \subseteq \mathrm{Rang}(\eta)$. Moreover, assume that (14) holds.

Then T has a fixed point in X.

P r o o f. (i) Set $\vartheta(s) = \eta(-\ln(s)), s \in [0, 1]$ and $F_{x,y}(t) = e^{-d(x,y)}$, for all $x, y \in X, t > 0$. Then $(F, M, *_p)$ is a complete PM-space, ϑ is u.s.c. function, $\mathrm{Rang}(\varphi) \subseteq \mathrm{Rang}(\eta) = \mathrm{Rang}(\vartheta)$ and inequality (15) obtains from inequality (14). Also since η^{-1} is a continuous strictly increasing function, $\eta^{-1} \circ \varphi$ is a l.s.c. function, so, $\vartheta^{-1} \circ \varphi = e^{-\eta^{-1} \circ \varphi}$ is an u.s.c. function, thus, Theorem 2.7-(i) and Remark 2.8 imply the existence of a fixed point.

(ii) Set $\vartheta(s) = \eta(1 - s), s \in [0, 1]$ and $F_{x,y}(t) = 1 - d(x, y)$, for all $x, y \in X, t > 0$. Then $(F, M, \mathfrak{L}_1)$ is a complete PM-space (see [10]), ϑ is u.s.c. and $\mathrm{Rang}(\varphi) \subseteq \mathrm{Rang}(\eta) = \mathrm{Rang}(\vartheta)$ and inequality (15) obtains from inequality (14). Also since η^{-1} is a l.s.c. strictly increasing function, $\eta^{-1} \circ \varphi$ is a l.s.c. function, so, $\vartheta^{-1} \circ \varphi = 1 - \eta^{-1} \circ \varphi$ is an u.s.c. function, thus, Theorem 2.7-(i) and Remark 2.8 imply the existence of a fixed point. □

Remark 2.10. The same argument in [1] can be applied to deduce Ekeland's variational principle, Takahashi's maximization theorem and equivalences among them in PM-spaces. Since the proof of them are similar to [1] except the following, we only state and prove the next theorem (for comparison see [1, Theorem 3.9]).

Theorem 2.11. (Ekeland's variational principle in complete PM-space) Let $(X, F, *)$ be a complete PM-space endowed with a right continuous and Archimedean t-norm, F be right continuous and that φ is a non-trivial and u.s.c. fuzzy set of X. Consider $v \in X$ such that $\varphi(v) \neq 0$. Then there exists $u \in X$ such that for every $t > 0$,

(i) $\varphi(u) * F_{v,u}(t) \geq \varphi(v)$,

(ii) $\varphi(x) * F_{u,x}(t) < \varphi(u)$ for all $x \in X, x \neq u$.

P r o o f. For any $x \in X$ denote the equivalence classes of x by

$$[x] = \{y \in X : x \preccurlyeq y\} = \{y \in X : \varphi(y) * F_{y,x}(t) \geq \varphi(x), \forall t > 0\}.$$

Define inductively a sequence $(x_n)_{n\geq 0}$ in X, stating with $x_0 = v$, suppose that x_n is known, then the defination of supremum allows us to choose $x_{n+1} \in [x_n]$ such that

$$2\varphi(x_{n+1}) - \varphi(x_n) \geq \sup_{z\in[x_n]} \varphi(z). \tag{18}$$

The same argument in the proof of Theorem 2.3-(i), can be applied to the increasing sequence $(x_n)_{n\geq 0}$ for proving the existence of upper bound $u \in X$ such that $\lim_{n\to+\infty} x_n = u$. Since $u \in [x_0]$, condition (i) holds, on the other hands, we need to show that if $x \neq u$ then $x \notin [u]$. Suppose it is not, as $x \in [u]$, we have $\varphi(x) * F_{u,x}(t) \geq \varphi(u)$, for all $t > 0$, so Archimedean condition implies that $\varphi(x) > \varphi(u)$. Indeed for all $n \in \mathbb{N}$, $x_n \preccurlyeq u$ and so it must be $x \in [x_n]$, consequently $\varphi(x) \leq \sup_{z\in[x_n]} \varphi(z)$. Also from (18) we obtain

$$\begin{aligned}\varphi(x) > \varphi(u) &\geq \limsup_{n\to+\infty} \varphi(x_n) \geq \liminf_{n\to+\infty}(2\varphi(x_{n+1}) - \varphi(x_n)) \\ &\geq \liminf_{n\to+\infty}(\sup_{z\in[x_n]} \varphi(z)) \geq \varphi(x),\end{aligned}$$

a contradiction, so condition (ii) holds too. □

3. CONCLUSION

Caristi's fixed point theorem and its equivalent theorems are very well-known theorems in nonlinear analysis and have extensive applications in various fields in mathematics, such as optimization, control theory, global analysis and geometric theory of Banach space. A number of generalizations of these results have been obtained by several authors. We have presented a generalization of Caristi's fixed point theorem, Ekeland's variational principle theorem in probabilistic metric spaces for the class of right continuous and Archimedean t-norms. Natural question arises in this article:

Question: Is there any non- Archimedean or non right continuous t-norm such that Theorem 2.3 remains true? notice that Example 3.8 in [1] shows that the answer is no for t-norm $*_\wedge$.

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