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# STABILITY OF PERTURBED DELAY HOMOGENEOUS SYSTEMS DEPENDING ON A PARAMETER

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In this paper, we analyze the stability of homogeneous delay systems based on the Lyapunov Razumikhin function in the presence of a varying parameter. In addition, we show the stability of perturbed time delay systems when the nominal part is homogeneous.

*Keywords:* nonlinear homogeneous system, varying delay, stability, Lyapunov Razumikhin function

*Classification:* 34D20

## 1. INTRODUCTION

The time delay dynamical systems have attracted the attention of many researchers because they were an interesting subject and very useful in many research areas such as chemistry, engineering, mechanics, physics, biology, etc. Consequently, the modern researches field show the strong presence of this type of systems in biological models such as the Nicholson equation for blowfly population growth in [18], the food-chain model in [11], the prey-predator model in [13]. In [12, 18], the authors considered the Mackey-Glass equation for the regulation of hematopoiesis as modelling of nonlinear systems. These mathematical models in biological population for a dynamical system with a constant delay have been studied in [9, 10, 11, 12, 13], or with two time varying delays such as [18]. Moreover, most of the authors considered the stochastic models [3, 20] as an application of this kind of systems.

The major problem is to analyze the stability and the stabilization for dynamical systems when the delay is dependent on time $t$.

A nonlinear time varying delay system can be described by the following form:

$$\dot{x} = f(t, x(t), x(t - \alpha(t))), \quad t \geq 0, \tag{1.1}$$

where the function $\alpha(t) \in [0, \tau]$, for $\tau > 0$, is continuous and nonnegative, $f : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a continuously differentiable function, $x(t) \in \mathbb{R}^n$ is the state vector and $f(t, 0, 0) = 0$, $\forall t \geq 0$.

Some delay systems can be described by the equation (1.1) and have more than one equilibrium point [11, 12, 13]. In the current work, we consider a class of time delay

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systems where the function $f$ is continuously differentiable and the delay system (1.1) has a unique equilibrium point which we look to study the stability behavior by using the Lyapunov-Razumikhin method.

The basic stability Lyapunov conditions are already developed in the two approaches of Lyapunov theory, the first one based on the Lyapunov Krasovskii functional [3, 5, 6, 16] which is useful only in the case of constant delay. However the second method based on the Lyapunov Razumikhin function [5, 6, 21] is more successful than the first one and is useful in the both cases: constant or varying delay. It is based essentially on Lyapunov functions satisfying additional conditions.

The parameterized families of dynamical systems hold an important position in practical and probably some authors choose to study the stability behavior by a Lyapunov function depending on a parameter [17].

Our purpose is to analyze the stability behavior of parametric time delay systems with a time-invariant parameter in a compact set which is described by the following form

$$\dot{x} = f(x(t), x(t-\alpha(t)), \theta), \quad t \geq 0,\ \theta \in \Omega, \tag{1.2}$$

where the function $f : \mathbb{R}^n \times \mathbb{R}^n \times \Omega \longrightarrow \mathbb{R}^n; (x, y, \theta) \mapsto f(x, y, \theta)$ is a continuous function, locally Lipschitz with respect to $(x, y)$ and homogeneous, $x(t) \in \mathbb{R}^n$ is the state vector, the function $\alpha(t)$ is continuous nonnegative and bounded for $t \geq 0$, $\theta$ is a time-invariant parameter in a given compact set $\Omega \subset \mathbb{R}^d$ and $f(0, 0, \theta) = 0,\ \forall \theta \in \Omega$.

Recently, some works [10, 11] studied the local stability behavior of the original delay system (1.1) by studying its linearization around the origin. Although, when the linearized approximation is null there are some research areas bring the light for a new notion for the dynamical systems and introduce one of the most important results based on homogeneous approximation [2, 7, 8, 14, 19, 22].

This concept is developed to characterize the behavior of the trivial solution of homogeneous time delay systems with constant delay in [4] and with varying delay in [1].

The purpose is to study the uniform asymptotic stability of the trivial solution of the system (1.2) under the presence of homogeneity notion.

This work is presented as follow: in section 2, we recall some preliminary results about time delay systems and the notion of homogeneity. In the third section, according to some results introduced by Aleksandrov [1], we study the asymptotic stability of the delay system (1.2) depending on a parameter $\theta$ which remains in a compact set $\Omega$ under the homogeneous effect. Then, we consider the system

$$\dot{x}(t) = f(x(t), x(t-\alpha(t)), \theta) + g(x(t), x(t-\alpha(t)), \theta), \tag{1.3}$$

where $f$ and $g$ are homogeneous functions of degree $d_0$ and $d_1$ respectively with respect to the same dilation under the hypothesis $d_0 < d_1$. By introducing sufficient conditions on the system $\dot{x}(t) = f(x(t), x(t-\alpha(t)))$, we prove the uniform asymptotic stability of the trivial solution of the system (1.3).

In section 3, we study the stability of the perturbed time delay system with varying parameter:

$$\dot{x} = f(x(t), x(t-\alpha(t)), \theta) + R(t, x(t-\alpha(t)), \theta) + P(t, x(t), \theta), \quad t \geq 0,\ \theta \in \Omega, \tag{1.4}$$

where $x \in \mathbb{R}^n$ is the state vector, the function $\alpha(t)$ is continuous nonnegative and bounded for $t \geq 0$ and $f$ is a continuous function, locally Lipschitz with respect to $(x, y)$ and homogeneous of degree $d_0 > 0$.

We assume that $R : \mathbb{R} \times \mathbb{R}^n \times \Omega \longrightarrow \mathbb{R}^n$ and $P : \mathbb{R} \times \mathbb{R}^n \times \Omega \longrightarrow \mathbb{R}^n$ are continuous functions. The obtained results are illustrated on numerical examples and we finish by a conclusion.

## 2. PRELIMINARIES

**Notations:**

1) For $x \in \mathbb{R}^n$, $\|x\| = (\sum\limits_{i=1}^{n} x_i^2)^{\frac{1}{2}}$ denotes the Euclidean norm on $\mathbb{R}^n$.

2) Let $x \in \mathbb{R}^n$, we denote $\|x\|_1 = \sum\limits_{i=1}^{n} |x_i|$.

3) Let $\mathcal{C} = \mathcal{C}([-\tau, 0], \mathbb{R}^n)$ where $\tau > 0$ the set of continuous mapping from $[-\tau, 0]$ to $\mathbb{R}^n$.

   For $\varphi : [-\tau, 0] \longrightarrow \mathbb{R}^n$, a continuous map, we define the uniform norm
$$\|\varphi\|_{\mathcal{C}} = \max_{-\tau \leq t \leq 0} \| \varphi(t) \| .$$

4) Let $(r_1, \ldots, r_n)$ a family of fixed positive reals where $(r_1, \ldots, r_n)^T$ is the vector of weight. Let the dilation map $\Lambda$ defined by:
$$\begin{array}{rlcl} \Lambda : & \mathbb{R}_+^* \times \mathbb{R}^n & \longrightarrow & \mathbb{R}^n \\ & (\lambda, x) & \longrightarrow & (\lambda^{r_1} x_1, \ldots, \lambda^{r_n} x_n), \end{array} \tag{2.1}$$
   we denote $\Lambda(\lambda, x) = \Lambda_\lambda(x)$.

5) For $x \in \mathbb{R}^n$, we denote $\rho(x) = \Big(\sum\limits_{i=1}^{n} \| x_i \|^{\frac{\epsilon}{r_i}} \Big)^{\frac{1}{\epsilon}}$, $\epsilon \geq \max\limits_{1 \leq i \leq n} r_i$, the homogeneous norm.

6) For each $p > 0$, we denote $S^p = \{x \in \mathbb{R}^n ; \rho(x) = p\}$ the homogeneous sphere.

In general, a varying time delay system can be described by:
$$\dot{x} = f(x(t), x(t - \alpha(t))), \quad t \geq 0, \tag{2.2}$$
where $\alpha(t) \in [0, \tau]$ for all $t \in \mathbb{R}_+$ is a continuous nonnegative function, $x : [t_0, +\infty) \longrightarrow \mathbb{R}^n$ and
$$\begin{array}{rcl} x_t : [-\tau, 0] & \longrightarrow & \mathbb{R}^n \\ \mu & \longmapsto & x(t + \mu). \end{array} \tag{2.3}$$

In other words, for $t_0 \in \mathbb{R}_+$, the initial condition in the state $x(t)$ in $[t_0 - \tau, t_0]$ is the function $\phi_0 \in \mathcal{C}$ such that

$$x_{t_0} = \phi_0. \tag{2.4}$$

In the next, we consider a class of nonlinear time delay systems

$$\dot{x} = f(x(t), x(t - \alpha(t)), \theta), \quad t \geq 0, \tag{2.5}$$

where the function $f : \mathbb{R}^n \times \mathbb{R}^n \times \Omega \longrightarrow \mathbb{R}^n$ is continuous and locally Lipschitz with respect to $(x, y)$, $x(t) \in \mathbb{R}^n$ is the state vector, $\alpha(t)$ is continuous and $\alpha(t) \in [0, \tau]$ for $t \geq 0$, $\tau > 0$, $\theta \in \Omega$ is a time-invariant parameter, $f(0, 0, \theta) = 0$, $\forall \theta \in \Omega$ and $\Omega \subset \mathbb{R}^d$ is a compact set.

Without loss of generality, the system (2.5) can be written as

$$\dot{x} = f(x(t), x_t(\mu), \theta), \quad t \geq 0, \ \ \theta \in \Omega, \ \ \mu \in [-\tau, 0], \tag{2.6}$$

where $x(t) \in \mathbb{R}^n$ is the vector state.

The corresponding free delay system can be written as:

$$\dot{x} = f(x(t), x(t), \theta), \quad t \geq 0, \quad \theta \in \Omega. \tag{2.7}$$

In the following, we will recall some homogeneity definitions for differential systems.

**Remark 2.1.** The homogeneous norm $\rho$ satisfies the following properties:

1) For $x \in \mathbb{R}^n \setminus \{0\}$, we have $\rho(\Lambda_\lambda(x)) = \lambda\rho(x)$, $\forall\ \lambda > 0$.

2) For $x \in S^1$, we have $\rho(\Lambda_\lambda(x)) = \lambda$, $\forall\ \lambda > 0$.

**Definition 2.2.** We say that:

- The function $h : \mathbb{R}^n \times \mathbb{R}^n \times \Omega \longrightarrow \mathbb{R}$ is homogeneous of degree $k$ if:

$$h(\Lambda_\lambda(x), \Lambda_\lambda(y), \theta) = \lambda^k h(x, y, \theta), \ \ \forall x, y \in \mathbb{R}^n, \ \forall \lambda > 0, \ \forall \theta \in \Omega.$$

- The function $f : \mathbb{R}^n \times \mathbb{R}^n \times \Omega \longrightarrow \mathbb{R}^n$ is homogeneous of degree $k$ if:

$$f(\Lambda_\lambda(x), \Lambda_\lambda(y), \theta) = \lambda^k \Lambda_\lambda(f(x, y, \theta)), \ \ \forall x, y \in \mathbb{R}^n, \ \forall \lambda > 0, \ \forall \theta \in \Omega.$$

**Definition 2.3.** The vector function $f(x, y, \theta) = (f_1(x, y, \theta), \ldots, f_n(x, y, \theta))^T$ is called locally Lipschitz with respect to $(x, y)$ if for all $(x_0, y_0) \in \mathbb{R}^n \times \mathbb{R}^n$, there exists $\mathcal{V} = \mathcal{V}(x_0, y_0)$ a neighborhood of $(x_0, y_0)$, there exists a positive number $L$ such that for all $((x', y'), (x", y")) \in \mathcal{V}^2$, one has

$$\|f(x', y', \theta) - f(x", y", \theta)\|_1 \leq L\|(x', y') - (x", y")\|_1, \ \forall \theta \in \Omega.$$

**Definition 2.4.** (Gu et al. [5]) The trivial solution of the system (2.2) is said to be:

- **stable** if $\forall t_0 \in \mathbb{R}$ and $\forall \varepsilon > 0$, there exists $\delta = \delta(t_0, \varepsilon) > 0$, such that

$$\|\phi_0\|_{\mathcal{C}} < \delta \Rightarrow \|x(t)\| < \varepsilon, \quad \forall t \geq t_0,$$

- **uniformly stable** if $\forall \varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that

$$\|\phi_0\|_{\mathcal{C}} < \delta \Rightarrow \|x(t)\| < \varepsilon, \quad \forall t_0 \in \mathbb{R}, \quad \forall t \geq t_0,$$

- **asymptotically stable** if it is stable and $\forall t_0 \in \mathbb{R}$ and $\forall \varepsilon > 0$, there exists $\delta = \delta(t_0, \varepsilon) > 0$ such that

$$\|\phi_0\|_{\mathcal{C}} < \delta \Rightarrow \lim_{t \longrightarrow \infty} x(t) = 0, \quad \forall t \geq t_0,$$

- **uniformly asymptotically stable** if it is uniformly stable and there exists $\delta > 0$ such that for all $\eta > 0$, there exists $T(\delta, \eta)$, satisfying

$$\|\phi_0\|_{\mathcal{C}} < \delta \Rightarrow \|x(t)\| < \eta, \qquad \forall t \geq t_0 + T(\delta, \eta), \quad \forall t_0 \in \mathbb{R},$$

- **globally (uniformly) asymptotically stable** if it is (uniformly) asymptotically stable and $\delta$ can be an arbitrarily large finite number.

**Definition 2.5.** The trivial solution of the delay system (2.6) is called:

1. uniformly stable with respect to the time-invariant parameter $\theta$ if for any $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that

$$\|\phi_0\|_{\mathcal{C}} < \delta \Rightarrow \|x(t)\| < \varepsilon, \quad \forall t_0 \in \mathbb{R}, \quad \forall t \geq t_0, \ \forall \theta \in \Omega.$$

2. uniformly asymptotically stable with respect to the time-invariant parameter $\theta$ if it is uniformly stable and there exists $\delta > 0$ such that for all $\eta > 0$ there exists $T(\delta, \eta)$ satisfying

$$\|\phi_0\|_{\mathcal{C}} < \delta \Rightarrow \|x(t)\| < \eta, \qquad \forall t \geq t_0 + T(\delta, \eta), \quad \forall t_0 \in \mathbb{R}, \ \forall \theta \in \Omega.$$

The study of asymptotic stability concept with the above definitions is difficult. Consequently, we recall other important theorems, called Lyapunov Razumikhin theorems [5, 6, 21], which are extensions of Lyapunov theory [15] and will simplify our study.

## 3. STABILITY OF TIME DELAY SYSTEMS DEPENDING ON A PARAMETER

### 3.1. Stability with homogeneous Lyapunov functions:

In the begging of this section, we recall the homogeneous Lyapunov function definition:

**Definition 3.1.** (Rosier [22]) Let $V : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a differentiable, positive definite function (i.e. $V(x) > 0,$ for $x \in \mathbb{R}^n \setminus \{0\}$ and $V(0) = 0$).
$V$ is called homogeneous of degree $k > 0$ with respect to the dilation $\lambda$, if

$$V(\Lambda_\lambda(x)) = \lambda^k V(x), \ \ \forall \lambda > 0, \ \ \forall x \in \mathbb{R}^n.$$

**Proposition 3.2.** (Sepulchre and Aeyels [23]) Let $V$ be a positive definite function, homogeneous of degree $k$.

Then for each $s > 0$, the following properties are satisfied:

*a*) The level set $V^s := \{x \text{ such that } V(x) = s\}$ of $V$ is homogeneous i.e. $V^s = \Lambda_{s^{\frac{1}{k}}}(V^1)$.

*b*) $V^s$ is homeomorphic to $S^{n-1}$.

*c*) For each $i \in \{1, \dots, n\}$, $\frac{\partial V}{\partial x^i}$ is homogeneous of degree $(k - r_i)$ i.e. $\frac{\partial V}{\partial x^i}(\Lambda_s(x)) = s^{k-r_i}\frac{\partial V}{\partial x^i}(x)$.

**Proposition 3.3.** (Zubov [24]) Let $V : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a $C^1$-Lyapunov function. If $V$ is homogeneous of degree $k > 0$, then there exist positive constants $a, b$ and $\gamma_i$, $i = 1, \dots, n$, such that the following estimations hold:

$$a\ \rho(x)^k \le V(x) \le b\ \rho(x)^k, \ \ \forall x \in \mathbb{R}^n,$$

and

$$\Big|\frac{\partial V}{\partial x^i}(x)\Big| \le \gamma_i\ \rho(x)^{k-r_i}, \ i = 1, \dots, n, \ \forall x \in \mathbb{R}^n.$$

**Remark 3.4.** The above results remain applicable for the Lyapunov function $V : \mathbb{R}^n \times \Omega \longrightarrow \mathbb{R}$ depending on the time-invariant parameter $\theta$ according to the study introduced in [15, 17].

For homogeneous functions, the following result was proved independently by Rosier in [2].

**Lemma 3.5.** Let $V : \mathbb{R}^n \times \Omega \longrightarrow \mathbb{R}$ be a $C^1$-Lyapunov function. If $V$ is homogeneous of degree $k > 0$ and $f$ is homogeneous of degree $d_0$ with respect to the same dilation $\Lambda$, then, the derivative of $V$ with respect to the system (2.7)

$$\omega_\theta(x) = \big\langle \nabla V(x, \theta), f(x, x, \theta)\big\rangle, \quad x \in \mathbb{R}^n, \ \theta \in \Omega,$$

is homogeneous of degree $d_0 + k$.

**Lemma 3.6.** If the system (2.7) is uniformly asymptotically stable, then there exists $m > 0$ such that the estimation

$$\omega_\theta(x) \leq -m\, \rho(x)^{d_0+k},$$

holds for all $x \in \mathbb{R}^n \setminus \{0\}$ and for all $\theta \in \Omega$.

Proof.

Let $\Lambda : \mathbb{R}_+^* \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ the dilation map. The system (2.7) is uniformly asymptotically stable, then there exists an homogeneous Lyapunov function of degree $k$, $V : \mathbb{R}^n \times \Omega \to \mathbb{R}$ of class $C^1$. Using the homogeneous fact of $f$ and $V$, we deduce the following:
for $x \neq 0$ is given, there exists a unique $y \in S^1$ such that $x = \Lambda_\lambda(y)$.

The derivative of $V$ along the trajectories of the system (2.7) is

$$\begin{aligned}
\omega_\theta(x) &= \big\langle \nabla V(x,\theta), f(x,x,\theta) \big\rangle \\
&= \langle \nabla V(\Lambda_\lambda(y),\theta), f(\Lambda_\lambda(y),\Lambda_\lambda(y),\theta) \rangle \\
&= \langle \lambda^k A^{-1}(\lambda) \nabla V(y,\theta), \lambda^{d_0} A(\lambda) f(y,y,\theta) \rangle \\
&= \lambda^{k+d_0} \langle \nabla V(y,\theta), f(y,y,\theta) \rangle = \lambda^{k+d_0} \omega_\theta(y),
\end{aligned}$$

where $A^{-1}(\lambda) = diag(\lambda^{-r_1}, \ldots, \lambda^{-r_n})$.

The uniform asymptotic stability of the system (2.7) implies that $\omega_\theta(y) < 0,\ \forall \theta \in \Omega$.

Under the assumptions that $\nabla V$ and $f$ are continuous functions and $S^1 \times \Omega$ is a compact set, we conclude that the constant $m$ defined by

$$m = \min_{y \in S^1, \theta \in \Omega} |\omega_\theta(y)|$$

is positive. The fact $\omega_\theta(y) < 0,\ \ \forall \theta \in \Omega$ and $y \in S^1$ implies that $\omega_\theta(y) \leq -m$.

Hence,

$$\omega_\theta(x) \leq -m \lambda^{k+d_0}, \ \ \forall \theta \in \Omega.$$

By the result that $\rho(\Lambda_\lambda(y)) = \lambda \rho(y)$ and since $y \in S^1$, we get $\rho(x) = \rho(\Lambda_\lambda(y)) = \lambda$. Therefore,

$$\omega_\theta(x) \leq -m \rho(x)^{k+d_0}, \ \ \forall \theta \in \Omega,\ x \in \mathbb{R}^n \setminus \{0\}.$$

This completes the proof. □

Now, we recall the Lyapunov Razumikhin Theorem which will be useful in the next:

**Theorem 3.7.** (Gu et al. [5]) Let $f : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$. Suppose that $\psi_1, \psi_2, \psi_3 : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ are continuous nondecreasing functions, and $\psi_1(s)$, $\psi_2(s)$ are positive for $s > 0$, $\psi_1(0) = \psi_2(0) = 0$ and $\psi_2$ is strictly increasing.

If there exists a continuous function

$$V : \mathbb{R}^n \longrightarrow \mathbb{R}_+,$$

such that

$$\psi_1(\|x(t)\|) \leq V(x(t)) \leq \psi_2(\|x(t)\|), \quad \forall t \geq 0, \forall x \in \mathbb{R}^n, \tag{3.1}$$

and the derivative of $V$ along the solutions $x(t)$ of the system (2.2) satisfies

$$\dot{V}(x(t)) \leq -\psi_3(\|x(t)\|) \text{ whenever } V(x(\xi)) \leq V(x(t)), \quad \forall t \geq 0, \ \forall \xi \in [t-\tau, t], \quad (3.2)$$

then the system (2.2) is uniformly stable.

If in addition, $\psi_3(s) > 0, \forall s > 0$ and there exists a continuous and nondecreasing function $p(s) > s$, $\forall s > 0$ such that the condition (3.2) is strengthened to

$$\dot{V}(x(t)) \leq -\psi_3(\|x(t)\|) \text{ whenever } V(x(\xi)) \leq p(V(x(t))), \quad \forall t \geq 0, \ \forall \xi \in [t-\tau, t], \ (3.3)$$

then the system (2.2) is uniformly asymptotically stable.

Moreover, if $\lim\limits_{s\to+\infty} \psi_1(s) = +\infty$, then system (2.2) is globally uniformly asymptotically stable.

To prove the stability behavior of the delay system (2.6), we need to introduce the following assumption.

**Assumption A1.** The trivial solution of the system (2.7) is uniformly asymptotically stable with respect to the parameter $\theta$.

**Theorem 3.8.** Let $f : \mathbb{R}^n \times \mathbb{R}^n \times \Omega \longrightarrow \mathbb{R}^n$ a continuous, locally Lipschitz with respect to $(x, y)$ and homogeneous function of degree $d_0 > 0$ with respect to the dilation $\Lambda$.

If the assumption $A1$ is fulfilled, then the trivial solution of the system (2.6) is locally uniformly asymptotically stable.

Proof. By assumption $A1$, the trivial solution of the system $\dot{x} = f(x, x, \theta)$ is uniformly asymptotically stable, then by the Lyapunov converse Theorem there exists $V : \mathbb{R}^n \times \Omega \longrightarrow \mathbb{R}$ a $C^1$-Lyapunov homogeneous function of degree $k$ with respect to the dilation $\Lambda$ and satisfying that there exists positive constants $a, b$ and $\gamma_i$, $i = 1, \dots, n$, such that:

$$a\ \rho(x)^k \leq V(x,\theta) \leq b\ \rho(x)^k \quad \text{and} \quad \left|\frac{\partial V}{\partial x^i}(x,\theta)\right| \leq \gamma_i\ \rho(x)^{k-r_i},$$

for $i = 1, \dots, n$, $\forall x \in \mathbb{R}^n$, $\forall \theta \in \Omega$.

To study the stability behavior of the time delay system, we need to prove that $V$ satisfies all conditions of the theorem 3.7. We choose positive numbers $\delta$ and $\sigma$ such that $\sigma > 1$ and we suppose that, for $x(t)$ is a solution of the system (2.6), $\rho(x(\xi)) < \delta$ and the Razumikhin condition (3.3) that we are introduced as

$$V(x(\xi), \theta) \leq \sigma V(x(t), \theta), \ \ \forall t \geq 0, x \in \mathbb{R}^n, \theta \in \Omega, \tag{3.4}$$

are fulfilled $\forall \xi \in [t-\tau, t], \tau = \sup\limits_{t\geq 0} \alpha(t)$.

Differentiating $V$ with respect to the system (2.6), we get

$$\dot{V}(x(t),\theta) \ \ = \ \ \omega_\theta(x(t)) + \sum_{i=1}^{n} \frac{\partial V}{\partial x^i}(x(t),\theta)\Big[f_i(x(t), x_t(\mu), \theta) - f_i(x(t), x(t), \theta)\Big]. \tag{3.5}$$

By the homogeneity of $V$, one has the first result of the proposition 3.3 and by applying to estimation (3.4), we deduce

$$a\ \rho(x(\xi))^k \leq V(x(\xi), \theta) \leq \sigma V(x(t), \theta) \leq \sigma b\ \rho(x(t))^k.$$

Hence, it is easy to get

$$\rho(x(\xi)) \leq (\frac{\sigma b}{a})^{\frac{1}{k}}\rho(x(t)), \ \ \forall \xi \in [t-\tau, t], x \in \mathbb{R}^n.$$

Let $B = \{x \in \mathbb{R}^n : \ \rho(x) \leq (\frac{\sigma b}{a})^{\frac{1}{k}}\}$.
For $x, x_t \in \mathbb{R}^n \setminus \{0\}$, we denote $\tilde{x} = (\tilde{x}^1, \ldots, \tilde{x}^n)^T$ and $\tilde{y} = (\tilde{y}^1, \ldots, \tilde{y}^n)^T$ two vector fields defined by:

$$\tilde{x}^i(t) = \frac{x^i(t)}{\rho(x(t))^{r_i}} \qquad \text{and} \qquad \tilde{y}^i(t) = \frac{x_t^i(\mu)}{\rho(x(t))^{r_i}}, \ \ \mu \in [-\alpha(t), 0], \ t \geq 0,$$

where $i = \{1, \ldots, n\}$ and $(r_1, \ldots, r_n) \in [0, +\infty)$ is the vector of weight.

It is clear that $\rho(\tilde{x}(t)) = 1$ and $\rho(\tilde{y}(t)) \leq (\frac{\sigma b}{a})^{\frac{1}{k}}$.
Through the homogeneity of the both functions $V$ and $f$ and by results given in proposition 3.3 and in lemma 3.6, we get:

$$\begin{aligned}
&\dot{V}(x(t), \theta) \\
\leq\ & -m\ \rho(x(t))^{d_0+k} + \sum_{i=1}^{n} \gamma_i \rho(x(t))^{k-r_i}\Big|f_i(x(t), x_t(\mu), \theta) - f_i(x(t), x(t), \theta)\Big| \\
\leq\ & -m\ \rho(x(t))^{d_0+k} + \sum_{i=1}^{n} \gamma_i \rho(x(t))^{k-r_i}\rho(x(t))^{d_0+r_i}\Big|f_i(\tilde{x}(t), \tilde{y}(t), \theta) - f_i(\tilde{x}(t), \tilde{x}(t), \theta)\Big| \\
\leq\ & -m\ \rho(x(t))^{d_0+k} + \sum_{i=1}^{n}\sum_{j=1}^{n} \gamma_i \rho(x(t))^{k+d_0} L\Big|\tilde{y}^j(t) - \tilde{x}^j(t)\Big| \\
\leq\ & -m\ \rho(x(t))^{d_0+k} + \sum_{i=1}^{n}\sum_{j=1}^{n} \gamma_i \rho(x(t))^{k+d_0} L\rho(x(t))^{-r_j}\Big|x_t^j(\mu) - x^j(t)\Big|.
\end{aligned}$$

By applying the mean value theorem, we get:

$$\Big|x_t^j(\mu) - x^j(t)\Big| \ \ = \ \ \alpha(t)\Big|\dot{x}^j(\epsilon_j(t))\Big|,$$

where $\epsilon_j(t) \in ]t-\alpha(t), t[$. Thus, we obtain:

$$\begin{aligned}
&\dot{V}(x(t), \theta) \\
\leq\ & -m\ \rho(x(t))^{d_0+k} + \sum_{i=1}^{n}\sum_{j=1}^{n} L\gamma_i\ \rho(x(t))^{k+d_0-r_j}\alpha(t)\Big|\dot{x}^j(\epsilon_j(t))\Big| \\
\leq\ & -m\ \rho(x(t))^{d_0+k} + \sum_{i=1}^{n}\sum_{j=1}^{n} L\gamma_i \rho(x(t))^{d_0+k-r_j}\tau\Big|f_j(x(\epsilon_j(t)), x(\epsilon_j(t) - \alpha(\epsilon_j(t))), \theta)\Big| \\
\leq\ & -m\ \rho(x(t))^{d_0+k} + \sum_{i=1}^{n}\sum_{j=1}^{n} L\gamma_i \rho(x(t))^{d_0+k-r_j}\tau\rho(x(t))^{d_0+r_j}\Big|f_j(\tilde{x}(\epsilon_j(t)), \tilde{y}(\epsilon_j(t)), \theta)\Big| \\
\leq\ & -m\ \rho(x(t))^{d_0+k} + \sum_{i=1}^{n}\sum_{j=1}^{n} L\gamma_i \tau\rho(x(t))^{2d_0+k}\tilde{b} \\
\leq\ & (-m\ + \tilde{a}\ \rho(x(t))^{d_0})\rho(x(t))^{d_0+k},
\end{aligned}$$

where $\tilde{a} = n\tau L\tilde{b}\sum_{i=1}^{n}\gamma_i$ and $\tilde{b}$ are positive real constants.

Let $\varepsilon_1 = \frac{m}{2}$ be a positive real constant, there exists $\tilde{\eta}_1$ where $\tilde{\eta}_1 \in [0, \delta]$ such that

$$\tilde{a}\ \rho(x(t))^{d_0} \leq \varepsilon_1.$$

Therefore

$$\dot{V}(x(t), \theta) \quad \leq \quad -\frac{m}{2}\ \rho(x(t))^{d_0+k}.$$

Hence, the solution of the system (2.6) is locally uniformly asymptotically stable with respect to $\theta$. □

### 3.2. Application to the stability of time delay systems depending on a parameter:

We consider the following system:

$$\dot{x}(t) = f(x(t), x(t-\alpha(t)), \theta) + g(x(t), x(t-\alpha(t)), \theta), \quad \theta \in \Omega,\ t \geq 0, \tag{3.6}$$

where $f : \mathbb{R}^n \times \mathbb{R}^n \times \Omega \longrightarrow \mathbb{R}^n; (x, y, \theta) \mapsto f(x, y, \theta)$ and $g : \mathbb{R}^n \times \mathbb{R}^n \times \Omega \longrightarrow \mathbb{R}^n; (x, y, \theta) \mapsto g(x, y, \theta)$ are continuous, locally Lipschitz with respect to $(x, y)$ and homogeneous functions of degree $d_0 > 0$ and $d_1 > 0$ respectively with respect to the dilation $\Lambda$ where $d_0 < d_1$. The function $\alpha(t)$ is continuous, nondecreasing and bounded with $\alpha(t) \in [0, \tau]$ for all $t \geq 0$, for a given $\tau > 0$.

To simplify, denote the system (3.6) by

$$\dot{x}(t) = f(x(t), x_t(\mu), \theta) + g(x(t), x_t(\mu), \theta), \ \ \mu \in [-\tau, 0], \ \ \theta \in \Omega,\ t \geq 0. \tag{3.7}$$

The corresponding free delay system is:

$$\dot{x}(t) = f(x(t), x(t), \theta) + g(x(t), x(t), \theta), \quad \theta \in \Omega,\ t \geq 0. \tag{3.8}$$

**Lemma 3.9.** Let $V : \mathbb{R}^n \times \Omega \longrightarrow \mathbb{R}$ be a $C^1$-Lyapunov function and homogeneous of degree $k > 0$ with respect to the dilation $\Lambda$.

If $g : \mathbb{R}^n \times \mathbb{R}^n \times \Omega \longrightarrow \mathbb{R}^n$ is a continuous and homogeneous function of degree $d_1 > 0$ with respect to the dilation $\Lambda$ and

$$\phi_\theta(x) = \langle \nabla V(x, \theta), g(x, x, \theta) \rangle.$$

Then,

$$|\phi_\theta(x)| \leq \varphi\rho(x)^{d_1+k},\ \forall\theta \in \Omega,\ x \in \mathbb{R}^n \setminus \{0\},$$

where $\varphi$ is a nonnegative real constant.

P r o o f. Let $\Lambda : \mathbb{R}_+^* \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ the dilation map and suppose that the both functions $V$ and $g$ are homogeneous of degree $k$ and $d_1$ respectively with respect to $\Lambda$.

For every fixed $x \neq 0$, there exists a unique $y \in S^1$ such that $x = \Lambda_\lambda(y)$ and for all $\theta \in \Omega$, we have

$$\begin{aligned}\phi_\theta(x) &= \langle \nabla V(x,\theta), g(x,x,\theta)\rangle = \langle \nabla V(\Lambda_\lambda(y),\theta), g(\Lambda_\lambda(y),\Lambda_\lambda(y),\theta)\rangle \\ &= \langle \lambda^k A^{-1}(\lambda)\nabla V(y,\theta), \lambda^{d_1} A(\lambda) g(y,y,\theta)\rangle \\ &= \lambda^{k+d_1}\langle \nabla V(y,\theta), g(y,y,\theta)\rangle,\end{aligned}$$

where $A^{-1}(\lambda) = diag(\lambda^{-r_1},\dots,\lambda^{-r_n})$.

Since the continuity of the both functions $\nabla V$ and $g$ and the compactness of $S^1 \times \Omega$, then there exists a nonnegative number $\varphi$ satisfying

$$\varphi = \sup_{y\in S^1,\theta\in\Omega} \Big|\langle \nabla V(y,\theta), g(y,y,\theta)\rangle\Big|.$$

On the other hand, the fact that $\rho(\Lambda_\lambda(y)) = \lambda\rho(y)$ and $y \in S^1$ implies that $\rho(x) = \lambda$. Then, we deduce that

$$\begin{aligned}\Big|\phi_\theta(x)\Big| &\le \Big|\lambda^{k+d_1}\langle \nabla V(y,\theta), g(y,y,\theta)\rangle\Big| \\ &\le \lambda^{k+d_1}\varphi \\ &\le \rho(x)^{k+d_1}\varphi, \quad \forall\theta\in\Omega,\ \forall x\in\mathbb{R}^n.\end{aligned}$$

□

**Assumption A2.** The trivial solution of the original system (2.6) is locally uniformly asymptotically stable with respect to $\theta$.

**Theorem 3.10.** Let $f : \mathbb{R}^n \times \mathbb{R}^n \times \Omega \longrightarrow \mathbb{R}^n$ and $g : \mathbb{R}^n \times \mathbb{R}^n \times \Omega \longrightarrow \mathbb{R}^n$.

If the assumption $A2$ is fulfilled, then the system (3.7) is locally uniformly asymptotically stable.

P r o o f. Let $V : \mathbb{R}^n \times \Omega \longrightarrow \mathbb{R}$ a $C^1$-Lyapunov function and homogeneous of degree $k > 0$ with respect to the dilation $\Lambda$ for the system (2.6). In the same way that we shown in the proof of theorem 3.8, we choose positive numbers $\delta$ and $\sigma$ such that $\sigma > 1$ and we suppose that, for $x(t)$ is a solution of the system (2.6), $\rho(x(\xi)) < \delta$ and the Razumikhin condition

$$V(x(\xi),\theta) \le \sigma V(x(t),\theta), \quad \forall t \ge 0, x\in\mathbb{R}^n, \theta\in\Omega,$$

are fulfilled $\forall\xi \in [t-\tau, t], \tau = \sup_{t\ge 0}\alpha(t)$.

Differentiating $V$ with respect to the system (3.7), we get

$$\begin{aligned}\dot{V}(x(t),\theta) &= \big\langle \nabla V(x(t),\theta), f(x(t),x_t(\mu),\theta)\big\rangle + \big\langle \nabla V(x(t),\theta), g(x(t),x(t),\theta)\big\rangle \\ &\quad + \sum_{i=1}^{n} \frac{\partial V}{\partial x^i}(x(t),\theta)\big(g_i(x(t),x_t(\mu)\theta) - g_i(x(t),x(t),\theta)\big) \\ &= \big\langle \nabla V(x(t),\theta), f(x(t),x_t(\mu),\theta)\big\rangle + \phi_\theta(x(t)) \\ &\quad + \sum_{i=1}^{n} \frac{\partial V}{\partial x^i}(x(t),\theta)\big(g_i(x(t),x_t(\mu),\theta) - g_i(x(t),x(t),\theta)\big).\end{aligned}$$

The assumption $A2$ implies that there exists a positive real constant $\tilde{c}$ such that

$$\big\langle \nabla V(x(t),\theta), f(x(t),x_t(\mu),\theta)\big\rangle \le -\tilde{c}\,\rho(x(t))^{d_0+k}, \quad \forall\, t \ge 0,\ x \in \mathbb{R}^n,\ \theta \in \Omega.$$

As the proof of theorem 3.8, under the hypothesis that $g$ is locally Lipschitz with respect to $(x,y)$ and homogeneous of degree $d_1$ and applying lemma 3.9, we obtain:

$$\begin{aligned}&\dot{V}(x(t),\theta) \\ \le\;& -\tilde{c}\,\rho(x(t))^{d_0+k} + \varphi\rho(x(t))^{d_1+k} + \sum_{i=1}^{n}\gamma_i\rho(x(t))^{k-r_i}\Big|g_i(x(t),x_t(\mu),\theta) - g_i(x(t),x(t),\theta)\Big| \\ \le\;& -\tilde{c}\,\rho(x(t))^{d_0+k} + \varphi\rho(x(t))^{d_1+k} + \sum_{i=1}^{n}\gamma_i\rho(x(t))^{k+d_1}\Big|g_i(\tilde{x}(t),\tilde{y}(t),\theta) - g_i(\tilde{x}(t),\tilde{x}(t),\theta)\Big| \\ \le\;& -\tilde{c}\,\rho(x(t))^{d_0+k} + \varphi\rho(x(t))^{d_1+k} + \sum_{i=1}^{n}\sum_{\ell=1}^{n}\gamma_i\rho(x(t))^{k+d_1}L'\Big|\tilde{y}^\ell(t) - \tilde{x}^\ell(t)\Big| \\ \le\;& -\tilde{c}\,\rho(x(t))^{d_0+k} + \varphi\rho(x(t))^{d_1+k} + \sum_{i=1}^{n}\sum_{\ell=1}^{n}\gamma_i\rho((x(t))^{k+d_1}L'\rho(x(t))^{-r_\ell}\Big|x_t^\ell(\mu) - x^\ell(t)\Big| \\ \le\;& -\tilde{c}\,\rho(x(t))^{d_0+k} + \varphi\rho(x(t))^{d_1+k} + \sum_{i=1}^{n}\sum_{\ell=1}^{n}L'\gamma_i\,\rho(x(t))^{k+d_1-r_\ell}\alpha(t)\Big|\dot{x}^\ell(\epsilon_\ell(t))\Big| \\ \le\;& -\tilde{c}\,\rho(x(t))^{d_0+k} + \varphi\rho(x(t))^{d_1+k} \\ &+ \sum_{i=1}^{n}\sum_{\ell=1}^{n}L'\gamma_i\rho(x(t))^{d_1+k-r_\ell}\tau\Big|g_\ell(x(\epsilon_\ell(t)),x(\epsilon_\ell(t)-\alpha(\epsilon_\ell(t))),\theta)\Big| \\ \le\;& -\tilde{c}\,\rho(x(t))^{d_0+k} + \varphi\rho(x(t))^{d_1+k} \\ &+ \sum_{i=1}^{n}\sum_{\ell=1}^{n}L'\gamma_i\rho(x(t))^{d_1+k-r_\ell}\tau\rho(x(t))^{d_1+r_\ell}\Big|g_\ell(\tilde{x}(\epsilon_\ell(t)),\tilde{y}(\epsilon_\ell(t)),\theta)\Big| \\ \le\;& \rho(x(t))^{k+d_0}(-\tilde{c} + \varphi\rho(x(t))^{d_1-d_0}) + \sum_{i=1}^{n} nL'\gamma_i\tau\tilde{c}_1\rho(x(t))^{2d_1+k} \\ \le\;& \rho(x(t))^{k+d_0}(-\tilde{c} + \varphi\rho(x(t))^{d_1-d_0} + \tilde{a}_1\rho(x(t))^{2d_1-d_0}),\end{aligned}$$

where $\tilde{a}_1 = nL'\tau\tilde{c}_1\sum_{i=1}^{n}\gamma_i$ and $\tilde{c}_1$ are positive real constants.

Let $\varepsilon_2 = \frac{\tilde{c}}{2} > 0$, there exists $\tilde{\eta}_2$ where $\tilde{\eta}_2 \in [0,\delta]$ such that

$$\varphi\rho(x(t))^{d_1-d_0} + \tilde{a}_1\rho(x(t))^{2d_1-d_0} \le \varepsilon_2.$$

This implies that

$$\dot{V}(x(t),\theta) \quad \leq \quad -\frac{\tilde{c}}{2}\,\rho(x(t))^{d_0+k}.$$

Therefore, this completes the proof. □

## 4. STABILITY OF PERTURBED TIME DELAY SYSTEMS WITH VARYING PARAMETER

In this part, we will look for conditions of the asymptotic stability of the solution when the original system (2.5) has some perturbations.

We consider the following perturbed system

$$\dot{x} = f(x(t), x(t-\alpha(t)), \theta) + R(t, x(t-\alpha(t)), \theta) + P(t, x(t), \theta), \quad t \geq 0,\ \theta \in \Omega, \quad (4.1)$$

where $x \in \mathbb{R}^n$ is the state vector, the function $\alpha(t)$ is continuous nonnegative and bounded for $t \geq 0$, $f$ is a continuous, locally Lipschitz with respect to $(x, y)$ and homogeneous of degree $d_0 > 0$ with respect to the dilation $\Lambda$, $R : \mathbb{R} \times \mathbb{R}^n \times \Omega \longrightarrow \mathbb{R}^n$ and $P : \mathbb{R} \times \mathbb{R}^n \times \Omega \longrightarrow \mathbb{R}^n$.

**Assumption A3.** Suppose that $R(t, z, \theta) = (R_1(t, z, \theta), \ldots, R_n(t, z, \theta))^T$ is a continuously differentiable function satisfying the following estimation:

$$\Big|R_i(t, z, \theta)\Big| \leq \Gamma_i \max_{\xi \in [t-\tau, t]} \rho(z(\xi))^{\tilde{\mu}+r_i}, \quad i = 1, \ldots, n,\ \forall \theta \in \Omega,\ \forall z \in \mathbb{R}^n,$$

where $\Gamma_i > 0$, $\tilde{\mu} > d_0 > 0$ and $\tau = \sup\limits_{t \geq 0} \alpha(t)$.

**Assumption A4.** Let $P : (t, x, \theta) \mapsto P(t, x(t), \theta) = (P_1(t, x(t), \theta), \ldots, P_n(t, x(t), \theta))^T$, $\theta \in \Omega$, $x \in \mathbb{R}^n$ and $t \geq 0$, a continuous function.

We assume that

$$\lim_{\lambda \to 0} \frac{\Big|P_i(t, \Lambda_\lambda(y), \theta)\Big|}{\lambda^{d_0+r_i}} = 0, \quad \forall t \geq 0,$$

uniformly on $S^1 \times \Omega$.

**Theorem 4.1.** If the assumptions $A2$, $A3$ and $A4$ are fulfilled, then the trivial solution of the perturbed system (4.1) is locally uniformly asymptotically stable.

Proof. Consider the $C^1-$Lyapunov and homogeneous function $V : \mathbb{R}^n \times \Omega \longrightarrow \mathbb{R}$ of degree $k$ with respect to $\theta$ which is constructed for time delay system (2.6).

Under assumption $A2$, we have the trivial solution of the system $\dot{x} = f(x, x_t, \theta)$ is locally uniformly asymptotically stable, then as in the proof of theorem 3.8, $V$ satisfies that there exists positive constants $\gamma_i$ and $\tilde{c}$, $i = 1, \ldots, n$ such that:
$\Big|\dfrac{\partial V}{\partial x^i}(x, \theta)\Big| \leq \gamma_i\, \rho(x)^{k-r_i}$ and $\dot{V}(x, \theta)_{|(2.6)} \leq -\tilde{c}\, \rho(x(t))^{d_0+k}$, $\forall x \in \mathbb{R}^n$, $\forall \theta \in \Omega$.

Hence, the derivative of $V$ with respect to the perturbed system (4.1) is described by

$$\begin{aligned}\dot{V}(x(t),\theta) &= \big\langle \nabla V(x(t),\theta), f(x(t),x_t(\mu),\theta)\big\rangle + \sum_{i=1}^{n}\frac{\partial V}{\partial x^i}(x(t),\theta)R_i(x(t),x_t(\mu),\theta)\\ &\quad + \sum_{i=1}^{n}\frac{\partial V}{\partial x^i}(x(t),\theta)P_i(t,x(t),\theta)\\ &\le -\tilde{c}\,\rho(x(t))^{d_0+k} + \sum_{i=1}^{n}\gamma_i\rho(x(t))^{k-r_i}\Gamma_i \max_{\xi\in[t-\tau,t]}\rho(x(\xi))^{\tilde{\mu}+r_i}\\ &\quad + \sum_{i=1}^{n}\gamma_i\rho(x(t))^{k-r_i}\Big|P_i(t,x(t),\theta)\Big|.\end{aligned}$$

We choose a positive number $\delta$ such that, for all $x(t)$ a solution of the system (2.6), the inequality $\rho(x(\xi)) < \delta$ is fulfilled $\forall\, \xi \in [t-\tau, t]$ and $\tau = \sup_{t\geq 0}\alpha(t)$.

Thus, it can be seen that

$$\dot{V}(x(t),\theta) \le -\tilde{c}\,\rho(x(t))^{d_0+k} + \sum_{i=1}^{n}\gamma_i\Gamma_i\rho(x(t))^{k+\tilde{\mu}} + \sum_{i=1}^{n}\gamma_i\rho(x(t))^{k+d_0}\frac{\Big|P_i(t,x(t),\theta)\Big|}{\rho(x(t))^{d_0+r_i}}.$$

For $x\in\mathbb{R}^n\setminus\{0\}$ is given, there exists a unique $y\in S^1$ such that $x=\Lambda_\lambda(y)$.

As $y\in S^1$, we get $\rho(y) = 1$ and on the other side, one has $\rho(\Lambda_\lambda(y)) = \lambda\rho(y)$, consequently $\rho(x)=\lambda$.

So, we obtain

$$\begin{aligned}\dot{V}(x(t),\theta) &\le -\tilde{c}\,\rho(\Lambda_\lambda(y(t)))^{d_0+k} + \sum_{i=1}^{n}\gamma_i\Gamma_i\rho(\Lambda_\lambda(y(t)))^{k+\tilde{\mu}}\\ &\quad + \sum_{i=1}^{n}\gamma_i\rho(\Lambda_\lambda(y(t)))^{k+d_0}\frac{\Big|P_i(t,\Lambda_\lambda(y(t)),\theta)\Big|}{\rho(\Lambda_\lambda(y(t)))^{d_0+r_i}}\\ &\le -\tilde{c}\,\lambda^{d_0+k} + \tilde{a}_2\lambda^{\tilde{\mu}+k} + \sum_{i=1}^{n}\gamma_i\lambda^{k+d_0}\frac{\Big|P_i(t,\Lambda_\lambda(y(t)),\theta)\Big|}{\lambda^{d_0+r_i}}.\end{aligned}$$

Since

$$\lim_{\lambda\to 0}\frac{\Big|P_i(t,\Lambda_\lambda(y(t)),\theta)\Big|}{\lambda^{d_0+r_i}} = 0,\quad \forall t\geq 0, \forall\theta\in\Omega,$$

uniformly on $S^1$, so for all $\beta>0$, there exists $\tilde{\lambda}>0$ where $\lambda\in[0,\tilde{\lambda}]$, such that one has

$$\frac{\Big|P_i(t,\Lambda_\lambda(y(t)),\theta)\Big|}{\lambda^{d_0+r_i}} < \beta,\ \forall t\geq 0, \forall\theta\in\Omega.$$

Then

$$\begin{aligned}\dot{V}(x(t),\theta) &\leq -\tilde{c}\,\lambda^{d_0+k} + \tilde{a}_2\lambda^{\tilde{\mu}+k} + \sum_{i=1}^{n}\gamma_i\lambda^{k+d_0}\beta \\ &\leq -\tilde{c}\,\rho(x(t))^{d_0+k} + \tilde{a}_2\rho(x(t))^{k+\tilde{\mu}} + \tilde{a}_3\rho(x(t))^{k+d_0} \\ &\leq \Big(-\tilde{c}\, + \tilde{a}_2\rho(x(t))^{\tilde{\mu}-d_0} + \tilde{a}_3\Big)\rho(x(t))^{d_0+k},\end{aligned}$$

where $\tilde{a}_2 = \sum_{i=1}^{n}\gamma_i\Gamma_i$ and $\tilde{a}_3 = \beta\sum_{i=1}^{n}\gamma_i$ are positive real constants for $\gamma_i,\ \beta$ and $\Gamma_i,\ i = 1,\dots,n$ are positive numbers.

For $\beta$ sufficiently small, let $\varepsilon_3$ be a positive constant and defined by the following equality

$$\varepsilon_3 = \frac{\tilde{c}}{2} - 2\tilde{a}_3,$$

there exists $\tilde{\eta}_3 \in [0,\delta]$ such that

$$\tilde{a}_2\rho(x(t))^{\tilde{\mu}-d_0} \leq \varepsilon_3.$$

Consequently, we deduce

$$\dot{V}(x(t),\theta) \quad\leq\quad -\tilde{n}\,\rho(x(t))^{d_0+k},$$

where $\tilde{n} = \frac{\tilde{c}}{2} + \tilde{a}_3 > 0$.

Finally, the system (4.1) is locally uniformly asymptotically stable with respect to $\theta$. □

## 5. EXAMPLES

**Example 5.1.** Let the time delay system described by

$$\dot{x}(t) = -4(2-\theta)^2x^3(t) + x^3(t)x^2(t-\tau) - (\sin(\theta)+2)^2x^5(t). \qquad (S_1)$$

Denote

$$f(x,y,\theta) = -4(2-\theta)^2x^3, \qquad (S_2)$$

and

$$g(x,y,\theta) = x^3y^2 - (\sin(\theta)+2)^2x^5, \qquad (S_3)$$

where $\theta \in [\frac{-3}{2},\frac{3}{2}] = \Omega$, $(x,y) \in \mathbb{R}^2$ and $\tau > 0$.

The function $f$ (respectively $g$) is homogeneous of degree 3 (respectively 5) with respect to the standard dilation i. e.:
$f(\lambda x,\lambda y,\theta) = \lambda^3 f(x,y,\theta),\ \forall(x,y,\theta) \in \mathbb{R}^2\times\Omega$;
$g(\lambda x,\lambda y,\theta) = \lambda^5 g(x,y,\theta),\ \forall(x,y,\theta) \in \mathbb{R}^2\times\Omega$.
Let $V(x,\theta) = \frac{1}{2}x^2$ be an homogeneous function of degree 2 with respect to the standard dilation.

For a given $\sigma > 1$, we choose $\psi_1(s) = (2\sigma-1)s,\ \psi_2(s) = e^{s^2} - 1$ and $p(s) = 2s$ $\forall s > 0$. It is clear that $\psi_1$ and $\psi_2$ satisfy the condition (3.1) and $p$ satisfies that $p(V(x(t),\theta)) \geq V(x(\xi),\theta)$, for all $t \geq 0,\ \theta \in \Omega$ and $\xi \in [t-\tau,t]$.

For all $x \in \mathbb{R}^n \setminus \{0\}$, the derivative of $V$ along the trajectories of the time delay system $(S_2)$ is

$$\langle \nabla V(x,\theta), f(x(t), x(t-\tau), \theta)\rangle = -4(2-\theta)^2 x^4 < 0.$$

In addition $\langle \nabla V(x,\theta), f(x,y,\theta)\rangle = 0$ implies that $x = 0$.

Then by the theorem 3.10, the system $(S_1)$ is locally uniformly asymptotically stable with respect to $\theta$.

The following figure illustrates the above example for $\theta = \frac{-\pi}{2}$, $\tau = 0.1$ and $t \in [0,1]$.

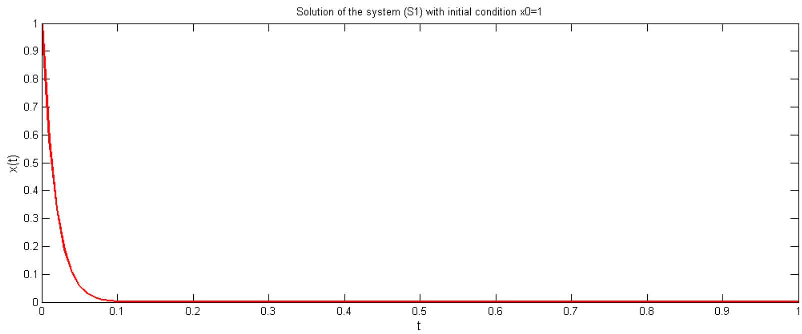


**Fig. 1.**

**Example 5.2.** Consider the following perturbed delay system depending on a parameter:

$$(S_4)\begin{cases} \dot{x}_1(t) = -x_1^3(t) - x_2(t)x_1^2(t-\alpha(t)) + \cos(\theta)x_1(t)x_2^2(t) + 2\theta x_1^3(t-\alpha(t)) \\ \qquad +(\cos(\theta) + (1+\theta)^2)x_3^4(t) \\ \dot{x}_2(t) = x_1(t)x_1^2(t-\alpha(t)) - x_2^3(t) - \cos(\theta)x_1^2(t)x_2(t) + \dfrac{x_2^2(t-\alpha(t))}{(\theta-1)^2} \\ \qquad +(2-\sin(\theta))^2 x_2^3(t)x_1(t) \\ \dot{x}_3(t) = -x_3^5(t) + \sqrt{(\sin^2(\frac{\theta}{2})+2)^3 x_3^2(t-\alpha(t))} + (\theta+1)^2 x_1^4(t) \end{cases}$$

We can rewrite the system $(S_4)$ in the following form:

$$\dot{x} = f(x,y,\theta) + R(t,y,\theta) + P(t,x,\theta),$$

where

$$f(x,y,\theta)=\begin{pmatrix} -x_1^3-x_2y_1^2+\cos(\theta)x_1x_2^2\\ x_1y_1^2-x_2^3-\cos(\theta)x_1^2x_2\\ -x_3^5 \end{pmatrix},\ R(t,y,\theta)=\begin{pmatrix} 2\theta y_1^3\\ \\ \dfrac{y_2^2}{(\theta-1)^2}\\ \\ \sqrt{(\sin^2(\frac{\theta}{2})+2)^3y_3^2} \end{pmatrix},$$

$$P(t,x,\theta)=\begin{pmatrix} (\cos(\theta)+(1+\theta)^2)x_3^4\\ (2-\sin(\theta))^2x_2^3x_1\\ (\theta+1)^2x_1^4 \end{pmatrix},$$

where $x=(x_1,x_2,x_3)\in\mathbb{R}^3$, $y=(y_1,y_2,y_3)\in\mathbb{R}^3$, $\lambda>0$ and $\theta\in[-\frac{1}{2},\frac{1}{2}]=\Omega$.

The function $f$ is homogeneous of degree 2 with respect to the dilation $\Lambda_\lambda(x)=(\lambda x_1,\lambda x_2,\lambda^{\frac{1}{2}}x_3)$.

Let $V(x,\theta)=\frac{1}{2}x_1^2+\frac{1}{2}x_2^2+\frac{1}{4}x_3^4$ which is a $C^1$-Lyapunov function and homogeneous of degree 2 with respect to the dilation $\Lambda$.

Let $\psi_1$, $\psi_2$ and $p$ be continuous and nondecreasing functions such that for a given $\sigma>1$, $\psi_1(s)=(\sigma-1)s$, $\psi_2(s)=(2\sigma-1)s^2$ and $p(s)=2s$, $\forall s>0$.

It is clear that $\psi_1$ and $\psi_2$ satisfy the condition (3.1) and $p(V(x(t),\theta))\geq V(x(\xi),\theta)$, for $t\geq0$, $\theta\in[-\frac{1}{2},\frac{1}{2}]$ and $\xi\in[t-\tau,t]$.

One has $\langle\nabla V(x,\theta),f(x,y,\theta)\rangle=-x_1^4-x_2^4-x_3^8\leq0$ and $\langle\nabla V(x,\theta),f(x,y,\theta)\rangle=0$ implies that $x_1=x_2=x_3=0$. Therefore, the assumption $A2$ is fulfilled.

As $r=(1,1,\frac{1}{2})$, the homogeneous norm is given by $\rho(y)=(|y_1|^\epsilon+|y_2|^\epsilon+|y_3|^{2\epsilon})^{\frac{1}{\epsilon}}$ for $\epsilon\geq1$, $y\in\mathbb{R}^3$ and as we known that for $x\in\mathbb{R}^3\setminus\{0\}$ there exists a unique $z\in S^1$ such that $x=\Lambda_\lambda(z)=(\lambda z_1,\lambda z_2,\lambda^{\frac{1}{2}}z_3)$.

Let $\tilde{\mu}=3$ and $\Gamma=(|1+\theta|,4(\theta+1),(|\frac{\theta}{2}|+2)^2)$, then we deduce that $R$ and $P$ satisfy assumptions $A3$ and $A4$ for $t\geq0$, $\lambda>0$ and $\theta\in[-\frac{1}{2},\frac{1}{2}]$.

By applying to theorem 4.1, one has that the system $(S_4)$ is locally uniformly asymptotically stable with respect to $\theta$.

## 6. CONCLUSION

In this paper, we have studied the asymptotic stability of time varying delay systems with presence of varying parameter. This work has been improved basically by the use of Razumikhin theorem and the homogeneous property of the considered system. The major importance of the resulting theorems is that we study the stability behavior of the time delay system depending on a parameter, which is in a compact set, by introducing hypothesis on a compact set $S^1$. Moreover, we have studied the stability of perturbed time delay systems depending on a parameter assuming some inequalities on the perturbation term. We finish by numerical examples which illustrate the given results.

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