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CONSTRUCTION METHODS FOR GAUSSOIDS

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The number of n-gaussoids is shown to be a double exponential function in n. The necessary bounds are achieved by studying construction methods for gaussoids that rely on prescribing 3-minors and encoding the resulting combinatorial constraints in a suitable transitive graph. Various special classes of gaussoids arise from restricting the allowed 3-minors.

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1. INTRODUCTION

Gaussoids are combinatorial structures that encode independence among Gaussian random variables, similar to how matroids encode independence in linear algebra. They fall into the larger class of *CI structures* which are arbitrary sets of conditional independence statements. The work of Fero Matúš is in particular concerned with special CI structures such as graphoids, pseudographoids, semigraphoids, separation graphoids, etc. In his works Matúš followed the idea that conditional independence can be abstracted away from concrete random variables to yield a combinatorial theory. This should happen in the same manner as matroid theory abstracts away the coefficients from linear algebra. His work [7] on minors of CI structures displays the inspiration from matroid theory very clearly.

In 2007, Lněnička and Matúš defined gaussoids [5] of dimension n as sets of symbols $(ij|K)$, denoting conditional independence statements, which satisfy the following Boolean formulas, called the *gaussoid axioms*:

$$(ij|L) \wedge (ik|jL) \Rightarrow (ik|L) \wedge (ij|kL), \tag{G1}$$

$$(ij|kL) \wedge (ik|jL) \Rightarrow (ij|L) \wedge (ik|L), \tag{G2}$$

$$(ij|L) \wedge (ik|L) \Rightarrow (ij|kL) \wedge (ik|jL), \tag{G3}$$

$$(ij|L) \wedge (ij|kL) \Rightarrow (ik|L) \vee (jk|L), \tag{G4}$$

for all distinct $i, j, k \in [n]$ and $L \subseteq [n] \setminus ijk$. Here and in the following, we use the efficient "Matúš set notation" where union is written as concatenation and singletons are written without curly braces. For example, ijk is shorthand for $\{i\} \cup \{j\} \cup \{k\}$.

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A gaussoid is *realizable* if its elements are exactly the conditional independence statements that are valid for some n-variate normal distribution. Realizability was characterized for $n = 4$ in [5] and a characterization for $n = 5$ is open. There is no general forbidden minor characterization for realizability of gaussoids [12, 14]. We therefore think about gaussoids as *synthetic conditional independence* in the sense of Felix Klein [4, Chapter V]. This view is inspired by the parallels to matroid theory. The algebra and geometry of gaussoids was developed with this in mind in [1]. Gaussoids are also the singleton-transitive compositional graphoids according to [11, Section 2.3].

In the present paper we view gaussoids as structured subsets of 2-faces of an n-cube, which is possible because the symbols $(ij|K)$ exactly index those faces. This point of view on CI statements was taken before in [3]. It readily simplifies the definition of a gaussoid, but it has several additional advantages. For example, it makes the formation of minors more effective, as this now corresponds to restricting to faces of the cube. To start, consider the usual 3-dimensional cube. A *knee* in the cube consists of two squares that share an edge. A *belt* consists of all but two opposing squares of the cube. We give a combinatorial definition of a gaussoid that uses knees and belts. It is equivalent to the original definition in [5] since both definitions only specify how a gaussoid looks in a 3-face of the n-cube and there both definitions yield the exact same 11 possibilities appearing in Figure 2.

Definition 1.1. An n-gaussoid is a set $\mathcal{G}$ of 2-faces of the n-cube such that for any 3-face C of the n-cube it holds:

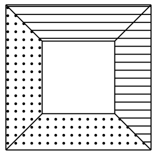
1. If $\mathcal{G}$ contains a knee of C, then it also contains the belt that contains that knee.
2. If $\mathcal{G}$ contains two opposing faces of C, then it also contains a belt that contains these two faces.

The *dimension* of $\mathcal{G}$ is n, the dimension of the ambient cube. $\mathfrak{G}_n$ is the set of n-dimensional gaussoids and $\mathfrak{G} := \bigcup_{n\geq 3} \mathfrak{G}_n$ the set of all gaussoids.

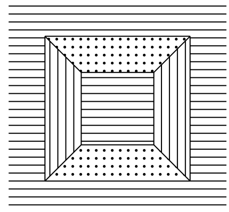
This definition is illustrated in Figure 1 where the premises are dot- and the conclusions are line-shaded. The "or" condition of the gaussoid axiom (G4) is reflected in the second item of the definition as there are two belts through two opposing faces of a 3-cube. As with the gaussoid axioms, Definition 1.1 applies certain closure rules in every 3-face of the n-cube, but whereas S_3 acts on the axes of the cube in the gaussoid axioms, the group acting on the two pictures in Figure 1 is the full symmetry group of the 3-cube, B_3. This bigger group conflates the first three axioms into the first picture.

There is a small caveat in our definition of dimension of a gaussoid as it equals that of the ambient cube. Due to this choice, the empty set, for example, can represent gaussoids of different dimension, just as there is a trivial matroid for every ground set. Usually, this will cause no confusion but it underlines the importance of keeping in mind that a gaussoid is always a *subset* of squares in some ambient cube.

The gaussoid axioms and also Definition 1.1 only work with 3-cubes. This locality can be expressed as in Lemma 3.3: For any $k \geq 3$, being an n-gaussoid is equivalent to all restrictions to k-faces being k-gaussoids. The aim of this work is to explore *gaussoid puzzling*, the reversal of this idea, that is, constructing n-gaussoids by prescribing their k-gaussoids. The implementation hinges on an understanding of how exactly the k-faces of



(G1)—(G3): Any knee in the cube is completed to the unique belt which contains it.



(G1)—(G3) $\circ$ (G4): Two opposite squares are completed to (at least) one of the two belts which contain them.

Fig. 1. The gaussoid axioms in the 3-cube. Premises of the axioms are dotted, conclusions are indicated by lines. Conclusions appearing together have matching line patterns. The pictures encode the gaussoid axioms mod B_3, the symmetry group of the 3-cube.

the n-cube intersect, because these intersections are obstructions to the free specification of k-gaussoids. In Section 3 we encode these obstructions in a graph and then Brooks's theorem gives access to large independent sets, where gaussoids can be freely placed. This yields a good estimate of the number of gaussoids in Theorem 3.12.

As gaussoids are defined by puzzling it is natural to ask what additional structures emerge if one restricts the available pieces. We do this in Section 4 where we explore classes of "special gaussoids" that arise by restricting the puzzling of 3-gaussoids to S_3-invariant subsets of the 11 available pieces. Consequently all of these classes can be axiomatized with S_n-invariant axioms that prescribe only the structure of 3-cubes (like the gaussoid axioms). We find natural connections to the theory of graphical models and other nice interpretations.

2. THE CUBE

Consider the face lattice $\mathcal{F}^n$ of the n-cube. This lattice contains $\emptyset$, the unique face of dimension $-\infty$. To specify a face of non-negative dimension k, one needs to specify the k dimensions in which the face extends, and then the location of the face in the remaining $n-k$ dimensions. We employ two natural ways to work with faces. The first is *string notation.* In this notation a face F is an element of $\{\mathtt{0},\mathtt{1},*\}^n$ where the $*$s indicate dimensions in which the face extends and the remaining binary string determines the location; a $\mathtt{1}$ at position p means that the face is translated along the pth axis inside the cube. This string notation naturally extends the binary string notation for the vertices of the n-cube: if $F \in \{\mathtt{0},\mathtt{1},*\}^n$, then its vertices are

$$\{a \in \{\mathtt{0},\mathtt{1}\}^n : a_i = F_i \text{ whenever } F_i \neq *\}\,.$$

The second choice is *set notation*. In this notation, a face $F = (I_F|K_F)$ of dimension $k = |I_F|$ is specified by two sets $I_F \subseteq [n]$ and $K_F \subseteq [n]\setminus I_F$, where $I_F = \{i \in [n] : F_i = *\}$ and $K_F = \{i \in [n] : F_i = \mathtt{1}\}$. In this context the symbol $\oplus$ denotes symmetric difference and $\overline{L}$ denotes the complement of L in $[n]$.

The set of k-faces of the n-cube is $\mathcal{F}^n_k$. As in [1], the squares of the n-cube are denoted by $\mathcal{A}_n := \mathcal{F}^n_2$. Of special interest in this article are also the 3-faces $\mathcal{C}_n := \mathcal{F}^n_3$. The constructions in Section 3 based on Lemma 3.3 frequently exploit the following

Fact. For $3 \le k \le m$, a k-face shares at most $\binom{k-1}{2}2^{k-3}$ squares with an m-face or is already included in it. In particular for $k = 3$, if a cube shares more than a single square with an m-face, then it is already contained in it.

Minors are important in matroid theory and gaussoid theory. When a simple matroid is represented as the geometric lattice of its flats, a minor corresponds to an interval of the lattice [18, Theorem 4.4.3], which is again a geometric lattice. For gaussoid minors the lattice is replaced by the set of squares in the hypercube and the lattice intervals are replaced by hypercube faces.

Minors for arbitrary CI structures have been studied for example in [7]. There, a *minor* of a CI structure $\mathcal{A} \subseteq \mathcal{A}_n$ is obtained by choosing two disjoint sets $L, M \subseteq [n]$ and performing *restriction* to LM followed by *contraction* of $\overline{L}$. Writing $\mathcal{A}_L = \{(ij|K) \in \mathcal{A}_n : ijK \subseteq L\}$ these are

$$\operatorname{contr}_L\mathcal{A} = \{(ij|K) \in \mathcal{A}_L : (ij|K\overline{L}) \in \mathcal{A}\} \subseteq \mathcal{A}_L,$$
$$\operatorname{restr}_L\mathcal{A} = \mathcal{A} \cap \mathcal{A}_L \subseteq \mathcal{A}_L.$$

In [1], minors were also defined specifically for gaussoids using statistical terminology with an emphasis on the parallels to matroid theory. A minor is every set of squares arising from a gaussoid via any sequence of *marginalization* and *conditioning*:

$$\operatorname{marg}_L\mathcal{A} = \{(ij|K) \in \mathcal{A} : L \subseteq \overline{ijK}\} \subseteq \mathcal{A}_{\overline{L}},$$
$$\operatorname{cond}_L\mathcal{A} = \{(ij|K) \in \mathcal{A}_{\overline{L}} : (ij|KL) \in \mathcal{A}\} \subseteq \mathcal{A}_{\overline{L}}.$$

These operations are dual to the ones defined by Matúš: $\operatorname{cond}_L = \operatorname{contr}_{\overline{L}}$ and $\operatorname{marg}_L = \operatorname{restr}_{\overline{L}}$. Furthermore, either operation can be the identity, $\operatorname{restr}_{[n]} = \mathrm{id}$ and $\operatorname{contr}_{[n]} = \mathrm{id}$, and finally, the two sets L and M in Matúš' definition of minor can be decoupled: $\operatorname{contr}_L\operatorname{restr}_{LM} = \operatorname{restr}_L\operatorname{contr}_{\overline{M}}$. Thus both notions of minor coincide.

Our aim is to provide a geometric intuition for the act of taking a gaussoid minor. A face $(L|M)$ of the n-cube is canonically isomorphic to the L-cube by deleting from the $[n]$-cube $\{\mathtt{0},\mathtt{1}\}^{[n]}$ all coordinates outside of L. This deletion is a lattice isomorphism $\pi_{(L|M)} : \mathcal{F}^n \cap (L|M) \leftrightarrow \mathcal{F}^L$, with the face lattice $\mathcal{F}^L$ of an $|L|$-dimensional cube. We can interpret taking the minor $\operatorname{restr}_L\operatorname{cond}_M$ as an operation in the hypercube.

Proposition 2.1. Let $\mathcal{A} \subseteq \mathcal{A}_n$, then $\operatorname{restr}_L\operatorname{cond}_M\mathcal{A} = \pi_{(L|M)}(\mathcal{A} \cap (L|M))$.

Proof. Take $(ij|K') \in \operatorname{restr}_L\operatorname{cond}_M\mathcal{A}$. Then ij and K' can be seen as subsets of $[n]$ and they satisfy $ijK' \subseteq L$ and $(ij|K'M) \in \mathcal{A}$. From this it is immediate that $ij \subseteq L$

and $K'M \subseteq LM$. Furthermore, $\overline{ijK'M} = \overline{ijK'} \cap \overline{M} \subseteq L\overline{LM}$, hence $(ij|K'M) \subseteq (L|M)$ and $(ij|K') \in \pi_{(L|M)}(\mathcal{A} \cap (L|M))$.

In the other direction, suppose that $(ij|K') \in \pi_{(L|M)}(\mathcal{A} \cap (L|M))$ and let $(ij|K)$ be its preimage under $\pi_{(L|M)}$. Then $(ij|K) \in \mathcal{A} \cap (L|M)$ and it follows $ij \subseteq L$, $K \subseteq LM$ and also $M \subseteq K$. Thus K decomposes into $K = K'M$ where naturally $K' \cap M = \emptyset$. This proves that $(ij|K') \in \mathrm{restr}_L \mathrm{cond}_M \mathcal{A}$. □

Proposition 2.1 compactly encodes the definitions of minor. The following definition introduces notation reflecting this as well as an opposite *embedding*, which mounts a set of squares from the L-cube into an $|L|$-dimensional face of a higher hypercube.

Definition 2.2. (1) For a set $\mathcal{A} \subseteq \mathcal{A}_n$ and $(L|M) \in \mathcal{F}_k^n$, the $(L|M)$*-minor* of $\mathcal{A}$ is the set $\mathcal{A}{\downarrow}(L|M) := \pi_{(L|M)}(\mathcal{A} \cap (L|M)) \subseteq \mathcal{A}_L$. A k*-minor* is an $(L|M)$-minor with $|L| = k$.

(2) For a set $\mathcal{A} \subseteq \mathcal{A}_L$ and $(L|M) \in \mathcal{F}_k^n$, the *embedding* of $\mathcal{A}$ *into* $(L|M)$ is the preimage $\mathcal{A}{\uparrow}(L|M) := \pi_{(L|M)}^{-1}(\mathcal{A}) \subseteq \mathcal{A}_n$.

3. GAUSSOID PUZZLES

Several theorems in matroid theory concern the (impossibility of a) characterization of classes of matroids in terms of forbidden minors. For CI structures such as gaussoids the definitions read as follows.

Definition 3.1. (1) A class $\mathfrak{A} \subseteq \bigcup_n 2^{\mathcal{A}_n}$ of sets of squares is *minor-closed* if with $\mathcal{A} \in \mathfrak{A}$ all minors of $\mathcal{A}$ belong to $\mathfrak{A}$.

(2) A set of squares $\mathcal{X}$ is a *forbidden minor* for a minor-closed class $\mathfrak{A}$ if it is minimal with the property that it does not belong to $\mathfrak{A}$, in the sense that all its proper minors do belong to $\mathfrak{A}$.

(3) The k-dimensional structures in a minor-closed class $\mathfrak{A}$ are its *compulsory* k*-minors*.

It is easy to see that gaussoids are minor-closed, i. e. any k-minor of an n-gaussoid is always a k-gaussoid. But even more is true: given any set of squares in the n-cube, if all of its k-minors, for any $k \geq 3$, are k-gaussoids, then the whole is an n-gaussoid. This claim is proved in Lemma 3.3. The present section uses this property to construct gaussoids by prescribing their k-minors. Section 4 investigates subclasses of gaussoids which have the same anatomy. We formalize this property in

Definition 3.2. A class $\mathfrak{A} = \bigcup_{n \geq n_0} \mathfrak{A}_n$ of sets of squares stratified by dimension, i. e. $\mathfrak{A}_n \subseteq 2^{\mathcal{A}_n}$, has a *puzzle property* if it is minor-closed and its nth stratum is generated via embeddings from the strata below n, i. e. if for some $\mathcal{A} \subseteq \mathcal{A}_n$ all its k-minors, $k < n$, are in $\mathfrak{A}_k$, then already $\mathcal{A} \in \mathfrak{A}_n$. The lowest stratum $\mathfrak{A}_{n_0}$ is the *basis* of $\mathfrak{A}$ and the puzzle property is *based* in dimension n_0.

Lemma 3.3. The set of gaussoids has a puzzle property based in dimension 3, whose basis are the eleven 3-gaussoids.

P r o o f. Let $\mathcal{G} \subseteq \mathcal{A}_n$ and $3 \le k \le n$. We show that $\mathcal{G}$ is an n-gaussoid if and only if $\mathcal{G} \downarrow D$ is a k-gaussoid for every $D \in \mathcal{F}_k^n$. First consider the case $k = 3$. The gaussoid axioms are quantified over arbitrary cubes $(ijk|L)$ together with an order on the set ijk and each axiom refers to squares inside the cube $(ijk|L)$ only. Confined to this cube, the axioms state precisely that this 3-minor is a 3-gaussoid. The case of $k > 3$ is reduced to the statement for $k = 3$. Indeed, all 3-minors of $\mathcal{G}$ are gaussoids if and only if all 3-minors of k-minors of $\mathcal{G}$ are gaussoids, because those two collections of minors both arise from the same set $\mathcal{C}_n$ of cubes of the n-cube. □

Turning Definition 3.2 upside down, the construction of an n-gaussoid can be seen as a high-dimensional jigsaw puzzle. The puzzle pieces are lower-dimensional gaussoids which are to be embedded into faces of the n-cube. The difficulty comes from the fact that every square is shared by $\binom{n-2}{k-2}$ k-faces. The minors must be chosen so that all of them agree on whether a shared square is an element of the n-gaussoid under construction or not. The incidence structure of k-faces in the n-cube is important. We study it via the following graph.

Definition 3.4. Let $Q(n, k, p, q)$, for $n \ge k \ge p \ge q$, be the undirected simple graph with vertex set $\mathcal{F}_k^n$ and an edge between $D, F \in \mathcal{F}_k^n$ if and only if there is a p-face S such that $\dim(D \cap S) \ge q$ and $\dim(F \cap S) \ge q$.

The idea behind this definition is that for suitable choices of p and q, the faces indexed by an *independent set* in these graphs will be just far enough away from each other in the n-cube to allow *free puzzling* of k-gaussoids without one minor choice creating constraints for other minors.

Theorem 3.5. The graph $Q(n, k, p, q)$ is transitive, hence regular. It is complete if and only if $n + q \le p + k$. The degree of any vertex can be calculated as follows:

$$\deg Q(n, k, p, q) = -1 + \sum_{m,j\ (\dagger)} \binom{k}{j} 2^{k-j} \binom{n-k}{k-j} \binom{n-2k+j}{m},$$

where the sum extends over $0 \le m \le n - k$ and $0 \le j \le k$ which satisfy the feasibility and connectivity conditions

$$n - 2k + j \ge m \quad \wedge \quad p \ge m + 2q - \min\{q, j\}. \tag{$\dagger$}$$

P r o o f. The symmetry group B_n acts on the n-cube as automorphisms of the face lattice. The group action is transitive on k-faces for any k and respects meet and join. Therefore B_n is a subgroup of the automorphisms of and acts transitively on the graph $Q(n, k, p, q)$.

The characterization of completeness rests on Lemma 3.6. Using the gap function ϱ_q defined there, it is shown that $\varrho_q(D, F) \le p$ is equivalent to the adjacency of D and F in $Q(n, k, p, q)$ and that if F' is a face with smaller gap, then F' is adjacent to D. Since $Q(n, k, p, q)$ is regular, it is complete if and only if some vertex is adjacent to all others. For that to happen, the vertex must be adjacent to one which has the largest gap to it.

As shown in the lemma, the maximum of ϱ_q is $n - k + q$ and hence completeness is equivalent to $n - k + q \le p$.

The exact degree also follows from Lemma 3.6. Fix any vertex D of $Q(n, k, p, q)$. By regularity it suffices to count the adjacent vertices F of D. We subdivide vertices F according to two parameters: $m = |(K_D \oplus K_F) \setminus I_D I_F|$ is a disagreement between D and F and $j = |I_D \cap I_F|$ is the number of common dimensions of D and F. A priori, m ranges in $[n - k]$ and j ranges in $[k]$, but not all combinations allow F to be a k-face adjacent to D. First, we determine the pairs (m, j) for which an adjacent k-face exists and then count how many of them exist for fixed parameters. Let $(m, j) \in [n - k] \times [k]$. For $j = |I_D \cap I_F|$ it must hold that $n \ge 2k - j$, since D and F are k-faces. Assuming this, F can be constructed if and only if the $k - j$ dimensions in $I_F \setminus I_D$ leave enough space to create the prescribed disagreement of size m. As an inequality this is $n - k \ge m + (k - j)$, or $n - 2k + j \ge m$. Together with $m \ge 0$, this inequality already entails the condition $n \ge 2k - j$ imposed by the choice of j. Thus it is sufficient to require $n - 2k + j \ge m$, which is the first condition in (†). Given a k-face F with parameters m and j, the existence of an edge between D and F in $Q(n, k, p, q)$ imposes the condition Lemma 3.6 (1), which is the right half of (†).

As for the counting, let D be a fixed k-face and let $(m, j) \in [n - k] \times [k]$ satisfy (†). We count the k-faces F with parameters m and j. There are $\binom{k}{j}$ ways to place the $*$s for $I_F \cap I_D$. On $I_D \setminus I_F$, there are $k - j$ independent choices from $\{\mathtt{0}, \mathtt{1}\}$. The choices so far fix F in I_D. There are now $\binom{n-k}{k-j}$ choices for the remaining $*$s in $I_F \setminus I_D$. Then I_F is fixed. Now to finish F, we may only place `0`s and `1`s in $[n] \setminus I_D I_F$ where D has only `0`s and `1`s as well. Among the remaining $n - 2k - j$ positions, a set of size m must be chosen, where F is already determined by the condition that it differs from D. On the remaining $n - 2k - j - m$ positions, F is determined by not differing from D. The feasibility of all the choices enumerated so far is guaranteed by (†). The tally is

$$\sum_{m,j\ (\dagger)} \binom{k}{j} 2^{k-j} \binom{n-k}{k-j} \binom{n-2k+j}{m}.$$

Since D is not adjacent to itself, which is uniquely described by the feasible parameters $j = k$ and $m = 0$, subtracting 1 concludes the proof. □

Lemma 3.6. Let D, F be k-faces and $\varrho_q(D, F) := m + 2q - \min\{q, j\}$, with $j = |I_D \cap I_F|$ and $m = |(K_D \oplus K_F) \setminus I_D I_F|$. The following hold:

(1) $\varrho_q(D, F) \le p$ if and only if D and F are adjacent in $Q(n, k, p, q)$,

(2) the range of ϱ_q is $[q, n - k + q]$,

(3) ϱ_q is strictly isotone with respect to q, i. e. $\varrho_q < \varrho_{q+1}$,

(4) for $D, D', F \in \mathcal{F}_k^n$ with $\varrho_q(D, D') \le \varrho_q(D, F)$, if D and F are adjacent in $Q(n, k, p, q)$, then so are D and D'.

Proof. Given two k-faces D and F, the ground set $[n]$ splits into three sets: (i) $(K_D \oplus K_F) \setminus I_D I_F$ of cardinality m where both have `0` and `1` symbols only but differ, (ii) $I_D \cap I_F$

of cardinality j of shared $*$ symbols, and (iii) everything else, i. e. positions where `0` and `1` patterns agree or where `0` and `1` are in one face and $*$ in the other. In order to connect two k-faces in $Q(n,k,p,q)$, there needs to be a p-face which intersects either of them in at least dimension q. Such a face has to cover the set of size m with $*$s, as otherwise it will not intersect both faces. Conversely, once m is covered, a 0-dimensional intersection with both faces is ensured by placing `0`s and `1`s appropriately. To achieve a q-dimensional intersection, q $*$s have to be placed on I_D and I_F each. By using the j shared $*$s, one needs at least $2q - \min\{q,j\}$ further $*$s to construct a connecting p-face. Thus $\varrho_q(D,F)$ is the minimum dimension p necessary to connect D and F in $Q(n,k,p,q)$. This proves claim (1).

It is clear that ϱ_q is minimal when m is minimal and j is maximal. This can be achieved simultaneously by choosing $F = D$, in which case $\varrho_q(D,D) = q$. Now consider the opposing face $D^\circ = (I_D, [n] \setminus K_D I_D)$ of D. The gap is $\varrho(D,D^\circ) = n - |I_D| + 2q - \min\{q, |I_D|\} = n - k + q$ assuming D is a vertex of $Q(n,k,p,q)$ where in particular $|I_D| = k \geq q$. Increasing this value would require reducing j since m is already maximal. Un-sharing $*$s with D consumes positions inside the block of `0`s and `1`s in d of size $n-k$ which reduces m by an equal amount. Hence $n-k+q$ is maximal. Furthermore, by varying m but keeping $j = k$, all values in the range $[q, n-k+q]$ can be attained, proving claim (2).

Claim (3) follows from a straightforward calculation:

$$\begin{aligned}\varrho_{q+1}(D,F) - \varrho_q(D,F) &= 2 - (\min\{q+1,j\} - \min\{q,j\}) \\ &= \begin{cases} 2, & j \leq q, \\ 1, & j \geq q+1. \end{cases}\end{aligned}$$

In the situation of claim (4), since D and F are adjacent in $Q(n,k,p,q)$, we have $\varrho_q(D,D') \leq \varrho_q(D,F) \leq p$ by (1). Applying (1) again in reverse proves the claim. □

Corollary 3.7.

(1) $Q(n,3,2,2)$ is complete for $n \leq 3$. Otherwise its degree is $6(n-3) \leq 6(n-2)$.

(2) $Q(n,3,3,2)$ is complete for $n \leq 4$. Otherwise its degree is $12(n-3)(n-4) + 7(n-3) \leq 12(n-1)(n-2)$.

Remark 3.8. For the theory of gaussoids the cases $k = 3, p = 2,3, q = 2$ are relevant. We consider it an interesting problem to study growth of the degree formula for other parameters. Certainly the graph can be complete, where the degree is as large as $\binom{n}{k}2^{n-k}$. To construct large independent sets, one wants smaller degrees. It is proved below that a maximal independent set in $Q(n,3,3,2)$ has cardinality in $\Theta(n2^n)$ of which one inequality follows from the degree formula.

Proposition 3.9. Let $\mathcal{F}$ be an independent set in $Q(n,k,3,2)$, then the following inequality holds: $|\mathfrak{G}_n| \geq |\mathfrak{G}_k|^{|\mathcal{F}|}$.

P r o o f. Let $D,F \in \mathcal{F}$. Since $\mathcal{F}$ is independent, there is no 3-cube sharing a square with D and with F. Since $k \geq 3$, also D and F share no square. Thus an assignment of

k-gaussoids $\alpha : \mathcal{F} \to \mathfrak{G}_k$ lifts to a well-defined set of squares $\mathcal{G} := \bigsqcup_{D \in \mathcal{F}} \alpha D \uparrow D \subseteq \mathcal{A}_n$. The map $\alpha \mapsto \mathcal{G}$ is injective.

To see that $\mathcal{G}$ is a gaussoid, we examine its 3-minors. Let $C \in \mathcal{C}_n$ be arbitrary. In case C is fully contained in some $D \in \mathcal{F}$, then clearly $\mathcal{G} \downarrow C = (\alpha D \uparrow D) \downarrow C \in \mathfrak{G}_3$ since $\alpha D \in \mathfrak{G}_k$. Otherwise C can share at most one square with any face in $\mathcal{F}$. If it shares no square with any element of $\mathcal{F}$, then $\mathcal{G} \downarrow C$ is empty, hence a gaussoid. Otherwise it shares a square with some face in $\mathcal{F}$ and thus cannot share a square with any other element of $\mathcal{F}$ because $\mathcal{F}$ is an independent set in $Q(n, k, 3, 2)$. In this case, $\mathcal{G} \downarrow C$ is a singleton or empty and hence a gaussoid. □

Proposition 3.10. Let $\mathcal{F}$ be an independent set in $Q(n, k, 2, 2)$ and c the maximum size of a set of mutually range-disjoint injections of $\mathfrak{G}_k$ into $2^{\mathcal{A}_k} \setminus \mathfrak{G}_k$. Then $\frac{2^{|\mathcal{A}_n|}}{|\mathfrak{G}_n|} \geq c^{|\mathcal{F}|}$.

P r o o f. The proof is analogous to Proposition 3.9 but uses the independent set to perturb any gaussoid injectively into $c^{|\mathcal{F}|}$ non-gaussoids. Again, since $q = 2$ and $\mathcal{F}$ is independent, an assignment $\alpha : \mathcal{F} \to 2^{\mathcal{A}_k}$ lifts uniquely via $\uparrow$ to a subset of $\mathcal{A}_n$. Let $\{f_i\}_{i \in [c]}$ be a set of range-disjoint injections as in the claim. Consider the maps $\alpha' : \mathcal{F} \to [c]$. To each $\mathcal{G} \in \mathfrak{G}_n$ associate $H_{\alpha'} := \bigsqcup_{D \in F} f_{\alpha' D}(\mathcal{G} \downarrow D) \uparrow D \subseteq \mathcal{A}_n$.

Because the ranges of the f_i are disjoint, the map $(\mathcal{G}, \alpha') \mapsto H_{\alpha'}$ is injective. None of the sets $H_{\alpha'}$ is a gaussoid since any $D \in \mathcal{F}$ certifies $H_{\alpha'} \downarrow D = f_{\alpha' D}(\mathcal{G} \downarrow D) \notin \mathfrak{G}_k$. □

Remark 3.11. The proofs of Propositions 3.9 and 3.10 exploit two properties of the class of gaussoids: (1) it has a puzzle property, and (2) the empty set and all singletons are in its basis. The same technique does not work for realizable gaussoids because they lack property (1) and not for graphical gaussoids (see Section 4) because they lack property (2). Indeed their numbers can be shown to be single exponential. For realizable gaussoids, this follows from Nelson's recent breakthrough: If a gaussoid is realizable with a positive-definite $n \times n$ covariance matrix Σ, then the $n \times 2n$ matrix $(I_n \, \Sigma)$ both defines a vector matroid and identifies the gaussoid. By [8, Theorem 1.1] there are only exponentially many realizable matroids and thus realizable gaussoids. Nelson's bound features a cubic polynomial in the exponent, while there are certainly $2^{\binom{n}{2}}$ realizable gaussoids coming from graphical models.

To get explicit bounds we apply the propositions for $k = 3$. To find suitable independent sets in $Q(n, 3, 3, 2)$ and $Q(n, 3, 2, 2)$ we use Brooks's Theorem [6] and the degree bounds from Corollary 3.7. Since the graphs are connected, have degree at least 3 but are not complete, there exists a proper $\deg Q(n, 3, 3, 2)$-coloring of $Q(n, 3, 3, 2)$, and we can pick a color class as an independent set $\mathcal{F}$. Its size is at least that of an average color class:

$$\frac{|\mathcal{F}_3^n|}{\deg Q(n, 3, 3, 2)} \geq \frac{n(n-1)(n-2)}{6 \cdot 12(n-1)(n-2)} 2^{n-3} = \frac{n}{6^2} 2^{n-4} = \frac{n}{9} 2^{n-6}.$$

For $Q(n, 3, 2, 2)$, we find analogously

$$\frac{|\mathcal{F}_3^n|}{\deg Q(n, 3, 2, 2)} \geq \frac{n(n-1)(n-2)}{6 \cdot 6(n-2)} 2^{n-3} = \frac{n(n-1)}{6^2} 2^{n-3} = \frac{n(n-1)}{9} 2^{n-5}.$$

Proposition 3.9 now shows, using $|\mathfrak{G}_3| = 11$ and $\log_2 11 \geq 3$, that there are at least $11^{\frac{n}{9}2^{n-6}} \geq 2^{\frac{n}{3}2^{n-6}}$ n-gaussoids. Similarly, Proposition 3.10 with $c = \left\lfloor \frac{64-11}{11} \right\rfloor = 4$ gives an upper bound on the ratio of n-gaussoids of $4^{\frac{n(n-1)}{9}2^{n-5}} = 2^{\frac{n(n-1)}{9}2^{n-4}}$. We have proved

Theorem 3.12. For $n \geq 5$, the number of n-gaussoids is bounded by

$$2^{\frac{1}{3}n2^{n-6}} \leq |\mathfrak{G}_n| \leq \frac{2^{|\mathcal{A}_n|}}{2^{\frac{4}{9}n(n-1)2^{n-6}}}.$$

Remark 3.13. A more direct way to obtain a weaker double exponential lower bound for the number of gaussoids was suggested to us by Peter Nelson, following a matroid construction of Piff and Welsh [10]. Let $\mathcal{R}_k$ be the set of all r-subsets S of $[n]$ for some $r < n$ such that $\sum_{i \in S} i \equiv k \pmod{n}$, for $0 \leq k < n$. We view $S \in \mathcal{R}_k$ as a 2-face $(ij|S \setminus ij)$ of the n-cube, where i, j are the minimal elements of S. In this way any subset of $\mathcal{R}_k$ defines a set of 2-faces which is vacuously a gaussoid: by construction, no gaussoid axiom has both of its premises in $\mathcal{R}_k$. Certainly k can be chosen so that $|\mathcal{R}_k| \geq \frac{1}{n}\binom{n}{r}$ and with $r = \lfloor n/2 \rfloor$ this gives at least $2^{\frac{1}{n}\binom{n}{r}} \in 2^{\Theta(n^{-3/2}2^n)}$ gaussoids.

Substituting $|\mathcal{A}_n| = \binom{n}{2}2^{n-2}$ in Theorem 3.12 gives an interval for the absolute number of n-gaussoids for $n \geq 5$. It shows $\log |\mathfrak{G}_n| \in \Omega(n2^n) \cap \mathcal{O}(n^2 2^n)$.

We conclude this section by showing that the $n2^n$ order lower bound is the best that the independent set construction in $Q(n,3,3,2)$ can do. The *independence number* $\alpha(G)$ of a graph G is the maximal size of an independent set in G. Similarly, the *clique number* $\omega(G)$ is the maximal size of a clique in G. Since $Q(n,3,3,2)$ is transitive, the following inequality holds [2, Lemma 7.2.2]:

$$\alpha(Q(n,3,3,2)) \leq \frac{|\mathcal{F}_3^n|}{\omega(Q(n,3,3,2))}.$$

Since $|\mathcal{F}_3^n| \in \Theta(n^3 2^n)$, it suffices to find a clique of size $\Omega(n^2)$ in every $Q(n,3,3,2)$. Take the set of cubes $\mathcal{J} := \{(1ij|) : ij \in \binom{[n]\setminus 1}{2}\}$. This set has cardinality $\binom{n-1}{2} \in \Theta(n^2)$ and any two elements $D = (1ij|)$, $F = (1kl|)$ in it are connected by an edge in $Q(n,3,3,2)$, since $\varrho_2(D,F) = m + 2 \cdot 2 - \min\{2,j\} = 4 - \min\{2,j\} \leq 3$ with $m = 0$ and $j \geq 1$.

4. SPECIAL GAUSSOIDS

Because of their puzzle property, gaussoids are the largest class of CI structures whose k-minors are k-gaussoids. The base case of this definition are the eleven 3-gaussoids arising from 3×3 covariance matrices of Gaussian distributions. The 3-gaussoids split into five symmetry classes modulo S_3 which we denote by letters `E`, `L`, `U`, `B`, and `F`. They are depicted in Figure 2.

The special S_n-invariant types of gaussoids in this section arise from choosing subsets of these five symmetry classes to base a puzzle property on. Each of the 32 sets of bases can be converted into axioms in the 3-cube similar to the gaussoid axioms (G1)—(G4). SAT solvers [16, 17, 13] were used on the resulting Boolean formulas to enumerate or count these classes and [15] was useful to create input files for the solver. The listings

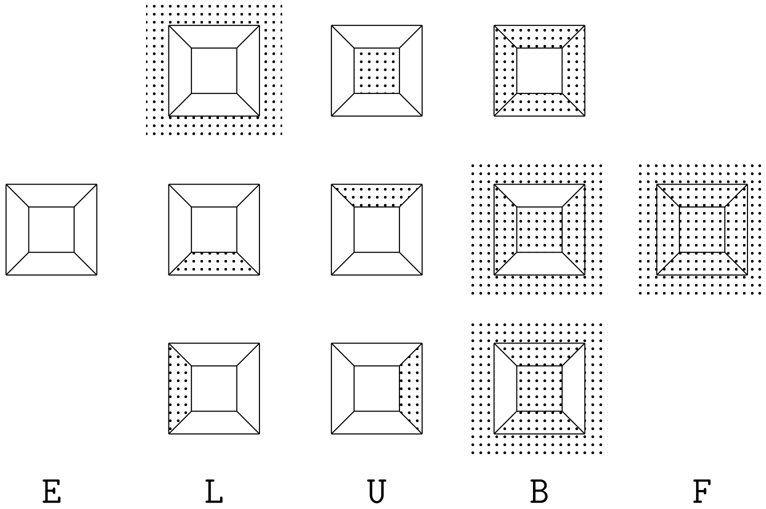


Fig. 2. The eleven 3-gaussoids in five symmetry classes mod S_3 organized in columns. From left to right: the empty gaussoid `E`, the lower singletons `L`, the upper singletons `U`, the belts `B` and the full gaussoid `F`.

can be found on our supplementary website gaussoids.de. For nine classes an entry in the OEIS [9] could be found. Table 1 is the main result of this section. It summarizes the different types of gaussoids that arise from the different bases.

The classes `E`, `B` and `F` are themselves closed under duality, while `L` and `U` are interchanged by it. It follows that any one of the 32 classes is invariant under duality if it contains either none of `L` and `U` or both of them. On the remaining classes, duality acts by swapping `L` with `U`. The combinatorial properties of the classes, e. g. the size, are unaffected by this action, hence `LB` and `UB` are conflated to {`L,U`}`B` in Table 1.

4.1. Fast-growing gaussoids

By Remark 3.11, the construction of doubly exponentially many members of a class of gaussoids requires that the class has a puzzle property and that its basis includes `ELU`. This explains the rapid growth of all four classes of this type.

4.2. Incompatible minors

As a consequence of Definition 3.2, if there is no gaussoid of dimension k in a class, there are no gaussoids of any dimension $\geq k$ in the class. Similarly, if the class contains only the empty or full gaussoid in dimension k, the members of dimension $\geq k$ are the empty or full gaussoid as well. Hence computations in small dimension suffice to explain these classes. Despite their simplicity, each of them provides higher compatibility axioms.

Name	Count in dim. $3, 4, 5, \ldots$	OEIS	Interpretation
Fast-growing			
ELUBF	11, 679, 60 212 776	—	Gaussoids
ELUB	10, 640, 59 348 930	—	—
ELUF	8, 522, 48 633 672	—	—
ELU	7, 513, 47 867 881	—	Required for Prop. 3.9
Incompatible			
LUB	9, 111, 0, 0	—	Vanishes for $n \geq 5$
LUF	7, 61, 1, 1	—	Only F for $n \geq 5$
LU	6, 60, 0, 0	—	Vanishes for $n \geq 5$
{L,U}B	6, 15, 0, 0	—	Vanishes for $n \geq 5$
{L,U}F	4, 1, 1, 1	—	Only F for $n \geq 4$
EF	2, 2, 2, 2	A007395	Only E or F for all n
Graphical			
E{L,U}BF	8, 64, 1 024, 32 768, 2 097 152	A006125	Undirected simple graphs
E{L,U}B	7, 41, 388, 5 789, 133 501	A213434	Graphs without 3-cycles
{L,U}BF	7, 34, 206, 1 486, 12 412	A011800	Forests of paths on $[n]$
E{L,U}F, EBF	5, 15, 52, 203, 877, 4 140	A000110	Partitions of $[n]$
E{L,U}, BF	4, 10, 26, 76, 232, 764, 2 620	A000085	Involutions on $[n]$
EB	4, 8, 16, 32, 64, 128, 256	A000079	Subsets of $[n-1]$
Exceptional			
LUBF	10, 142, 1 166, 12 796, 183 772, 3 221 660	—	—

Tab. 1. 26 classes of special gaussoids categorized into four types. The remaining six classes are described by one or zero letters of $\{\mathtt{E}, \mathtt{L}, \mathtt{U}, \mathtt{B}, \mathtt{F}\}$ and belong to the Incompatible type, as each of them is a subclass of a class found to be Incompatible.

For example the annihilation of LUB in dimension 5 implies that every 5-minor of a gaussoid contains an empty or a full 3-minor. Or: a graphical 4-gaussoid with no belts is full or contains an empty 3-minor.

4.3. Graphical gaussoids

Each undirected simple graph $G = ([n], E)$ defines a CI structure $\langle\!\langle G \rangle\!\rangle := \{(ij|K) \in \mathcal{A}_n : K \text{ separates } i \text{ and } j\}$, where two vertices i and j are *separated* by a set K if every path between i and j intersects K. These are the *separation graphoids* of [7]. They fulfill a localized version of the global Markov property. According to [5, Remark 2], separation graphoids are exactly the gaussoids satisfying the *ascension axiom*:

$$(ij|L) \Rightarrow (ij|kL), \qquad \forall\, i, j, k \in [n], L \subseteq [n] \setminus ijk. \tag{A}$$

Therefore we refer to them as *ascending gaussoids*. The operation $G \mapsto \langle\!\langle G \rangle\!\rangle$ is a bijection whose inverse recovers the graph via its edges $E = \{ij : (ij|*) \notin \langle\!\langle G \rangle\!\rangle\}$, where

$(ij|*)$ abbreviates $(ij|[n] \setminus ij)$. Any gaussoid in this section is of the form $\langle\!\langle G \rangle\!\rangle$ for some undirected simple graph G.

Since (A) uses only 2-faces of a single 3-face of the n-cube, being an ascending gaussoid is a puzzle property based in dimension 3. Its basis are the ascending 3-gaussoids. This was shown by Matúš [7, Proposition 2] and in our terminology it can be restated as follows

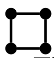
Lemma 4.1. A gaussoid is ascending if and only if L is a forbidden minor.

This shows that EUBF are the ascending gaussoids. Their duals are ELBF and it is easy to see that their axiomatization replaces (A) by the *descension axiom*

$$(ij|kL) \Rightarrow (ij|L), \qquad \forall\, i,j,k \in [n], L \subseteq [n] \setminus ijk. \tag{D}$$

EUBF-gaussoids arise from undirected graphs via vertex separation, i. e. $(ij|K) \in \langle\!\langle G \rangle\!\rangle$ if and only if i and j are in different connected components of $G \setminus K$. Their duals contain $(ij|K)$ if and only if i and j are in different connected components in the induced subgraph on ijK. Therefore we call elements of EUBF∪ELBF *graphical* gaussoids. For our classification purposes it is sufficient to study the "Upper" half of dual pairs.

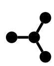
Our technique to understand EUBF and its subclasses has already been used in [7]: since the presence of an edge ij in G is encoded by the non-containment $(ij|*) \notin \langle\!\langle G \rangle\!\rangle$, the compulsory minors of $\langle\!\langle G \rangle\!\rangle$ of the form $\langle\!\langle G \rangle\!\rangle \downarrow (ijk|*)$ prescribe induced subgraphs on vertex triples ijk. In the opposite direction, however, the induced 3-subgraphs of a graph do not in general reveal the types of all minors $\langle\!\langle G \rangle\!\rangle \downarrow (ijk|L)$ in its corresponding gaussoid.

Example 4.2. Consider the cycle  corresponding to the gaussoid $\{(13|24), (24|13)\}$. Its 3-minors are exclusively E and U. The U minors arise precisely in the 3-cubes

$$\{1{*}{*}{*}\}, \{{*}1{*}{*}\}, \{{*}{*}1{*}\}, \{{*}{*}{*}1\}.$$

All other 3-minors are E. This means that the 4-cycle is contained in EUB, EUF, and EU. To match with Table 1, check that the 4-cycle has no induced 3-cycle, corresponds to the partition 13|24 of $\{1, 2, 3, 4\}$, and the involution $(1\ 3)(2\ 4) \in S_4$.

This graph shows that the class of a gaussoid cannot be determined by looking only at the induced subgraphs of G. All 3-minors observable from induced subgraphs are U, but the smallest class to which this gaussoid belongs is EU.

Example 4.3. Consider the star  with interior node 1 and leaves $2, 3, 4$. It corresponds to the gaussoid

$$\{(23|1), (23|14), (24|1), (24|13), (34|1), (34|12)\}.$$

Because the right-hand side of every element of the gaussoid contains 1, this gaussoid has the minor F in $1{*}{*}{*}$, E in the opposite face $0{*}{*}{*}$ and U everywhere else.

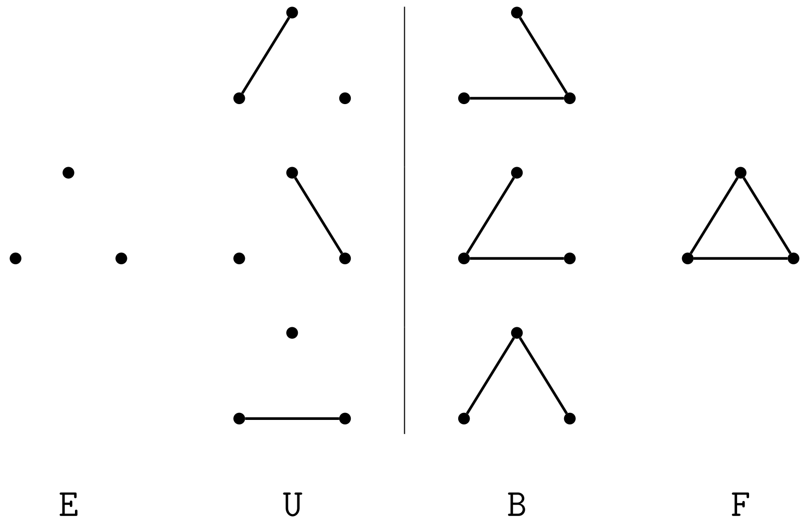


Fig. 3. The complementary graphs G^c of 3-gaussoids $\langle\langle G\rangle\rangle$ organized in symmetry classes mod S_3 according to Figure 2. E, U, B, F index a partition of the S_3 orbits of all graphs on 3 vertices. To obtain the diagram of graphs G, flip the pictures over the vertical axis.

We now establish relationships of subclasses of EUBF with known combinatorial objects. For some the graph G is more convenient, for others it is the complement graph G^c which is more natural. Figure 3 shows the complement graphs corresponding to E, U, B and F and is useful to keep in mind for the proof of Theorem 4.4.

Theorem 4.4. The gaussoids in the class EUBF are in bijection with the simple undirected graphs on n vertices. The subclasses distribute as follows:
(1) EUB contains exactly the gaussoids $\langle\langle G\rangle\rangle$ such that G^c is K_3-free.
(2) UBF contains exactly the gaussoids $\langle\langle G\rangle\rangle$ such that each connected component of G is a path.
(3) EUF contains exactly the gaussoids $\langle\langle G\rangle\rangle$ such that in G^c each connected component is a clique, and hence corresponds to partitions of the vertex set $[n]$.
(4) EU is EUF where additionally every connected component of G^c has at most two vertices.

Proof. The first statement summarizes the discussion in the beginning of this section. (1) The graphs G^c for $\langle\langle G\rangle\rangle \in$ EUB are free of triangles, as seen in Figure 3. If conversely G^c is triangle-free, then $\langle\langle G\rangle\rangle$ does not have F among its minors $(ijk|*)$. By ascension, the cardinality of $\langle\langle G\rangle\rangle\downarrow(ijk|L)$ is monotone in L and thus no minor of $\langle\langle G\rangle\rangle$ is F.

(2) For $\langle\langle G\rangle\rangle \in$ UBF we first show that every vertex of G has degree at most two. Suppose a vertex i was adjacent to three distinct vertices j, k, l. The subgraph induced on $ijkl$ is the star discussed in Example 4.3 since i has degree three in this subgraph

but none of its induced 3-subgraphs can be complete. The corresponding gaussoid has $\mathtt{E}$ as a minor and therefore this situation cannot arise in G. Therefore G is a disjoint union of cycles and paths. If G contains a cycle, let i, j, k be vertices of that cycle. Since cycles are 2-connected, neither $(ij|k)$, nor $(ik|j)$, nor $(jk|i)$ is in $\langle\langle G\rangle\rangle$. Consequently, the minor $\langle\langle G\rangle\rangle\downarrow(ijk|) = \mathtt{E}$ and thus G contains no cycles.

Let now G be a forest of paths. Consider any three vertices i, j, k. If they are not all in the same connected component, say i, j are in different connected components, then $(ij|), (ij|k) \in \langle\langle G\rangle\rangle\downarrow(ijk|)$ and thus this minor is not $\mathtt{E}$. If i, j, k are in the same connected component, then, after suitable renaming, i and j on this path become disconnected after removing k. Then $(ij|k) \in \langle\langle G\rangle\rangle\downarrow(ijk|)$ and this minor is not $\mathtt{E}$. In both cases, with ascension, it follows that for every $L \subseteq [n] \setminus ijk$ the minor $\langle\langle G\rangle\rangle\downarrow(ijk|L)$ is not $\mathtt{E}$.

(3) Let $\langle\langle G\rangle\rangle \in \mathtt{EUF}$. The induced subgraphs of G^c on three vertices are precisely those which are closed under the reachability relation within that subgraph. It is then clear that every two vertices in the neighborhood of a fixed vertex are connected by an edge, hence every connected component is a clique.

Let G^c be a disjoint union of cliques and $i, j, k \in [n]$. If they lie in pairwise different connected components, then the $(ijk|*)$-minor of $\langle\langle G\rangle\rangle$ is $\mathtt{E}$; if exactly two of them are in one component, then that minor is $\mathtt{U}$. By ascension, none of the minors $(ijk|L)$ can be $\mathtt{B}$ in these cases. Finally suppose that i, j, k are in the same connected component and that $\langle\langle G\rangle\rangle\downarrow(ijk|L)$ is a belt containing, say, $(ij|L)$ and $(ik|L)$ but not $(jk|L)$. Then G contains a path from j to k avoiding L. Because jk is an edge in G^c, this path contains another vertex $l \in [n] \setminus Lijk$ which is adjacent to j. Since jl is a non-edge in G^c and i and j are in the same clique, i and l are adjacent in G. This provides a path from i over l to k in G which avoids L, contradicting the assumption.

(4) Since $\mathtt{EU} = \mathtt{EUF} \cap \mathtt{EUB}$, every component of G^c, for $\langle\langle G\rangle\rangle \in \mathtt{EU}$, is a clique but since there are also no induced 3-cliques, the claim follows. □

Remark 4.5. Motivated by the theory of databases, Matúš [7, Consequence 4] also considered ascending gaussoids of chordal graphs. These have one forbidden 4-minor in addition to the compulsory 3-minors $\mathtt{EUBF}$. In general, classes of graphs with prescribed induced subgraphs on vertex sets I can be studied from the gaussoid perspective by choosing appropriate compulsory $(I|*)$-minors.

The only graphical classes left are the subclasses of $\mathtt{EBF} = \mathtt{EUBF} \cap \mathtt{ELBF}$. These *bi-monotone* gaussoids are simultaneously ascending and descending because $\mathtt{L}$ and $\mathtt{U}$ are forbidden. A bi-monotone gaussoid $\langle\langle G\rangle\rangle$ is fixed by the symbols $(ij|)$ it contains. Such gaussoids can be seen as irreflexive, symmetric, binary relations on $[n]$.

Lemma 4.6. $\mathtt{EBF}$-gaussoids are in bijection with the partitions of $[n]$.

Proof. Consider the gaussoid axioms under bi-monotonicity. Axioms (G1)–(G3) are trivial in the presence of ascension and descension axioms, and (G4) becomes $(ij|) \Rightarrow (ik|) \vee (jk|)$. In terms of binary relations, this is transitivity of the complement of $\langle\langle G\rangle\rangle$. Hence $\mathtt{EBF}$-gaussoids are complements of equivalence relations on $[n]$. □

A subclass of bi-monotone gaussoids is obtained by forbidding the empty minor in addition to the forbidden singletons. The resulting $\mathtt{BF}$-gaussoids only have 3-minors of cardinality at least four and are called *dense* gaussoids.

Lemma 4.7. The dense gaussoids BF correspond to involutions on $[n]$.

Proof. Let ι be an involution and $\langle\!\langle G \rangle\!\rangle$ the EBF-gaussoid associated, by Lemma 4.6, to ι's disjoint cycle decomposition. Since ι is an involution, every cycle is either a fixed point or a transposition. Take any two disjoint cycles $(i\ j)$ and $(k\ l)$ in ι. Since $ij \cap kl = \emptyset$, no two symbols of the form $(ij|K)$ and $(kl|M)$ appear in the same 3-face, for any choice of K and M. This implies that every 3-minor of $\langle\!\langle G \rangle\!\rangle$ can miss at most a single pair of opposite squares, which shows density.

Conversely, let $\langle\!\langle G \rangle\!\rangle$ be a dense gaussoid. Consider the partition corresponding to $\langle\!\langle G \rangle\!\rangle$ as an EBF-gaussoid. Assume there is a block containing at least three distinct elements i, j, k, then $\langle\!\langle G \rangle\!\rangle$ would not contain $(ij|)$, $(ik|)$ and $(jk|)$, which is a contradiction to $\langle\!\langle G \rangle\!\rangle$ being dense at the $(ijk|)$-minor. □

Lemma 4.8. An EB-gaussoid is defined by its characteristic vector with respect to $(12|), (13|), (14|), \ldots, (1n|)$ and every such vector defines an EB-gaussoid.

Proof. Let $\langle\!\langle G \rangle\!\rangle$ be an EB-gaussoid and $i, j \neq 1$ be distinct. Consider the $(1ij|)$-minor of $\langle\!\langle G \rangle\!\rangle$. Looking up $(1i|)$ and $(1j|)$ in the characteristic vector, we can decide whether $\langle\!\langle G \rangle\!\rangle \downarrow (1ij|)$ is empty or a belt. In either case the containment of $(ij|)$ in $\langle\!\langle G \rangle\!\rangle$ is determined by the status of $(1i|)$ and $(1j|)$. Vice versa, this reconstruction method freely defines a gaussoid all whose minors are necessarily E or B. □

Remark 4.9. We consider it an interesting topic to determine properties beyond combinatorics of the tamer graphical classes. For example, the EUBF-gaussoids are precisely the positively orientable gaussoids (see [1, Section 5] for the precise definition), their duals ELBF are the negatively orientable ones. It can also be shown that a BF-gaussoid $\langle\!\langle G \rangle\!\rangle$ has exactly 2^t orientations where t is the number of transpositions in the involution associated with $\langle\!\langle G \rangle\!\rangle$. All graphical gaussoids are realizable.

4.4. The exceptional class

The class LUBF remains mysterious. We have tried various arithmetic operations to transform the counts before searching OEIS, but nothing emerged. Unlike graphical gaussoids, there exist non-orientable and hence non-realizable LUBF-gaussoids. The following table lists their counts, for which likewise no interpretation is known to the authors.

n	3	4	5	6	7	8
all LUBF	10	142	1 166	12 796	183 772	3 221 660
non-orientable	0	42	210	1 260	14 700	355 740

It is remarkable that all these numbers are divisible by 42 and the numbers for dimensions ≥ 5 even by 210. As a first step towards understanding whether some structure underlies these numbers, we pose

Challenge 4.10. Find or disprove the existence of a finite forbidden minor characterization for non-orientable LUBF-gaussoids.

Our taxonomy of special gaussoids displays a trichotomy of growth behaviors: double exponential, single exponential or bounded in the dimension. We do not know where the growth of `LUBF` falls. Since the number of `LUBF`-gaussoids appears to grow slower than the number of ascending gaussoids, we make the following

Conjecture 4.11. There is a single exponential upper bound for the number of `LUBF`-gaussoids in dimension n.

Support for this conjecture comes from the fact that forbidding `E` as a minor leads to a high density, that is many squares, in the resulting gaussoids. To see this take an independent set in $Q(n,3,2,2)$. Each of the minors indexed by that set contains at least one 2-face and the independence ensures that no 2-face is counted twice. Thus an `LUBF`-gaussoid has at least $\alpha(Q(n,3,2,2)) \geq \delta n^2 2^n$ elements, with a positive constant δ independent of n. We suspect that containing a positive fraction of all squares is sufficient for `LUBF` to have single exponential size.

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