KYBERNETIKA — VOLUME *56* (2020), NUMBER 5, PAGES 916–933

VIOLATIONS OF THE INGLETON INEQUALITY AND REVISING THE FOUR-ATOM CONJECTURE

Nigel Boston and Ting-Ting Nan

The entropy region is a fundamental object of study in mathematics, statistics, and information theory. On the one hand, it involves pure group theory, governing inequalities satisfied by subgroup indices, whereas on the other hand, computing network coding capacities amounts to a convex optimization over this region. In the case of four random variables, the points in the region that satisfy the Ingleton inequality (corresponding to abelian groups and to linear network codes) form a well-understood polyhedron, and so attention has turned to Ingleton-violating points in the region. How far these points extend is measured by their Ingleton score, where points with positive score are Ingleton-violating. The Four-Atom Conjecture stated that the Ingleton score cannot exceed 0.089373, but this was disproved by Matúš and Csirmaz. In this paper we employ two methods to investigate Ingleton-violating points and thereby produce the currently largest known Ingleton scores.

First, we obtain many Ingleton-violating examples from non-abelian groups. Factorizability appears in many of those and is used to propose a systematic way to produce more.

Second, we rephrase the problem of maximizing Ingleton score as an optimization question and introduce a new Ingleton score function, which is a limit of Ingleton scores with maximum unchanged. We use group theory to exploit symmetry in these new Ingleton score functions and the relations between them. Our approach yields some large Ingleton scores and, using this methodology, we find that there are entropic points with score 0.0925000777, currently the largest known score.

Keywords: entropy vectors, information inequalities, subgroup indices

Classification: 94A15, 20B35

1. INTRODUCTION

In 2000 Ahlswede, Cai, Li, and Yeung [1] showed that deploying coding at each node in a network communication system can increase the network capacity in comparison with deploying routing alone. When coding at nodes in network systems, linear codes are generally preferred. It has been proved that linear coding and random linear coding achieve the multicast capacity when the alphabet size is large enough [8, 12, 15]. Furthermore, the linear coding capacity can be achieved if the alphabet size is sufficiently large [16]. In generic multicast networks [16], the linear coding capacity is equal to the coding capacity.

DOI: 10.14736/kyb-2020-5-0916

People have been searching for connections between networks and coding solutions. R. Dougherty, C. Freiling, and K. Zeger introduced matroidal networks [10] with respect to representable matroids over finite fields and provided an algorithm to construct a matroidal network from a representable matroid together with a scalar linear coding solution. It was shown that a scalar linear coding solution exists on such a network. Conversely, if a network has a scalar linear coding solution, then this network is matroidal. V. Muralidharan and B. S. Rajan introduced discrete polymatroidal networks [24]. They built the connection between discrete polymatroidal networks and vector linear coding solutions. The construction of discrete polymatroidal networks was also presented.

The linear coding capacity, however, does not equal the coding capacity on arbitrary networks [9], and some networks can never have an optimal linear coding solution. For the case of four random variables this difference between coding capacity and linear coding capacity is related to the Ingleton score of the variables. The main question addressed in this paper is to investigate the supremum of the Ingleton scores over all such cases. The Four-Atom Conjecture [11] originally proposed that this supremum is 0.089373, but that was disproven by Matúš and Csirmaz [22].

Ingleton scores are defined in terms of the corresponding entropy region. Background on entropy regions, their group-theoretic characterization, and information inequalities are described in Section 1. Section 2 elaborates on the Ingleton inequality, the Ingleton score, the Four-Atom conjecture, and the question which concerns us the most, namely how large Ingleton scores can get.

There are two major approaches to this question. The first idea is to use group theory to find Ingleton-violating examples, which we describe in Section 3. Unlike the families of violating examples discovered by W. Mao, M. Thill and B. Hassibi [18], whose Ingleton scores approach zero, we find many group violating examples with significantly large Ingleton scores. Among the Ingleton-violating examples we have found, a special phenomenon, factorizability, appears in many of them. Moreover, we suggest a construction that uses the factorizability to produce Ingleton-violating examples quickly using group theory.

The second idea is to turn the Ingleton scores into multivariable functions by using partitions of unity. Then the problem we care about becomes a question in optimization. The Ingleton score functions are not convex or concave, but they do have a lot of symmetry, which helps us solve such optimization questions. Furthermore, group theory allows us to build relations between the Ingleton score functions and to reduce the number of variables. Our method yields a record Ingleton score of 0.0925000777, slightly larger than that found by Matúš and Csirmaz [22]. We explore this idea further in Section 4.

1.1. Entropies

In this paper, all random variables are discrete. Given a collection of random variables, consider their joint entropies as below.

Definition 1.1. Let $X_1, \ldots, X_n$ be n random variables over sets $\mathcal{X}_1, \ldots, \mathcal{X}_n$, with joint probability mass function p. Then the *joint entropy* of $(X_1, \ldots, X_n)$ is defined to be

$$H(X_1, \ldots, X_n) = - \sum_{(x_1,\ldots,x_n) \in \mathcal{X}_1 \times \cdots \times \mathcal{X}_n} p(x_1, \ldots, x_n) \log_2 p(x_1, \ldots, x_n).$$

The following example will turn out be very important.

Example 1.2. (Dougherty et al. [11]) Let $n = 4$ and $X_1, \ldots, X_4$ be binary random variables with joint distribution given by $p(0,0,0,0) = p(1,1,1,1) = \alpha$ and $p(0,1,0,1) = p(0,1,1,0) = 0.5 - \alpha$ where $\alpha = 0.350457$. Then $H(X_1, \ldots, X_4) = 1.88017$.

Many notions in information theory such as conditional entropy, mutual information, and conditional mutual information can be expressed in terms of joint entropies. In particular, for three random variables X, Y, Z, these can be expressed as

$$\begin{aligned} H(X|Y) =& H(XY) - H(Y). \\ I(X;Y) =& H(X) + H(Y) - H(XY). \\ I(X;Y|Z) =& H(XZ) + H(YZ) - H(XYZ) - H(Z). \end{aligned} \tag{1}$$

The interested reader is encouraged to consult [29] for an extensive discussion of these topics.

1.2. Entropy regions and information inequalities

The entropy region is the arena in which to address many entropy related problems. It considers all possible joint entropies together for multiple random variables.

Throughout this paper, n is a positive integer, $k = 2^n - 1$, $N_n = \{1, \ldots, n\}$, α is a nonempty subset of N_n, and $\mathbf{h}$ is a k-dimensional vector indexed by $\alpha \subseteq N_n$. For random variables $X_1, \ldots, X_n$, let $X_\alpha = (X_i : i \in \alpha)$.

Definition 1.3. A k-dimensional vector $\mathbf{h}$ is *entropic* if there exist n random variables $X_1, \ldots, X_n$ such that the coordinate of $\mathbf{h}$ at α is

$$\mathbf{h}_\alpha = H(X_\alpha),$$

where $H(X_\alpha)$ denotes the joint entropy of $(X_i : i \in \alpha)$. We will use the notation $\mathbf{h}_\alpha$ for the joint entropy $H(X_\alpha)$.

The *entropy region*, denoted by Γ_n^*, is the subset of k-dimensional real space consisting of all entropic vectors.

We call an inequality involving entropies an *information inequality* if it holds for every set of n random variables. For any n random variables $X_1, X_2, \ldots, X_n$ and any nonempty subsets α, β, γ of N_n, some examples of information inequalities include:

$$\begin{aligned} H(X_\alpha) \geq 0, \quad & H(X_\alpha | X_\beta) \geq 0, \\ I(X_\alpha; X_\beta) \geq 0, \quad & I(X_\alpha; X_\beta | X_\gamma) \geq 0, \end{aligned} \tag{2}$$

since any of these familiar terms are non-negative [29].

An information inequality is called *Shannon-type* if it can be derived from the information inequalities in (2), i. e. it can be expressed as a linear combination of such information inequalities with non-negative coefficients. An information inequality is *non-Shannon-type* if it is not Shannon-type.

Entropy regions are almost well characterized for fewer than 4 random variables [30] by Shannon-type information inequalities alone- the region's closure is well characterized for 3 random variables. However, for 4 or more random variables non-Shannon-type information inequalities are necessary to cut out the entropy regions.

Let Γ_n denote the subset of k-dimensional real space cut out by all Shannon-type information inequalities for n random variables, and $\overline{\Gamma_n^*}$ is the closure of Γ_n^*. We also call vectors in $\overline{\Gamma_n^*}$ *almost entropic*.

Theorem 1.4. (Zhang and Yeung [30]) Let n be a positive integer.

1. Γ_n is a pointed convex cone with point at the origin.
2. $\Gamma_n^* \subseteq \overline{\Gamma_n^*} \subseteq \Gamma_n$.
3. $\Gamma_2^* = \Gamma_2$, i. e., Shannon-type information inequalities are enough to describe the entropy region for two random variables.
4. $\Gamma_3^* \subsetneq \overline{\Gamma_3^*} = \Gamma_3$, i. e. Shannon-type information inequalities are enough to describe the entropy region for three random variables except for the boundaries.

The first non-Shannon-type inequalities associated with 4 random variables were found by Zhang and Yeung [31] in 1998. It follows that $\overline{\Gamma_4^*} \subsetneq \Gamma_4$. Afterwards, several more non-Shannon-type inequalities were discovered by K. Makarycheve, L. Makarycheve, A. E. Romashchenko, and N. K. Vereshchagin [17], Zhang and Yeung [30] and Matúš [21, 23, 19, 20].

Theorem 1.5. (Matúš [21]) $\overline{\Gamma_n^*}$ is not a polyhedral cone when $n \geq 4$.

1.3. Group theory characterizations of entropy regions

Group theory seems, a priori, to be a subject unrelated to information theory. However, it turns out to be an important and helpful tool in characterizing the entropy region [7]. We recommend [14] for further background on finite group theory.

Definition 1.6. We say a vector $\mathbf{h}$ of dimension $k = 2^n - 1$ is *group characterizable* if there is a finite group G with n subgroups $G_1, G_2, \ldots, G_n$ such that

$$\mathbf{h}_\alpha = \log_2 \frac{|G|}{|G_\alpha|}$$

for every nonempty subset $\alpha \subseteq N_n$ where $G_\alpha = \cap_{i\in\alpha} G_i$.

We denote by γ_n the set consisting of all group characterizable vectors of dimension k.

Theorem 1.7. (Chen and Yeung [7]) Every group characterizable vector is entropic. Therefore, $\gamma_n \subseteq \Gamma_n^*$. Moreover, the closure of the convex hull of γ_n is exactly $\overline{\Gamma_n^*}$.

2. THE FOUR-ATOM CONJECTURE

2.1. The Ingleton inequality

In 1971, Ingleton [13] showed that the rank function ρ of any representable matroid satisfies a specific inequality. It turns out to be an important inequality in network coding.

Definition 2.1. (The Ingleton inequality) For four random variables X_1, X_2, X_3, X_4, the Ingleton inequality is

$$\begin{aligned} &H(X_{123}) + H(X_{124}) + H(X_{34}) + H(X_1) + H(X_2) \\ \leq &H(X_{12}) + H(X_{13}) + H(X_{14}) + H(X_{23}) + H(X_{24}). \end{aligned} \tag{3}$$

In spite of the failure of the Ingleton inequality for some sets of four random variables, the following result demonstrates its importance. Furthermore, any linear code on a network satisfies the Ingleton inequality.

Theorem 2.2. (Matúš and Studený [23]) Any vector in Γ_4 that satisfies all six of the Ingleton inequalities, up to permutation of X_1, X_2, X_3, X_4, is almost entropic.

This means that, for four variables, the part of the entropic region satisfying all six of the Ingleton inequalities is entirely described by Shannon-type inequalities and so is well understood and polyhedral.

It follows that the Ingleton inequality plays a fundamental role in network coding. In particular, the following measures of how far entropic points are from satisfying the inequality give an indication of how much capacity is lost by using only linear network coding.

2.2. The Ingleton score

One way to measure violation of the Ingleton inequality is by considering *the Ingleton difference* for 4 random variables X_1, X_2, X_3, X_4

$$d(X_1, X_2, X_3, X_4) = \mathbf{h}_{123} + \mathbf{h}_{124} + \mathbf{h}_{34} + \mathbf{h}_1 + \mathbf{h}_2 - \mathbf{h}_{12} - \mathbf{h}_{13} - \mathbf{h}_{14} - \mathbf{h}_{23} - \mathbf{h}_{24}. \tag{4}$$

(Note that here $\mathbf{h}_{123}$ is shorthand for $\mathbf{h}_{\{1,2,3\}}$ etc.) In particular, $d > 0$ if and only if the Ingleton inequality is violated.

However, the Ingleton difference can be arbitrarily large. Given four random variables with positive Ingleton difference, taking multiple identical independent copies of them increases the Ingleton difference arbitrarily. It follows that we should normalize the Ingleton difference appropriately.

Definition 2.3. (Dougherty et al. [11]) For any four random variables X_1, X_2, X_3, X_4, the *Ingleton score* $s(X_1, X_2, X_3, X_4)$ is defined to be

$$s(X_1, X_2, X_3, X_4) = \frac{d(X_1, X_2, X_3, X_4)}{\mathbf{h}_{1234}}. \tag{5}$$

Note that $s > 0$ if and only if the Ingleton inequality is violated.

Thanks to this normalization, the Ingleton score has an upper bound. For example, an upper bound for the Ingleton score is found to be $\frac{1}{4}$ by linear programming, using Shannon-type information inequalities. Incorporating some known non-Shannon-type inequalities yields a tighter upper bound of $\frac{3}{19} = 0.15789\ldots$. See [11] for more details. Example 2 in this paper yields an Ingleton score of about 0.089373 and the authors of [11] note that they were unable to find any examples with larger score, which leads to the following conjecture.

2.3. The four-atom conjecture

The supremum of the Ingleton score over any four random variables provides a way to measure the gap between coding capacities and linear coding capacities. Informally speaking, the higher the Ingleton score is, the more rate we may obtain when deploying a non-linear, rather than linear, coding scheme on a network.

Conjecture 2.4. (The four-atom conjecture [11]) For any four random variables X_1, X_2, X_3, X_4, the Ingleton score $s(X_1, X_2, X_3, X_4)$ cannot exceed 0.089373.

As indicated above, the distribution in Example 2 attains this value. However, this conjecture was disproved by F. Matúš and L. Csirmaz [22]. They showed examples exist with Ingleton score 0.09243. In this paper we show examples exist with Ingleton score 0.0925000777. Both results go via a slight variant of Ingleton score, which we now define.

Definition 2.5. For any four random variables X_1, X_2, X_3, X_4, the *new Ingleton score* $\tilde{s}(X_1, X_2, X_3, X_4)$ is defined to be

$$\tilde{s}(X_1, X_2, X_3, X_4) = \frac{d(X_1, X_2, X_3, X_4)}{\mathbf{h}_{123} + \mathbf{h}_{124} + \mathbf{h}_{34} - 2\mathbf{h}_{1234}}. \tag{6}$$

Theorem 2.6. (Matúš and Csirmaz [22]) Let X_1, X_2, X_3, X_4 be random variables. If (X_1, X_2, X_3, X_4) violates the Ingleton inequality, then there exist sequences of random variables (Y_1, Y_2, Y_3, Y_4) whose scores $s(Y_1, Y_2, Y_3, Y_4)$ are arbitrarily close to the new Ingleton score $\tilde{s}(X_1, X_2, X_3, X_4)$.

Note that under the circumstances of Theorem 2.6, the new Ingleton score is no less than the Ingleton score. They have the same positive numerator and the denominator in (6) is never greater than the one in (5), as can be derived by Shannon-type information inequalities.

3. GROUP THEORY APPROACH

Let G be a finite group and G_1, G_2, G_3, G_4 be subgroups of G. By the group characterization of the entropy region (1.6), the Ingleton inequality in terms of groups is

$$|G_{12}||G_{13}||G_{14}||G_{23}||G_{24}| \le |G_{123}||G_{124}||G_{34}||G_1||G_2|, \tag{7}$$

where $G_\alpha = \cap_{i\in\alpha} G_i$ for any $\alpha \subseteq N_4$.

Theorem 3.1. (Chen [6]) The Ingleton inequality in group terminology (7) holds if the finite group G is abelian.

More generally, it holds when G_1G_2 is a subgroup of G, where $G_1G_2 = \{g_1g_2 : g_i \in G_i\}$.

Several group properties have been discovered that imply the Ingleton inequality holds for group-characterizable vectors. When inclusion happens between subgroups, i. e., $G_i \leq G_j$ for distinct $i, j \in N_4$, or G_i is trivial for some i, the Ingleton inequality holds. Readers are referred to [18] for similar such properties.

3.1. The Ingleton score in group terminology

Definition 3.2. For a finite group G with four subgroups $G_i, i \in N_4$, the *Ingleton ratio* r is

$$r = \frac{|G_{12}||G_{13}||G_{14}||G_{23}||G_{24}|}{|G_{123}||G_{124}||G_{34}||G_1||G_2|}. \tag{8}$$

$r > 1$ if and only if the Ingleton inequality is violated. For a group characterizable vector the Ingleton difference d is

$$d = \log_2 r.$$

Note that the Ingleton ratio does not depend on the size of G. One goal is to find large Ingleton ratios in relatively small groups. Direct products of G and G_i's can give us arbitrarily large Ingleton ratios, but the size of the groups involved increases exponentially as well. Therefore, the Ingleton ratio is not the best measurement for the Ingleton violation and normalization is necessary, as was the case for the Ingleton score defined in (5).

The Ingleton score s in terms of groups for a finite group G and its four subgroups G_1, G_2, G_3, G_4 is

$$s = \frac{\log_2 r}{\log_2 \frac{|G|}{|G_{1234}|}},$$

where r is the Ingleton ratio.

Likewise, the new Ingleton score has a group-theoretical formulation by simply replacing the denominator of (5) by that of (6).

3.2. The first example in groups that violates the Ingleton inequality

W. Mao, M. Thill and B. Hassibi [18] found examples in non-abelian groups that violate the Ingleton inequality. They discovered that the smallest Ingleton-violating example arises in Sym(5) with Ingleton ratio 16/15 and Ingleton score 0.01348. Note that there is one and only one quadruple of subgroups, up to conjugacy, in Sym(5) violating the Ingleton inequality. Furthermore, G_1 and G_2 in this example are Frobenius groups (Frob(3) and Frob(5) respectively), which we define below.

Definition 3.3. A group G is *Frobenius* if there is a nontrivial proper subgroup H of G such that for any $g \in G - H$, H and $g^{-1}Hg$ intersect trivially.

For example, the group Frob(q) of affine linear maps from $\mathbf{F}_q$ to itself is Frobenius. Note that Frob(q) has order $q(q-1)$ and that Frob(3) and Frob(4) are isomorphic to Sym(3) and Alt(4) respectively. See Chapter 6 of [14] for further theory of these groups. This text also explains more about the (standard) notation below.

In their paper [18], $G_1 = \langle(3,4,5),(1,2)(4,5)\rangle, G_2 = \langle(1,2,3,4,5),(1,4,3,5)\rangle, G_3 = \langle(2,3),(1,3,4,2)\rangle$, and $G_4 = \langle(2,4),(1,2,5,4)\rangle$. Note that G_3 and G_4 are conjugate. Moreover, they analyzed this example and extended it to produce Ingleton-violating quadruples of subgroups in other groups such as general linear groups $GL(2,q)$ and projective general linear groups $PGL(n,q)$, where q is a power of a prime number. Note that $PGL(2,5)$ is isomorphic to Sym(5). In their examples, the Ingleton ratio $r = \frac{4(q-1)}{3q}$, which is always less than $\frac{4}{3}$. However, the group size grows as q^4 and so the Ingleton scores tend to 0 as q grows.

3.3. More Ingleton-violating examples

Inspired by Mao, Thill, and Hassibi's example in Sym(5), we [4] found Ingleton-violating examples in Sym(n), the alternating groups Alt(n), and some other groups using MAGMA [3].

A group G has an *Ingleton-violating example* if there exist four subgroups G_1, G_2, G_3, G_4 such that they violate the Ingleton inequality and generate the whole group G. To better organize our search for small groups with large Ingleton score, we borrow the following notion from that of efficiency frontiers in economics. In that case there are competing goals, namely high return and low risk, and we seek strategies such that no other strategy has higher return for lower risk. Here, the competing goals are large Ingleton score and small order, inspiring the following definition.

Definition 3.4. We call an example *efficient* if there is no group of smaller order with a larger Ingleton score.

Our work identifies the following examples. Note that several years of computation have produced these examples. We are highly confident that these are efficient, but a proof of that would be prohibitively computationally hard.

Example 3.5. The first example is that of Sym(5), given above, where the group order is 120, the Ingleton ratio is 16/15, and the Ingleton score is 0.01348.

All our examples have subgroups G_1, G_2, G_3, and G_4 of a particular form, namely $G_1 = \langle a, a^x\rangle, G_2 = \langle b, b^x\rangle, G_3 = \langle a, b\rangle, G_4 = \langle a^x, b^x\rangle$. For example, the last example arises with e. g. $a = (1,2)(3,4), b = (1,2,4,3), x = (3,4,5)$. We present the examples in this way.

Example 3.6. The second example is the group Alt(4) $\times$ Alt(4) of order 144 with Ingleton ratio 9/8 and Ingleton score 0.02370. The subgroups satisfy $G_1 \cong G_2 \cong$ Frob(4), $G_3 \cong G_4 \cong C_3 \times C_3$ and are presented as above with a, b, x as follows.

$$\begin{aligned} a &= (1,3,4)(5,6,8), \\ b &= (1,3,4)(5,8,6), \\ x &= (2,3,4)(5,7,6). \end{aligned}$$

Example 3.7. The third example is that of the following group of order 320 with Ingleton ratio 32/25 and Ingleton score 0.04279. The subgroups satisfy $G_1 \cong G_2 \cong \text{Frob}(5), G_3 \cong G_4 \cong H$, where H is SmallGroup(16,3) in the MAGMA [3] database of groups of order 16, and are presented as above with a, b, x as follows.

$$\begin{aligned} a =&(2,4,10,8)(3,7,9,5), \\ b =&(2,4,5,3)(7,9,10,8), \\ x =&(1,7,8,9,10)(2,3,4,5,6). \end{aligned}$$

Example 3.8. The fourth example is that of the following group of order 2058 with Ingleton ratio 72/49 and Ingleton score 0.05044. The subgroups satisfy $G_1 \cong G_2 \cong G_3 \cong G_4 \cong \text{Frob}(7)$ and are presented as above with a, b, x as follows.

$$\begin{aligned} a =&(1,35,26,42,9,15)(2,32,28,36,13,17)(3,29,23,37,10,19) \\ &(4,33,25,38,14,21)(5,30,27,39,11,16)(6,34,22,40,8,18) \\ &(7,31,24,41,12,20)(43,45,46)(44,49,48), \\ b =&(1,39,15,12,24,45)(2,36,17,13,28,47)(3,40,19,14,25,49) \\ &(4,37,21,8,22,44)(5,41,16,9,26,46)(6,38,18,10,23,48) \\ &(7,42,20,11,27,43)(29,34,33)(30,31,35), \\ x =&(1,10)(2,11)(3,12)(4,13)(5,14)(6,8)(7,9)(15,44)(16,45)(17,46) \\ &(18,47)(19,48)(20,49)(21,43)(22,40)(23,41)(24,42)(25,36)(26,37) \\ &(27,38)(28,39). \end{aligned} \tag{9}$$

Example 3.9. The fifth example is the group $\text{Frob}(8) \times \text{Frob}(8)$ of order 3136 with Ingleton ratio 49/32 and Ingleton score 0.05293. The subgroups satisfy $G_1 \cong G_2 \cong \text{Frob}(8), G_3 \cong G_4 \cong C_7 \times C_7$ and are presented as above with a, b, x as follows.

$$\begin{aligned} a =&(1,5,6,8,2,3,4)(10,14,15,12,11,13,16), \\ b =&(1,2,5,3,6,4,8)(10,15,11,16,14,12,13), \\ x =&(1,4,8,3,5,7,6)(9,11,12,16,14,15,10). \end{aligned} \tag{10}$$

Example 3.10. There is a further example, which is almost efficient in the sense that $|G|/|G_{1234}| = 1800$ and its Ingleton ratio is 25/18, yielding Ingleton score 0.04383.

If there were such a group of order 1800, then it would go between the third and fourth examples above and surely be efficient. We can, however, show that there is no such group of order 1800 since it would need to have a subgroup G_1 of order 30 generated by two elements of order 5 and no such group exists.

For our example, G has order 7200 and G_{1234} order 4. The subgroups are presented as above with a, b, x as follows.

$$\begin{aligned} a =&(1,23,13,15,18,4,9,19,21,12)(2,24,14,16,17,3,10,20,22,11)(5,8)(6,7), \\ b =&(1,16,13,24,18,11,9,20,21,3)(2,15,14,23,17,12,10,19,22,4)(5,7)(6,8), \\ x =&(1,2)(3,4)(5,14)(6,13)(7,19)(8,20)(9,21)(10,22)(11,12)(15,23)(16,24)(17,18). \end{aligned} \tag{11}$$

The challenge is now to find further efficient examples. Direct generalizations of the above examples do not succeed. We did, however, find examples, such as the next two, that suggest a systematic approach that we develop in subsection 3.4.

Example 3.11. In the best example we found in Sym(7), G_1 and G_2 are both isomorphic to Frob(7), whereas G_3 and G_4 are of order 36 and each isomorphic to Sym(3) $\times$ Sym(3). Their Ingleton ratio is 72/49 and Ingleton score is 0.04514. Here we can take, for instance:

$$G_1 = \langle (1,2,3,4,5,6), (1,3,6,7,4,2) \rangle, \;\; G_2 = \langle (1,2,3,6,5,4), (1,3,4,7,6,2) \rangle,$$
$$G_3 = \langle (1,2,3,4,5,6), (1,2,3,6,5,4) \rangle, \;\; G_4 = \langle (1,3,6,7,4,2), (1,3,4,7,6,2) \rangle.$$

Example 3.12. In the second best example in Sym(7) with Ingleton ratio 10/7 and Ingleton score 0.04184, G_1 is isomorphic to Frob(7), whereas G_2, G_3, and G_4 are all conjugate to each other and are isomorphic to Sym(5) acting transitively on 6 letters as the following:

$$G_1 = \langle (1,2,3,4,5,6,7), (1,3,2,6,4,5) \rangle, \;\; G_2 = \langle (2,4,6,7,5,3), (2,5)(3,7)(4,6) \rangle,$$
$$G_3 = \langle (1,6,3,7,4,5), (1,5)(3,4)(6,7) \rangle, \;\; G_4 = \langle (1,3,2,6,4,5), (1,4)(2,3)(5,6) \rangle.$$

Both examples are similar to the violating example in Sym(5) since G_1 is Frobenius and G_3 and G_4 are conjugate. However, both of them have higher Ingleton ratio than any example in [18].

3.3.1. Nilpotent and solvable groups

We found the first example of an Ingleton-violating solvable group, namely $G = \mathrm{Alt}(4) \times \mathrm{Alt}(4)$ above. This example is important because it shows that the Ingleton inequality does not always hold for solvable groups. In fact, whereas much work on finding Ingleton-violating examples originally focused on nonsolvable groups, note that all the efficient examples above, except for the first, are solvable.

For nilpotent groups, Stancu and Oggier [27] claimed that the Ingleton inequality holds. However, a mistake was found in their proof. Later, Paajanen [26] proved that the Ingleton inequality holds for p-groups of class less than p by embedding subgroup lattices of such p-groups into subgroup lattices of abelian groups. Thus, if every Sylow p-subgroup of a nilpotent group has class less than p, then the Ingleton inequality holds for this nilpotent group, since every nilpotent group is the direct product of its Sylow p-subgroups.

3.4. Factorizability

In many of our Ingleton-violating examples, their Ingleton ratios happen to be equal to $|G_3G_4|/|G_1G_2|$. This formula simplifies the one for the Ingleton ratio significantly, and accelerates the search for Ingleton-violating examples.

Definition 3.13. A group G is *factorizable* if there exist two nontrivial proper subgroups A, B of G such that $G = AB = \{ab : a \in A, b \in B\}$. This factorization is called *exact* if $A \cap B = 1$.

Definition 3.14. (Boston and Nan [5]) Let $\mathbf{h}$ be a group characterizable vector associated with a finite group G and subgroups G_1, G_2, G_3, G_4. We call $\mathbf{h}$ factorizable if G_3 and G_4 admit factorizations as

$$G_3 = (G_1 \cap G_3)(G_2 \cap G_3), \quad G_4 = (G_1 \cap G_4)(G_2 \cap G_4). \tag{12}$$

Lemma 3.15. For subgroups G_1, G_2, G_3, G_4 of a group G, if the Ingleton ratio is equal to $\frac{|G_3G_4|}{|G_1G_2|}$, then G_3 and G_4 admit factorizations as (12).

Proof. $G_{13}G_{23}$ is a subset of G_3, and thus $|G_{13}G_{23}| \leq |G_3|$, and likewise $|G_{14}G_{24}| \leq |G_4|$. Setting the Ingleton ratio and $|G_3G_4|/|G_1G_2|$ equal to each other, we have

$$\frac{|G_3||G_4||G_{12}|}{|G_1||G_2||G_{34}|} = \frac{|G_{12}||G_{13}||G_{14}||G_{23}||G_{24}|}{|G_{123}||G_{124}||G_{34}||G_1||G_2|}.$$

This implies that

$$|G_3||G_4| = \frac{|G_{13}||G_{23}|}{|G_{123}|} \cdot \frac{|G_{14}||G_{24}|}{|G_{124}|} = |G_{13}G_{23}||G_{14}G_{24}|.$$

Thus, $|G_3| = |G_{13}G_{23}|, |G_4| = |G_{14}G_{24}|$, and G_3, G_4 admit factorizations. □

Theorem 3.16. If a group characterizable vector $\mathbf{h}$ is factorizable, then its Ingleton ratio simplifies to

$$r = |G_3G_4|/|G_1G_2|. \tag{13}$$

Proof. Rearranging the formula for the Ingleton ratio (8), we can rewrite it as:

$$\begin{aligned} r =& \frac{|G_{12}|}{|G_1||G_2|} \cdot \frac{|G_{13}||G_{23}|}{|G_{123}|} \cdot \frac{|G_{14}||G_{24}|}{|G_{124}|} \cdot \frac{1}{|G_{34}|} \\ =& \frac{|G_{13}G_{23}||G_{14}G_{24}|}{|G_1G_2||G_{34}|} \\ =& \frac{|G_3||G_4|}{|G_1G_2||G_{34}|}. \\ =& \frac{|G_3G_4|}{|G_1G_2|}. \end{aligned}$$

□

The fact that some of our best examples exhibit factorizability and have G_3, G_4 conjugate lead us to develop the following systematic method for producing plenty of Ingleton-violating examples in $\mathrm{Sym}(n)$.

3.5. A fast search

Suppose that there is a factorizable subgroup AB of $\mathrm{Sym}(n)$ with $A \cap B = \{1\}$. Choose a $g \in G$ such that $\langle A, g^{-1}Ag\rangle$ and $\langle B, g^{-1}Bg\rangle$ are relatively small subgroups of $\mathrm{Sym}(n)$. Namely, $\langle A, g^{-1}Ag\rangle \cap\ AB = A$ and $\langle B, g^{-1}Bg\rangle \cap\ AB = B$. Setting

$$\begin{array}{ll} G_1 = \langle A, g^{-1}Ag\rangle, & G_2 = \langle B, g^{-1}Bg\rangle, \\ G_3 = AB, & G_4 = g^{-1}G_3 g \end{array} \tag{14}$$

will often produce a large Ingleton ratio. Note that G_3, G_4 are factorizable as (12) and conjugate in $\mathrm{Sym}(n)$.

Example 3.17. We observe that $\mathrm{Sym}(3) \times \mathrm{Sym}(3)$ has an exact factorization into two cyclic subgroups of order 6. Embedding this group into $\mathrm{Sym}(7)$ by taking the two subgroups to be $A = \langle(1,2,3,4,5,6)\rangle$ and $B = \langle(1,2,3,6,5,4)\rangle$ and letting g be the permutation $(1,2)(4,6)(5,7)$, then the above fast search method yields Example 3.12.

Example 3.18. Likewise, the second best example for $\mathrm{Sym}(7)$ arises from the exact factorization of $PGL(2,5)$ into a cyclic subgroup of order 6 and a Frobenius group of order 20. We found many good examples by starting with bicyclic groups, namely groups that factor as a product AB of cyclic groups of orders r and s, where each is one less than a prime, say $r = p-1$ and $s = q-1$. If we can find a common element g such that $\langle A, g^{-1}Ag\rangle$ and $\langle B, g^{-1}Bg\rangle$ are Frobenius groups of order $p(p-1)$ and $q(q-1)$ respectively, then this often yields a large Ingleton ratio. Example 3.12 is one such example with $r = s = 6$. The $PGL(2,p)$ examples in [18] come from $r = 2, s = p-1$.

4. OPTIMIZATION APPROACH

The main goal of this paper is to try to repair the Four-Atom Conjecture by finding the supremum of all Ingleton scores. In this section, we introduce another approach to study the supremum of the Ingleton score, defined to be

$$\sup_{X_1,X_2,X_3,X_4} s(X_1,X_2,X_3,X_4). \tag{15}$$

By Theorem 2.6, whereby the new Ingleton score is no less than the Ingleton score for Ingleton-violating examples, finding the supremum of the new Ingleton scores provides an easier way to tackle our problem. We therefore focus on the new Ingleton score rather than the Ingleton score. Our problem is restated as wanting to find

$$\sup_{X_1,X_2,X_3,X_4} \tilde{s}(X_1,X_2,X_3,X_4). \tag{16}$$

4.1. Nondecreasing

Let $\tilde{s}_n$ denote the maximum of the new Ingleton score over any quadruple of n-ary random variables (X_1,X_2,X_3,X_4), i.e.,

$$\tilde{s}_n = \max_{X_1,X_2,X_3,X_4 \text{ over } N_n} \tilde{s}(X_1,X_2,X_3,X_4), \tag{17}$$

where $N_n = \{1, \dots, n\}$.

Since a random variable X over N_n can be viewed as a random variable over N_m for any $m \geq n$, we know that

Proposition 4.1. $\tilde{s}_n$ is non-decreasing. Therefore,

$$\sup_{X_1,X_2,X_3,X_4} \tilde{s}(X_1, X_2, X_3, X_4) = \lim_{n\to\infty} \tilde{s}_n.$$

Under certain circumstances, $\tilde{s}(X_1, X_2, X_3, X_4)$ remains the same as the size of the sample space grows.

Definition 4.2. For two random variables X, Y over $\mathcal{X}, \mathcal{Y}$ with joint probability mass function p, the *transition matrix* $W(X|Y)$ *of* X *given* Y is a matrix indexed by $\mathcal{X} \times \mathcal{Y}$ with entries

$$p(x|y) = \frac{p(x, y)}{p(y)}$$

for all pairs $(x, y) \in \mathcal{X} \times \mathcal{Y}$, assuming $p(y) > 0$.

Theorem 4.3. Let X_1, X_2, X_3, X_4 be random variables over $\mathcal{N}_n$ and Y_1, Y_2, Y_3, Y_4 random variables over $\mathcal{N}_m$ with $n \geq m$. If the transition matrix satisfies

$$W(X_1, X_2, X_3, X_4|Y_1, Y_2, Y_3, Y_4) = W(X_1|Y_1) \otimes \cdots \otimes W(X_4|Y_4)$$

and $H(Y_i|X_i) = 0$ for all i, then

$$\tilde{s}(X_1, X_2, X_3, X_4) = \tilde{s}(Y_1, Y_2, Y_3, Y_4).$$

Here, $\otimes$ denotes the usual Kronecker (tensor) product of matrices.

Proof. A straightforward calculation by proving the equality of numerators of $\tilde{s}(X_1, X_2, X_3, X_4)$ and $\tilde{s}(Y_1, Y_2, Y_3, Y_4)$. Same with denominators. Details are given in [25]. □

4.2. Optimization with symmetry and invariance groups

Let n be a positive integer and $M_n = (\mathbb{Z}/n\mathbb{Z})^4$ be the free module of rank four over $\mathbb{Z}/n\mathbb{Z}$. The probability mass function p of four n-ary random variables can be treated as a map from M_n to $[0, 1]$:

$$\begin{array}{rccc} p: & M_n & \longrightarrow & [0,1] \\ & i & \mapsto & p_i, \end{array}$$

where $\sum_{i\in M_n} p_i = 1$.

One can employ an optimization algorithm with $n^4 - 1$ variables to find local solutions of (17) but this is not efficient when n is large. The entropy of a random variable X is invariant under symmetries of its sample space, and the new Ingleton scores possess even more symmetries. Introducing symmetry helps us find optima since it reduces dimensionality. Here we show how to exploit the symmetry of the new Ingleton scores.

Definition 4.4. Let I be an index set and f be a function of $\overline{x} = (x_i : i \in I)$. The *invariance group of* f is the subgroup H of $\mathrm{Sym}(I)$ consisting of permutations $\sigma \in \mathrm{Sym}(I)$ such that

$$f(\sigma(\overline{x})) = f(\overline{x}),$$

where $\sigma(\overline{x}) = (x_{\sigma^{-1}(i)} : i \in I)$.

Definition 4.5. Let A, H be two groups with H a subgroup of $\mathrm{Sym}(n)$, and let $K = \prod_{i=1}^{n} A$. The *wreath product* $A \wr H$ of A and H is the semidirect product $K \rtimes H$ of K and H with H acting on the indices of K via its embedding in $\mathrm{Sym}(n)$, i. e.,

$$\begin{array}{rccc} h \in H: & K & \to & K \\ & (a_i) & \mapsto & (a_{h \cdot i}). \end{array}$$

Theorem 4.6. The invariance group G_n of the new Ingleton score $\tilde{s}_n$ contains

$$G_n = (\mathrm{Sym}(n) \wr C_2)^2.$$

P r o o f. The entropy $H(X)$ of a random variable X with underlying set $\mathcal{X}$ of size n is invariant under any permutation in $\mathrm{Sym}(n)$. Thus, the invariant group of $\tilde{s}_n$ contains the direct product of four $\mathrm{Sym}(n)$. Furthermore, the Ingleton inequality is invariant under swapping X_1, X_2. This means that C_2 acts on the first two copies of $\mathrm{Sym}(n)$ by permuting the first and second coordinates. Likewise, the Ingleton inequality is also invariant under swapping X_3, X_4. Thus, the invariance group G_n contains $(\mathrm{Sym}(n) \wr C_2)^2$. □

Remark 4.7. The invariance group of s_n also contains G_n.

In fact, G_n acts on M_n as below. For $j \in N_n$, j_i indicates the j at the ith coordinate of M_n. Also for $\sigma \in \mathrm{Sym}(n)$, σ_i denotes the permutation σ acting on the ith coordinate of M_n. For instance in G_2, $(0_3, 1_3)$, also denoted as $(0,1)_3$, permutes the 0 and the 1 at the third coordinate and fixes other positions.

For the first copy of $\mathrm{Sym}(n) \wr C_2$ in G_n, let φ_{12} denote the group action of C_2 on $\mathrm{Sym}(n) \times \mathrm{Sym}(n)$ such that $\varphi_{12}(\sigma_1) = \sigma_2$. The element $(\varphi_{12}, (\sigma_1, \sigma_2))$ in $\mathrm{Sym}(n) \wr C_2$ is denoted as ω_{12}, and likewise ω_{34} for the second copy. For instance in G_2, $\omega_{12} = (0_1, 0_2)(1_1, 1_2)$ and $\omega_{34} = (0_3, 0_4)(1_3, 1_4)$.

4.3. Critical subgroups

It is not always the case that in optimization problems where functions are symmetric, their maxima and minima occur when the variables are equal on orbits [28]. As noted above, the new Ingleton scores have rich symmetry groups as in Theorem 4.6. However, a point maximizing $\tilde{s}_n$ is not necessarily symmetric under the full invariance group.

Maximizing functions under certain symmetries is the same as maximizing over a subspace, thereby reducing the number of variables. However, overusing symmetries, which is same as considering too small a subspace, may not yield solutions to the problem because of the issues noted in the last paragraph. Thus, identifying the right symmetries becomes critical.

Definition 4.8. Let $\overline{x} = (x_i : i \in I)$ be variables indexed by I, and H be a subgroup of $\mathrm{Sym}(I)$. We say $\overline{x}$ is *invariant under* H if $\sigma(\overline{x}) = \overline{x}$ for all $\sigma \in H$, where $\sigma(\overline{x}) = (x_{\sigma^{-1}(i)} : i \in I)$.

Definition 4.9. We call a subgroup H of G_n *critical* if H consists of those permutations leaving a maximizer for $\tilde{s}_n$ invariant.

Example 4.10. For $n = 2$ there are 16 isolated maxima which attain the largest new Ingleton score 0.09103635. For example,

$$\begin{gathered} p_{0110} = p_{1001} = 0.33985,\ p_{0000} = p_{0011} = 0.14886, \\ p_{0100} = p_{0111} = p_{1000} = p_{1011} = 0.00404, \\ p_{0001} = p_{0010} = 0.00303,\ p_{1100} = p_{1111} = 0.00018, \\ p_{1101} = p_{1110} = 0.00000375,\ p_{0101} = p_{1010} = 0.0000000198. \end{gathered}$$

All the maxima form one orbit under G_2 with stabilizers

$$H_2 =< (0_3, 1_4)(0_4, 1_3), \omega_{12}\omega_{34} >$$

and its conjugates. Note that G_2 has order 64 and H_2 order 4, explaining the $64/4 = 16$ maxima. Therefore, H_2 is a critical subgroup for the Ingleton score $\tilde{s}_2$.

Example 4.11. For $n = 4$ there are 82944 isolated maxima which attain the largest new Ingleton score 0.0925000777, and they form one orbit under G_4 with stabilizers

$$\begin{aligned} H_4 =<&(0,2,1,3)_1(0,3,1,2)_2(0,2,1,3)_3(0,3,1,2)_4 \\ &\quad (0,3)_1(1,2)_1(2,3)_2(0,3)_3(1,2)_3(2,3)_4, \\ &\omega_{12}\omega_{34} > \end{aligned}$$

and its conjugates. Note that G_4 has order 1327104 and H_4 order 16 and $1327104/16 = 82944$. Therefore, H_4 is a critical subgroup for the new Ingleton score $\tilde{s}_4$.

5. MAXIMA FOR 8 VARIABLES

Extensive investigations indicate that any maxima for $n = 5, 6$, or 7 derive from the maxima for $n = 2$ and 4. To search for truly new maxima, we focused on the case of $n = 8$, which is considerably harder. We started with a randomly chosen feasible point (i. e. probability distribution), uniformly chosen from the set of $8^4 = 4096$ nonnegative real numbers adding to 1. We then applied random walks to search for local solutions (a zig-zag search). For each iteration, among coordinates in $\overline{x}$ we randomly chose a coordinate x_i ($i \in M_n$) and then added or subtracted a small randomly chosen perturbation with normalization applied to ensure $\overline{x}$ remained a partition of unity. We kept doing so until a feasible point was obtained with a larger value of $\tilde{s}_n$, and then updated $\overline{x}$. After enough such updates to $\overline{x}$, we obtained a local maximizer.

Such processes take a long time when the number of variables is large (here 4096 coordinates). Therefore, reduction of the number of variables was desirable to expedite the process. We did this by finding local solutions invariant under a particular subgroup, so that rather than perturbing the value at one coordinate, we perturbed coordinates in one orbit equally. Thus, the true number of variables was, in effect, the number of orbits of the subgroup. This could be particularly effective if our chosen subgroup happened to be a subgroup of a critical subgroup.

For $\tilde{s}_2$ (Example 4.10), the critical subgroup H_2 is of order 4 while the size of G_2 is 64. Whilst the number of variables is 16, there are just 7 orbits.. For $\tilde{s}_4$ (Example 4.11), the critical subgroup H_4 is of order 16 while the size of G_4 is 1327104. Whilst the number of variables is 256, there are just 20 orbits. Such reduction in the number of variables accelerates searching for local solutions.

Based on the observations from those two examples, we attempted to search for Ingleton scores larger than 0.0925 in G_8. There are, up to conjugation, 255 subgroups of G_8 of order 2, and so 255 cases to try.

We hoped that calculating local solutions invariant under some subgroup of order 2 would reveal a new Ingleton score larger than 0.0925. The University of Wisconsin-Madison Center For High Throughput Computing (CHTC) was used to do such massive computations in parallel. For about 100 of the 255 cases, we found a local maximum of 0.09103635, which was just Example 42 spread over more coordinates. Very occasionally we reached a local maximum of 0.0925000777, which likewise was a spread out version of Example 43.

In fact, despite conducting many millions of parallel searches, the highest score found was 0.0925000777, and so we did no better than for $n = 4$. Either there is no better (when the Four-Atom Conjecture should be replaced by a 256-Atom Conjecture) or the true maxima are extremely elusive.

6. CONCLUSION

In this paper we investigate two ways of finding unusually large Ingleton scores systematically. We produce many Ingleton-violating examples using non-abelian groups, including the best ones coming from small groups, extending the work of [18]. We develop new methods for producing examples with large Ingleton ratio and Ingleton score and successfully apply our methods to obtain an Ingleton score as large as 0.0925000777, which is now the largest known Ingleton score. Note that the recent paper [2] found an example with Ingleton score 0.092499.

ACKNOWLEDGEMENT

Supported by the Simons Foundation grant MSN179747.

(Received February 27, 2019)

REFERENCES

[1] R. Ahlswede, N. Cai, S.-Y. R. Li, and R. W. Yeung: Network information flow. IEEE Trans. Inform. Theory *46* (2007), 1204–1216. DOI:10.1109/18.850663

[2] S. Alam, S. Thakor, and S. Abbas: On Enumerating Distributions for Associated Vectors in the Entropy Space. arXiv:1807.08573, 2018.

[3] W. Bosma, J. Cannon, and C. Playoust: The Magma algebra system. The user language, J. Symbolic Comput. *24* (1997), 235–265. DOI:10.1006/jsco.1996.0125

[4] N. Boston and T.-T. Nan: Large violations of the Ingleton inequality. In: 50th Annual Allerton Conference on Communication, Control, and Computing, Allerton 2012. DOI:10.1109/allerton.2012.6483410

[5] N. Boston and T.-T. Nan: A refinement of the four-atom conjecture. In: International Symposium on Network Coding (NetCod), 2013. DOI:10.1109/netcod.2013.6570833

[6] T. Chen: Group characterizable entropiy functions. In: Proc. of the 2005 IEEE International Symposium on Information Theory, Nice 2007, pp. 507–510. DOI:10.1109/isit.2007.4557275

[7] T. H. Chen and R. W. Yeung: On a relation between information inequalities and group theory. IEEE Trans. Inform. Theory *48* (2002), 1992–1995. DOI:10.1109/tit.2002.1013138

[8] P. A. Chou, Y. Wu, and K. Jain: Practical network coding. In: Proc. 2003 Allerton Conf. on Commun., Control, and Computing, 2003, pp. 40–49.

[9] R. Dougherty, C. Freiling, and K. Zeger: Insufficiency of linear coding in network information flow. IEEE Trans. Inform. Theory *51* (2005), 2745–2759. DOI:10.1109/tit.2005.851744

[10] R. Dougherty, C. Freiling, and K. Zeger: Networks, matroids, and non-Shannon information inequalities. IEEE Trans. Inform. Theory *53* (2007), 1949–1969. DOI:10.1109/tit.2007.896862

[11] R. Dougherty, C. Freiling, and K. Zeger: Non-Shannon information inequalities in four random variables. arXiv:1104.3602v1, 2011.

[12] T. Ho, M. Médard, R. Koetter, D. R. Karger, M. Effros, J. Shi, and B. Leong: A random linear network coding approach to multicast. IEEE Trans. Inform. Theory *52* (2006), 4413–4430. DOI:10.1109/tit.2006.881746

[13] A. Ingleton: Representation of matroids. In: Combinatorial Mathematics and its Applications, 1971, pp. 149–167.

[14] I. M.Isaacs: Finite Group Theory. American Mathematical Society, 2008. DOI:10.1090/gsm/092

[15] R. Koetter and M. Médard: An algebraic approach to network coding. IEEE/ACM Trans. Network. *2* (2003), 782–795. DOI:10.1109/tnet.2003.818197

[16] R. Li, R. Yeung, and N. Cai: Linear network coding. IEEE Trans. Inform. Theory *49* (2003), 371–381. DOI:10.1109/tit.2002.807285

[17] K. Makarycheve, L. Makarycheve, A. E. Romashchenko, and N. K. Vereshchagin: A new class of non-Shannon type inequalities for entropies. Com. Inform. Syst. *2* (2002), 147–166. DOI:10.4310/cis.2002.v2.n2.a3

[18] W. Mao, M. Thill, and B. Hassibi: On Ingleton-violating finite groups. IEEE Trans. Inform. Theory *63* (2017), 183–200. DOI:10.1109/tit.2016.2627530

[19] F. Matúš: Conditional independences among four random variables II. Combin. Probab. Comput. *4* (1995), 407–417. DOI:10.1017/s0963548300001747

[20] F. Matúš: Conditional independences among four random variables III: Final conclusion. Combin. Probab. Comput. *8* (1999), 269–276. DOI:10.1017/s0963548399003740

[21] F. Matúš: Infinitely many information inequalities. In: IEEE International Symposium on Information Theory, IEEE 2007, pp. 41–44. DOI:10.1109/isit.2007.4557201

[22] F. Matúš and L. Csirmaz: Entropy region and convolution. IEEE Trans. Inform. Theory *62* (2016), 6007–6018. DOI:10.1109/tit.2016.2601598

[23] F. Matúš and M. Studený: Conditional independences among four random variables I. Combin. Probab. Comput. *4* (1995), 269–278. DOI:10.1017/s0963548300001644

[24] V. T. Muralidharan and B. S. Rajan: On the vector linear solvability of networks and discrete polymatroids. GLOBECOM 2013, pp. 1979–1984.

[25] T.-T. Nan: Entropy Regions and the Four-Atom Conjecture. PhD. Dissertation, Department of Mathematics, University of Wisconsin–Madison, 2015.

[26] P. Paajanen: Finite p-groups, entropy vectors and the ingleton inequality for nilpotent groups. IEEE Trans. Inform. Theory *60* (2014), 3821–3824.

[27] R. Stancu and F. Oggier: Finite nilpotent and metacyclic groups never violate the Ingleton inequality. 2012 International Symposium on Network Coding (NetCod), 2012, pp. 25–30. DOI:10.1109/netcod.2012.6261879

[28] W. Waterhouse: Do symmetric problems have symmetric solutions? Amer. Math. Monthly *90* (1983), 378–387. DOI:10.1080/00029890.1983.11971235

[29] R. W. Yeung: Information Theory and Network Coding. Springer, 2008.

[30] Z. Zhang and R. W. Yeung: A non-Shannon type conditional inequality of information quantities. IEEE Trans. Inform. Theory *43* (1997), 1982–1986. DOI:10.1109/18.641561

[31] Z. Zhang and R. W. Yeung: On characterization of entropy function via information inequalities. IEEE Trans. Inform. Theory *44* (1998), 1440–1452. DOI:10.1109/18.681320

Nigel Boston, Department of Mathematics, University of Wisconsin–Madison, WI, 53706, U. S. A.
e-mail: boston@math.wisc.edu

Ting-Ting Nan,
e-mail: ayanaminan@gmail.com