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CONTROLLABILITY OF LINEAR IMPULSIVE SYSTEMS —AN EIGENVALUE APPROACH

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This article considers a class of finite-dimensional linear impulsive time-varying systems for which various sufficient and necessary algebraic criteria for complete controllability, including matrix rank conditions are established. The obtained controllability results are further synthesised for the time-invariant case, and under some special conditions on the system parameters, we obtain a Popov-Belevitch-Hautus (PBH)-type rank condition which employs eigenvalues of the system matrix for the investigation of their controllability. Numerical examples are provided that demonstrate—for the linear impulsive systems, null controllability need not imply their complete controllability, unlike for the non-impulsive linear systems.

Keywords: eigenvalues, impulses, controllability

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1. INTRODUCTION

Many of the evolution processes exhibit impulsive behaviour, that is, the states of such system are subjected to short-term perturbations at certain moments of time. Dynamical systems which show an impulsive behaviour are often encountered in artificial intelligence, biological systems, population dynamics, neural networks, robotics, telecommunications, information science etc. Readers can refer a monograph by Lakshmikantham et al [6] for a detailed study on impulsive systems. Controllability is one of the fundamental properties of the dynamical control system, which simply means the system can be steered from any of its initial state to any desired final state within its state-space in some finite-time, by using a set of admissible control functions.

The investigation on controllability of impulsive systems has begun in 1993 by Leela et al [7], where, it is analysed an effect of impulsive control on system controllability. Later the research on these systems took a rapid growth when many other control theorists started investigating the controllability of different types of impulsive systems. Some of the remarkable contributions were made by Benzaid and Sznaier [1], George et al [2], Guan et al ([3], [4]), Xie and Wang [10], Zhao and Sun ([11], [12]), Han et al [5] and others. In [1], a homogeneous linear impulsive system is considered and its global null controllability is established. In [2], the authors investigated the controllability of impulsive systems with nonlinear perturbations. In [3], various necessary and

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sufficient controllability conditions are obtained for the linear impulsive systems of both autonomous and nonautonomous type. Xie and Wang [10] considered linear piecewise constant impulsive systems and obtained their controllability results, which are generalised in [11] for the time-varying case, and further simplifications are performed in [5]. But it is worth pointing out here that, in all these papers the authors actually investigated the null controllability (i.e. controllable to the origin from any initial state) of impulsive systems, which is not equivalent to their complete controllability unlike the linear systems without impulses.

As we know for the time-invariant case of a linear system without impulses, in addition to Kalman's matrix rank condition, an equivalent Popov-Belevitch-Hautus (PBH) rank condition which adopts the eigenvalues of system matrix is one of the easily verifiable and a powerful tool in the analysis of their controllability (Terrell [9]); and our literature survey shows that in none of the articles on impulsive control systems, a PBH-type rank condition is discovered so far.

This article provides various sufficient and necessary criteria for complete controllability of linear impulsive systems. The accomplished results are further reduced to the corresponding time-invariant case and subsequently obtained Kalman's type and PBH-type rank conditions under some special properties of the system parameters. When we specialise the obtained complete controllability conditions to that for null controllability, the results coincide with those in [3].

We organise this manuscript in four sections. In Section 2, some of the preliminaries and a class of linear impulsive control systems whose controllability are to be investigated are presented. The main results of the paper begins with Section 3, where we discuss various sufficient and necessary conditions for controllability of the considered system. Also a numerical example is introduced to support the claim—the null controllability need not imply the complete controllability for the linear impulsive systems. In Section 4, the controllability results which are obtained is Section 3 are applied to the corresponding time-invariant case of the system, and subsequently the conditions are reduced in terms of the system matrices and their eigenvalues. Finally the paper terminates with some concluding remarks.

2. PRELIMINARIES AND SYSTEM DESCRIPTION

Throughout the paper $\mathbb{R}$ denotes the set of all real numbers and $\mathbb{N}$ the positive integer set. For any fixed $m, n \in \mathbb{N}$, $\mathbb{R}^{n\times m}$ defines the real vector space of all $(n \times m)$-real matrices under the component-wise addition and component-wise scalar multiplication of matrices; in particular for $m = 1$, $\mathbb{R}^n$ collects all $(n \times 1)$-real matrices (also called as vectors), which represents an n-dimensional real vector space over a real field, and is also a real Banach space endowed with an Euclidean norm $\|\cdot\|_{\mathbb{R}^n}$. Let $\mathbf{I}_n$ symbolises an identity matrix of size $(n \times n)$. The spectrum $\sigma(\mathbf{A})$ defines the set of all eigenvalues of square matrix $\mathbf{A}$. The transpose of $\mathbf{A} \in \mathbb{R}^{n\times m}$ is denoted by $\mathbf{A}^T \in \mathbb{R}^{m\times n}$. Denote $\mathcal{C}(A; B)$ for the set of all continuous functions from set A to set B. For $1 \leqq m \leqq n$, the reverse product of square matrices (of same size) is defined by $\prod_{j=n}^{m} \mathbf{A}_j := \mathbf{A}_n\mathbf{A}_{n-1}\ldots\mathbf{A}_m$. The zero matrix of any size is denoted by $\mathbf{O}$ and zero vector by $\mathbf{0}$.

The natural space to work on the controllability of finite-dimensional linear impulsive systems is the real Banach space:

$$\mathcal{B} := \Big\{ \mathbf{x}(\cdot) \,\big|\, \mathbf{x}(\cdot) : [t_0, \infty) \to \mathbb{R}^n \text{ is continuous and bounded on } [t_0, \infty) \setminus \{t_k : k = 1, 2, \ldots\} \text{ and differentiable a.e. on } [t_0, \infty) \text{ such that } \exists\, \mathbf{x}(t_k^-) := \lim_{t \uparrow t_k} \mathbf{x}(t) \text{ and } \mathbf{x}(t_k^+) := \lim_{t \downarrow t_k} \mathbf{x}(t) \text{ with } \mathbf{x}(t_k^-) = \mathbf{x}(t_k), \text{ and } \mathbf{x}(t_0) = \lim_{t \downarrow t_0} \mathbf{x}(t) \Big\}$$

endowed with sup-norm

$$\big\|\mathbf{x}(\cdot)\big\|_{\mathcal{B}} := \sup_{t \in [t_0, \infty)} \big\|\mathbf{x}(t)\big\|_{\mathbb{R}^n}.$$

We also need the following real Banach space:

$$\mathcal{PC} := \Big\{ \mathbf{u}(\cdot) \,\big|\, \mathbf{u}(\cdot) : [t_0, \infty) \to \mathbb{R}^m \text{ is bounded piecewise continuous function on } [t_0, \infty) \Big\}$$

endowed with the sup-norm as in $\mathcal{B}$.

In this work, we consider the class of dynamical control systems modelled by the following n-dimensional linear impulsive ordinary differential equations:

$$\left.\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t), \quad t \in [t_0, \infty) \setminus \{t_k : k = 1, 2, 3, \ldots\}, \\ \mathbf{x}(t_0) &= \mathbf{x}_0, \\ \Delta\mathbf{x}(t_k) &= \mathbf{E}_k\mathbf{x}(t_k) + \mathbf{F}_k\mathbf{u}(t_k), \end{aligned}\right\} \tag{1}$$

where

(i) the state function $\mathbf{x}(\cdot) \in \mathcal{B}$ with a known initial state $\mathbf{x}(t_0) = \mathbf{x}_0$, and the control function $\mathbf{u}(\cdot) \in \mathcal{PC}$,

(ii) $\mathbf{A}(\cdot) \in \mathcal{C}\big([t_0, \infty); \mathbb{R}^{n\times n}\big)$ and $\mathbf{B}(\cdot) \in \mathcal{C}\big([t_0, \infty); \mathbb{R}^{n\times m}\big)$ are the known matrix valued functions,

(iii) $\Delta\mathbf{x}(t_k) := \mathbf{x}(t_k^+) - \mathbf{x}(t_k)$ is an impulse in the state at known times t_k,

(iv) $\mathbf{E}_k \in \mathbb{R}^{n\times n}$ and $\mathbf{F}_k \in \mathbb{R}^{n\times m}$ are the known constant matrices.

The solution to system (1) is given in the following lemma.

Lemma 1. By assuming there are M-impulses, $M \in \mathbb{N}$ in the time interval $[t_0, t_f]$, the solution to the system (1) in the time-duration $(t_k, t_{k+1}]$, $k = 1, 2, \ldots, M$ is given by

$$\begin{aligned} \mathbf{x}(t) = \mathbf{\Phi}(t, t_k)\Bigg\{ &\prod_{j=k}^{1} (\mathbf{I}_n + \mathbf{E}_j)\mathbf{\Phi}(t_j, t_{j-1})\mathbf{x}_0 \\ &+ \sum_{i=1}^{k} \Bigg(\prod_{j=k}^{i+1} (\mathbf{I}_n + \mathbf{E}_j)\mathbf{\Phi}(t_j, t_{j-1}) \Bigg)(\mathbf{I}_n + \mathbf{E}_i) \int_{t_{i-1}}^{t_i} \mathbf{\Phi}(t_i, s)\mathbf{B}(s)\mathbf{u}(s)\mathrm{d}s \\ &+ \sum_{i=2}^{k} \prod_{j=k}^{i} (\mathbf{I}_n + \mathbf{E}_j)\mathbf{\Phi}(t_j, t_{j-1})\mathbf{F}_{i-1}\mathbf{u}(t_{i-1}) + \mathbf{F}_k\mathbf{u}(t_k) \Bigg\} \\ &+ \int_{t_k}^{t} \mathbf{\Phi}(t, s)\mathbf{B}(s)\mathbf{u}(s)\mathrm{d}s, \end{aligned} \tag{2}$$

where $\mathbf{\Phi}(t,s) := \mathbf{\Phi}(t)\mathbf{\Phi}^{-1}(s)$ is the state-transition matrix associated with $\mathbf{A}(t)$, and $\mathbf{\Phi}(t)$ is the fundamental matrix (solution) to the homogeneous system: $\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t)$; it is convention to assume $\prod_{j=k}^{k+1}(\mathbf{I}_n + \mathbf{E}_j)\mathbf{\Phi}(t_j, t_{j-1}) = \mathbf{I}_n$ and $t_f = t_{M+1}$.

Proof. The solution to the system (1) in $[t_0, t_1]$ is given by using the method of variation of parameters as

$$\mathbf{x}(t) = \mathbf{\Phi}(t, t_0)\mathbf{x}_0 + \int_{t_0}^{t} \mathbf{\Phi}(t,s)\mathbf{B}(s)\mathbf{u}(s)\mathrm{d}s. \tag{3}$$

This gives

$$\mathbf{x}(t_1) = \mathbf{\Phi}(t_1, t_0)\mathbf{x}_0 + \int_{t_0}^{t_1} \mathbf{\Phi}(t_1,s)\mathbf{B}(s)\mathbf{u}(s)\mathrm{d}s,$$

so that

$$\begin{aligned}\mathbf{x}(t_1^+) &= (\mathbf{I}_n + \mathbf{E}_1)\mathbf{x}(t_1) + \mathbf{F}_1\mathbf{u}(t_1)\\ &= (\mathbf{I}_n + \mathbf{E}_1)\left\{\mathbf{\Phi}(t_1, t_0)\mathbf{x}_0 + \int_{t_0}^{t_1} \mathbf{\Phi}(t_1,s)\mathbf{B}(s)\mathbf{u}(s)\mathrm{d}s\right\} + \mathbf{F}_1\mathbf{u}(t_1),\end{aligned}$$

using which we can write the solution to system (1) in $(t_1, t_2]$ as

$$\begin{aligned}\mathbf{x}(t) &= \mathbf{\Phi}(t, t_1)\mathbf{x}(t_1^+) + \int_{t_1}^{t} \mathbf{\Phi}(t,s)\mathbf{B}(s)\mathbf{u}(s)\mathrm{d}s\\ &= \mathbf{\Phi}(t, t_1)\Big\{(\mathbf{I}_n + \mathbf{E}_1)\mathbf{\Phi}(t_1, t_0)\mathbf{x}_0 + (\mathbf{I}_n + \mathbf{E}_1)\int_{t_0}^{t_1} \mathbf{\Phi}(t_1,s)\mathbf{B}(s)\mathbf{u}(s)\mathrm{d}s\\ &\qquad + \mathbf{F}_1\mathbf{u}(t_1)\Big\} + \int_{t_1}^{t} \mathbf{\Phi}(t,s)\mathbf{B}(s)\mathbf{u}(s)\mathrm{d}s,\end{aligned}$$

which is same as

$$\begin{aligned}\mathbf{x}(t) = \mathbf{\Phi}(t, t_1)\Bigg\{&\prod_{j=1}^{1}(\mathbf{I}_n + \mathbf{E}_j)\mathbf{\Phi}(t_j, t_{j-1})\mathbf{x}_0\\ &+ \sum_{i=1}^{1}\left(\prod_{j=1}^{i+1}(\mathbf{I}_n + \mathbf{E}_j)\mathbf{\Phi}(t_j, t_{j-1})\right)(\mathbf{I}_n + \mathbf{E}_i)\int_{t_{i-1}}^{t_i} \mathbf{\Phi}(t_i,s)\mathbf{B}(s)\mathbf{u}(s)\mathrm{d}s\\ &+ \sum_{i=2}^{1}\prod_{j=1}^{i}(\mathbf{I}_n + \mathbf{E}_j)\mathbf{\Phi}(t_j, t_{j-1})\mathbf{F}_{i-1}\mathbf{u}(t_{i-1}) + \mathbf{F}_1\mathbf{u}(t_1)\Bigg\}\\ &+ \int_{t_1}^{t} \mathbf{\Phi}(t,s)\mathbf{B}(s)\mathbf{u}(s)\mathrm{d}s,\end{aligned} \tag{4}$$

from which we get

$$\mathbf{x}(t_2) = \mathbf{\Phi}(t_2, t_1)\left\{(\mathbf{I}_n + \mathbf{E}_1)\mathbf{\Phi}(t_1, t_0)\mathbf{x}_0 + (\mathbf{I}_n + \mathbf{E}_1)\int_{t_0}^{t_1} \mathbf{\Phi}(t_1,s)\mathbf{B}(s)\mathbf{u}(s)\mathrm{d}s + \mathbf{F}_1\mathbf{u}(t_1)\right\}$$

$$+\int_{t_1}^{t_2}\mathbf{\Phi}(t_2,s)\mathbf{B}(s)\mathbf{u}(s)\mathrm{d}s,$$

so that

$$\begin{aligned}\mathbf{x}(t_2^+) &= (\mathbf{I}_n+\mathbf{E}_2)\mathbf{x}(t_2)+\mathbf{F}_2\mathbf{u}(t_2)\\ &= (\mathbf{I}_n+\mathbf{E}_2)\mathbf{\Phi}(t_2,t_1)(\mathbf{I}_n+\mathbf{E}_1)\mathbf{\Phi}(t_1,t_0)\mathbf{x}_0+\\ &\quad+(\mathbf{I}_n+\mathbf{E}_2)\mathbf{\Phi}(t_2,t_1)(\mathbf{I}_n+\mathbf{E}_1)\int_{t_0}^{t_1}\mathbf{\Phi}(t_1,s)\mathbf{B}(s)\mathbf{u}(s)\mathrm{d}s\\ &\quad+(\mathbf{I}_n+\mathbf{E}_2)\mathbf{\Phi}(t_2,t_1)\mathbf{F}_1\mathbf{u}(t_1)+(\mathbf{I}_n+\mathbf{E}_2)\int_{t_1}^{t_2}\mathbf{\Phi}(t_2,s)\mathbf{B}(s)\mathbf{u}(s)\mathrm{d}s+\mathbf{F}_2\mathbf{u}(t_2),\end{aligned}$$

using which the solution to system (1) in $(t_2,t_3]$ is given by

$$\begin{aligned}\mathbf{x}(t) &= \mathbf{\Phi}(t,t_2)\mathbf{x}(t_2^+)+\int_{t_2}^{t}\mathbf{\Phi}(t,s)\mathbf{B}(s)\mathbf{u}(s)\mathrm{d}s\\ &= \mathbf{\Phi}(t,t_2)\Big\{(\mathbf{I}_n+\mathbf{E}_2)\mathbf{\Phi}(t_2,t_1)(\mathbf{I}_n+\mathbf{E}_1)\mathbf{\Phi}(t_1,t_0)\mathbf{x}_0+\\ &\quad+(\mathbf{I}_n+\mathbf{E}_2)\mathbf{\Phi}(t_2,t_1)(\mathbf{I}_n+\mathbf{E}_1)\int_{t_0}^{t_1}\mathbf{\Phi}(t_1,s)\mathbf{B}(s)\mathbf{u}(s)\mathrm{d}s\\ &\quad+(\mathbf{I}_n+\mathbf{E}_2)\mathbf{\Phi}(t_2,t_1)\mathbf{F}_1\mathbf{u}(t_1)+(\mathbf{I}_n+\mathbf{E}_2)\int_{t_1}^{t_2}\mathbf{\Phi}(t_2,s)\mathbf{B}(s)\mathbf{u}(s)\mathrm{d}s\\ &\quad+\mathbf{F}_2\mathbf{u}(t_2)\Big\}+\int_{t_2}^{t}\mathbf{\Phi}(t,s)\mathbf{B}(s)\mathbf{u}(s)\mathrm{d}s,\end{aligned}$$

that is

$$\begin{aligned}\mathbf{x}(t) = \mathbf{\Phi}(t,t_2)\Big\{&\prod_{j=2}^{1}(\mathbf{I}_n+\mathbf{E}_j)\mathbf{\Phi}(t_j,t_{j-1})\mathbf{x}_0\\ &+\sum_{i=1}^{2}\left(\prod_{j=2}^{i+1}(\mathbf{I}_n+\mathbf{E}_j)\mathbf{\Phi}(t_j,t_{j-1})\right)(\mathbf{I}_n+\mathbf{E}_i)\int_{t_{i-1}}^{t_i}\mathbf{\Phi}(t_i,s)\mathbf{B}(s)\mathbf{u}(s)\mathrm{d}s\\ &+\sum_{i=2}^{2}\prod_{j=2}^{i}(\mathbf{I}_n+\mathbf{E}_j)\mathbf{\Phi}(t_j,t_{j-1})\mathbf{F}_{i-1}\mathbf{u}(t_{i-1})+\mathbf{F}_2\mathbf{u}(t_2)\Big\}+\int_{t_2}^{t}\mathbf{\Phi}(t,s)\mathbf{B}(s)\mathbf{u}(s)\mathrm{d}s.\end{aligned}\tag{5}$$

By observing eqs (4) and (5), and using the principle of mathematical induction, we can claim that the solution to system (1) in $(t_k,t_{k+1}]$ is given by eq (2). □

Lemma 2. If each $\mathbf{E}_k$ commutes with the state-transition matrix $\mathbf{\Phi}(t,s)$, that is

$$\mathbf{E}_k\mathbf{\Phi}(t,s)=\mathbf{\Phi}(t,s)\mathbf{E}_k,\ \forall\,k=1,2,\dots,M,$$

then the solution to system (1) given in eq (2) reduces to

$$\begin{aligned}\mathbf{x}(t) = \prod_{j=k}^{1}(\mathbf{I}_n+\mathbf{E}_j)\mathbf{\Phi}(t,t_0)\mathbf{x}_0 + \sum_{i=1}^{k}\prod_{j=k}^{i}(\mathbf{I}_n+\mathbf{E}_j)\int_{t_{i-1}}^{t_i}\mathbf{\Phi}(t,s)\mathbf{B}(s)\mathbf{u}(s)\mathrm{d}s \\ + \sum_{i=2}^{k}\prod_{j=k}^{i}(\mathbf{I}_n+\mathbf{E}_j)\mathbf{\Phi}(t,t_{i-1})\mathbf{F}_{i-1}\mathbf{u}(t_{i-1}) + \mathbf{\Phi}(t,t_k)\mathbf{F}_k\mathbf{u}(t_k) \\ + \int_{t_k}^{t}\mathbf{\Phi}(t,s)\mathbf{B}(s)\mathbf{u}(s)\mathrm{d}s, \quad t\in(t_k,t_{k+1}],\end{aligned} \tag{6}$$

where as usual we mean $\prod_{j=k}^{k+1}(\mathbf{I}_n+\mathbf{E}_j)\mathbf{\Phi}(t_j,t_{j-1}) = \mathbf{I}_n$.

P r o o f. Since this lemma discusses the special case of Lemma 1 under the condition that $\mathbf{\Phi}(\cdot,\cdot)$ commutes with each $\mathbf{E}_k$, so with each $(\mathbf{I}_n+\mathbf{E}_k)$, and by using the semigroup property of $\mathbf{\Phi}(\cdot,\cdot)$, the solution to system (1) in the time-duration $(t_k,t_{k+1}]$ given in eq (2) reduces to

$$\begin{aligned}\mathbf{x}(t) &= \mathbf{\Phi}(t,t_k)\prod_{j=k}^{1}(\mathbf{I}_n+\mathbf{E}_j)\mathbf{\Phi}(t_j,t_{j-1})\mathbf{x}_0 \\ &\quad + \sum_{i=1}^{k}\mathbf{\Phi}(t,t_k)\left(\prod_{j=k}^{i+1}(\mathbf{I}_n+\mathbf{E}_j)\mathbf{\Phi}(t_j,t_{j-1})\right)(\mathbf{I}_n+\mathbf{E}_i)\int_{t_{i-1}}^{t_i}\mathbf{\Phi}(t_i,s)\mathbf{B}(s)\mathbf{u}(s)\mathrm{d}s \\ &\quad + \sum_{i=2}^{k}\mathbf{\Phi}(t,t_k)\prod_{j=k}^{i}(\mathbf{I}_n+\mathbf{E}_j)\mathbf{\Phi}(t_j,t_{j-1})\mathbf{F}_{i-1}\mathbf{u}(t_{i-1}) + \mathbf{\Phi}(t,t_k)\mathbf{F}_k\mathbf{u}(t_k) \\ &\quad + \int_{t_k}^{t}\mathbf{\Phi}(t,s)\mathbf{B}(s)\mathbf{u}(s)\mathrm{d}s \\ &= \mathbf{\Phi}(t,t_k)(\mathbf{I}_n+\mathbf{E}_k)\mathbf{\Phi}(t_k,t_{k-1})\dots(\mathbf{I}_n+\mathbf{E}_1)\mathbf{\Phi}(t_1,t_0)\mathbf{x}_0 \\ &\quad + \sum_{i=1}^{k}\Bigg\{\mathbf{\Phi}(t,t_k)(\mathbf{I}_n+\mathbf{E}_k)\mathbf{\Phi}(t_k,t_{k-1})\dots(\mathbf{I}_n+\mathbf{E}_{i+1})\mathbf{\Phi}(t_{i+1},t_i)(\mathbf{I}_n+\mathbf{E}_i) \\ &\quad \times\int_{t_{i-1}}^{t_i}\mathbf{\Phi}(t_i,s)\mathbf{B}(s)\mathbf{u}(s)\mathrm{d}s\Bigg\} \\ &\quad + \sum_{i=2}^{k}\Big\{\mathbf{\Phi}(t,t_k)(\mathbf{I}_n+\mathbf{E}_k)\mathbf{\Phi}(t_k,t_{k-1})\dots(\mathbf{I}_n+\mathbf{E}_i)\mathbf{\Phi}(t_i,t_{i-1})\mathbf{F}_{i-1}\mathbf{u}(t_{i-1})\Big\} \\ &\quad + \mathbf{\Phi}(t,t_k)\mathbf{F}_k\mathbf{u}(t_k) + \int_{t_k}^{t}\mathbf{\Phi}(t,s)\mathbf{B}(s)\mathbf{u}(s)\mathrm{d}s \\ &= \prod_{j=k}^{1}(\mathbf{I}_n+\mathbf{E}_j)\mathbf{\Phi}(t,t_0)\mathbf{x}_0 + \sum_{i=1}^{k}\prod_{j=k}^{i}(\mathbf{I}_n+\mathbf{E}_j)\int_{t_{i-1}}^{t_i}\mathbf{\Phi}(t,s)\mathbf{B}(s)\mathbf{u}(s)\mathrm{d}s\end{aligned}$$

$$+\sum_{i=2}^{k}\prod_{j=k}^{i}(\mathbf{I}_n+\mathbf{E}_j)\mathbf{\Phi}(t,t_{i-1})\mathbf{F}_{i-1}\mathbf{u}(t_{i-1})+\mathbf{\Phi}(t,t_k)\mathbf{F}_k\mathbf{u}(t_k)$$
$$+\int_{t_k}^{t}\mathbf{\Phi}(t,s)\mathbf{B}(s)\mathbf{u}(s)\mathrm{d}s,$$

which is exactly same as eq (6). □

Remark 1. As a special case, when system (1) is autonomous and if $\mathbf{A}$ commutes with each $\mathbf{E}_k$, that is $\mathbf{A}\mathbf{E}_k=\mathbf{E}_k\mathbf{A}$, $\forall\, k=1,2,\ldots,M$, then state-transition matrix will also commute with each $\mathbf{E}_k$, as $\mathbf{\Phi}(t,s)=e^{\mathbf{A}(t-s)}$.

3. CONTROLLABILITY RESULTS

In this section, we obtain several sufficient and necessary conditions associated with the controllability of system (1) under various assumptions on the system parameters. First we recall the definition of complete controllability as follows.

Definition 1. (Complete controllability) The system (1) is said to be completely controllable in $\mathbb{R}^n$, over $[t_0,t_f]$, $0\leqq t_0<t_f<\infty$, if for each pair $(\mathbf{x}_0,\mathbf{x}_f)\in\mathbb{R}^n\times\mathbb{R}^n$, there exists some $\mathbf{u}(\cdot)\in\mathcal{PC}$ such that the corresponding solution of the system (1) with an initial state $\mathbf{x}(t_0)=\mathbf{x}_0$ satisfies $\mathbf{x}(t_f)=\mathbf{x}_f$. Such $\mathbf{u}(\cdot)$ is one of the control function to system (1) for driving its state from $\mathbf{x}_0$ to $\mathbf{x}_f$ over $[t_0,t_f]$.

Definition 2. (Null controllability) If $\mathbf{x}_f=\mathbf{0}$ in the above definition of complete controllability, then system (1) is said to be null controllable over $[t_0,t_f]$.

Remark 2. It is clear that, if system (1) is completely controllable over $[t_0,t_f]$, then it is also null controllable over $[t_0,t_f]$. But the converse need not be true (see Example 1).

Theorem 1. (Sufficient conditions) If one of the following conditions hold true, then system (1) is completely controllable in $\mathbb{R}^n$, over $[t_0,t_f]$.

(i) There exists some $l\in\{1,2,\ldots,(M-1)\}$, $M\geqq 2$ and an $(m\times n)$-matrix $\mathbf{F}'_l$ such that $\mathbf{F}_l\mathbf{F}'_l=\mathbf{I}_n$, and $(\mathbf{I}_n+\mathbf{E}_{l+1})$, $(\mathbf{I}_n+\mathbf{E}_{l+2}),\ldots,(\mathbf{I}_n+\mathbf{E}_M)$ are invertible.

(ii) There exists an $(m\times n)$-matrix $\mathbf{F}'_M$ such that $\mathbf{F}_M\mathbf{F}'_M=\mathbf{I}_n$.

(iii) There exists some $p\in\{1,2,\ldots,M\}$ such that $(\mathbf{I}_n+\mathbf{E}_p)$, $(\mathbf{I}_n+\mathbf{E}_{p+1}),\ldots,(\mathbf{I}_n+\mathbf{E}_M)$ are invertible, and $\int_{t_{p-1}}^{t_p}\big(\mathbf{\Phi}(t_p,s)\mathbf{B}(s)\big)\big(\mathbf{\Phi}(t_p,s)\mathbf{B}(s)\big)^T\mathrm{d}s$ is positive-definite.

(iv) $\int_{t_M}^{t_f}\big(\mathbf{\Phi}(t_f,s)\mathbf{B}(s)\big)\big(\mathbf{\Phi}(t_f,s)\mathbf{B}(s)\big)^T\mathrm{d}s$ is positive-definite.

P r o o f. (a) First we consider case (i). Without loss of generality, suppose there exists a $l\in\{1,2,\ldots,(M-1)\}$, $M\geqq 2$ and an $(m\times n)$-matrix $\mathbf{F}'_l$ such that $\mathbf{F}_l\mathbf{F}'_l=\mathbf{I}_n$, and $(\mathbf{I}_n+\mathbf{E}_{l+1})$, $(\mathbf{I}_n+\mathbf{E}_{l+2}),\dots,(\mathbf{I}_n+\mathbf{E}_M)$ are invertible. Then given an initial state $\mathbf{x}_0\in\mathbb{R}^n$ and a desired final state $\mathbf{x}_f\in\mathbb{R}^n$ of the system (1), by using a function

$$\mathbf{u}(t):=\begin{cases}\mathbf{F}'_l\left(\prod_{j=M}^{l+1}(\mathbf{I}_n+\mathbf{E}_j)\mathbf{\Phi}(t_j,t_{j-1})\right)^{-1}\mathbf{\Phi}(t_M,t_f)\\ \quad\times\left\{\mathbf{x}_f-\mathbf{\Phi}(t_f,t_M)\prod_{j=M}^{1}(\mathbf{I}_n+\mathbf{E}_j)\mathbf{\Phi}(t_j,t_{j-1})\mathbf{x}_0\right\}, & t=t_l,\\ \mathbf{0}, \quad t\in[t_0,t_f]\setminus\{t_l\},\end{cases}\tag{7}$$

the solution to system (1) given in eq (2) satisfies $\mathbf{x}(t_f)=\mathbf{x}_f$.

(b) Now we consider case (ii). Let there exists an $(m\times n)$-matrix $\mathbf{F}'_M$ such that $\mathbf{F}_M\mathbf{F}'_M=\mathbf{I}_n$. Then with an initial state $\mathbf{x}_0\in\mathbb{R}^n$ and a desired final state $\mathbf{x}_f\in\mathbb{R}^n$ of the system (1), by taking

$$\mathbf{u}(t):=\begin{cases}\mathbf{F}'_M\mathbf{\Phi}(t_M,t_f)\left\{\mathbf{x}_f-\mathbf{\Phi}(t_f,t_M)\prod_{j=M}^{1}(\mathbf{I}_n+\mathbf{E}_j)\mathbf{\Phi}(t_j,t_{j-1})\mathbf{x}_0\right\}, & t=t_M,\\ \mathbf{0}, \quad t\in[t_0,t_f]\setminus\{t_M\},\end{cases}\tag{8}$$

we can verify that its state given in eq (2) satisfies $\mathbf{x}(t_f)=\mathbf{x}_f$.

(c) To prove the case (iii), let there exists a $p\in\{1,2,\dots,M\}$ such that $(\mathbf{I}_n+\mathbf{E}_p)$, $(\mathbf{I}_n+\mathbf{E}_{p+1}),\dots,(\mathbf{I}_n+\mathbf{E}_M)$ are all invertible and $\mathbf{W}=\int_{t_{p-1}}^{t_p}\big(\mathbf{\Phi}(t_p,s)\mathbf{B}(s)\big)\big(\mathbf{\Phi}(t_p,s)\mathbf{B}(s)\big)^T\mathrm{d}s$ is positive-definite. Then the following function drives the state of system (1) given in eq (2) from $\mathbf{x}_0$ to $\mathbf{x}_f$ over $[t_0,t_f]$.

$$\mathbf{u}(t):=\begin{cases}\big(\mathbf{\Phi}(t_p,t)\mathbf{B}(t)\big)^T\mathbf{W}^{-1}(\mathbf{I}_n+\mathbf{E}_p)^{-1}\left(\prod_{j=M}^{p+1}(\mathbf{I}_n+\mathbf{E}_j)\mathbf{\Phi}(t_j,t_{j-1})\right)^{-1}\mathbf{\Phi}(t_M,t_f)\\ \quad\times\left\{\mathbf{x}_f-\mathbf{\Phi}(t_f,t_M)\prod_{j=M}^{1}(\mathbf{I}_n+\mathbf{E}_j)\mathbf{\Phi}(t_j,t_{j-1})\mathbf{x}_0\right\}, & t\in(t_{p-1},t_p),\\ \mathbf{0}, \quad t\in[t_0,t_f]\setminus(t_{p-1},t_p).\end{cases}\tag{9}$$

(d) Finally to prove case (iv), let $\mathbf{W}=\int_{t_M}^{t_f}\big(\mathbf{\Phi}(t_f,s)\mathbf{B}(s)\big)\big(\mathbf{\Phi}(t_f,s)\mathbf{B}(s)\big)^T\mathrm{d}s$ be positive-definite matrix. Then plugging the following function

$$\mathbf{u}(t):=\begin{cases}\big(\mathbf{\Phi}(t_f,t)\mathbf{B}(t)\big)^T\mathbf{W}^{-1}\left\{\mathbf{x}_f-\mathbf{\Phi}(t_f,t_M)\prod_{j=M}^{1}(\mathbf{I}_n+\mathbf{E}_j)\mathbf{\Phi}(t_j,t_{j-1})\mathbf{x}_0\right\}, & t\in(t_M,t_f],\\ \mathbf{0}, \quad t\in[t_0,t_f]\setminus(t_M,t_f],\end{cases}\tag{10}$$

into solution eq (2) of system (1), we get $\mathbf{x}(t_f)=\mathbf{x}_f$. All these show that, the functions used in eq (7), (8), (9) and (10) are the control functions to system (1) for driving its state from $\mathbf{x}_0$ to $\mathbf{x}_f$ over $[t_0,t_f]$ under the considered cases. □

Remark 3. As in the introduction we mentioned—if a linear impulsive system is null controllable, then it is not necessarily completely controllable, unlike the non-impulsive linear systems, hence we analyse the null controllability of system (1) rigorously. Though the conditions stated in Theorem 1 are sufficient for the complete controllability, and so, for null controllability of system (1), but one can still weaken these conditions for the null controllability case by waiving the invertibility of $(\mathbf{I}_n + \mathbf{E}_k)$'s unlike in Theorem 1.

Theorem 2. (Sufficient conditions) If one of the following conditions hold true, then system (1) is null controllable in $\mathbb{R}^n$, over $[t_0, t_f]$.

(i) There exists some $p \in \{1, 2, \ldots, M\}$ and an $(m \times n)$-matrix $\mathbf{F}'_p$ such that $\mathbf{F}_p \mathbf{F}'_p = \mathbf{I}_n$.

(ii) There exists a positive-definite matrix $\mathbf{W} = \int_{t_{q-1}}^{t_q} \big(\mathbf{\Phi}(t_q, s)\mathbf{B}(s)\big)\big(\mathbf{\Phi}(t_q, s)\mathbf{B}(s)\big)^T \mathrm{d}s$, for some $q \in \{1, 2, \ldots, (M+1)\}$.

P r o o f. (a) First we consider case (i). Without loss of generality, suppose there exists a $p \in \{1, 2, \ldots, M\}$ and an $(m \times n)$-matrix $\mathbf{F}'_p$ such that $\mathbf{F}_p \mathbf{F}'_p = \mathbf{I}_n$. Then given an initial state $\mathbf{x}_0 \in \mathbb{R}^n$, the function given by

$$\mathbf{u}(t) := \begin{cases} -\,\mathbf{F}'_p \displaystyle\prod_{j=p}^{1} (\mathbf{I}_n + \mathbf{E}_j)\mathbf{\Phi}(t_j, t_{j-1})\mathbf{x}_0, & t = t_p, \\ \mathbf{0}, \quad t \in [t_0, t_f] \setminus \{t_p\}, \end{cases} \tag{11}$$

is a control to system (1) for steering its state given in eq (2) from $\mathbf{x}_0$ to $\mathbf{0}$, over $[t_0, t_f]$.

(b) To prove the case (ii), let $\mathbf{W} = \int_{t_{q-1}}^{t_q} \big(\mathbf{\Phi}(t_q, s)\mathbf{B}(s)\big)\big(\mathbf{\Phi}(t_q, s)\mathbf{B}(s)\big)^T \mathrm{d}s$ be positive-definite matrix for some $q \in \{1, 2, \ldots, (M+1)\}$. Then the following is a control function to system (1) for driving its state from $\mathbf{x}_0$ to $\mathbf{0}$, over $[t_0, t_f]$.

$$\mathbf{u}(t) := \begin{cases} -\,\big(\mathbf{\Phi}(t_q, t)\mathbf{B}(t)\big)^T \mathbf{W}^{-1} \mathbf{\Phi}(t_q, t_{q-1}) \displaystyle\prod_{j=q-1}^{1} (\mathbf{I}_n + \mathbf{E}_j)\mathbf{\Phi}(t_j, t_{j-1})\mathbf{x}_0, \ t \in (t_{q-1}, t_q), \\ \mathbf{0}, \quad t \in [t_0, t_f] \setminus (t_{q-1}, t_q), \end{cases} \tag{12}$$

where $\prod_{j=0}^{1} (\mathbf{I}_n + \mathbf{E}_j)\mathbf{\Phi}(t_j, t_{j-1}) = \mathbf{I}_n$. □

The following example shows that a null controllable linear impulsive system is not completely controllable.

Example 1. Consider a LTI-system with two impulses:

$$\left.\begin{aligned} \dot{\mathbf{x}}(t) &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \mathbf{x}(t), \quad t \in [0, 3] \setminus \{1, 2\}, \\ \mathbf{x}(0) &= \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \\ \Delta\mathbf{x}(1) &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \end{bmatrix} \mathbf{u}(1), \\ \Delta\mathbf{x}(2) &= \begin{bmatrix} 1 & 4 \\ 3 & 5 \end{bmatrix} \mathbf{x}(2). \end{aligned}\right\} \tag{13}$$

In this system $n = 2$, $m = 3$, and $\mathbf{F}_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \end{bmatrix}$. Note that there exists a matrix $\mathbf{F}_1' = \begin{bmatrix} 1 & 1 & -2 \\ 0 & 0.5 & 0 \end{bmatrix}^T$ such that $\mathbf{F}_1\mathbf{F}_1' = \mathbf{I}_2$, therefore by condition (i) of Theorem 2, we conclude that system (13) is null controllable over $[0, 3]$. Further, the solution to system (13) is obtained from eq (2) as

$$\mathbf{x}(t) = \begin{cases} e^t \begin{bmatrix} 1 \\ 0 \end{bmatrix}, & t \in [0,1], \\ e^t \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} + e^{-1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \end{bmatrix} \mathbf{u}(1) \right\}, & t \in (1,2], \\ e^t \left\{ \begin{bmatrix} 2 \\ 3 \end{bmatrix} + e^{-1} \begin{bmatrix} 2 & 8 & 4 \\ 3 & 12 & 6 \end{bmatrix} \mathbf{u}(1) \right\}, & t \in (2,3]. \end{cases} \tag{14}$$

One of the control that drives the system (13) to the zero state can be computed from eq (11), and is found to be

$$\mathbf{u}(t) = \begin{cases} \begin{bmatrix} -e & -e & 2e \end{bmatrix}^T, & t = 1, \\ \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}^T, & t \in [0,3] \setminus \{1\}, \end{cases} \tag{15}$$

and for this $\mathbf{u}(t)$, the components $x_1(t)$ and $x_2(t)$ of the controlled trajectory $\mathbf{x}(t)$ computed in eq (14) are plotted in Figure 1. If the desired final state is $\begin{bmatrix} 1 & 1 \end{bmatrix}^T$, then

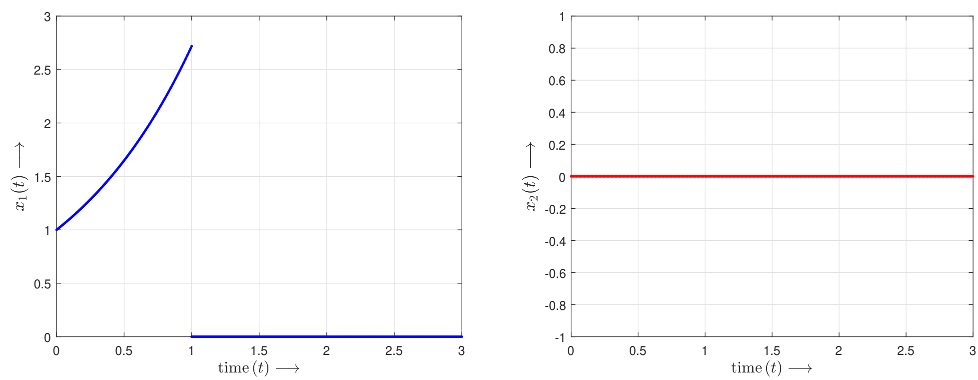

Fig. 1. Components of controlled trajectory in system (13) with the control given in (15)

the state function given in eq (14) reduces to: $\begin{bmatrix} 1 & 1 \end{bmatrix}^T = \mathbf{x}(3) = c\begin{bmatrix} 2 & 3 \end{bmatrix}^T$, that is $1 = 2c$ and $1 = 3c$, for any control $\mathbf{u}(1) = \begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix}^T$, where $a_1, a_2, a_3 \in \mathbb{R}$ and $c = e^3\big(1 + e^{-1}(a_1 + 4a_2 + 2a_3)\big)$. But the equations: $1 = 2c$ and $1 = 3c$ do not have a common solution for c. Hence we conclude that, there exists no control function for steering the state of system (13) from $\begin{bmatrix}1 & 0\end{bmatrix}^T$ to $\begin{bmatrix}1 & 1\end{bmatrix}^T$, showing that this system is not completely controllable in $\mathbb{R}^2$, over $[0,3]$.

Corollary 1. (Sufficient conditions) If in system (1), each $\mathbf{E}_k$ commutes with the state-transition matrix, i.e. $\mathbf{E}_k\mathbf{\Phi}(t,s) = \mathbf{\Phi}(t,s)\mathbf{E}_k$, $\forall\, k = 1,2,\dots,M$, then the sufficient conditions for complete controllability in $\mathbb{R}^n$, over $[t_0,t_f]$ of the system (1) given in Theorem 1 reduces to the following conditions:

(i) There exists some $l \in \{1,2,\dots,(M-1)\}$, $M \geqq 2$ and an $(m\times n)$-matrix $\mathbf{F}'_l$ such that $\mathbf{F}_l\mathbf{F}'_l = \mathbf{I}_n$, and $(\mathbf{I}_n+\mathbf{E}_{l+1})$, $(\mathbf{I}_n+\mathbf{E}_{l+2}),\dots,(\mathbf{I}_n+\mathbf{E}_M)$ are invertible.

(ii) There exists an $(m\times n)$-matrix $\mathbf{F}'_M$ such that $\mathbf{F}_M\mathbf{F}'_M = \mathbf{I}_n$.

(iii) There exists some $p \in \{1,2,\dots,M\}$ such that $(\mathbf{I}_n+\mathbf{E}_p),(\mathbf{I}_n+\mathbf{E}_{p+1}),\dots,(\mathbf{I}_n+\mathbf{E}_M)$ are invertible, and $\mathbf{W} = \int_{t_{p-1}}^{t_p} \big(\mathbf{\Phi}(t_f,s)\mathbf{B}(s)\big)\big(\mathbf{\Phi}(t_f,s)\mathbf{B}(s)\big)^T \mathrm{d}s$ is positive-definite.

(iv) $\mathbf{W} = \int_{t_M}^{t_f} \big(\mathbf{\Phi}(t_f,s)\mathbf{B}(s)\big)\big(\mathbf{\Phi}(t_f,s)\mathbf{B}(s)\big)^T \mathrm{d}s$ is positive-definite.

Proof. The proof is similar to that of Theorem 1, hence we skip the details. However we provide one of the control function that steers the state of system (1) given in eq (6), from $\mathbf{x}_0$ to $\mathbf{x}_f$. These are as follows. For the case (i),

$$\mathbf{u}(t) := \begin{cases} \mathbf{F}'_l\mathbf{\Phi}(t_l,t_f)\left(\displaystyle\prod_{j=M}^{l+1}(\mathbf{I}_n+\mathbf{E}_j)\right)^{-1}\left\{\mathbf{x}_f - \displaystyle\prod_{j=M}^{1}(\mathbf{I}_n+\mathbf{E}_j)\mathbf{\Phi}(t_f,t_0)\mathbf{x}_0\right\}, & t = t_l,\\ \mathbf{0}, \quad t \in [t_0,t_f]\setminus\{t_l\}, \end{cases} \tag{16}$$

for the case (ii),

$$\mathbf{u}(t) := \begin{cases} \mathbf{F}'_M\mathbf{\Phi}(t_M,t_f)\left\{\mathbf{x}_f - \displaystyle\prod_{j=M}^{1}(\mathbf{I}_n+\mathbf{E}_j)\mathbf{\Phi}(t_f,t_0)\mathbf{x}_0\right\}, & t = t_M,\\ \mathbf{0}, \quad t \in [t_0,t_f]\setminus\{t_M\}, \end{cases} \tag{17}$$

for the case (iii),

$$\mathbf{u}(t) := \begin{cases} \big(\mathbf{\Phi}(t_f,t)\mathbf{B}(t)\big)^T\mathbf{W}^{-1}\left(\displaystyle\prod_{j=M}^{p}(\mathbf{I}_n+\mathbf{E}_j)\right)^{-1}\left\{\mathbf{x}_f - \displaystyle\prod_{j=M}^{1}(\mathbf{I}_n+\mathbf{E}_j)\mathbf{\Phi}(t_f,t_0)\mathbf{x}_0\right\},\\ \qquad\qquad\qquad\qquad\qquad t \in (t_{p-1},t_p),\\ \mathbf{0}, \quad t \in [t_0,t_f]\setminus(t_{p-1},t_p), \end{cases} \tag{18}$$

and finally for the case (iv), it is

$$\mathbf{u}(t) := \begin{cases} \big(\mathbf{\Phi}(t_f,t)\mathbf{B}(t)\big)^T\mathbf{W}^{-1}\left\{\mathbf{x}_f - \displaystyle\prod_{j=M}^{1}(\mathbf{I}_n+\mathbf{E}_j)\mathbf{\Phi}(t_f,t_0)\mathbf{x}_0\right\}, & t \in (t_M,t_f],\\ \mathbf{0}, \quad t \in [t_0,t_f]\setminus(t_M,t_f]. \end{cases} \tag{19}$$

□

Corollary 2. (Sufficient conditions) If $\mathbf{E}_k\mathbf{\Phi}(t,s) = \mathbf{\Phi}(t,s)\mathbf{E}_k$, $\forall\, k = 1,2,\ldots,M$ in system (1), then the sufficient conditions for its null controllability in $\mathbb{R}^n$, over $[t_0, t_f]$ given in Theorem 2 reduces to the following conditions:

(i) There exists some $p \in \{1,2,\ldots,M\}$ and an $(m\times n)$-matrix $\mathbf{F}_l'$ such that $\mathbf{F}_l\mathbf{F}_l' = \mathbf{I}_n$.

(ii) The matrix $\mathbf{W} = \int_{t_{q-1}}^{t_q} \big(\mathbf{\Phi}(t_f,s)\mathbf{B}(s)\big)\big(\mathbf{\Phi}(t_f,s)\mathbf{B}(s)\big)^T \mathrm{d}s$ is positive-definite for some $q \in \{1,2,\ldots,(M+1)\}$.

Proof. The proof is similar to Theorem 2. Under case (i), we can consider the control function

$$\mathbf{u}(t) := \begin{cases} -\mathbf{F}_l' \displaystyle\prod_{j=l}^{1}(\mathbf{I}_n + \mathbf{E}_j)\mathbf{\Phi}(t_l,t_0)\mathbf{x}_0, & t = t_l, \\ \mathbf{0}, \quad t \in [t_0,t_f] \setminus \{t_l\}, \end{cases} \tag{20}$$

to steer the state of system (1) given in eq (6) from $\mathbf{x}_0$ to $\mathbf{0}$.

For case (ii), we can take the control function as

$$\mathbf{u}(t) := \begin{cases} -\big(\mathbf{\Phi}(t_f,t)\mathbf{B}(t)\big)^T\mathbf{W}^{-1} \displaystyle\prod_{j=q-1}^{1}(\mathbf{I}_n + \mathbf{E}_j)\mathbf{\Phi}(t_f,t_0)\mathbf{x}_0, & t \in (t_{q-1},t_q), \\ \mathbf{0}, \quad t \in [t_0,t_f] \setminus (t_{q-1},t_q), \end{cases} \tag{21}$$

to verify $\mathbf{x}(t_f) = \mathbf{0}$ in eq (6). □

The theorems and corollaries introduced so far in this section provide the sufficient conditions under which the system (1) is controllable. We now investigate the necessary and sufficient controllability condition for the system (1) by letting $(\mathbf{I}_n + \mathbf{E}_k)$ as nonsingular matrices for every $k = 1,2,\ldots,M$. First we introduce the following positivesemidefinite $(n \times n)$-matrices:

$$\left.\begin{aligned}
\mathbf{W}_k = \mathbf{W}(t_{k-1}, t_k) &:= \int_{t_{k-1}}^{t_k} \left\{ \left(\prod_{j=k}^{1} (\mathbf{I}_n + \mathbf{E}_j) \boldsymbol{\Phi}(t_j, t_{j-1}) \right)^{-1} (\mathbf{I}_n + \mathbf{E}_k) \boldsymbol{\Phi}(t_k, s) \mathbf{B}(s) \right\} \\
&\qquad \times \left\{ \left(\prod_{j=k}^{1} (\mathbf{I}_n + \mathbf{E}_j) \boldsymbol{\Phi}(t_j, t_{j-1}) \right)^{-1} (\mathbf{I}_n + \mathbf{E}_k) \boldsymbol{\Phi}(t_k, s) \mathbf{B}(s) \right\}^T \mathrm{d}s, \\
\mathbf{W}_{M+1} = \mathbf{W}(t_M, t_f) &:= \int_{t_M}^{t_f} \left\{ \left(\prod_{j=M}^{1} (\mathbf{I}_n + \mathbf{E}_j) \boldsymbol{\Phi}(t_j, t_{j-1}) \right)^{-1} \boldsymbol{\Phi}(t_M, s) \mathbf{B}(s) \right\} \\
&\qquad \times \left\{ \left(\prod_{j=M}^{1} (\mathbf{I}_n + \mathbf{E}_j) \boldsymbol{\Phi}(t_j, t_{j-1}) \right)^{-1} \boldsymbol{\Phi}(t_M, s) \mathbf{B}(s) \right\}^T \mathrm{d}s, \\
\mathbf{V}_1 = \mathbf{V}(t_0, t_1) &:= \int_{t_0}^{t_1} \big(\boldsymbol{\Phi}(t_0, s) \mathbf{B}(s) \big) \big(\boldsymbol{\Phi}(t_0, s) \mathbf{B}(s) \big)^T \mathrm{d}s, \\
\mathbf{V}_{k+1} = \mathbf{V}(t_k, t_{k+1}) &:= \int_{t_k}^{t_{k+1}} \left\{ \left(\prod_{j=k}^{1} (\mathbf{I}_n + \mathbf{E}_j) \right)^{-1} \boldsymbol{\Phi}(t_0, s) \mathbf{B}(s) \right\} \\
&\qquad \times \left\{ \left(\prod_{j=k}^{1} (\mathbf{I}_n + \mathbf{E}_j) \right)^{-1} \boldsymbol{\Phi}(t_0, s) \mathbf{B}(s) \right\}^T \mathrm{d}s, \\
\mathbf{G}_k &:= \left\{ \left(\prod_{j=k}^{1} (\mathbf{I}_n + \mathbf{E}_j) \boldsymbol{\Phi}(t_j, t_{j-1}) \right)^{-1} \mathbf{F}_k \right\} \left\{ \left(\prod_{j=k}^{1} (\mathbf{I}_n + \mathbf{E}_j) \boldsymbol{\Phi}(t_j, t_{j-1}) \right)^{-1} \mathbf{F}_k \right\}^T, \\
\mathbf{H}_k &:= \left\{ \left(\prod_{j=k}^{1} (\mathbf{I}_n + \mathbf{E}_j) \right)^{-1} \boldsymbol{\Phi}(t_0, t_k) \mathbf{F}_k \right\} \left\{ \left(\prod_{j=k}^{1} (\mathbf{I}_n + \mathbf{E}_j) \right)^{-1} \boldsymbol{\Phi}(t_0, t_k) \mathbf{F}_k \right\}^T,
\end{aligned}\right\} \tag{22}$$

where $k = 1, 2, \ldots, M$ and $t_{M+1} = t_f$.

Theorem 3. (Necessary and Sufficient condition) If each $(\mathbf{I}_n + \mathbf{E}_k)$ is nonsingular, then system (1) is completely controllable in $\mathbb{R}^n$, over $[t_0, t_f]$ if and only if

$$\operatorname{rank}\big(\mathbf{W}_1,\ \mathbf{W}_2, \ldots, \mathbf{W}_{M+1},\ \mathbf{G}_1,\ \mathbf{G}_2, \ldots, \mathbf{G}_M\big) = n. \tag{23}$$

In addition, if each $\mathbf{E}_k$ commutes with the state-transition matrix $\boldsymbol{\Phi}(t, s)$, then the above necessary and sufficient condition for the complete controllability reduces to

$$\operatorname{rank}\big(\mathbf{V}_1,\ \mathbf{V}_2, \ldots, \mathbf{V}_{M+1},\ \mathbf{H}_1,\ \mathbf{H}_2, \ldots, \mathbf{H}_M\big) = n. \tag{24}$$

P r o o f. First we prove that the condition in (23) is necessary as well as sufficient for the system (1) to be completely controllable, where it is given that all $(\mathbf{I}_n + \mathbf{E}_k)$'s are nonsingular. The necessity of this condition can be proved by contradiction. For this, let the system (1) be completely controllable over $[t_0, t_f]$, but assume that

$$\operatorname{rank}\big(\mathbf{W}_1,\, \mathbf{W}_2, \ldots, \mathbf{W}_{M+1},\, \mathbf{G}_1,\, \mathbf{G}_2, \ldots, \mathbf{G}_M\big) < n.$$

Then a homogeneous system:

$$\begin{pmatrix} \mathbf{W}_1 & \mathbf{W}_2 & \ldots & \mathbf{W}_{M+1} & \mathbf{G}_1 & \mathbf{G}_2 & \ldots & \mathbf{G}_M \end{pmatrix}^T \mathbf{z} = \mathbf{0} \in \mathbb{R}^{(2M+1)n}$$

have at least one nonzero solution $\mathbf{z} \in \mathbb{R}^n$. Further this $\mathbf{z}$ also satisfies the equations:

$$\mathbf{W}_k \mathbf{z} = \mathbf{0},\ \mathbf{W}_{M+1}\mathbf{z} = \mathbf{0}, \text{ and } \mathbf{G}_k\, \mathbf{z} = \mathbf{0},\ \forall\, k = 1, 2, \ldots, M,$$

from which we can obtain

$$\begin{cases} \displaystyle\int_{t_{k-1}}^{t_k} \left\| \mathbf{z}^T \left(\prod_{j=k}^{1} (\mathbf{I}_n + \mathbf{E}_j)\mathbf{\Phi}(t_j, t_{j-1}) \right)^{-1} (\mathbf{I}_n + \mathbf{E}_k)\mathbf{\Phi}(t_k, s)\mathbf{B}(s) \right\|_{\mathbb{R}^{1\times m}}^2 \mathrm{d}s = 0, \\ \displaystyle\int_{t_M}^{t_f} \left\| \mathbf{z}^T \left(\prod_{j=M}^{1} (\mathbf{I}_n + \mathbf{E}_j)\mathbf{\Phi}(t_j, t_{j-1}) \right)^{-1} \mathbf{\Phi}(t_M, s)\mathbf{B}(s) \right\|_{\mathbb{R}^{1\times m}}^2 \mathrm{d}s = 0, \\ \displaystyle\left\| \mathbf{z}^T \left(\prod_{j=k}^{1} (\mathbf{I}_n + \mathbf{E}_j)\mathbf{\Phi}(t_j, t_{j-1}) \right)^{-1} \mathbf{F}_k \right\|_{\mathbb{R}^{1\times m}}^2 = 0, \end{cases}$$

for all $k = 1, 2, \ldots, M$. Since the integrands in the above integrals are non-negative continuous functions over their domains, hence it follows that

$$\begin{cases} \displaystyle\mathbf{z}^T \left(\prod_{j=k}^{1} (\mathbf{I}_n + \mathbf{E}_j)\mathbf{\Phi}(t_j, t_{j-1}) \right)^{-1} (\mathbf{I}_n + \mathbf{E}_k)\mathbf{\Phi}(t_k, s)\mathbf{B}(s) = \mathbf{O} \in \mathbb{R}^{1\times m},\ \forall\, s \in (t_{k-1}, t_k), \\ \displaystyle\mathbf{z}^T \left(\prod_{j=M}^{1} (\mathbf{I}_n + \mathbf{E}_j)\mathbf{\Phi}(t_j, t_{j-1}) \right)^{-1} \mathbf{\Phi}(t_M, s)\mathbf{B}(s) = \mathbf{O} \in \mathbb{R}^{1\times m},\ \forall\, s \in (t_M, t_f], \\ \displaystyle\mathbf{z}^T \left(\prod_{j=k}^{1} (\mathbf{I}_n + \mathbf{E}_j)\mathbf{\Phi}(t_j, t_{j-1}) \right)^{-1} \mathbf{F}_k = \mathbf{O} \in \mathbb{R}^{1\times m}, \end{cases} \tag{25}$$

for all $k = 1, 2, \ldots, M$. Here for convention we take (t_{k-1}, t_k) as $[t_0, t_1)$ when $k = 1$. Now as the system (1) is completely controllable over $[t_0, t_f]$, hence in particular it is null controllable, and therefore there exists a control function $\mathbf{u}(\cdot) \in \mathcal{PC}$ that steers the state of system (1) given in eq (2) from $\mathbf{x}(t_0) = \mathbf{z}$ to $\mathbf{x}(t_f) = \mathbf{0}$; that is to say

$$\begin{aligned} \mathbf{0} = \mathbf{x}(t_f) = \mathbf{\Phi}(t_f, t_M) \Bigg\{ & \prod_{j=M}^{1} (\mathbf{I}_n + \mathbf{E}_j)\mathbf{\Phi}(t_j, t_{j-1})\mathbf{z} \\ & + \sum_{i=1}^{M} \left(\prod_{j=M}^{i+1} (\mathbf{I}_n + \mathbf{E}_j)\mathbf{\Phi}(t_j, t_{j-1}) \right) (\mathbf{I}_n + \mathbf{E}_i) \int_{t_{i-1}}^{t_i} \mathbf{\Phi}(t_i, s)\mathbf{B}(s)\mathbf{u}(s)\mathrm{d}s \end{aligned}$$

$$
\begin{aligned}
&+\sum_{i=2}^{M}\prod_{j=M}^{i}(\mathbf{I}_n+\mathbf{E}_j)\boldsymbol{\Phi}(t_j,t_{j-1})\mathbf{F}_{i-1}\mathbf{u}(t_{i-1})+\mathbf{F}_M\mathbf{u}(t_M)\Bigg\}\\
&+\int_{t_M}^{t_f}\boldsymbol{\Phi}(t_f,s)\mathbf{B}(s)\mathbf{u}(s)\mathrm{d}s\\
=\;&\boldsymbol{\Phi}(t_f,t_M)\prod_{j=M}^{1}(\mathbf{I}_n+\mathbf{E}_j)\boldsymbol{\Phi}(t_j,t_{j-1})\\
&\times\Bigg\{\mathbf{z}+\sum_{k=1}^{M}\int_{t_{k-1}}^{t_k}\left(\prod_{j=k}^{1}(\mathbf{I}_n+\mathbf{E}_j)\boldsymbol{\Phi}(t_j,t_{j-1})\right)^{-1}(\mathbf{I}_n+\mathbf{E}_k)\boldsymbol{\Phi}(t_k,s)\mathbf{B}(s)\mathbf{u}(s)\mathrm{d}s\\
&+\int_{t_M}^{t_f}\left(\prod_{j=M}^{1}(\mathbf{I}_n+\mathbf{E}_j)\boldsymbol{\Phi}(t_j,t_{j-1})\right)^{-1}\boldsymbol{\Phi}(t_M,s)\mathbf{B}(s)\mathbf{u}(s)\mathrm{d}s\\
&+\sum_{k=1}^{M}\left(\prod_{j=k}^{1}(\mathbf{I}_n+\mathbf{E}_j)\boldsymbol{\Phi}(t_j,t_{j-1})\right)^{-1}\mathbf{F}_k\mathbf{u}(t_k)\Bigg\}.
\end{aligned}
$$

Premultiply the above step with $\mathbf{z}^T\big(\prod_{j=M}^{1}(\mathbf{I}_n+\mathbf{E}_j)\boldsymbol{\Phi}(t_j,t_{j-1})\big)^{-1}\boldsymbol{\Phi}(t_M,t_f)$ and using the results of eq (25), we obtain $\mathbf{z}^T\mathbf{z}=0$ which implies $\mathbf{z}=\mathbf{0}$, a contradiction. Therefore if the system (1) is completely controllable, then $\text{rank}\big(\mathbf{W}_1,\ldots,\mathbf{W}_{M+1},\,\mathbf{G}_1,\ldots,\mathbf{G}_M\big)=n$.

To prove the sufficiency of condition in (23), let $\text{rank}\big(\mathbf{W}_1,\ldots,\mathbf{W}_{M+1},\,\mathbf{G}_1,\ldots,\mathbf{G}_M\big)=n$. Denote $\mathbf{W}=\mathbf{W}_1+\mathbf{W}_2+\cdots+\mathbf{W}_{M+1}+\mathbf{G}_1+\mathbf{G}_2+\cdots+\mathbf{G}_M$, then

$$\text{rank}\big(\mathbf{W}_1,\,\mathbf{W}_2,\ldots,\mathbf{W}_{M+1},\,\mathbf{G}_1,\,\mathbf{G}_2,\ldots,\mathbf{G}_M\big)=\text{rank}(\mathbf{W}),$$

so $\mathbf{W}$ is positive-definite (refer Lemma 3.1 in Muni and George [8]). Now for a given initial state $\mathbf{x}_0 \in \mathbb{R}^n$ and a final state $\mathbf{x}_f \in \mathbb{R}^n$ of system (1), define a function:

$$\mathbf{u}(t) := \begin{cases} \left\{ \left(\prod_{j=k}^{1} (\mathbf{I}_n + \mathbf{E}_j)\mathbf{\Phi}(t_j, t_{j-1}) \right)^{-1} (\mathbf{I}_n + \mathbf{E}_k)\mathbf{\Phi}(t_k, t)\mathbf{B}(t) \right\}^T \mathbf{W}^{-1} \\ \quad \times \left\{ -\mathbf{x}_0 + \left(\prod_{j=M}^{1} (\mathbf{I}_n + \mathbf{E}_j)\mathbf{\Phi}(t_j, t_{j-1}) \right)^{-1} \mathbf{\Phi}(t_M, t_f)\mathbf{x}_f \right\}, \quad t \in (t_{k-1}, t_k), \\ \left\{ \left(\prod_{j=M}^{1} (\mathbf{I}_n + \mathbf{E}_j)\mathbf{\Phi}(t_j, t_{j-1}) \right)^{-1} \mathbf{\Phi}(t_M, t)\mathbf{B}(t) \right\}^T \mathbf{W}^{-1} \\ \quad \times \left\{ -\mathbf{x}_0 + \left(\prod_{j=M}^{1} (\mathbf{I}_n + \mathbf{E}_j)\mathbf{\Phi}(t_j, t_{j-1}) \right)^{-1} \mathbf{\Phi}(t_M, t_f)\mathbf{x}_f \right\}, \quad t \in (t_M, t_f], \\ \left\{ \left(\prod_{j=k}^{1} (\mathbf{I}_n + \mathbf{E}_j)\mathbf{\Phi}(t_j, t_{j-1}) \right)^{-1} \mathbf{F}_k \right\}^T \mathbf{W}^{-1} \\ \quad \times \left\{ -\mathbf{x}_0 + \left(\prod_{j=M}^{1} (\mathbf{I}_n + \mathbf{E}_j)\mathbf{\Phi}(t_j, t_{j-1}) \right)^{-1} \mathbf{\Phi}(t_M, t_f)\mathbf{x}_f \right\}, \quad t = t_k, \end{cases} \tag{26}$$

where $k = 1, 2, \ldots, M$. Here for convention we take (t_{k-1}, t_k) as $[t_0, t_1)$ when $k = 1$. Now plug-in (26) in eq (2), we obtain $\mathbf{x}(t_f) = \mathbf{x}_f$, showing that system (1) is completely controllable over $[t_0, t_f]$.

Now we show that the condition given in (24) is necessary and sufficient for the complete controllability of (1) over $[t_0, t_f]$, under the assumptions that $(\mathbf{I}_n + \mathbf{E}_k)$'s are invertible and $\mathbf{E}_k\mathbf{\Phi}(t, s) = \mathbf{\Phi}(t, s)\mathbf{E}_k$, for all k. Similar to the first part of this theorem, the necessity of this condition can be proved by contradiction. For this, let the system (1) be completely controllable over $[t_0, t_f]$, but assume

$$\operatorname{rank}\big(\mathbf{V}_1,\, \mathbf{V}_2, \ldots, \mathbf{V}_{M+1},\, \mathbf{H}_1,\, \mathbf{H}_2, \ldots, \mathbf{H}_M\big) < n.$$

But then, there exists a nonzero vector $\mathbf{z} \in \mathbb{R}^n$ such that

$$\mathbf{V}_1\mathbf{z} = \mathbf{0},\ \mathbf{V}_{k+1}\,\mathbf{z} = \mathbf{0}, \text{ and } \mathbf{H}_k\,\mathbf{z} = \mathbf{0},$$

which implies

$$\begin{cases} \int_{t_0}^{t_1} \left\| \mathbf{z}^T\mathbf{\Phi}(t_0, s)\mathbf{B}(s) \right\|^2_{\mathbb{R}^{1\times m}} \mathrm{d}s = 0, \\ \int_{t_k}^{t_{k+1}} \left\| \mathbf{z}^T \left(\prod_{j=k}^{1} (\mathbf{I}_n + \mathbf{E}_j) \right)^{-1} \mathbf{\Phi}(t_0, s)\mathbf{B}(s) \right\|^2_{\mathbb{R}^{1\times m}} \mathrm{d}s = 0, \\ \left\| \mathbf{z}^T \left(\prod_{j=k}^{1} (\mathbf{I}_n + \mathbf{E}_j) \right)^{-1} \mathbf{\Phi}(t_0, t_k)\mathbf{F}_k \right\|^2_{\mathbb{R}^{1\times m}} = 0, \end{cases}$$

for all $k = 1, 2, \ldots, M$. Since the integrands in the above integrals are non-negative continuous functions over their domains, hence it follows that

$$\begin{cases} \mathbf{z}^T \mathbf{\Phi}(t_0, s)\mathbf{B}(s) = \mathbf{O} \in \mathbb{R}^{1\times m}, \quad \forall\, s \in [t_0, t_1), \\ \mathbf{z}^T \left(\prod_{j=k}^{1} (\mathbf{I} + \mathbf{E}_j)\right)^{-1} \mathbf{\Phi}(t_0, s)\mathbf{B}(s) = \mathbf{O} \in \mathbb{R}^{1\times m}, \quad \forall\, s \in (t_k, t_{k+1}), \\ \mathbf{z}^T \left(\prod_{j=k}^{1} (\mathbf{I} + \mathbf{E}_j)\right)^{-1} \mathbf{\Phi}(t_0, t_k)\mathbf{F}_k = \mathbf{O} \in \mathbb{R}^{1\times m}, \end{cases} \tag{27}$$

for all $k = 1, 2, \ldots, M$. Now as the system (1) is completely controllable over $[t_0, t_f]$, hence in particular it is null controllable. Therefore there exists a control function $\mathbf{u}(\cdot) \in \mathcal{PC}$ that steers the state of system (1) given in eq (6) from $\mathbf{x}(t_0) = \mathbf{z}$ to $\mathbf{x}(t_f) = \mathbf{0}$. That is,

$$\begin{aligned} \mathbf{0} = {} & \prod_{j=M}^{1} (\mathbf{I}_n + \mathbf{E}_j)\mathbf{\Phi}(t_f, t_0)\mathbf{z} + \sum_{i=1}^{M} \prod_{j=M}^{i} (\mathbf{I} + \mathbf{E}_j) \int_{t_{i-1}}^{t_i} \mathbf{\Phi}(t_f, s)\mathbf{B}(s)\mathbf{u}(s)\mathrm{d}s \\ & + \sum_{i=2}^{M} \prod_{j=M}^{i} (\mathbf{I}_n + \mathbf{E}_j)\mathbf{\Phi}(t_f, t_{i-1})\mathbf{F}_{i-1}\mathbf{u}(t_{i-1}) + \mathbf{\Phi}(t_f, t_M)\mathbf{F}_M\mathbf{u}(t_M) \\ & + \int_{t_M}^{t_f} \mathbf{\Phi}(t_f, s)\mathbf{B}(s)\mathbf{u}(s)\mathrm{d}s \\ = {} & \prod_{j=M}^{1} (\mathbf{I}_n + \mathbf{E}_j)\mathbf{\Phi}(t_f, t_0)\Bigg\{ \mathbf{z} + \int_{t_0}^{t_1} \mathbf{\Phi}(t_0, s)\mathbf{B}(s)\mathbf{u}(s)\mathrm{d}s \\ & + \sum_{k=1}^{M} \int_{t_k}^{t_{k+1}} \left(\prod_{j=k}^{1} (\mathbf{I}_n + \mathbf{E}_j)\right)^{-1} \mathbf{\Phi}(t_0, s)\mathbf{B}(s)\mathbf{u}(s)\mathrm{d}s \\ & + \sum_{k=1}^{M} \left(\prod_{j=k}^{1} (\mathbf{I}_n + \mathbf{E}_j)\right)^{-1} \mathbf{\Phi}(t_0, t_k)\mathbf{F}_k\mathbf{u}(t_k) \Bigg\}. \end{aligned}$$

Premultiply the above expression with $\mathbf{z}^T\mathbf{\Phi}(t_0, t_f)\big(\prod_{j=M}^{1}(\mathbf{I}+\mathbf{E}_j)\big)^{-1}$ and using eq (27), we obtain $0 = \mathbf{z}^T\mathbf{z} \Longrightarrow \mathbf{z} = \mathbf{0}$, a contradiction. Therefore if the system (1) is completely controllable, then $\operatorname{rank}\big(\mathbf{V}_1,\, \mathbf{V}_2, \ldots, \mathbf{V}_{M+1},\, \mathbf{H}_1,\, \mathbf{H}_2, \ldots, \mathbf{H}_M\big) = n$.

For the converse, let $\operatorname{rank}\big(\mathbf{V}_1,\, \mathbf{V}_2, \ldots, \mathbf{V}_{M+1},\, \mathbf{H}_1,\, \mathbf{H}_2, \ldots, \mathbf{H}_M\big) = n$, so that $\mathbf{W} = \mathbf{V}_1 + \mathbf{V}_2 + \cdots + \mathbf{V}_{M+1} + \mathbf{H}_1 + \mathbf{H}_2 + \cdots + \mathbf{H}_M$ is positive-definite. Now in order to steer the state of system (1) given in eq (6) from $\mathbf{x}_0$ to $\mathbf{x}_f$, we apply the following control:

$$\mathbf{u}(t) := \begin{cases} \big(\mathbf{\Phi}(t_0,t)\mathbf{B}(t)\big)^T\mathbf{W}^{-1}\Bigg\{-\mathbf{x}_0 + \left(\prod_{j=M}^{1}(\mathbf{I}_n+\mathbf{E}_j)\mathbf{\Phi}(t_f,t_0)\right)^{-1}\mathbf{x}_f\Bigg\}, & t\in[t_0,t_1), \\ \Bigg\{\left(\prod_{j=k}^{1}(\mathbf{I}_n+\mathbf{E}_j)\right)^{-1}\mathbf{\Phi}(t_0,t)\mathbf{B}(t)\Bigg\}^T\mathbf{W}^{-1} & \\ \quad\times\Bigg\{-\mathbf{x}_0 + \left(\prod_{j=M}^{1}(\mathbf{I}_n+\mathbf{E}_j)\mathbf{\Phi}(t_f,t_0)\right)^{-1}\mathbf{x}_f\Bigg\}, & t\in(t_k,t_{k+1}), \\ \Bigg\{\left(\prod_{j=k}^{1}(\mathbf{I}_n+\mathbf{E}_j)\right)^{-1}\mathbf{\Phi}(t_0,t_k)\mathbf{F}_k\Bigg\}^T\mathbf{W}^{-1} & \\ \quad\times\Bigg\{-\mathbf{x}_0 + \left(\prod_{j=M}^{1}(\mathbf{I}_n+\mathbf{E}_j)\mathbf{\Phi}(t_f,t_0)\right)^{-1}\mathbf{x}_f\Bigg\}, & t=t_k, \end{cases} \tag{28}$$

where $k = 1, 2, \ldots, M$. □

4. CONTROLLABILITY RESULTS FOR A TIME-INVARIANT SYSTEM

In this section, we reduce the controllability conditions obtained in Section 3 to that for the time-invariant case of the system (1) under some assumptions on the system components. The following theorems accomplish this. Here necessary and sufficient conditions for the complete controllability are given separately.

Theorem 4. (Necessary conditions) Let all $(\mathbf{I}+\mathbf{E}_k)$'s be nonsingular matrices and each $\mathbf{E}_k$ commutes with $\mathbf{A}$. If system (1) is completely controllable in $\mathbb{R}^n$, over $[t_0, t_f]$, then the following conditions hold true:

(i) $\operatorname{rank}(\mathrm{P}) = n$, where

$$\begin{aligned} \mathrm{P} := \Bigg\{ & \mathbf{B},\ \mathbf{AB},\ \mathbf{A}^2\mathbf{B},\ldots,\mathbf{A}^{n-1}\mathbf{B}, \\ & (\mathbf{I}_n+\mathbf{E}_1)^{-1}\big(\mathbf{B},\ \mathbf{AB},\ \mathbf{A}^2\mathbf{B},\ldots,\mathbf{A}^{n-1}\mathbf{B}\big),\ldots, \\ & \left(\prod_{j=M}^{1}(\mathbf{I}_n+\mathbf{E}_j)\right)^{-1}\big(\mathbf{B},\ \mathbf{AB},\ \mathbf{A}^2\mathbf{B},\ldots,\mathbf{A}^{n-1}\mathbf{B}\big), \\ & (\mathbf{I}_n+\mathbf{E}_1)^{-1}\big(\mathbf{F}_1,\ \mathbf{AF}_1,\ \mathbf{A}^2\mathbf{F}_1,\ldots,\mathbf{A}^{n-1}\mathbf{F}_1\big),\ldots, \\ & \left(\prod_{j=M}^{1}(\mathbf{I}_n+\mathbf{E}_j)\right)^{-1}\big(\mathbf{F}_M,\ \mathbf{AF}_M,\ \mathbf{A}^2\mathbf{F}_M,\ldots,\mathbf{A}^{n-1}\mathbf{F}_M\big)\Bigg\}. \end{aligned} \tag{29}$$

(ii) $\operatorname{rank}(\mathrm{Q}) = n$, $\forall\,\lambda \in \sigma(\mathbf{A})$, where

$$
\begin{aligned}
\mathrm{Q} := \Bigg\{ & (\lambda \mathbf{I}_n - \mathbf{A}),\ \ \mathbf{B},\ \ \big(\lambda \mathbf{I}_n - (\mathbf{I}_n + \mathbf{E}_1)^{-1}\mathbf{A}\big),\ \ \big((\mathbf{I}_n + \mathbf{E}_1)^{-1}\mathbf{B}\big), \\
& \left(\lambda \mathbf{I}_n - \left(\prod_{j=2}^{1}(\mathbf{I}_n + \mathbf{E}_j)\right)^{-1}\mathbf{A}\right),\ \left(\left(\prod_{j=2}^{1}(\mathbf{I}_n + \mathbf{E}_j)\right)^{-1}\mathbf{B}\right), \dots, \\
& \left(\lambda \mathbf{I}_n - \left(\prod_{j=M}^{1}(\mathbf{I}_n + \mathbf{E}_j)\right)^{-1}\mathbf{A}\right),\ \left(\left(\prod_{j=M}^{1}(\mathbf{I}_n + \mathbf{E}_j)\right)^{-1}\mathbf{B}\right), \\
& \mathbf{F}_1,\ \big((\mathbf{I}_n + \mathbf{E}_1)^{-1}\mathbf{F}_1\big),\ \mathbf{F}_2,\ \left(\left(\prod_{j=2}^{1}(\mathbf{I}_n + \mathbf{E}_j)\right)^{-1}\mathbf{F}_2\right), \dots, \\
& \mathbf{F}_M,\ \left(\left(\prod_{j=M}^{1}(\mathbf{I}_n + \mathbf{E}_j)\right)^{-1}\mathbf{F}_M\right)\Bigg\}.
\end{aligned}
\tag{30}
$$

P r o o f. First let us show that condition (i) is necessary for the complete controllability of the system (1), by letting $\operatorname{rank}(\mathrm{P}) < n$. Then there exists a nonzero vector $\mathbf{z} \in \mathbb{R}^n$ such that

$$
\begin{cases}
\mathbf{z}^T \mathbf{A}^l \mathbf{B} = \mathbf{O} \in \mathbb{R}^{1\times m}, \\
\mathbf{z}^T \left(\prod_{j=k}^{1}(\mathbf{I}_n + \mathbf{E}_j)\right)^{-1} \mathbf{A}^l \mathbf{B} = \mathbf{O} \in \mathbb{R}^{1\times m}, \\
\mathbf{z}^T \left(\prod_{j=k}^{1}(\mathbf{I}_n + \mathbf{E}_j)\right)^{-1} \mathbf{A}^l \mathbf{F}_k = \mathbf{O} \in \mathbb{R}^{1\times m},
\end{cases}
\tag{31}
$$

for all $l = 0, 1, 2, \dots, (N-1)$ and $k = 1, 2, \dots, M$. From (22), and using (31), we have

$$
\begin{aligned}
\mathbf{z}^T \mathbf{V}_1 &= \int_{t_0}^{t_1} \mathbf{z}^T \big(e^{\mathbf{A}(t_0 - s)}\mathbf{B}\big)\big(e^{\mathbf{A}(t_0 - s)}\mathbf{B}\big)^T \mathrm{d}s \\
&= \int_{t_0}^{t_1} \left(\sum_{l=0}^{n-1} f_l(t_0 - s)\mathbf{z}^T \mathbf{A}^l \mathbf{B}\right)\big(e^{\mathbf{A}(t_0 - s)}\mathbf{B}\big)^T \mathrm{d}s = \mathbf{O} \in \mathbb{R}^{1\times n}, \\
\mathbf{z}^T \mathbf{V}_{k+1} &= \int_{t_k}^{t_{k+1}} \mathbf{z}^T \left(\left(\prod_{j=k}^{1}(\mathbf{I}_n + \mathbf{E}_j)\right)^{-1} e^{\mathbf{A}(t_0 - s)}\mathbf{B}\right)\left(\left(\prod_{j=k}^{1}(\mathbf{I}_n + \mathbf{E}_j)\right)^{-1} e^{\mathbf{A}(t_0 - s)}\mathbf{B}\right)^T \mathrm{d}s \\
&= \int_{t_k}^{t_{k+1}} \left(\sum_{l=0}^{n-1} f_l(t_0 - s)\mathbf{z}^T \left(\prod_{j=k}^{1}(\mathbf{I}_n + \mathbf{E}_j)\right)^{-1} \mathbf{A}^l \mathbf{B}\right) \\
&\qquad \times \left(\left(\prod_{j=k}^{1}(\mathbf{I}_n + \mathbf{E}_j)\right)^{-1} e^{\mathbf{A}(t_0 - s)}\mathbf{B}\right)^T \mathrm{d}s
\end{aligned}
$$

$$
\begin{aligned}
&= \mathbf{O} \in \mathbb{R}^{1\times n},\\
\mathbf{z}^T\mathbf{H}_k &= \mathbf{z}^T\left(\left(\prod_{j=k}^{1}(\mathbf{I}_n+\mathbf{E}_j)\right)^{-1} e^{\mathbf{A}(t_0-t_k)}\mathbf{F}_k\right)\left(\left(\prod_{j=k}^{1}(\mathbf{I}_n+\mathbf{E}_j)\right)^{-1} e^{\mathbf{A}(t_0-t_k)}\mathbf{F}_k\right)^T\\
&= \left(\sum_{l=0}^{n-1} f_l(t_0-s)\mathbf{z}^T\left(\prod_{j=k}^{1}(\mathbf{I}_n+\mathbf{E}_j)\right)^{-1}\mathbf{A}^l\mathbf{F}_k\right)\left(\left(\prod_{j=k}^{1}(\mathbf{I}_n+\mathbf{E}_j)\right)^{-1} e^{\mathbf{A}(t_0-t_k)}\mathbf{F}_k\right)^T\\
&= \mathbf{O} \in \mathbb{R}^{1\times n}.
\end{aligned}
$$

Therefore we proved $\mathbf{z}^T\big(\mathbf{V}_1,\, \mathbf{V}_2, \dots, \mathbf{V}_{M+1},\, \mathbf{H}_1,\, \mathbf{H}_2, \dots, \mathbf{H}_M\big) = \mathbf{O} \in \mathbb{R}^{1\times(2M+1)n}$ for some nonzero vector $\mathbf{z}$, which implies $\operatorname{rank}\big(\mathbf{V}_1,\, \mathbf{V}_2, \dots, \mathbf{V}_{M+1},\, \mathbf{H}_1,\, \mathbf{H}_2, \dots, \mathbf{H}_M\big) < n$, so that by Theorem 3, system (1) is not completely controllable over $[t_0, t_f]$.

Next to show the necessity of condition (ii) for the complete controllability of system (1), assume there exists some $\lambda \in \sigma(\mathbf{A})$ such that $\operatorname{rank}(\mathrm{Q}) < n$. But then there exists nonzero $\mathbf{z} \in \mathbb{R}^n$ such that

$$
\begin{cases}
\mathbf{z}^T(\lambda\mathbf{I}_n - \mathbf{A}) = \mathbf{O} \in \mathbb{R}^{1\times n},\\
\mathbf{z}^T\mathbf{B} = \mathbf{O} \in \mathbb{R}^{1\times m},\\
\mathbf{z}^T\left\{\lambda\mathbf{I}_n - \left(\prod_{j=k}^{1}(\mathbf{I}_n+\mathbf{E}_j)\right)^{-1}\mathbf{A}\right\} = \mathbf{O} \in \mathbb{R}^{1\times n},\\
\mathbf{z}^T\left(\prod_{j=k}^{1}(\mathbf{I}_n+\mathbf{E}_j)\right)^{-1}\mathbf{B} = \mathbf{O} \in \mathbb{R}^{1\times m},\\
\mathbf{z}^T\mathbf{F}_k = \mathbf{O} \in \mathbb{R}^{1\times m},\\
\mathbf{z}^T\left(\prod_{j=k}^{1}(\mathbf{I}_n+\mathbf{E}_j)\right)^{-1}\mathbf{F}_k = \mathbf{O} \in \mathbb{R}^{1\times m},
\end{cases}
\tag{32}
$$

for all $k = 1, 2, \dots, M$. With the repeated use of equations given in (32), one can finally arrive at $\operatorname{rank}(\mathrm{P}) < n$, proving that system (1) is not completely controllable over $[t_0, t_f]$ by condition (i). □

Remark 4. The rank conditions given in Theorem 4 are necessary for the complete controllability of system (1) under the said assumptions, but not sufficient, as the following examples confirm.

Example 2. Consider a LTI-system with single impulse as

$$
\left.
\begin{aligned}
\dot{\mathbf{x}}(t) &= \begin{bmatrix}1 & 1\\ 0 & 2\end{bmatrix}\mathbf{x}(t) + \begin{bmatrix}1\\0\end{bmatrix}\mathbf{u}(t), \quad t \in [0,2]\setminus\{1\},\\
\mathbf{x}(0) &= \begin{bmatrix}0\\0\end{bmatrix},\\
\Delta(\mathbf{x}(1)) &= \begin{bmatrix}1 & 2\\ 0 & 3\end{bmatrix}\mathbf{x}(1) + \begin{bmatrix}2\\0\end{bmatrix}\mathbf{u}(1).
\end{aligned}
\right\}
\tag{33}
$$

In this system $\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\mathbf{E}_1 = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$, and $\mathbf{F}_1 = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$. Clearly $\mathbf{AE}_1 = \mathbf{E}_1\mathbf{A}$ and $(\mathbf{I}_2 + \mathbf{E}_1)$ is invertible. Further $\sigma(\mathbf{A}) = \{1, 2\}$. Now one can verify that

$$\operatorname{rank}\big(\lambda\mathbf{I}_2 - \mathbf{A},\ \mathbf{B},\ \big(\lambda\mathbf{I}_2 - (\mathbf{I}_2 + \mathbf{E}_1)^{-1}\mathbf{A}\big),\ \big((\mathbf{I}_2 + \mathbf{E}_1)^{-1}\mathbf{B}\big),\ \mathbf{F}_1,\ \big((\mathbf{I}_2 + \mathbf{E}_1)^{-1}\mathbf{F}_1\big)\big) = 2,$$

for both $\lambda = 1$ and 2, i.e. the validation of condition (ii) in Theorem 4. However we obtain, $\operatorname{rank}\big(\mathbf{B},\ \mathbf{AB},\ (\mathbf{I}_2+\mathbf{E}_1)^{-1}\{\mathbf{B},\ \mathbf{AB}\},\ (\mathbf{I}_2+\mathbf{E}_1)^{-1}\{\mathbf{F}_1,\ \mathbf{AF}_1\}\big) = 1 < 2$, implying by Theorem 4–condition (i) that, the system (33) is not completely controllable in $\mathbb{R}^2$, over $[0, 2]$. This example also shows that the rank condition (ii) need not imply the rank condition (i) in Theorem 4.

Example 3. Consider another LTI-system with single impulse:

$$\left.\begin{aligned} \dot{\mathbf{x}}(t) &= \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \mathbf{x}(t), \quad t \in [0, 2] \setminus \{1\}, \\ \mathbf{x}(0) &= \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \\ \Delta(\mathbf{x}(1)) &= \begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix} \mathbf{x}(1) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \mathbf{u}(1). \end{aligned}\right\} \tag{34}$$

In this system we have

$$\mathbf{A} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \mathbf{E}_1 = \begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix}, \text{ and } \mathbf{F}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Clearly $\mathbf{AE}_1 = \mathbf{E}_1\mathbf{A}$ and $(\mathbf{I}_2 + \mathbf{E}_1)$ is invertible, and we see that

$$\operatorname{rank}\big(\mathbf{B},\ \mathbf{AB},\ (\mathbf{I}_2 + \mathbf{E}_1)^{-1}\{\mathbf{B},\ \mathbf{AB}\},\ (\mathbf{I}_2 + \mathbf{E}_1)^{-1}\{\mathbf{F}_1,\ \mathbf{AF}_1\}\big) = 2,$$

i.e. the validation of condition (i) in Theorem 4. Now, the state of system (34) at any time $t \in (1, 2]$ is obtained by using eq (2) as

$$\mathbf{x}(t) = e^{\mathbf{A}(t-1)} \begin{bmatrix} \mathbf{u}(1) \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ (t-1) & 1 \end{bmatrix} \begin{bmatrix} \mathbf{u}(1) \\ 0 \end{bmatrix} = \begin{bmatrix} \mathbf{u}(1) \\ (t-1)\mathbf{u}(1) \end{bmatrix}.$$

Clearly there is no $\mathbf{u}(t)$ that steers the above state function from $\begin{bmatrix} 0 & 0 \end{bmatrix}^T$ to $\begin{bmatrix} 1 & 2 \end{bmatrix}^T$, implying that system (34) is not completely controllable in $\mathbb{R}^2$, over $[0, 2]$.

Theorem 5. (Sufficient conditions) Under one of the following conditions, system (1) is completely controllable in $\mathbb{R}^n$, over $[t_0, t_f]$.

(i) $\operatorname{rank}\big(\mathbf{B},\ \mathbf{AB},\ \mathbf{A}^2\mathbf{B}, \dots, \mathbf{A}^{n-1}\mathbf{B}\big) = n$.

(ii) $\operatorname{rank}\big(\lambda\mathbf{I}_n - \mathbf{A},\ \mathbf{B}\big) = n$, for every $\lambda \in \sigma(\mathbf{A})$.

Proof. (a) First we prove case (i). Let $\operatorname{rank}\big(\mathbf{B},\ \mathbf{AB},\ \mathbf{A}^2\mathbf{B}, \dots, \mathbf{A}^{n-1}\mathbf{B}\big) = n$, but assume that system (1) is not completely controllable. Then $\int_{t_M}^{t_f} \big(e^{\mathbf{A}(t_f-s)}\mathbf{B}\big)\big(e^{\mathbf{A}(t_f-s)}\mathbf{B}\big)^T \mathrm{d}s$

is singular by case (iv) of Theorem 1, therefore there exists a nonzero vector, say $\mathbf{z} \in \mathbb{R}^n$ such that

$$\mathbf{z}^T\left(\int_{t_M}^{t_f} \left(e^{\mathbf{A}(t_f-s)}\mathbf{B}\right)\left(e^{\mathbf{A}(t_f-s)}\mathbf{B}\right)^T \mathrm{d}s\right)\mathbf{z} = 0,$$

which can be written as

$$\int_{t_M}^{t_f} \left\|\mathbf{z}^T e^{\mathbf{A}(t_f-s)}\mathbf{B}\right\|^2_{\mathbb{R}^{1\times m}} \mathrm{d}s = 0.$$

The integrand in the above integral is a continuous non-negative function on $(t_M, t_f]$, therefore

$$\mathbf{z}^T e^{\mathbf{A}(t_f-s)}\mathbf{B} = \mathbf{O} \in \mathbb{R}^{1\times m},\ \forall\, s \in (t_M, t_f].$$

At $s = t_f$, we have $\mathbf{z}^T\mathbf{B} = \mathbf{O}$. Further, differentiating the above equation successively with respect to 's' and substituting $s = t_f$ in each, we get $\mathbf{z}^T\mathbf{A}\mathbf{B} = \mathbf{z}^T\mathbf{A}^2\mathbf{B} = \ldots = \mathbf{z}^T\mathbf{A}^{n-1}\mathbf{B} = \mathbf{O}$. Hence

$$\mathbf{z}^T\left(\mathbf{B},\ \mathbf{A}\mathbf{B},\ \mathbf{A}^2\mathbf{B}, \ldots, \mathbf{A}^{n-1}\mathbf{B}\right) = \mathbf{O} \in \mathbb{R}^{1\times mn}.$$

This implies $\operatorname{rank}\left(\mathbf{B},\ \mathbf{A}\mathbf{B},\ \mathbf{A}^2\mathbf{B}, \ldots, \mathbf{A}^{n-1}\mathbf{B}\right) < n$, which is a contradiction. Hence the system (1) must be completely controllable.

(b) Now consider case (ii). Here we show that

$$\operatorname{rank}\left(\lambda\mathbf{I}_n - \mathbf{A},\ \ \mathbf{B}\right) = n,\ \forall\, \lambda \in \sigma(\mathbf{A}),$$

is equivalent to

$$\operatorname{rank}\left(\mathbf{B},\ \mathbf{A}\mathbf{B},\ \mathbf{A}^2\mathbf{B}, \ldots, \mathbf{A}^{n-1}\mathbf{B}\right) = n.$$

To show $\operatorname{rank}\left(\lambda\mathbf{I}_n - \mathbf{A},\ \mathbf{B}\right) = n$ implies $\operatorname{rank}\left(\mathbf{B},\ \mathbf{A}\mathbf{B},\ \mathbf{A}^2\mathbf{B}, \ldots, \mathbf{A}^{n-1}\mathbf{B}\right) = n$, assume

$$0 < \operatorname{rank}\left(\mathbf{B},\ \mathbf{A}\mathbf{B},\ \mathbf{A}^2\mathbf{B}, \ldots, \mathbf{A}^{n-1}\mathbf{B}\right) = r < n,$$

and prove that there exists some $\lambda \in \sigma(\mathbf{A})$ such that $\operatorname{rank}\left(\lambda\mathbf{I}_n - \mathbf{A},\ \mathbf{B}\right) < n$. This is accomplished as follows. Let $\mathbf{T}$ be a nonsingular operator such that the transformation $\mathbf{y}(t) = \mathbf{T}^{-1}\mathbf{x}(t)$ converts the system (1) into normal form (see pp. 95 of Terrell [9]). To this end, we have

$$\left.\begin{aligned} \dot{\mathbf{y}}(t) &= \mathbf{T}^{-1}\dot{\mathbf{x}}(t) = \mathbf{T}^{-1}\left(\mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)\right) = (\mathbf{T}^{-1}\mathbf{A}\mathbf{T})\mathbf{y}(t) + \left(\mathbf{T}^{-1}\mathbf{B}\right)\mathbf{u}(t), \\ \Delta\mathbf{y}(t_k) &= \mathbf{T}^{-1}\Delta\mathbf{x}(t_k) = \mathbf{T}^{-1}\left(\mathbf{E}_k\mathbf{x}(t_k) + \mathbf{F}_k\mathbf{u}(t_k)\right) = \left(\mathbf{T}^{-1}\mathbf{E}_k\right)\mathbf{x}(t_k) + \left(\mathbf{T}^{-1}\mathbf{F}_k\right)\mathbf{u}(t_k). \end{aligned}\right\} \tag{35}$$

The system (35) is in normal form, provided if we

(i) assume that $\mathbf{T}^{-1}\mathbf{A}\mathbf{T}$ and $\mathbf{T}^{-1}\mathbf{B}$ are of the form

$$\mathbf{T}^{-1}\mathbf{A}\mathbf{T} = \begin{pmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{O} & \mathbf{A}_{22} \end{pmatrix} \text{ and } \mathbf{T}^{-1}\mathbf{B} = \begin{pmatrix} \mathbf{B}_{11} \\ \mathbf{O} \end{pmatrix}, \tag{36}$$

such that $\mathbf{A}_{11}$ is $(r \times r)$-matrix, $\mathbf{B}_{11}$ is $(r \times m)$-matrix with $r < n$, and

(ii) show that $\operatorname{rank}\big(\mathbf{B}_{11},\ \mathbf{A}_{11}\mathbf{B}_{11},\ldots,\mathbf{A}_{11}^{r-1}\mathbf{B}_{11}\big)=r$.

Now, from eq (36) we have

$$\mathbf{T}^{-1}\mathbf{A}\mathbf{B} \;=\begin{pmatrix}\mathbf{A}_{11} & \mathbf{A}_{12}\\ \mathbf{O} & \mathbf{A}_{22}\end{pmatrix}\mathbf{T}^{-1}\mathbf{B}=\begin{pmatrix}\mathbf{A}_{11} & \mathbf{A}_{12}\\ \mathbf{O} & \mathbf{A}_{22}\end{pmatrix}\begin{pmatrix}\mathbf{B}_{11}\\ \mathbf{O}\end{pmatrix}=\begin{pmatrix}\mathbf{A}_{11}\mathbf{B}_{11}\\ \mathbf{O}\end{pmatrix}. \tag{37}$$

Since

$$\mathbf{T}^{-1}\mathbf{A}^2\mathbf{T}=(\mathbf{T}^{-1}\mathbf{A}\mathbf{T})^2=(\mathbf{T}^{-1}\mathbf{A}\mathbf{T})(\mathbf{T}^{-1}\mathbf{A}\mathbf{T})=\begin{pmatrix}\mathbf{A}_{11}^2 & \mathbf{A}_{11}\mathbf{A}_{12}+\mathbf{A}_{12}\mathbf{A}_{22}\\ \mathbf{O} & \mathbf{A}_{22}^2\end{pmatrix},$$

so

$$\mathbf{T}^{-1}\mathbf{A}^2\mathbf{B}=\begin{pmatrix}\mathbf{A}_{11}^2 & \mathbf{A}_{11}\mathbf{A}_{12}+\mathbf{A}_{12}\mathbf{A}_{22}\\ \mathbf{O} & \mathbf{A}_{22}^2\end{pmatrix}\mathbf{T}^{-1}\mathbf{B}=\begin{pmatrix}\mathbf{A}_{11}^2 & \mathbf{A}_{11}\mathbf{A}_{12}+\mathbf{A}_{12}\mathbf{A}_{22}\\ \mathbf{O} & \mathbf{A}_{22}^2\end{pmatrix}\begin{pmatrix}\mathbf{B}_{11}\\ \mathbf{O}\end{pmatrix},$$

that is

$$\mathbf{T}^{-1}\mathbf{A}^2\mathbf{B}=\begin{pmatrix}\mathbf{A}_{11}^2\mathbf{B}_{11}\\ \mathbf{O}\end{pmatrix}. \tag{38}$$

Continuing this computation, in general we have

$$\mathbf{T}^{-1}\mathbf{A}^{n-1}\mathbf{B}=\begin{pmatrix}\mathbf{A}_{11}^{n-1}\mathbf{B}_{11}\\ \mathbf{O}\end{pmatrix}. \tag{39}$$

Therefore

$$\begin{aligned}\mathbf{T}^{-1}\big(\mathbf{B},\ \mathbf{A}\mathbf{B},\ldots,\mathbf{A}^{n-1}\mathbf{B}\big)&=\big(\mathbf{T}^{-1}\mathbf{B},\ \mathbf{T}^{-1}\mathbf{A}\mathbf{B},\ldots,\mathbf{T}^{-1}\mathbf{A}^{n-1}\mathbf{B}\big)\\ &=\begin{pmatrix}\mathbf{B}_{11} & \mathbf{A}_{11}\mathbf{B}_{11} & \cdots & \mathbf{A}_{11}^{n-1}\mathbf{B}_{11}\\ \mathbf{O} & \mathbf{O} & \cdots & \mathbf{O}\end{pmatrix}.\end{aligned}$$

Hence,

$$\operatorname{rank}\big(\mathbf{T}^{-1}\big(\mathbf{B},\ \mathbf{A}\mathbf{B},\ldots,\mathbf{A}^{n-1}\mathbf{B}\big)\big)=\operatorname{rank}\begin{pmatrix}\mathbf{B}_{11} & \mathbf{A}_{11}\mathbf{B}_{11} & \ldots & \mathbf{A}_{11}^{n-1}\mathbf{B}_{11}\\ \mathbf{O} & \mathbf{O} & \ldots & \mathbf{O}\end{pmatrix},$$

which implies $\operatorname{rank}\big(\mathbf{B}_{11},\ \mathbf{A}_{11}\mathbf{B}_{11},\ldots,\mathbf{A}_{11}^{n-1}\mathbf{B}_{11}\big)=\operatorname{rank}\big(\mathbf{B},\ \mathbf{A}\mathbf{B},\ldots,\mathbf{A}^{n-1}\mathbf{B}\big)=r$. Since $\mathbf{A}_{11}$ is $(r\times r)$-matrix and $\mathbf{B}_{11}$ is $(r\times m)$-matrix, therefore from Cayley–Hamilton theorem we have

$$\operatorname{rank}\big(\mathbf{B}_{11},\ \mathbf{A}_{11}\mathbf{B}_{11},\ldots,\mathbf{A}_{11}^{r-1}\mathbf{B}_{11}\big)=r.$$

This proves that system (35) is in normal form. Now, let $\boldsymbol{\omega}_0\in\mathbb{R}^{n-r}$ be an eigenvector of $\mathbf{A}_{22}^T$ corresponds to its eigenvalue λ, i.e. $\mathbf{A}_{22}^T\boldsymbol{\omega}_0=\lambda\boldsymbol{\omega}_0$. Also note that λ is an eigenvalue of $\mathbf{A}_{22}$ too, so an eigenvalues of $\mathbf{A}$. By defining a vector $\boldsymbol{\omega}^{\boldsymbol{T}}:=\big(\mathbf{O}\ \ \boldsymbol{\omega}_0^T\big)\mathbf{T}^{-1}\neq\mathbf{O}\in\mathbb{R}^{1\times n}$, we compute

$$\boldsymbol{\omega}^T\mathbf{B}=\big(\mathbf{O}\ \ \boldsymbol{\omega}_0^T\big)\mathbf{T}^{-1}\mathbf{B}=\big(\mathbf{O}\ \ \boldsymbol{\omega}_0^T\big)\begin{pmatrix}\mathbf{B}_{11}\\ \mathbf{O}\end{pmatrix}=\mathbf{O}\in\mathbb{R}^{1\times m},$$

$$\text{and } \boldsymbol{\omega}^T\mathbf{A} = \big(\mathbf{O}\ \ \boldsymbol{\omega}_0^T\big)\mathbf{T}^{-1}\mathbf{A} = \big(\mathbf{O}\ \ \boldsymbol{\omega}_0^T\big)\begin{pmatrix}\mathbf{A}_{11} & \mathbf{A}_{12}\\ \mathbf{O} & \mathbf{A}_{22}\end{pmatrix}\mathbf{T}^{-1} = \big(\mathbf{O}\ \ \boldsymbol{\omega}_0^T\mathbf{A}_{22}\big)\mathbf{T}^{-1}$$
$$= \big(\mathbf{O}\ \ \lambda\boldsymbol{\omega}_0^T\big)\mathbf{T}^{-1}$$
$$= \lambda\big(\mathbf{O}\ \ \boldsymbol{\omega}_0^T\big)\mathbf{T}^{-1} = \lambda\boldsymbol{\omega}^T.$$

This proves that, there exists an eigenvalue λ of $\mathbf{A}$ such that $\mathbf{B}^T\boldsymbol{\omega} = \mathbf{0} \in \mathbb{R}^m$ and $\mathbf{A}^T\boldsymbol{\omega} = \lambda\boldsymbol{\omega}$. Combining these two results, we can write $\begin{pmatrix}\lambda\mathbf{I}_n - \mathbf{A}^T\\ \mathbf{B}^T\end{pmatrix}\boldsymbol{\omega} = \mathbf{0} \in \mathbb{R}^{m+n}$ with $\boldsymbol{\omega} \neq \mathbf{0}$. This implies that $\operatorname{rank}\big(\lambda\mathbf{I}_n - \mathbf{A},\ \mathbf{B}\big) < n$.

Conversely, to prove $\operatorname{rank}\big(\mathbf{B},\ \mathbf{AB}, \dots, \mathbf{A}^{n-1}\mathbf{B}\big) = n$ implies $\operatorname{rank}\big(\lambda\mathbf{I}_n - \mathbf{A},\ \mathbf{B}\big) = n$, we assume that

$$0 < \operatorname{rank}\big(\lambda\mathbf{I}_n - \mathbf{A},\ \mathbf{B}\big) < n, \text{ for some } \lambda \in \sigma(\mathbf{A}),$$

and show that $\operatorname{rank}\big(\mathbf{B},\ \mathbf{AB}, \dots, \mathbf{A}^{n-1}\mathbf{B}\big) < n$. But this assumption forces us to write

$$0 < \operatorname{rank}\begin{pmatrix}\lambda\mathbf{I}_n - \mathbf{A}^T\\ \mathbf{B}^T\end{pmatrix} < n,$$

which implies that the homogeneous system:

$$\begin{pmatrix}\lambda\mathbf{I}_n - \mathbf{A}^T\\ \mathbf{B}^T\end{pmatrix}\boldsymbol{\omega} = \mathbf{0} \in \mathbb{R}^{m+n}$$

has a nontrivial solution $\boldsymbol{\omega} \in \mathbb{R}^n$. That is, with some nonzero vector $\boldsymbol{\omega}$, we have

$$\boldsymbol{\omega}^T\mathbf{A} = \lambda\boldsymbol{\omega}^T \text{ and } \boldsymbol{\omega}^T\mathbf{B} = \mathbf{O} \in \mathbb{R}^{1\times m}. \tag{40}$$

With the repeated use of eq (40), one would arrive at

$$\boldsymbol{\omega}^T\big(\mathbf{B},\ \mathbf{AB}, \dots, \mathbf{A}^{n-1}\mathbf{B}\big) = \big(\boldsymbol{\omega}^T\mathbf{B},\ \boldsymbol{\omega}^T\mathbf{AB}, \dots, \boldsymbol{\omega}^T\mathbf{A}^{n-1}\mathbf{B}\big) = \big(\mathbf{O},\ \mathbf{O}, \dots, \mathbf{O}\big)$$
$$= \mathbf{O} \in \mathbb{R}^{1\times mn}, \text{ with } \boldsymbol{\omega} \neq \mathbf{0}.$$

This is equivalent to saying that an augmented matrix $\big(\mathbf{B},\ \mathbf{AB}, \dots, \mathbf{A}^{n-1}\mathbf{B}\big)$ has linearly dependent rows, and so that $\operatorname{rank}\big(\mathbf{B},\ \mathbf{AB}, \dots, \mathbf{A}^{n-1}\mathbf{B}\big) < n$. Hence, finally we proved $\operatorname{rank}\big(\lambda\mathbf{I}_n - \mathbf{A},\ \mathbf{B}\big) = n$, $\forall\, \lambda \in \sigma(\mathbf{A})$ is equivalent to $\operatorname{rank}\big(\mathbf{B},\ \mathbf{AB},\ \mathbf{A}^2\mathbf{B}, \dots, \mathbf{A}^{n-1}\mathbf{B}\big) = n$. □

Remark 5. Here we give a procedure to compute the nonsingular operator $\mathbf{T}$ used in Theorem 5 that converts system (35) into normal form. Let $\mathbf{T}^{-1} = \big(\boldsymbol{\tau}_1^T\ \ \boldsymbol{\tau}_2^T\ \ \dots\ \ \boldsymbol{\tau}_n^T\big)^T$, where $\boldsymbol{\tau}_1, \boldsymbol{\tau}_2, \dots, \boldsymbol{\tau}_n \in \mathbb{R}^{1\times n}$ are linearly independent row matrices to be determined. Since

$$\mathbf{T}^{-1}\mathbf{B} = \begin{pmatrix}\mathbf{B}_{11}\\ \mathbf{O}\end{pmatrix} \Longrightarrow \begin{pmatrix}\boldsymbol{\tau}_1\mathbf{B}\\ \vdots\\ \boldsymbol{\tau}_r\mathbf{B}\\ \boldsymbol{\tau}_{r+1}\mathbf{B}\\ \vdots\\ \boldsymbol{\tau}_n\mathbf{B}\end{pmatrix} = \begin{pmatrix}\mathbf{B}_{11}\\ \mathbf{O}\end{pmatrix},$$

therefore $\begin{pmatrix} \boldsymbol{\tau}_1 \mathbf{B} \\ \vdots \\ \boldsymbol{\tau}_r \mathbf{B} \end{pmatrix} = \mathbf{B}_{11}$ and $\begin{pmatrix} \boldsymbol{\tau}_{r+1} \mathbf{B} \\ \vdots \\ \boldsymbol{\tau}_n \mathbf{B} \end{pmatrix} = \mathbf{O} \in \mathbb{R}^{(n-r)\times m}$, from which we can determine $\boldsymbol{\tau}_1, \boldsymbol{\tau}_2, \dots, \boldsymbol{\tau}_n$, and hence $\mathbf{T}^{-1}$.

Remark 6. The conditions given in Theorem 5 are sufficient for the complete controllability of system (1), but not necessary, as the following example confirms.

Example 4. Consider a LTI-system with a single impulse as

$$\left.\begin{aligned} \dot{\mathbf{x}}(t) &= \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 0 & 0 & 0 \\ 1 & 2 & 3 \end{bmatrix} \mathbf{u}(t), \quad t \in [0,2] \setminus \{1\}, \\ \mathbf{x}(0) &= \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \\ \Delta(\mathbf{x}(1)) &= \begin{bmatrix} 1 & 1 \\ 0 & 3 \end{bmatrix} \mathbf{x}(1) + \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 1 \end{bmatrix} \mathbf{u}(1). \end{aligned}\right\} \tag{41}$$

In this system $\mathbf{A} = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 2 & 3 \end{bmatrix}$, $\mathbf{E}_1 = \begin{bmatrix} 1 & 1 \\ 0 & 3 \end{bmatrix}$, and $\mathbf{F}_1 = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 1 \end{bmatrix}$. We observe that there exists a (3×2)-matrix $\mathbf{F}_1' = \begin{bmatrix} 1 & 3 & -1 \\ 0 & -2 & 1 \end{bmatrix}^T$ such that $\mathbf{F}_1 \mathbf{F}_1' = \mathbf{I}_2$, so the system (41) is completely controllable on $[0,2]$ by condition (ii) of Theorem 1. However, observe that $\operatorname{rank}\big(\mathbf{B},\ \mathbf{AB}\big) = 1 < 2$.

Concluding Remarks: In this article, a class of dynamical control systems modelled with n-dimensional linear impulsive ordinary differential equations are considered. Various sufficient and necessary conditions for their controllability are investigated. The established results are further reduced to the corresponding time-invariant case of the system, and subsequently obtained Kalman's type and PBH-type matrix rank conditions under various assumptions on the system components. Further it is proved that for the linear impulsive systems, the null controllability need not imply their complete controllability, unlike for the non-impulsive linear systems. Numerical examples are provided to substantiate the theoretical results.

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