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CONVERGENCE OF THE TAIL PROBABILITY FOR WEIGHTED SUMS OF NEGATIVELY ORTHANT DEPENDENT RANDOM VARIABLES

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In this research, strong convergence properties of the tail probability for weighted sums of negatively orthant dependent random variables are discussed. Some sharp theorems for weighted sums of arrays of rowwise negatively orthant dependent random variables are established. These results not only extend the corresponding ones of Cai [4], Wang et al. [19] and Shen [13], but also improve them, respectively.

Keywords: negatively orthant dependent random variables, the tail probability, strong convergence

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1. INTRODUCTION

By weakening the assumptions of validity for strong convergence, we provide an extension for possible applications of probability theory to various fields, especially to statistical research areas. In many statistical theoretical frameworks, we assume that variables are independent. But in real studies, this assumption is not plausible. Then, many statisticians have revised this assumption in order also to consider dependent cases, such as negative dependence structures, positive dependence structures and many others.

First, let us briefly restate some concepts of negative dependent structures.

Definition 1.1. A finite family of random variables $\{X_i, 1 \leq i \leq n\}$ is said to be negatively associated (NA) if for every pair of disjoint subsets A_1 and A_2 of $\{1, 2, \ldots, n\}$ and any real non-decreasing functions f_1 on $\mathbb{R}^{A_1}$ and f_2 on $\mathbb{R}^{A_2}$,

$$\operatorname{Cov}\left(f_1\left(X_i, i \in A_1\right), f_2\left(X_j, j \in A_2\right)\right) \leq 0, \tag{1.1}$$

whenever the covariance exists. An infinite family of random variables $\{X_n, n \geq 1\}$ is NA if every finite sub-family is NA.

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Definition 1.2. A finite family of random variables $\{X_i, 1 \le i \le n\}$ is said to be negatively orthant dependent (NOD) if all $x_1, x_2, \ldots, x_n \in \mathbb{R}$,

$$P(X_1 \le x_1, X_2 \le x_2, \ldots, X_n \le x_n) \le \prod_{j=1}^{n} P(X_j \le x_j), \tag{1.2}$$

and

$$P(X_1 > x_1, X_2 > x_2, \ldots, X_n > x_n) \le \prod_{j=1}^{n} P(X_j > x_j). \tag{1.3}$$

An infinite family of random variables $\{X_n, n \ge 1\}$ is said to be NOD if every finite subfamily is NOD. An array of random variables $\{X_{ni}, 1 \le i \le n, n \ge 1\}$ is called rowwise NOD if for every $n \ge 1$, $\{X_{ni}, 1 \le i \le n\}$ is a sequence of NOD random variables.

The concept of NA was introduced by Joag-Dev and Proschan [9], while the notion of NOD was given by Ebrahimi and Ghosh [6]. Obviously, a sequence of independent random variables is NA, and is also NOD. However, by taking

$$f\left(X_1, X_2, \ldots, X_{n-1}\right) = I\left(X_1 \le x_1, X_2 \le x_2, \ldots, X_{n-1} \le x_{n-1}\right), g\left(X_n\right) = I\left(X_n \le x_n\right),$$

and

$$\bar{f}\left(X_1, X_2, \ldots, X_{n-1}\right) = I\left(X_1 > x_1, X_2 > x_2, \ldots, X_{n-1} > x_{n-1}\right), \bar{g}\left(X_n\right) = I\left(X_n > x_n\right).$$

It easily follows that NA implies (1.2) and (1.3), neither NOD implies being NA. There are two specific examples presented, which are NOD but are not NA.

Example 1. (Wu [21]) Let X_i be a binary random variable such that $P\left(X_i = 0\right) = P\left(X_i = 1\right) = 0.5$ for $i = 1, 2, 3$, and let (X_1, X_2, X_3) be the values $(1,1,1)$, $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$, each with the probability value 0.25. It follows that all the conditions of NOD hold. However,

$$P\left(X_1 + X_3 \le 1, X_2 \le 0\right) = \frac{4}{8} > P\left(X_1 + X_3 \le 1\right) P\left(X_2 \le 0\right) = \frac{3}{8}.$$

Hence, X_1, X_2 and X_3 are not NA.

In addition, let $X = (X_1, X_2, X_3, X_4)$ be NOD, Joag-Dev and Proschan [9] introduced the following example.

Example 2. (Joag-Dev and Proschan [9]) Let X_i be a binary random variable such that $P\left(X_i = 1\right) = 0.5$ for $i = 1, 2, 3, 4$. Let (X_1, X_2) and (X_3, X_4) be the same bivariate distributions, and let $X = (X_1, X_2, X_3, X_4)$ be joint distribution as shown in the following table.

It follows that all the conditions of NOD hold. However,

$$P\left(X_i = 1, i = 1, 2, 3, 4\right) > P\left(X_1 = X_2 = 1\right) P\left(X_3 = X_4 = 1\right).$$

Hence, X_1, X_2, X_3 and X_4 are also not NA.

		(X_1, X_2)				
		(0,0)	(0,1)	(1,0)	(1,1)	Marginal
	(0,0)	.0577	.0623	.0623	.0577	.24
	(0,1)	.0623	.0677	.0677	.0623	.26
(X_3, X_4)	(1,0)	.0623	.0677	.0677	.0623	.26
	(1,1)	.0577	.0623	.0623	.0577	.24
	Marginal	.24	.26	.26	.24	

Tab. 1.

From the above, NOD is a very wide scope family, which contains NA and independent cases. A number of well-known multivariate distributions possess the NA property. Hence, extending and improving the convergence properties of NA to the wider NOD class is of interest in studying issues of theoretical and practical significance. Since the concept of NOD random variables introduced, there are many applications with respect to NOD random variables. We refer the readers to Volodin [16], Asadian et al. [2], Wu [20, 21], Amini and Bozorgnia [1], Wu and Zhu [24], Kuczmaszewska [10], Zarei and Jabbari [25], Huang and Wang [8], Wang et al. [17, 18, 19], Gan and Chen [7], Shen [13], Qiu et al. [11, 12], Zhang and Wang [26] and Sung [15] among others.

In the following four theorems, assume that $\{X_{ni}, 1 \le i \le n, n \ge 1\}$ is an array of rowwise NOD random variables which is stochastically dominated by a random variable X, and $EX_{ni} = 0$ for $1 < \alpha < 2$ (or $1 < \alpha \le 2$). Let $\{a_{ni}, i \ge 1, n \ge 1\}$ be an array of real numbers. $b_n = n^{1/\alpha}(\log n)^{1/\gamma}$ for some $\gamma > 0$.

Wang et al. [19] extended the result of Cai [4] for NA random variables to NOD case without assumption of identical distribution.

Theorem 1.3. Suppose that $\{a_{ni}, i \ge 1, n \ge 1\}$ is an array of real numbers satisfying $\sum_{i=1}^{n} |a_{ni}|^{\alpha} = O\left(n^{\delta}\right)$ for some δ with $0 < \delta < 1$ and some α with $0 < \alpha < 2$. If $E\exp(h|X|^{\gamma}) < \infty$ for some $h > 0$ and $\gamma > 0$, then for $\alpha p \ge 1$

$$\sum_{n=1}^{\infty} n^{\alpha p-2} P\left(\max_{1\le j\le n}\left|\sum_{i=1}^{j} a_{ni}X_{ni}\right| > \varepsilon b_n\right) < \infty \quad \text{for} \quad \forall\, \varepsilon > 0. \tag{1.4}$$

Theorem 1.4. Suppose that $\{a_{ni}, i \ge 1, n \ge 1\}$ is an array of real numbers satisfying $\sum_{i=1}^{n} |a_{ni}|^{\alpha} = O\left(n\right)$ for some α with $0 < \alpha < 2$. If $E\exp(h|X|^{\gamma}) < \infty$ for some $h > 0$ and $\gamma > 0$, then

$$\sum_{n=1}^{\infty} n^{-1} P\left(\max_{1\le j\le n}\left|\sum_{i=1}^{j} a_{ni}X_{ni}\right| > \varepsilon b_n\right) < \infty \quad \text{for} \quad \forall\, \varepsilon > 0. \tag{1.5}$$

Recently, Shen [13] also obtained two theorems of weighted sums for array of rowwise NOD random variables.

Theorem 1.5. Suppose that $\{a_{ni}, i \ge 1, n \ge 1\}$ is an array of real numbers satisfying $\sum_{i=1}^{n} |a_{ni}|^{\alpha} = O\left(n\right)$ for some α with $0 < \alpha \le 2$. Then the following statements hold:

(1) If $\alpha > \gamma$, then $E|X|^\alpha < \infty$ implies

$$\sum_{n=1}^{\infty} n^{-1} P\left(\left|\sum_{i=1}^{n} a_{ni} X_{ni}\right| > \varepsilon b_n\right) < \infty \quad \text{for } \forall \varepsilon > 0. \tag{1.6}$$

(2) If $\alpha = \gamma$, then $E|X|^\alpha \log^+ |X| < \infty$ implies (1.6).

(3) If $\alpha < \gamma$, then $E|X|^\gamma < \infty$ implies (1.6).

Theorem 1.6. Under the assumption conditions of Theorem 1.5, if $E|X|^\beta < \infty$ for $\beta > \alpha + 2$, then (1.5) holds.

In this work, our main purpose is to further study the strong convergence properties of the tail probability for weighed sums of NOD random variables. We establish some sharp theorems for weighted sums of arrays of rowwise negatively orthant dependent random variables under some mild moment conditions. These results not only extend the corresponding ones of Cai [4], Wang et al. [19] and Shen [13], but also improve them, respectively.

Throughout this paper, the symbol C represents positive constant which may be different in various places, $a_n = O(b_n)$ stands for $a_n \le C b_n$, $I(A)$ be the indicator function of a set A. This article is organized as follows. Some important lemmas are provided in Section 2. The main results and their proofs are presented in Section 3 and Section 4.

2. PRELIMINARIES

In this section, we state the following definition of stochastical domination and some important lemmas, which are applied to prove the main results.

Definition 2.1. An array of random variables $\{X_{ni}, 1 \le i \le n, n \ge 1\}$ is said to be stochastically dominated by a random variable X if there exists some positive constant C such that

$$\sup_{i \ge 1, n \ge 1} P(|X_{ni}| > x) \le C P(|X| > x) \quad \text{for } \forall x \ge 0. \tag{2.1}$$

Lemma 2.2. (Bozorgnia, Patterson and Taylor [3]) Let $\{X_n, n \ge 1\}$ be a sequence of NOD random variables, and let $\{f_n, n \ge 1\}$ be a sequence of all non-decreasing (or all non-increasing) functions, then $\{f_n(X_n), n \ge 1\}$ is still a sequence of NOD random variables.

Lemma 2.3. (Asadian, Fakoor and Bozorgnia [2]; Wu [21]) Let $\{X_n, n \ge 1\}$ be a sequence of NOD random variables with mean zero and $E|X_n|^M < \infty$ for some $M \ge 2$ and all $n \ge 1$. Then there exists a positive constant C depending only on M such that for all $n \ge 1$,

$$E\left(\left|\sum_{i=1}^{n} X_i\right|^M\right) \le C\left(\sum_{i=1}^{n} E|X_i|^M + \left(\sum_{i=1}^{n} EX_i^2\right)^{M/2}\right), \tag{2.2}$$

$$E\left(\max_{1\le j\le n}\left|\sum_{i=1}^{j} X_i\right|^M\right) \le C(\log 2n)^M\left(\sum_{i=1}^{n} E|X_i|^M + \left(\sum_{i=1}^{n} EX_i^2\right)^{M/2}\right). \quad (2.3)$$

Lemma 2.4. (Wu [22]) Let $\{X_{ni}, 1 \le i \le n, n \ge 1\}$ be an array of random variables which is stochastically dominated by a random variable X. For all $u > 0$ and $t > 0$, the following two statements hold:

$$E|X_{ni}|^u I(|X_{ni}| \le t) \le C(E|X|^u I(|X| \le t) + t^u P(|X| > t)), \quad (2.4)$$

$$E|X_{ni}|^u I(|X_{ni}| > t) \le CE|X|^u I(|X| > t). \quad (2.5)$$

Lemma 2.5. Let $\{X_{ni}, 1 \le i \le n, n \ge 1\}$ be an array of random variables which is stochastically dominated by a random variable X, and let $\{a_{ni}, i \ge 1, n \ge 1\}$ be an array of real constants. Then for all $u > 0$ and $t > 0$, the following statements hold:

$$E|a_{ni}X_{ni}|^\mu I\left(|a_{ni}X_{ni}| \le t\right) \le C\left(E|a_{ni}X|^\mu I\left(|a_{ni}X| \le t\right) + t^\mu P\left(|a_{ni}X| > t\right)\right), \quad (2.6)$$

$$E|a_{ni}X_{ni}|^\mu I\left(|a_{ni}X_{ni}| > t\right) \le CE|a_{ni}X|^\mu I\left(|a_{ni}X| > t\right). \quad (2.7)$$

Proof. If $a_{ni} = 1$, i.e. the above statements are Lemma 2.4; if $a_{ni} = 0$, the above statements are apparent. If $a_{ni} \ne 1$ and $a_{ni} \ne 0$, it follows from Lemma 2.4 that

$$\begin{aligned}
&E|a_{ni}X_{ni}|^\mu I\left(|a_{ni}X_{ni}| \le t\right) = |a_{ni}|^\mu E|X_{ni}|^\mu I\left(|X_{ni}| \le \frac{t}{|a_{ni}|}\right)\\
&\le |a_{ni}|^\mu C\left(E|X|^\mu I\left(|X| \le \frac{t}{|a_{ni}|}\right) + \left(\frac{t}{|a_{ni}|}\right)^\mu P\left(|X| > \frac{t}{|a_{ni}|}\right)\right)\\
&= C\left(E|a_{ni}X|^\mu I\left(|a_{ni}X| \le t\right) + t^\mu P\left(|a_{ni}X| > t\right)\right).
\end{aligned}$$

Similarly, the proof of (2.7) is apparent. □

Lemma 2.6. (Wu, Sung and Volodin [23]) Suppose that $\{a_{ni}, i \ge 1, n \ge 1\}$ is an array of real constants satisfying $\sum_{i=1}^{n} |a_{ni}|^\alpha = O(n)$ for some $\alpha > 0$. Let X be a random variable, $b_n = n^{1/\alpha}(\log n)^{1/\gamma}$ for some $\gamma > 0$. Then

$$\sum_{n=1}^{\infty} \frac{1}{nb_n^\alpha} \sum_{i=1}^{n} E|a_{ni}X|^\alpha I\left(|a_{ni}X| > b_n\right) \le \begin{cases} CE|X|^\alpha, & \text{for} \quad \alpha > \gamma,\\ CE|X|^\alpha \log(1+|X|), & \text{for} \quad \alpha = \gamma,\\ CE|X|^\gamma, & \text{for} \quad \alpha < \gamma. \end{cases}$$

Lemma 2.7. (Wu, Sung and Volodin [23]) Suppose that $\{a_{ni}, i \ge 1, n \ge 1\}$ is an array of real constants satisfying $\sum_{i=1}^{n} |a_{ni}|^\alpha = O(n)$ for some $\alpha > 0$. Let X be a random variable, $b_n = n^{1/\alpha}(\log n)^{1/\gamma}$ for some $\gamma > 0$. If $q > \max\{\alpha, \gamma\}$, then

$$\sum_{n=1}^{\infty} \frac{1}{nb_n^q} \sum_{i=1}^{n} E|a_{ni}X|^q I\left(|a_{ni}X| \le b_n\right) \le \begin{cases} CE|X|^\alpha, & \text{for} \quad \alpha > \gamma,\\ CE|X|^\alpha \log(1+|X|), & \text{for} \quad \alpha = \gamma,\\ CE|X|^\gamma, & \text{for} \quad \alpha < \gamma. \end{cases}$$

3. COMPLETE CONVERGENCE

In the following sections, assume that $\{X_{ni}, 1 \le i \le n, n \ge 1\}$ is an array of rowwise NOD random variables which is stochastically dominated by a random variable X, and $EX_{ni} = 0$ for $1 < \alpha \le 2$. Let $\{a_{ni}, i \ge 1, n \ge 1\}$ be an array of real numbers. $b_n = n^{1/\alpha}(\log n)^{1/\gamma}$ for some $\gamma > 0$. We establish some sharp results on complete convergence of the maximum weighted sums for arrays of rowwise NOD random variables without assumption of identical distribution. The idea is mainly inspired by Shen [13], Shen and Wu [14].

Theorem 3.1. Suppose that $\{a_{ni}, i \ge 1, n \ge 1\}$ is an array of real numbers satisfying $\sum_{i=1}^{n} |a_{ni}|^{\alpha} = O\left(n^{\delta}\right)$ for some δ with $0 < \delta < 1$ and some α with $0 < \alpha \le 2$. If $E|X|^q < \infty$ for $q > \max\{\alpha^2 p, \alpha + 2, \alpha + \alpha(\alpha p - 1)/(1-\delta), \alpha(\alpha p - 1) + 2\delta\}$ and $\alpha p \ge 1$, then

$$\sum_{n=1}^{\infty} n^{\alpha p - 2} P\left(\max_{1\le j\le n}\left|\sum_{i=1}^{j} a_{ni}X_{ni}\right| > \varepsilon b_n\right) < \infty \quad \text{for } \forall\, \varepsilon > 0. \tag{3.1}$$

P r o o f. Without loss of generality, assume that $a_{ni} \ge 0$. For fixed $n \ge 1$, define

$$Y_{ni} = -b_n I(X_{ni} < -b_n) + X_{ni} I(|X_{ni}| \le b_n) + b_n I(X_{ni} > b_n),\ i \ge 1,$$

$$T_{nj} = \sum_{i=1}^{j} \left(a_{ni}Y_{ni} - Ea_{ni}Y_{ni}\right),\ j = 1, 2, \ldots, n.$$

Obviously, for fixed $n \ge 1$, $\{Y_{ni}, i \ge 1\}$ and $\{Y_{ni} - EY_{ni}, i \ge 1\}$ are still two sequences of NOD random variables by Lemma 2.2. For $\forall \varepsilon > 0$, noting that

$$\left(\max_{1\le j\le n}\left|\sum_{i=1}^{j} a_{ni}X_{ni}\right| > \varepsilon b_n\right) \subset \left(\max_{1\le j\le n}\left|\sum_{i=1}^{j} a_{ni}Y_{ni}\right| > \varepsilon b_n\right) \bigcup \left(\bigcup_{i=1}^{n} (|X_{ni}| > b_n)\right),$$

which implies

$$\begin{aligned} P\left(\max_{1\le j\le n}\left|\sum_{i=1}^{j} a_{ni}X_{ni}\right| > \varepsilon b_n\right) &\le P\left(\max_{1\le j\le n}\left|\sum_{i=1}^{j} a_{ni}Y_{ni}\right| > \varepsilon b_n\right) + P\left(\bigcup_{i=1}^{n} (|X_{ni}| > b_n)\right) \\ &= P\left(\max_{1\le j\le n} |T_{nj}| > \varepsilon b_n - \max_{1\le j\le n}\left|\sum_{i=1}^{j} Ea_{ni}Y_{ni}\right|\right) \\ &\quad + P\left(\bigcup_{i=1}^{n} (|X_{ni}| > b_n)\right). \end{aligned} \tag{3.2}$$

It follows from $\max_{1\le i\le n} |a_{ni}|^{\alpha} \le \sum_{i=1}^{n} |a_{ni}|^{\alpha} = O\left(n^{\delta}\right)$ for $0 < \delta < 1$ that

$$\sum_{i=1}^{n} |a_{ni}|^{k} = \sum_{i=1}^{n} |a_{ni}|^{\alpha} |a_{ni}|^{k-\alpha} \le C n^{\delta} n^{\delta(k-\alpha)/\alpha} = C n^{\delta k/\alpha} \quad \text{for } \forall k \ge \alpha. \tag{3.3}$$

If $0 < \alpha \leq 1$, by (2.6) of Lemma 2.5, (3.3) (for $k = 1$) and the Markov's inequality, we have

$$\begin{aligned}
& b_n^{-1} \max_{1\leq j\leq n} \left| \sum_{i=1}^{j} E a_{ni} Y_{ni} \right| \leq C b_n^{-1} \sum_{i=1}^{n} |E a_{ni} Y_{ni}| \\
& \leq C b_n^{-1} \sum_{i=1}^{n} |a_{ni}| \, E\,|X|\, I\left(|X| \leq b_n\right) + C \sum_{i=1}^{n} |a_{ni}|\, P\left(|X| > b_n\right) \\
& \leq C b_n^{-1} n^{\delta/\alpha} E\,|X|\, I\left(|X| \leq b_n\right) + C n^{\delta/\alpha} P\left(|X| > b_n\right) \\
& \leq C b_n^{-1} n^{\delta/\alpha} \sum_{k=2}^{n} E\,|X|\, I\left(b_{k-1} < |X| \leq b_k\right) + C n^{\delta/\alpha} b_n^{-q} E|X|^q \\
& \leq C b_n^{-1} n^{\delta/\alpha} \sum_{k=2}^{n} b_k P\left(|X| > b_{k-1}\right) + C n^{\delta/\alpha} n^{-q/\alpha} (\log n)^{-q/\gamma} E|X|^q \\
& \leq C b_n^{-1} n^{\delta/\alpha} \sum_{k=2}^{n} b_k E|X|^q b_{k-1}^{-q} + C n^{\delta/\alpha} n^{-q/\alpha} (\log n)^{-q/\gamma} \\
& \leq C b_n^{-1} n^{\delta/\alpha} \sum_{k=2}^{n} k^{1/\alpha} (\log k)^{1/\gamma} (k-1)^{-q/\alpha} (\log(k-1))^{-q/\gamma} \\
& + C n^{\delta/\alpha} n^{-q/\alpha} (\log n)^{-q/\gamma} \\
& \leq C n^{\delta/\alpha + 1 - q/\alpha} (\log n)^{-1/\gamma} + C n^{\delta/\alpha - q/\alpha} (\log n)^{-q/\gamma} \to 0 \quad \text{as } n \to \infty.
\end{aligned} \tag{3.4}$$

Moreover, $\sum_{i=1}^{n} |a_{ni}|^\alpha = O\left(n^\delta\right)$ and the Hölder inequality imply that

$$\sum_{i=1}^{n} |a_{ni}|^k \leq \left(\sum_{i=1}^{n} \left(|a_{ni}|^k\right)^{\frac{\alpha}{k}} \right)^{\frac{k}{\alpha}} \left(\sum_{i=1}^{n} 1 \right)^{\frac{\alpha-k}{\alpha}} \leq Cn \quad \text{for} \quad 1 \leq k < \alpha. \tag{3.5}$$

If $1 < \alpha \leq 2$, by $EX_{ni} = 0$, (2.7) of Lemma 2.5, (3.5) and $E|X|^q < \infty$, we also have

$$\begin{aligned}
& b_n^{-1} \max_{1\leq j\leq n} \left| \sum_{i=1}^{j} E a_{ni} Y_{ni} \right| \leq C b_n^{-1} \max_{1\leq j\leq n} \left| \sum_{i=1}^{j} E a_{ni} X_{ni} I(|X_{ni}| \leq b_n) \right| \\
& \qquad + C \sum_{i=1}^{m} |a_{ni}| P(|X_{ni}| > b_n) \\
& \leq C b_n^{-1} \sum_{i=1}^{n} |a_{ni}| E|X| I(|X| > b_n) + C \sum_{i=1}^{n} |a_{ni}| P(|X| > b_n) \\
& \leq C b_n^{-1} n E|X| I(|X| > b_n) + C n P(|X| > b_n) \\
& = C b_n^{-1} n \sum_{k=n}^{\infty} E|X| I(b_k < |X| \leq b_{k+1}) + C n \frac{E|X|^q}{b_n^q} \\
& \leq C b_n^{-1} n \sum_{k=n}^{\infty} b_{k+1} P(|X| > b_k) + C n n^{-q/\alpha} (\log n)^{-q/\gamma} \\
& \leq C b_n^{-1} n \sum_{k=n}^{\infty} b_{k+1} \frac{E|X|^q}{b_k^q} + C n^{1-q/\alpha} (\log n)^{-q/\gamma} \\
& \leq C n^{2-q/\alpha} (\log n)^{-1/\gamma} + C n^{1-q/\alpha} (\log n)^{-q/\gamma} \to 0 \quad \text{as } n \to \infty.
\end{aligned} \tag{3.6}$$

Therefore, $\max\limits_{1\le j\le n}\left|\sum_{i=1}^{j}Ea_{ni}Y_{ni}\right| \le \frac{\varepsilon b_n}{2}$ holds for all n large enough, which implies

$$P\left(\max_{1\le j\le n}\left|\sum_{i=1}^{j}a_{ni}X_{ni}\right|>\varepsilon b_n\right)\le\sum_{i=1}^{n}P\left(|X_{ni}|>b_n\right)+P\left(\max_{1\le j\le n}|T_{nj}|>\frac{\varepsilon b_n}{2}\right). \quad (3.7)$$

To prove (3.1), it suffices to show that

$$I\triangleq\sum_{n=1}^{\infty}n^{\alpha p-2}\sum_{i=1}^{n}P\left(|X_{ni}|>b_n\right)<\infty, \quad (3.8)$$

and

$$J\triangleq\sum_{n=1}^{\infty}n^{\alpha p-2}P\left(\max_{1\le j\le n}|T_{nj}|>\frac{\varepsilon b_n}{2}\right)<\infty. \quad (3.9)$$

By the Markov's inequality, $E|X|^q<\infty$ and standard computations, it follows that

$$\begin{aligned} I &\le C\sum_{n=1}^{\infty}n^{\alpha p-2}\sum_{i=1}^{n}P(|X|>b_n)\\ &\le C\sum_{n=1}^{\infty}n^{\alpha p-1}\frac{E|X|^q}{b_n^q}\\ &\le C\sum_{n=1}^{\infty}n^{\alpha p-1}n^{-q/\alpha}(\log n)^{-q/\gamma}<\infty. \end{aligned} \quad (3.10)$$

For J, by the Markov's inequality (for $M\ge 2$) and Lemma 2.3,

$$\begin{aligned} J &\le C\sum_{n=1}^{\infty}n^{\alpha p-2}b_n^{-M}E\left(\max_{1\le j\le n}|T_{nj}|^M\right)\\ &\le C\sum_{n=1}^{\infty}n^{\alpha p-2}b_n^{-M}(\log 2n)^M\sum_{i=1}^{n}|a_{ni}|^M E|Y_{ni}|^M\\ &\quad +C\sum_{n=1}^{\infty}n^{\alpha p-2}b_n^{-M}(\log 2n)^M\left(\sum_{i=1}^{n}|a_{ni}|^2E|Y_{ni}|^2\right)^{M/2}\\ &\triangleq J_1+J_2. \end{aligned} \quad (3.11)$$

Take some suitable constant M such that $\max\left\{2,\frac{\alpha(\alpha p-1)}{1-\delta}\right\}<M<\min\left\{q-\alpha,\frac{q-\alpha^2p+\alpha}{\delta}\right\}$, which implies

$$q>\alpha+M,\quad \frac{q}{\alpha}-\frac{M}{\alpha}>1,\quad q>\alpha^2p-\alpha+M\delta,$$

and

$$\frac{q}{\alpha}-\alpha p+2-\frac{M\delta}{\alpha}>1,\quad \alpha p-2+\frac{M\delta}{\alpha}-\frac{M}{\alpha}<-1,\quad M>\alpha.$$

By (3.3), (2.6) of Lemma 2.5 and the Markov's inequality, we obtain

$$\begin{aligned}
J_1 &\le C\sum_{n=1}^{\infty} n^{\alpha p-2} b_n^{-M}(\log 2n)^M \left(\sum_{i=1}^{n}|a_{ni}|^M \left(E|X_{ni}|^M I\left(|X_{ni}|\le b_n\right)+b_n^M P(|X_{ni}|>b_n)\right)\right)\\
&\le C\sum_{n=1}^{\infty} n^{\alpha p-2} b_n^{-M}(\log 2n)^M \left(\sum_{i=1}^{n}|a_{ni}|^M \left(E|X|^M I\left(|X|\le b_n\right)+b_n^M P(|X|>b_n)\right)\right)\\
&\quad +C\sum_{n=1}^{\infty} n^{\alpha p-2}(\log 2n)^M n^{M\delta/\alpha} P(|X|>b_n)\\
&\le C\sum_{n=1}^{\infty} n^{\alpha p-2+M\delta/\alpha} b_n^{-M}(\log 2n)^M \sum_{k=2}^{n} E|X|^M I(b_{k-1}<|X|\le b_k)\\
&\quad +C\sum_{n=1}^{\infty} n^{\alpha p-2+M\delta/\alpha}(\log 2n)^M \frac{E|X|^q}{b_n^q}\\
&\le C\sum_{k=2}^{\infty} b_k^M P(|X|>b_{k-1}) \sum_{n=k}^{\infty} n^{\alpha p-2+M\delta/\alpha-M/\alpha}(\log n)^{-M/\gamma}(\log 2n)^M\\
&\quad +C\sum_{n=1}^{\infty} \frac{n^{\alpha p-2+M\delta/\alpha}(\log 2n)^M}{n^{q/\alpha}(\log n)^{q/\gamma}}\\
&\le C\sum_{k=3}^{\infty} \frac{k^{M/\alpha}(\log k)^{M/\gamma}}{(k-1)^{q/\alpha}(\log(k-1))^{q/\gamma}} + C\sum_{n=1}^{\infty} \frac{n^{\alpha p-2+M\delta/\alpha}(\log 2n)^M}{n^{q/\alpha}(\log n)^{q/\gamma}} < \infty.
\end{aligned} \tag{3.12}$$

For J_2, by (2.6) of Lemma 2.5, (3.3) and the proof of (3.12), it also follows that

$$\begin{aligned}
J_2 &\le C\sum_{n=1}^{\infty} n^{\alpha p-2} b_n^{-M}(\log 2n)^M \left(\sum_{i=1}^{n}|a_{ni}|^2 \left(E|X|^2 I(|X|\le b_n)+b_n^2 P(|X|>b_n)\right)\right)^{M/2}\\
&\le C\sum_{n=1}^{\infty} n^{\alpha p-2} b_n^{-M}(\log 2n)^M n^{M\delta/\alpha} \left(\left(E|X|^2 I(|X|\le b_n)+b_n^2 P(|X|>b_n)\right)\right)^{M/2}\\
&\le C\sum_{n=1}^{\infty} n^{\alpha p-2} b_n^{-M}(\log 2n)^M n^{M\delta/\alpha} \left(E|X|^2 I(|X|\le b_n)\right)^{M/2}\\
&\quad +C\sum_{n=1}^{\infty} n^{\alpha p-2}(\log 2n)^M n^{M\delta/\alpha} \left(P(|X|>b_n)\right)^{M/2}\\
&\le C\sum_{n=1}^{\infty} n^{\alpha p-2} b_n^{-M}(\log 2n)^M n^{M\delta/\alpha} E|X|^M I(|X|\le b_n)\\
&\quad +C\sum_{n=1}^{\infty} n^{\alpha p-2}(\log 2n)^M n^{M\delta/\alpha} P(|X|>b_n)\\
&< \infty.
\end{aligned} \tag{3.13}$$

The proof of Theorem 3.1 is completed. □

The next theorem treats the case $\alpha p = 1$. The proof is analogous to that of Theorem 3.1. Here will omit the details.

Theorem 3.2. Suppose that $\{a_{ni}, i \geq 1, n \geq 1\}$ is an array of real constants satisfying $\sum_{i=1}^{n} |a_{ni}|^{\alpha} = O(n)$ for some α with $0 < \alpha \leq 2$. If $E|X|^q < \infty$ for $q > \alpha + 2$, then

$$\sum_{n=1}^{\infty} n^{-1} P\left(\max_{1\leq j\leq n}\left|\sum_{i=1}^{j} a_{ni}X_{ni}\right| > \varepsilon b_n\right) < \infty \quad \text{for} \quad \forall\, \varepsilon > 0. \tag{3.14}$$

Remark 3.3. The moment condition $E|X|^q < \infty$ in Theorem 3.1 and Theorem 3.2 is much weaker than the corresponding moment condition $E\exp(h|X|^{\gamma}) < \infty$ of Wang et al. [19], Cai [4]. Since NA implies NOD, Theorem 3.1 and Theorem 3.2 extend and improve the corresponding ones of Wang et al. [19], Cai [4](by letting X_{ni} instead of X_i and $\sum_{i=1}^{n} |a_{ni}|^{\alpha} = O(n)$ for $0 < \alpha \leq 2$), respectively. In addition, it is worth pointing out that the method in the proof of Theorem 3.1 is different from those of Cai [4] and Wang et al. [19].

4. COMPLETE MOMENT CONVERGENCE

In this section, we will discuss the complete moment convergence for weighted sums of NOD random variables. The concept of complete moment convergence was introduced by Chow [5] as follows: let $\{X_n, n \geq 1\}$ be a sequence of random variables, and $a_n > 0$, $b_n > 0$, $q > 0$. If for all $\varepsilon \geq 0$, $\sum_{n=1}^{\infty} a_n E\left(b_n^{-1}|X_n| - \varepsilon\right)_+^q < \infty$, then $\{X_n; n \geq 1\}$ is said to be in the sense of complete moment convergence.

Theorem 4.1. Suppose that $\{a_{ni}, i \geq 1, n \geq 1\}$ is an array of real constants satisfying $\sum_{i=1}^{n} |a_{ni}|^{\alpha} = O(n)$ for some $0 < \alpha \leq 2$. Then the following statements hold:

(i) If $\alpha > \gamma$, then $E|X|^{\alpha} < \infty$ implies

$$\sum_{n=1}^{\infty} \frac{1}{n} E\left(\frac{1}{b_n}\left|\sum_{i=1}^{n} a_{ni}X_{ni}\right| - \varepsilon\right)_+^{\alpha} < \infty \quad \text{for} \quad \forall\varepsilon > 0. \tag{4.1}$$

(ii) If $\alpha = \gamma$, then $E|X|^{\alpha}\log(1 + |X|) < \infty$ implies (4.1).

(iii) If $\alpha < \gamma$, then $E|X|^{\gamma} < \infty$ implies (4.1).

Proof. For $\forall\varepsilon > 0$, it is easy to check that

$$\begin{aligned}
\sum_{n=1}^{\infty} \tfrac{1}{n} E\left(\tfrac{1}{b_n}\left|\sum_{i=1}^{n} a_{ni}X_{ni}\right| - \varepsilon\right)_+^{\alpha} &= \sum_{n=1}^{\infty} \tfrac{1}{n} \int_0^{\infty} P\left(\tfrac{1}{b_n}\left|\sum_{i=1}^{n} a_{ni}X_{ni}\right| - \varepsilon > t^{1/\alpha}\right) \mathrm{d}t \\
&= \sum_{n=1}^{\infty} \tfrac{1}{n} \int_0^{1} P\left(\tfrac{1}{b_n}\left|\sum_{i=1}^{n} a_{ni}X_{ni}\right| > \varepsilon + t^{1/\alpha}\right) \mathrm{d}t \\
&\quad + \sum_{n=1}^{\infty} \tfrac{1}{n} \int_1^{\infty} P\left(\tfrac{1}{b_n}\left|\sum_{i=1}^{n} a_{ni}X_{ni}\right| > \varepsilon + t^{1/\alpha}\right) \mathrm{d}t \\
&\leq \sum_{n=1}^{\infty} \tfrac{1}{n} P\left(\left|\sum_{i=1}^{n} a_{ni}X_{ni}\right| > \varepsilon b_n\right) \\
&\quad + \sum_{n=1}^{\infty} \tfrac{1}{n} \int_1^{\infty} P\left(\left|\sum_{i=1}^{n} a_{ni}X_{ni}\right| > b_n t^{1/\alpha}\right) \mathrm{d}t \\
&\triangleq I + J.
\end{aligned} \tag{4.2}$$

To prove (4.1), it suffices to prove that $I < \infty$ and $J < \infty$. By the corresponding theorems in Shen [13], $I < \infty$ follows directly.

Assume that $a_{ni} \geq 0$. For fixed $n \geq 1$, $i \geq 1$ and all $t \geq 1$, define

$$Y_{ni} = -b_n t^{1/\alpha} I\left(a_{ni}X_{ni} < -b_n t^{1/\alpha}\right) + a_{ni}X_{ni} I\left(|a_{ni}X_i| \leq b_n t^{1/\alpha}\right) + b_n t^{1/\alpha} I\left(a_{ni}X_{ni} > b_n t^{1/\alpha}\right),$$

$$Z_{ni} = \left(a_{ni}X_{ni} + b_n t^{1/\alpha}\right) I\left(a_{ni}X_{ni} < -b_n t^{1/\alpha}\right) + \left(a_{ni}X_{ni} - b_n t^{1/\alpha}\right) I\left(a_{ni}X_{ni} > b_n t^{1/\alpha}\right).$$

Obviously, for $\forall \varepsilon > 0$,

$$\begin{aligned} P\left(\left|\sum_{i=1}^{n} a_{ni}X_{ni}\right| > b_n t^{1/\alpha}\right) &\leq P\left(\left|\sum_{i=1}^{n} Y_{ni}\right| > b_n t^{1/\alpha}\right) + P\left(\bigcup_{i=1}^{n}\left(|a_{ni}X_{ni}| > b_n t^{1/\alpha}\right)\right) \\ &\leq P\left(\left|\sum_{i=1}^{n} (Y_{ni} - EY_{ni})\right| > b_n t^{1/\alpha} - \left|\sum_{i=1}^{n} EY_{ni}\right|\right) \\ &\quad + \sum_{i=1}^{n} P\left(|a_{ni}X_{ni}| > b_n t^{1/\alpha}\right). \end{aligned}$$

Firstly, we prove

$$\sup_{t\geq 1} \frac{1}{b_n t^{1/\alpha}} \left|\sum_{i=1}^{n} EY_{ni}\right| \to 0 \quad \text{as} \quad n \to \infty. \tag{4.3}$$

If $0 < \alpha \leq 1$, by (2.6) of Lemma 2.5, the C_r inequality, the Markov's inequality and $E|X|^\alpha < \infty$, we have

$$\begin{aligned} \sup_{t\geq 1} \tfrac{1}{b_n t^{1/\alpha}} \left|\sum_{i=1}^{n} EY_{ni}\right| &\leq C\sup_{t\geq 1} \tfrac{1}{b_n t^{1/\alpha}} \sum_{i=1}^{n} |EY_{ni}| \\ &\leq C\sup_{t\geq 1} \tfrac{1}{b_n t^{1/\alpha}} \sum_{i=1}^{n} E|a_{ni}X_{ni}| I\left(|a_{ni}X_{ni}| \leq b_n t^{1/\alpha}\right) \\ &\quad + C\sup_{t\geq 1} \sum_{i=1}^{n} P\left(|a_{ni}X_{ni}| > b_n t^{1/\alpha}\right) \\ &\leq C\sup_{t\geq 1} \tfrac{1}{b_n^\alpha t} \sum_{i=1}^{n} a_{ni}^\alpha E|X|^\alpha I\left(|a_{ni}X| \leq b_n t^{1/\alpha}\right) \\ &\quad + C\sup_{t\geq 1} \tfrac{1}{b_n^\alpha t} \sum_{i=1}^{n} a_{ni}^\alpha E|X|^\alpha \\ &\leq C(\log n)^{-\alpha/\gamma} E|X|^\alpha \to 0 \quad \text{as} \quad n \to \infty. \end{aligned} \tag{4.4}$$

Noting that $0 < Z_{ni} = a_{ni}X_{ni} - b_n t^{1/\alpha} < a_{ni}X_{ni}$ if $a_{ni}X_{ni} > b_n t^{1/\alpha}$; $a_{ni}X_{ni} < Z_{ni} = a_{ni}X_{ni} + b_n t^{1/\alpha} < 0$ if $a_{ni}X_{ni} < -b_n t^{1/\alpha}$. Hence, $|Z_{ni}| < |a_{ni}X_{ni}| I\left(|a_{ni}X_{ni}| > b_n t^{1/\alpha}\right)$.

For $1 < \alpha \leq 2$, by $EX_{ni} = 0$, (2.7) of Lemma 2.5, the C_r inequality and $E|X|^\alpha < \infty$, we also have

$$\begin{aligned}\sup_{t\geq 1}\frac{1}{b_n t^{1/\alpha}}\left|\sum_{i=1}^{n}EY_{ni}\right| &= \sup_{t\geq 1}\frac{1}{b_n t^{1/\alpha}}\left|\sum_{i=1}^{n}EZ_{ni}\right| \\ &\leq C\sup_{t\geq 1}\frac{1}{b_n t^{1/\alpha}}\sum_{i=1}^{n}E\left|Z_{ni}\right| \\ &\leq C\sup_{t\geq 1}\frac{1}{b_n t^{1/\alpha}}\sum_{i=1}^{n}E\left|a_{ni}X\right|I\left(\left|a_{ni}X\right|>b_n t^{1/\alpha}\right) \\ &\leq C\sup_{t\geq 1}\frac{1}{b_n^{\alpha} t}\sum_{i=1}^{n}a_{ni}^{\alpha}E|X|^{\alpha}I\left(\left|a_{ni}X\right|>b_n t^{1/\alpha}\right) \\ &\leq C(\log n)^{-\alpha/\gamma}E|X|^{\alpha}\to 0 \quad \text{as} \quad n\to\infty.\end{aligned} \tag{4.5}$$

Hence, $\left|\sum_{i=1}^{n}EY_{ni}\right|\leq\frac{b_n t^{1/\alpha}}{2}$ holds uniformly for n large enough and all $t\geq 1$.

$$P\left(\left|\sum_{i=1}^{n}a_{ni}X_{ni}\right|>b_n t^{1/\alpha}\right)\leq P\left(\left|\sum_{i=1}^{n}(Y_{ni}-EY_{ni})\right|>\frac{b_n t^{1/\alpha}}{2}\right)$$
$$+\sum_{i=1}^{n}P\left(\left|a_{ni}X_{ni}\right|>b_n t^{1/\alpha}\right).$$

To prove $J<\infty$, it suffices to show that

$$J_1\triangleq\sum_{n=1}^{\infty}\frac{1}{n}\int_1^{\infty}P\left(\left|\sum_{i=1}^{n}(Y_{ni}-EY_{ni})\right|>\frac{b_n t^{1/\alpha}}{2}\right)\mathrm{d}t<\infty, \tag{4.6}$$

$$J_2\triangleq\sum_{n=1}^{\infty}\frac{1}{n}\int_1^{\infty}\sum_{i=1}^{n}P\left(\left|a_{ni}X_{ni}\right|>b_n t^{1/\alpha}\right)\mathrm{d}t<\infty. \tag{4.7}$$

By the Markov's inequality and (2.2) of Lemma 2.3,

$$\begin{aligned}J_1 &\leq C\sum_{n=1}^{\infty}\frac{1}{n}\int_1^{\infty}\frac{1}{b_n^M t^{M/\alpha}}E\left(\left|\sum_{i=1}^{n}(Y_{ni}-EY_{ni})\right|^M\right)\mathrm{d}t \qquad \left(M>\max\left\{2,\frac{2\gamma}{\alpha}\right\}\right) \\ &\leq C\sum_{n=1}^{\infty}\frac{1}{n}\int_1^{\infty}\frac{1}{b_n^M t^{M/\alpha}}\left(\sum_{i=1}^{n}E|Y_{ni}-EY_{ni}|^M+\left(\sum_{i=1}^{n}E|Y_{ni}-EY_{ni}|^2\right)^{M/2}\right)\mathrm{d}t \\ &\leq C\sum_{n=1}^{\infty}\frac{1}{nb_n^M}\int_1^{\infty}\frac{1}{t^{M/\alpha}}\sum_{i=1}^{n}E|Y_{ni}|^M dt+C\sum_{n=1}^{\infty}\frac{1}{nb_n^M}\int_1^{\infty}\frac{1}{t^{M/\alpha}}\left(\sum_{i=1}^{n}E|Y_{ni}|^2\right)^{M/2}\mathrm{d}t \\ &\triangleq J_{11}+J_{12}.\end{aligned} \tag{4.8}$$

For J_{12}, by (2.6) of Lemma 2.5, the C_r inequality, $0<\alpha\leq 2$ and $M>\frac{2\gamma}{\alpha}$,

$$\begin{aligned}J_{12} &\leq C\sum_{n=1}^{\infty}\frac{1}{n}\int_1^{\infty}\left(\sum_{i=1}^{n}\frac{E|a_{ni}X|^{\alpha}}{b_n^{\alpha}t}\right)^{M/2}\mathrm{d}t \\ &\leq C\sum_{n=1}^{\infty}\frac{1}{n}\int_1^{\infty}t^{-M/2}\left(b_n^{-\alpha}\sum_{i=1}^{n}\left(E|a_{ni}X|^{\alpha}\right)\right)^{M/2}\mathrm{d}t \\ &\leq C\sum_{n=1}^{\infty}\frac{1}{n}(\log n)^{-\alpha M/2\gamma}\left(E|X|^{\alpha}\right)^{M/2}<\infty.\end{aligned} \tag{4.9}$$

For J_{11}, by (2.6) of Lemma 2.5 and (2.1), we have

$$\begin{aligned}
J_{11} &\le C\sum_{n=1}^{\infty}\frac{1}{n}\int_1^{\infty}\sum_{i=1}^{n}P\left(|a_{ni}X_{ni}|>b_n t^{1/\alpha}\right)\mathrm{d}t \\
&\quad +C\sum_{n=1}^{\infty}\frac{1}{nb_n^M}\int_1^{\infty}\frac{1}{t^{M/\alpha}}\sum_{i=1}^{n}E|a_{ni}X_{ni}|^M I\left(|a_{ni}X_{ni}|\le b_n t^{1/\alpha}\right)\mathrm{d}t \\
&\le C\sum_{n=1}^{\infty}\frac{1}{n}\int_1^{\infty}\sum_{i=1}^{n}P\left(|a_{ni}X|>b_n t^{1/\alpha}\right)\mathrm{d}t \\
&\quad +C\sum_{n=1}^{\infty}\frac{1}{nb_n^M}\int_1^{\infty}\frac{1}{t^{M/\alpha}}\sum_{i=1}^{n}E|a_{ni}X|^M I\left(|a_{ni}X|\le b_n t^{1/\alpha}\right)\mathrm{d}t \\
&= C\sum_{n=1}^{\infty}\frac{1}{n}\int_1^{\infty}\sum_{i=1}^{n}P\left(|a_{ni}X|>b_n t^{1/\alpha}\right)\mathrm{d}t \\
&\quad +C\sum_{n=1}^{\infty}\frac{1}{nb_n^M}\int_1^{\infty}\frac{1}{t^{M/\alpha}}\sum_{i=1}^{n}E|a_{ni}X|^M I\left(|a_{ni}X|\le b_n\right)\mathrm{d}t \\
&\quad +C\sum_{n=1}^{\infty}\frac{1}{nb_n^M}\int_1^{\infty}\frac{1}{t^{M/\alpha}}\sum_{i=1}^{n}E|a_{ni}X|^M I\left(b_n<|a_{ni}X|\le b_n t^{1/\alpha}\right)\mathrm{d}t \\
&\triangleq J_{111}+J_{112}+J_{113}.
\end{aligned} \tag{4.10}$$

For J_{111}, by $\sum_{i=1}^{n}|a_{ni}|^\alpha = O(n)$ and Lemma 2.6,

$$\begin{aligned}
J_{111} &= C\sum_{n=1}^{\infty}\frac{1}{n}\int_1^{\infty}\sum_{i=1}^{n}P\left(|a_{ni}X|>b_n t^{1/\alpha}\right)\mathrm{d}t \\
&\le C\sum_{n=1}^{\infty}\frac{1}{n}\sum_{i=1}^{n}\int_1^{\infty}P\left(|a_{ni}X|>b_n t^{1/\alpha}\right)\mathrm{d}t \\
&= C\sum_{n=1}^{\infty}\frac{1}{n}\sum_{i=1}^{n}\int_1^{\infty}P\left(\frac{|a_{ni}X|^\alpha}{b_n^\alpha}>t\right)\mathrm{d}t \\
&\le C\sum_{n=1}^{\infty}\frac{1}{nb_n^\alpha}\sum_{i=1}^{n}E|a_{ni}X|^\alpha I\left(|a_{ni}X|>b_n\right)<\infty.
\end{aligned} \tag{4.11}$$

For J_{112}, by Lemma 2.7 and $M>2\ge\alpha$,

$$\begin{aligned}
J_{112} &= C\sum_{n=1}^{\infty}\frac{1}{n}\sum_{i=1}^{n}\frac{E|a_{ni}X|^M}{b_n^M}I\left(|a_{ni}X|\le b_n\right)\int_1^{\infty}\frac{1}{t^{M/\alpha}}\mathrm{d}t \\
&\le C\sum_{n=1}^{\infty}\frac{1}{nb_n^M}\sum_{i=1}^{n}E|a_{ni}X|^M I\left(|a_{ni}X|\le b_n\right)<\infty.
\end{aligned} \tag{4.12}$$

Take $t = x^\alpha$. By $M > 2 \geq \alpha$ and Lemma 2.6,

$$\begin{aligned}
J_{113} &= C \sum_{n=1}^{\infty} \frac{1}{nb_n^M} \int_1^{\infty} \frac{1}{t^{M/\alpha}} \sum_{i=1}^{n} E|a_{ni}X|^M I\left(b_n < |a_{ni}X| \leq b_n t^{1/\alpha}\right)\mathrm{d}t \\
&\leq C \sum_{n=1}^{\infty} \frac{1}{nb_n^M} \int_1^{\infty} x^{\alpha-M-1} \sum_{i=1}^{n} E|a_{ni}X|^M I\left(b_n < |a_{ni}X| \leq b_n x\right)\mathrm{d}x \\
&= C \sum_{n=1}^{\infty} \frac{1}{nb_n^M} \sum_{m=1}^{\infty} \int_m^{m+1} x^{\alpha-M-1} \sum_{i=1}^{n} E|a_{ni}X|^M I\left(b_n < |a_{ni}X| \leq b_n x\right)\mathrm{d}x \\
&\leq C \sum_{n=1}^{\infty} \frac{1}{nb_n^M} \sum_{m=1}^{\infty} m^{\alpha-M-1} \sum_{i=1}^{n} E|a_{ni}X|^M I\left(b_n < |a_{ni}X| \leq b_n\left(m+1\right)\right) \\
&= C \sum_{n=1}^{\infty} \frac{1}{nb_n^M} \sum_{i=1}^{n} \sum_{s=1}^{\infty} E|a_{ni}X|^M I\left(b_n s < |a_{ni}X| \leq b_n\left(s+1\right)\right) \sum_{m=s}^{\infty} m^{\alpha-M-1} \\
&\leq C \sum_{n=1}^{\infty} \frac{1}{nb_n^M} \sum_{i=1}^{n} \sum_{s=1}^{\infty} E|a_{ni}X|^M I\left(b_n s < |a_{ni}X| \leq b_n\left(s+1\right)\right) s^{\alpha-M} \\
&\leq C \sum_{n=1}^{\infty} \frac{1}{nb_n^\alpha} \sum_{i=1}^{n} \sum_{s=1}^{\infty} \left(s+1\right)^{M-\alpha} s^{\alpha-M} E|a_{ni}X|^\alpha I\left(b_n s < |a_{ni}X| \leq b_n\left(s+1\right)\right) \\
&\leq C \sum_{n=1}^{\infty} \frac{1}{nb_n^\alpha} \sum_{i=1}^{n} E|a_{ni}X|^\alpha I\left(|a_{ni}X| > b_n\right) < \infty.
\end{aligned} \tag{4.13}$$

The proof of Theorem 4.1 is completed. □

Remark 4.2. Under the conditions of Theorem 4.1,

$$\begin{aligned}
\infty &> \sum_{n=1}^{\infty} \frac{1}{n} E\left(\frac{1}{b_n}\left|\sum_{i=1}^{n} a_{ni}X_{ni}\right| - \varepsilon\right)_+^\alpha \\
&= \sum_{n=1}^{\infty} \frac{1}{n} \int_0^{\infty} P\left(\frac{1}{b_n}\left|\sum_{i=1}^{n} a_{ni}X_{ni}\right| - \varepsilon > t^{1/\alpha}\right)\mathrm{d}t \\
&= \sum_{n=1}^{\infty} \frac{1}{n} \int_0^{\varepsilon^\alpha} P\left(\left|\sum_{i=1}^{n} a_{ni}X_{ni}\right| > b_n t^{1/\alpha} + b_n\varepsilon\right)\mathrm{d}t \\
&\quad + \sum_{n=1}^{\infty} \frac{1}{n} \int_{\varepsilon^\alpha}^{\infty} P\left(\left|\sum_{i=1}^{n} a_{ni}X_{ni}\right| > b_n t^{1/\alpha} + b_n\varepsilon\right)\mathrm{d}t \\
&\geq C \sum_{n=1}^{\infty} \frac{1}{n} \int_0^{\varepsilon^\alpha} P\left(\left|\sum_{i=1}^{n} a_{ni}X_{ni}\right| > 2b_n\varepsilon\right)\mathrm{d}t \\
&\geq C\varepsilon^\alpha \sum_{n=1}^{\infty} \frac{1}{n} P\left(\left|\sum_{i=1}^{n} a_{ni}X_{ni}\right| > 2b_n\varepsilon\right).
\end{aligned} \tag{4.14}$$

Since $\varepsilon > 0$ is arbitrary. From (4.14), it is clear to see that the complete moment convergence implies the complete convergence. Hence, Theorem 4.1 improves the corresponding result of Shen [13] listed in references under the same assumption conditions.

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