KYBERNETIKA — VOLUME *56* (2020), NUMBER 3, PAGES 543–558

DELAY-DEPENDENT STABILITY OF LINEAR MULTI-STEP METHODS FOR LINEAR NEUTRAL SYSTEMS

Guang-Da Hu and Lizhen Shao

In this paper, we are concerned with numerical methods for linear neutral systems with multiple delays. For delay-dependently stable neutral systems, we ask what conditions must be imposed on linear multi-step methods in order that the numerical solutions display stability property analogous to that displayed by the exact solutions. Combining with Lagrange interpolation, linear multi-step methods can be applied to the neutral systems. Utilizing the argument principle, a sufficient condition is derived for linear multi-step methods with preserving delay-dependent stability. Numerical examples are given to illustrate the main results.

Keywords: neutral systems with multiple delays, delay-dependent stability, linear multi-step method, Lagrange interpolation, argument principle

Classification: 65L05, 65L07, 65L20

1. INTRODUCTION

We are concerned with linear neutral systems with multiple delays described by

$$\begin{cases} \dot{u}(t) = Lu(t) + \sum_{j=1}^{m}[M_j u(t-\tau_j) + N_j \dot{u}(t-\tau_j)], \quad \text{for} \quad t \geq 0, \\ u(t) = \phi(t) \quad \text{for} \quad t \leq 0, \end{cases} \tag{1}$$

with the condition

$$\sum_{j=1}^{m} \|N_j\| < 1, \tag{2}$$

where parameter matrices $L, M_j, N_j \in \mathcal{R}^{d\times d}$, $\tau_j > 0$ are given positive constants for $j = 1, \cdots, m$, $\tau_m > \tau_{m-1} > \cdots \tau_1 > 0$, and $u(t) \in \mathcal{R}^d$ is the unknown vector-valued function.

There are a number of applications [1, 4, 5, 14, 18, 20] where one has to consider delay differential equations. In general it is impossible to obtain an exact solution of delay systems. It is necessary to seek a numerical solution of a delay system, for instance, in order to study the rise time, maximum overshoot and setting time (the transient performance) of a closed-loop delay control system. Numerical analysis is an important tool to investigate delay control systems [13].

DOI: 10.14736/kyb-2020-3-0543

Stability of delay and neutral systems can be divided into two categories according to its dependence upon the size of delays. The stability which does not depend on delays is called delay-independent, whereas it depends on delays is referred to as delay-dependent. The stability of numerical methods is also divided into delay-independent and delay-dependent according to they are applied to delay-independently stable and delay-dependently stable systems, respectively. In this paper, we are concerned with delay-dependent stability of numerical methods for neutral systems with multiple delays.

Numerical methods has been studied in [2, 8, 10, 13, 17, 19] for delay and neutral differential equations. Delay-independent stability of numerical methods for delay and neutral differential equations has been investigated in [2, 8, 19]. For delay-dependent stability of numerical methods for a linear delay system with a single delay, only few works have been reported, for example, [2, 10, 17]. Recently, combining with Lagrange interpolation, delay-dependent stability of Runge-Kutta methods is investigated [7] for system (1). Linear multi-step methods need less computational effort than Runge-Kutta those. It is necessary to study stability of linear multi-step methods for neutral differential equations.

In this paper, along the line of [7], we investigate delay-dependent stability of linear multi-step methods combined with Lagrange interpolation for linear neutral systems with multiple delays. To the best of the authors' knowledge, delay-dependent stability of linear multi-step methods for neutral systems with multiple delays has not been reported in the literature.

This paper is organized as follows. Several lemmas are reviewed in section 2. In section 3, delay-dependent stability of linear multi-step methods is studied. Numerical examples are provided to illustrate the main results in section 4. Conclusions are given in section 5.

Throughout the paper, $||A||$ stands for the matrix norm. The j^{th} eigenvalue of A is denoted by $\lambda_j(A)$. The symbol $\rho(A)$ represents the spectral radius.

2. PRELIMINARIES

In this section, several lemmas are reviewed. They will be used in sections 3 and 4, respectively.

We consider the asymptotic stability of system (1) satisfying condition (2). The characteristic equation of system (1) is

$$P(s) = \det[sI - L - \sum_{j=1}^{m}(M_j \exp(-\tau_j s) + sN_j \exp(-\tau_j s))] = 0. \tag{3}$$

The asymptotic stability of system (1) satisfying condition (2) is determined by the position of the characteristic roots. System (1) is asymptotically stable if and only if all characteristic roots lie in the open left complex half-plane [5].

Now we will review the results in [7] which are concerned with stability of (1) satisfying condition (2).

Lemma 2.1. (Hu [7]) For system (1), assume that condition (2) holds. Let ξ be an unstable characteristic root of Eq. (3), then

$$|\xi| \le \beta = \frac{\|L\| + \sum_{j=1}^{m} \|M_j\|}{1 - \sum_{j=1}^{m} \|N_j\|}. \tag{4}$$

We need the following definition to present a stability criterion of system (1).

Definition 2.2. (Hu [7]) Assume that the conditions of Lemma 2.1 hold. The set D is defined by

$$D = \{s : \operatorname{Re} s \ge 0 \quad \text{and} \quad |s| \le \beta\},$$

and its boundary is denoted by C. Here β is given by (4) in Lemma 2.1.

The following lemma will exclude all the unstable characteristic root of Eq. (3) from the set D. A necessary and sufficient condition for asymptotic stability of system (1) satisfying condition (2) is given by the argument principle.

Lemma 2.3. (Hu [7]) System (1) satisfying condition (2) is asymptotically stable if and only if

$$P(s) \ne 0 \quad \text{for} \quad s \in C \tag{5}$$

and

$$\triangle_C \arg P(s) = 0 \tag{6}$$

hold, where $\arg P(s)$ stands for the argument of $P(s)$ and $\triangle_C \arg P(s)$ change of the argument of $P(s)$ along the curve C.

In Appendix, an algorithm is provided to check the delay-dependent stability of analytical solutions due to Lemma 2.3. It can be found in [7].

3. NUMERICAL STABILITY OF LINEAR MULTI-STEP METHODS

In this section, delay-dependent stability of linear multi-step (LM) methods for linear neutral systems with multiple delays is discussed. Based on the argument principle, a sufficient condition for delay-dependent stability of LM methods is presented.

First, we assume the numerical solution we are now discussing gives a sequence of approximate values $\{u_0, u_1, \ldots, u_n, \ldots\}$ of $\{u(0), u(t_1), \ldots, u(t_n), \ldots\}$ of (1) on certain equidistant step-values $\{t_j = jh\}$ with the step-size h. The following definition of delay-dependent stability of numerical methods for system (1) is an extension of that in [2].

Definition 3.1. Assume that system (1) satisfying condition (2) is asymptotically stable for given matrices L, M_j, N_j and delays τ_j. A numerical method is called delay-dependently stable for system (1) if there exists a step-size h and the numerical solution u_n with h satisfies

$$u_n \to 0 \quad \text{as} \quad n \to \infty$$

for any initial function.

When applied to ODEs

$$\dot{y}(t) = f(t, y(t)), \quad \text{for} \quad t \geq 0, \tag{7}$$

a linear k-step method is given as follows (e. g., [15]):

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h \sum_{j=0}^{k} \beta_j f_{n+j}, \tag{8}$$

where h stands for the step-size, $f_j = f(t_j, y_j)$, α_j, β_j are the formula parameters satisfying $\alpha_k = 1$ and $|\alpha_0| + |\beta_0| \neq 0$.

In order to solve numerically an asymptotically stable system (1) with (2), we want to determine a step-size h such that the resulting difference system from the LM method is asymptotically stable, i. e.

$$u_n \to 0 \quad \text{as} \quad n \to \infty$$

for any initial function $u(t) = \phi(t)$ with $-\tau_m \leq t \leq 0$.

For the application to the linear system (1) with (2), we introduce the same reformulation given by [8]. That is, with an auxiliary variable $v(t)$ the system (1) is rewritten as the simultaneous system:

$$\dot{u}(t) = v(t) \tag{9}$$

and

$$v(t) = Lu(t) + \sum_{i=1}^{m} (M_i u(t - \tau_i) + N_i v(t - \tau_i)). \tag{10}$$

Then, an application of the method (8) to (9) and (10) yields

$$\sum_{j=0}^{k} \alpha_j u_{n+j} = h \sum_{j=0}^{k} \beta_j v_{n+j}, \tag{11}$$

and

$$v_{n+j} = Lu_{n+j} + \sum_{i=1}^{m} (M_i u^h(t_{n+j} - \tau_i) + N_i v^h(t_{n+j} - \tau_i)). \tag{12}$$

for $n = 1, 2, \cdots$. Here $u^h(t)$ and $v^h(t)$ with $t \geq 0$ are defined by Lagrange interpolation [8], respectively, i. e.

$$u^h(t_l + \delta h) = \phi(t_l + \delta h) \quad \text{with} \quad t_l + \delta h \leq 0,$$

$$u^h(t_l + \delta h) = \sum_{p=-r}^{q} L_j(\delta) u_{l+p} \quad \text{with} \quad t_l + \delta h \geq 0,$$

$$v^h(t_l + \delta h) = \dot{\phi}(t_l + \delta h) \quad \text{with} \quad t_l + \delta h \leq 0,$$

and

$$v^h(t_l + \delta h) = \sum_{p=-r}^{q} L_j(\delta) v_{l+p} \quad \text{with} \quad t_l + \delta h \geq 0,$$

for $0 \le \delta < 1, l = 0, 1 \cdots$, and

$$L_p(\delta) = \prod_{k=-r, k \neq p}^{q} \frac{\delta - k}{p - k}, \tag{13}$$

where $r, q \ge 0$ are integers and $r \le q \le r+2$. Let $l_i = [\tau_i h^{-1}], \delta_i = l_i - \tau_i h^{-1}, 0 \le \delta_i < 1$ for $i = 1, \cdots, m$ and $l_m \ge l_{m-1} \ge \cdots \ge l_1 \ge q+1$, where $[\sigma]$ denotes the smallest integer that is greater than or equal to $\sigma \in \mathcal{R}$. We have that

$$\tau_i = l_i h - \delta_i h, \quad \text{and} \quad t_{n+j} - \tau_i = (n+j)h - (l_i h - \delta_i h) = (n+j-l_i)h + \delta_i h.$$

Hence

$$u^h(t_{n+j} - \tau_i) = u^h(t_{n+j-l_i} + \delta h) = \sum_{p=-r}^{q} L_p(\delta_i) u_{l+j-l_i+p} \quad \text{with} \quad t_{n+j-l_i} + \delta h \ge 0 \tag{14}$$

and

$$v^h(t_{n+j} - \tau_i) = v^h(t_{n+j-l_i} + \delta h) = \sum_{p=-r}^{q} L_p(\delta_i) v_{l+j-l_i+p} \quad \text{with} \quad t_{n+j-l_i} + \delta h \ge 0. \tag{15}$$

Lemma 3.2. The characteristic polynomial of the resulting difference system (11) and (12) from LM method (8) is given by

$$P_{LM}(z) =$$

$$\det \left[\sum_{j=0}^{k} \left((\alpha_j I - h\beta_j L) z^{l_m + r} - \sum_{i=1}^{m} \left((\alpha_j N_i + h\beta_j M_i) \sum_{p=-r}^{q} L_p(\delta_i) z^{l_m - l_i + r + p} \right) \right) z^j \right]. \tag{16}$$

P r o o f. Eq. (11) derives

$$hN_i L_p(\delta_i) \sum_{j=0}^{k} \beta_j v_{n+j-l_i+p} = N_i L_p(\delta_i) \sum_{j=0}^{k} \alpha_j u_{n+j-l_i+p}.$$

Hence

$$h \sum_{j=0}^{k} \beta_j \left(\sum_{i=1}^{m} N_i \sum_{p=-r}^{q} L_p(\delta_i) v_{n+j-l_i+p} \right) = \sum_{j=0}^{k} \alpha_j \left(\sum_{i=1}^{m} N_i \sum_{p=-r}^{q} L_p(\delta_i) u_{n+j-l_i+p} \right)$$

which, together with (11), (12), (14) and (15) leads

$$\sum_{j=0}^{k} \alpha_j u_{n+j} = h \sum_{j=0}^{k} \beta_j \left(L u_{n+j} + \sum_{i=1}^{m} \left(M_i \sum_{p=-r}^{q} L_p(\delta_i) u_{n+j-l_i+p} + N_i \sum_{p=-r}^{q} L_p(\delta_i) v_{n+j-l_i+p} \right) \right)$$

$$= h\sum_{j=0}^{k}\beta_j\left(Lu_{n+j}+\sum_{i=1}^{m}\left(M_i\sum_{p=-r}^{q}L_p(\delta_i)u_{n+j-l_i+p}\right)\right)+\sum_{j=0}^{k}\alpha_j\left(\sum_{i=1}^{m}N_i\sum_{p=-r}^{q}L_p(\delta_i)u_{n+j-l_i+p}\right).$$

Taking its Z-transform and putting $Z\{u_{n-l_m-r}\}=U(z)$, we obtain

$$\left(\sum_{j=0}^{k}\alpha_j z^{l_m+r+j}\right)U(z)=(\sum_{j=0}^{k}h\beta_j(Lz^{l_m+r+j}+\sum_{i=1}^{m}M_i\sum_{p=-r}^{q}L_p(\delta_i)z^{l_m-l_i+p+r+j})$$
$$+\alpha_j\sum_{i=1}^{m}N_i\sum_{p=-r}^{q}L_p(\delta_i)z^{l_m-l_i+p+r+j})U(z).$$

This means

$$\left[\sum_{j=0}^{k}\left((\alpha_jI-h\beta_jL)z^{l_m+r}-\sum_{i=1}^{m}\left((\alpha_jN_i+h\beta_jM_i)\sum_{p=-r}^{q}L_p(\delta_i)z^{l_m-l_i+r+p}\right)\right)z^j\right]U(z)=0.$$

Hence, the characteristic polynomial is given as desired. □

For delay-dependent stability of an LM method, we have the following result.

Theorem 3.3. Assume that

(i) system (1) satisfying condition (2) is asymptotically stable for given matrices L, M_i, N_i and delay τ_i, for $i=1,\cdots,m$, i. e. Lemma 2.3 holds;

(ii) there exists a step-size h such that the matrix $I-h\beta_kL$ in the underlying LM method is nonsingular;

(iii) define

$$Q_{LM}(z)=\det G(z), \tag{17}$$

where

$$G(z)=\left[\sum_{j=0}^{k}\left((\alpha_jI-h\beta_jL)-\sum_{i=1}^{m}\left((\alpha_jN_i+h\beta_jM_i)\sum_{p=-r}^{q}L_p(\delta_i)z^{-l_i+p}\right)\right)z^j\right], \tag{18}$$

for the step-size h, the rational function $Q_{LM}(z)$ never vanishes on the unit circle $\mu=\{z:|z|=1\}$ and its change of argument satisfies

$$\frac{1}{2\pi}\triangle_\mu \arg Q_{LM}(z)=kd. \tag{19}$$

Then the resulting difference system (11) and (12) from the LM method combined with Lagrange interpolation is delay-dependently stable.

P r o o f. The difference system (11) and (12) is asymptotically stable if and only if all the characteristic roots of $P_{LM}(z) = 0$ lie in the inside of the unit circle, i. e.

$$P_{LM}(z) = 0 \quad \text{implies} \quad |z| < 1, \tag{20}$$

where $P_{LM}(z)$ is given by (16) in Lemma 3.2. Because of the condition (ii) the parameter matrix $\alpha_k I - h\beta_k L$ of the term z^{k+r+l_m} in (16) is nonsingular for $\alpha_k = 1$. We know that the degree of $P_{LM}(z)$ is $d(k+r+l_m)$. Hence $P_{LM}(z) = 0$ has $d(k+r+l_m)$ roots. From (17),

$$P_{LM}(z) = z^{d(l_m+r)} Q_{LM}(z). \tag{21}$$

Condition (iii) and (21) means that $P_{LM}(z)$ has neither zeros nor poles on the unit circle μ. Since

$$\arg P_{LM}(z) = \arg z^{d(l_m+r)} Q_{LM}(z) = \arg z^{d(l_m+r)} + \arg Q_{LM}(z), \tag{22}$$

we have

$$\frac{1}{2\pi} \triangle_\mu \arg P_{LM}(z) = \frac{1}{2\pi} \triangle_\mu \arg z^{d(l_m+r)} + \frac{1}{2\pi} \triangle_\mu \arg Q_{LM}(z). \tag{23}$$

According to (19) and (23),

$$\begin{aligned} &\frac{1}{2\pi} \triangle_\mu \arg P_{LM}(z) \\ = \; &\frac{1}{2\pi} \triangle_\mu \arg z^{d(l_m+r)} + \frac{1}{2\pi} \triangle_\mu \arg Q_{LM}(z) = d(l_m+r) + dk = d(k+r+l_m). \end{aligned} \tag{24}$$

By means of the argument principle and (24), the condition (iii) means that all the $d(k+r+l_m)$ roots of $P_{LM}(z) = 0$ lie in the inside of the unit circle $|z| = 1$. □

Remark 3.4. When LM methods are explicit, $\beta_k = 0$ [15], we have that $\alpha_k I - h\beta_k L = I$ for $\alpha_k = 1$, i. e., condition (ii) of Theorem 3.1 holds automatically.

Remark 3.5. In general, it is necessary to use an one-step method to produce the additional starting values (a starting procedure) to implement LM methods for the neutral delay systems.

Now we can describe an algorithm to check the delay-dependent stability of numerical solutions due to Theorem 3.3.

Algorithm 1

Step 1. Taking a sufficiently big $n \in \mathbb{N}$, we have $n+1$ nodes $\{z_0, z_1, \ldots, z_n\}$ upon the unit circle μ of z-plane such that $\arg z_\ell = (2\pi)\ell/n$. From the definition of $G(z)$, (18), for each $z_\ell (\ell = 0, 1 \ldots n)$, we have

$$Q_{LM}(z_\ell) = \det G(z_\ell),$$

where

$$G(z_\ell) = \left[\sum_{j=0}^{k} \left((\alpha_j I - h\beta_j L) - \sum_{i=1}^{m} \left((\alpha_j N_i + h\beta_j M_i) \sum_{p=-r}^{q} L_p(\delta_i) z_\ell^{-l_i+p} \right) \right) z_\ell^j \right].$$

Since $G(z_\ell)$ is a numerical matrix, we can evaluate $Q_{LM}(z_\ell)$ by calculating the determinant of matrix $G(z_\ell)$ through the elementary row (or column) operations. Also we decompose $Q_{LM}(z_\ell)$ into its real and imaginary parts.

Step 2. We examine whether $Q_{LM}(z_\ell) = 0$ holds for each z_ℓ ($\ell = 0, 1, \ldots, n$) by checking its magnitude satisfies $|Q_{LM}(z_\ell)| \leq \delta$ with the preassigned small positive tolerance δ. If it holds, i. e., $z_\ell \in \mu$ is a root of $Q_{LM}(z)$, then the numerical scheme for the neutral system is not asymptotically stable and stop the algorithm. Otherwise, go to the next step.

Step 3. We compute the change of the argument along the sequence

$$\{Q_{LM}(z_0), Q_{LM}(z_1), \ldots, Q_{LM}(z_n)\}.$$

If it equals to kd, then the LM scheme combined with Lagrange interpolation for the neutral system is asymptotically stable, otherwise not asymptotically stable.

Remark 3.6. From the above theorem, in order to solve numerically an asymptotically stable delay differential system of neutral type by a LM method combined with Lagrange interpolation, it is enough to choose a step-size h such that the resulting difference system is asymptotically stable.

Remark 3.7. Both Schur-Cohn and Jury stability criteria [12] need information of all the coefficients of the characteristic polynomial $P_{LM}(z)$. It is an ill-posed problem to compute all the coefficients of the characteristic polynomial for a high dimensional matrix [11]. Although Schur-Cohn and Jury stability criteria can be applied to the resulting difference systems from LM methods in theoretical sense, they can not work well in practice when l_m or d are big. Algorithm 1 does not compute the coefficients of the characteristic polynomial $P_{LM}(z)$, it evaluates the determinant of $G(z_\ell)$ through the elementary row (or column) operations which are relatively efficient ways [11].

Remark 3.8. In [7], a similar algorithm is presented to check delay-dependent stability of Runge-Kutta methods for linear neutral systems with multiple delays.

4. NUMERICAL EXAMPLES

In this section, two numerical examples are given to demonstrate the main results in section 3. The 2-matrix norm $F = \sqrt{\lambda_{\max}(F^T F)}$ is used. The underlying scheme is the fourth-order explicit Adams–Bashforth method [15]. The coefficients of the LM method are as follows.

$$[\alpha_4, \alpha_3, \alpha_2, \alpha_1, \alpha_0] = [1, -1, 0, 0, 0] \quad \text{and}$$
$$[\beta_4, \beta_3, \beta_2, \beta_1, \beta_0] = \left[0, \frac{55}{24}, \frac{-59}{24}, \frac{37}{24}, \frac{-9}{24}\right].$$

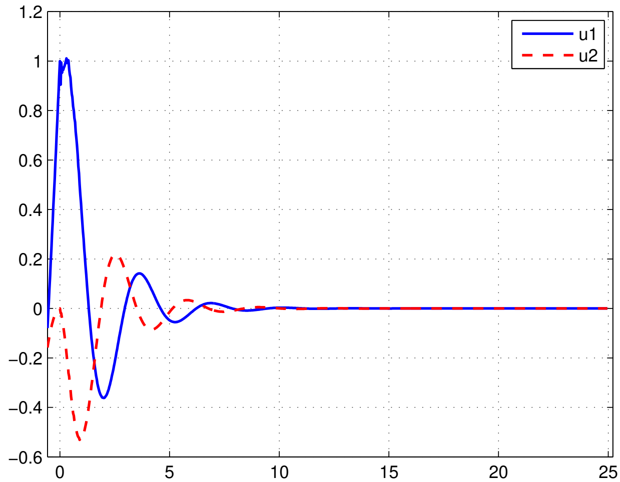

Fig. 1. Numerical solutions are asymptotically stable when $h = 0.03$ for $\tau_1 = 0.3, \tau_2 = 0.4, \tau_3 = 0.5$. in Example 4.1.

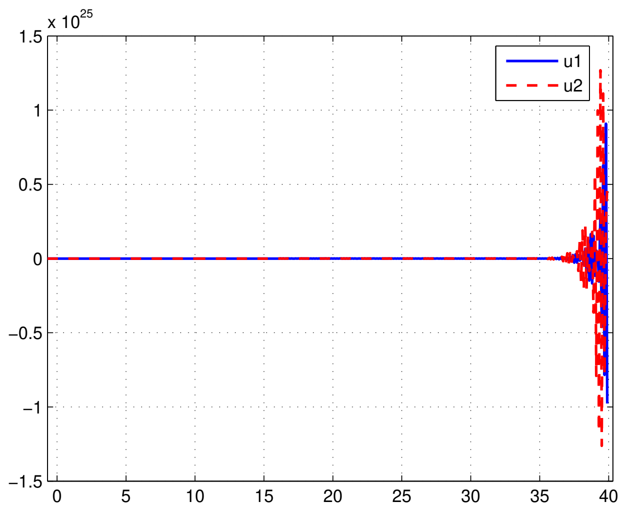

Fig. 2. Numerical solutions are not stable when $h = 0.1$ for $\tau_1 = 0.3, \tau_2 = 0.4, \tau_3 = 0.5$. in Example 4.1.

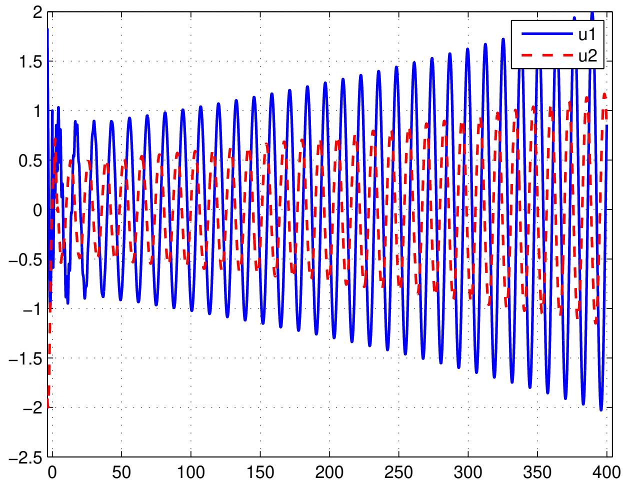

Fig. 3. Numerical solutions are not stable when $h = 0.03$ for $\tau_1 = 1.3, \tau_2 = 1.5, \tau_3 = 3.5$ in Example 4.1.

Example 4.1. Consider system (1) with

$$L = \begin{bmatrix} -2 & 0 \\ 0 & -4 \end{bmatrix}, \quad M_1 = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix},$$

$$M_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad M_3 = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix},$$

$$N_1 = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.2 \end{bmatrix}, \quad N_2 = \begin{bmatrix} 0.3 & 0.1 \\ 0 & 0.1 \end{bmatrix} \quad \text{and} \quad N_3 = \begin{bmatrix} 0.1 & 0.1 \\ 0 & 0.1 \end{bmatrix}.$$

Let the initial vector function be

$$u(t) = \begin{bmatrix} 2\sin t + 1 \\ \cos t - 1 \end{bmatrix} \quad \text{for} \quad t \leq 0.$$

Since $\Sigma_{j=1}^{3} \|N_j\| = 0.6798 < 1$, the condition (2) holds. We have $\beta = 25.7186$.

The case of $\tau_1 = 0.3, \tau_2 = 0.4, \tau_3 = 0.5$. First we analyze the stability of the system by Lemma 2.3. We have that $P(s) \neq 0$ for $s \in C$ and $\triangle_C \arg P(s) = 0$ along the curve C. Lemma 2.3 tells that the system with the given parameter matrices is asymptotically stable. Now we investigate the numerical stability of the LM scheme combined with Lagrange interpolation by Theorem 3.3. When $h = 0.03$, we obtain that $Q_{LM}(z) \neq 0$ for $z \in \mu$ and $\triangle_\mu \arg Q_{LM}(z) = dk = 2 \times 4 = 8$. Theorem 3.3 asserts the LM method for the system is asymptotically stable. The numerical solution is converging to 0, is depicted in Figure 1. However, when $h = 0.1$, we obtain that $Q_{LM}(z) \neq 0$ for $z \in \mu$ and $\triangle_\mu \arg Q_{LM}(z) = 6 \neq dk = 8$, the theorem does not hold. The numerical solution is divergent and its behaviour is shown in Figure 2.

The case of $\tau_1 = 1.3, \tau_2 = 1.5, \tau_3 = 3.5$. First we analyze the stability of the system by Lemma 2.3. We have that $P(s) \neq 0$ for $s \in C$ and $\triangle_C \arg P(s) = 2 \neq 0$ along the curve C. Lemma 2.3 tells that the system with the given parameter matrices is not asymptotically stable. Then the assumptions of Theorem 3.3 do not hold and the numerical solution is not guaranteed to be asymptotically stable. In fact, its figure given in Figure 3 shows a divergence for $h = 0.03$. We also carry out experiments for $h = 0.01$ and $h = 0.001$, respectively, the numerical solutions are still divergent.

Example 4.2. Consider system (1) with

$$L = \begin{bmatrix} -4 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -5 \end{bmatrix}, M_1 = \begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & -1 \\ 0 & -1 & 2 \end{bmatrix},$$

$$M_2 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 1 & -1 & 1 \end{bmatrix}, M_3 = \begin{bmatrix} -1 & 0 & 1 \\ -1 & -1 & 1 \\ 1 & 0 & 1 \end{bmatrix},$$

$$N_1 = \begin{bmatrix} 0.1 & 0 & 0.1 \\ 0 & 0.3 & 0 \\ 0.1 & 0 & 0.1 \end{bmatrix}, \quad N_2 = \begin{bmatrix} 0.1 & 0 & 0 \\ 0.2 & 0.1 & 0 \\ 0.1 & 0 & 0.1 \end{bmatrix} \quad \text{and} \quad N_3 = \begin{bmatrix} 0.1 & 0.1 & 0 \\ 0 & 0.1 & 0 \\ 0 & 0 & 0.2 \end{bmatrix}.$$

Let the initial vector function be

$$u(t) = \begin{bmatrix} 2\sin t + 1 \\ \cos t - 1 \\ \sin 2t + 1 \end{bmatrix} \quad \text{for} \quad t \leq 0.$$

Since $\Sigma_{j=1}^3 \|N_j\| = 0.7618 < 1$, the condition (2) holds. We have $\beta = 51.4434$.

The case of $\tau_1 = 0.3, \tau_2 = 0.4, \tau_3 = 0.5$. First we analyze the stability of the system by Lemma 2.3. We have that $P(s) \neq 0$ for $s \in C$ and $\triangle_C \arg P(s) = 0$ along the curve C. Lemma 2.3 tells that the system with the given parameter matrices is asymptotically stable. Now we investigate the numerical stability of the LM scheme combined with Lagrange interpolation by Theorem 3.3. When $h = 0.03$, we obtain that $Q_{LM}(z) \neq 0$ for $z \in \mu$ and $\triangle_\mu \arg Q_{LM}(z) = dk = 3 \times 4 = 12$. Theorem 3.3 asserts the LM method for the system is asymptotically stable. The numerical solution is converging to 0, is depicted in Figure 4. However, when $h = 0.07$, we obtain that $Q_{LM}(z) \neq 0$ for $z \in \mu$ and $\triangle_\mu \arg Q_{LM}(z) = 11 \neq dk = 12$, the theorem does not hold. The numerical solution is divergent and its behaviour is shown in Figure 5.

The case of $\tau_1 = 0.3, \tau_2 = 0.4, \tau_3 = 1.5$. First we analyze the stability of the system by Lemma 2.3. We have that $P(s) \neq 0$ for $s \in C$ and $\triangle_C \arg P(s) = 2 \neq 0$ along the curve C. Lemma 2.3 tells that the system with the given parameter matrices is

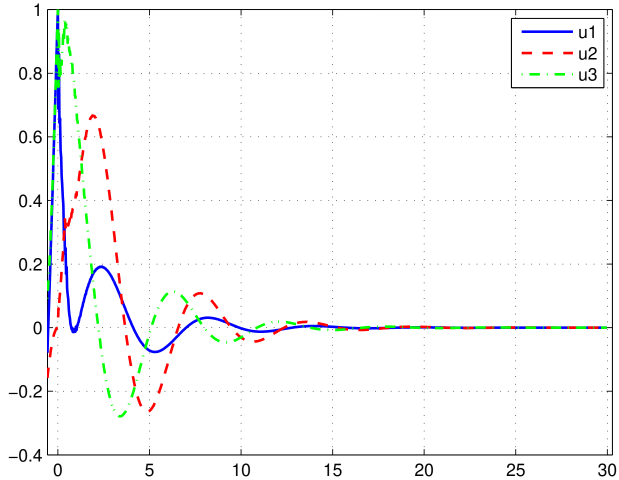

Fig. 4. Numerical solutions are asymptotically stable when $h = 0.03$ for $\tau_1 = 0.3, \tau_1 = 0.4, \tau_1 = 0.5$ in Example 4.2.

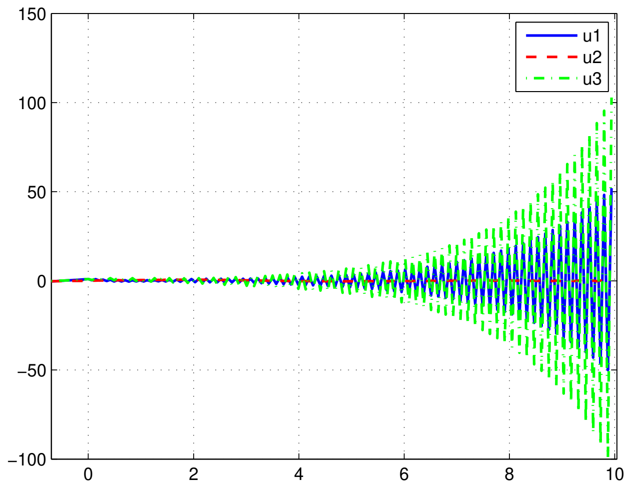

Fig. 5. Numerical solutions are not stable when $h = 0.07$ for $\tau_1 = 0.3, \tau_1 = 0.4, \tau_1 = 0.5$ in Example 4.2.

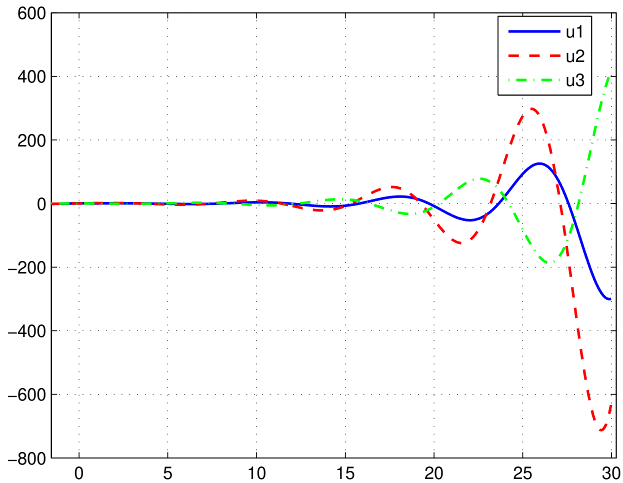

Fig. 6. Numerical solutions are not stable when $h = 0.03$ for $\tau_1 = 0.3, \tau_1 = 0.4, \tau_1 = 1.5$ in Example 4.2.

not asymptotically stable. Then the assumptions of Theorem 3.3 do not hold and the numerical solution is not guaranteed to be asymptotically stable. In fact, its figure given in Figure 6 shows a divergence for $h = 0.03$. We also carry out experiments for $h = 0.01$ $h = 0.001$, respectively, the numerical solutions are still divergent.

Remark 4.3. The two numerical examples show that the main results are valid for actual computation. The main results explain that our following experiences are reasonable: In order to solve numerically an asymptotically stable system (1), if one chooses a small step-size h, it is possible that the resulting difference system from LM method combined with Lagrange interpolation is asymptotically stable.

5. CONCLUSIONS

A sufficient condition is presented for delay-dependent stability of LM methods. It shows that it is possible that there is a step-size h such that the resulting difference system from a LM method combined with Lagrange interpolation is asymptotically stable for an asymptotically stable neutral system. An algorithm is provided for checking delay-dependent stability of LM methods.

APPENDIX

Algorithm 2

Step 0. Compute β by (4) and determine the curve C which consists of two parts, i. e., the segment $\{s = it;\ -\beta \le t \le \beta\}$ and the half-circle $\{s;\ |s| = \beta \text{ and } -\pi/2 \le \arg s \le \pi/2\}$.

Step 1. Take a sufficiently large integer $n \in \mathbb{N}$ and distribute n node points $\{s_j\}$ $(j = 0, 1, \ldots, n)$ on C as uniformly as possible. Let

$$R(s) = sI - L - \sum_{i=1}^{m}(M_i \exp(-\tau_i s) + sN_i \exp(-\tau_i s)), \tag{25}$$

then

$$P(s) = \det R(s). \tag{26}$$

From (26), for each s_j $(j = 0, 1, \ldots, n)$, we have

$$P(s_j) = \det R(s_j), \tag{27}$$

where

$$R(s_j) = s_j I - L - \sum_{i=1}^{m}(M_i \exp(-\tau_i s_j) - s_j N_i \exp(-\tau_i s_j)). \tag{28}$$

Since $R(s_j)$ is a numerical matrix for each j, we can evaluate $P(s_j)$ by calculating the determinant of matrix $R(s_j)$ through the elementary row (or column) operations. Also we decompose $P(s_j)$ into its real and imaginary parts.

Step 2. We examine whether $P(s_j) = 0$ holds for each s_j $(j = 0, 1, \ldots, n)$ by checking its magnitude satisfies $|P(s_j)| \le \delta$ with the preassigned tolerance δ. If it holds, i. e., $s_j \in C$ is a root of $P(s)$, then the neutral system is not asymptotically stable and stop the algorithm. Otherwise, go to the next step.

Step 3. We compute the change of the argument along the sequence

$$\{P(s_0), P(s_1), \ldots, P(s_n)\}.$$

If it is zero, then the system is asymptotically stable, otherwise not asymptotically stable.

ACKNOWLEDGEMENT

This work was supported by the National Natural Science Foundation of China (11871330).

The authors wish to thank Miss Fangyuan Zhao for implementing the numerical experiments. The authors are greatly indebted to the associate editor and the anonymous reviewer for careful reviews and valuable suggestions which are helpful to improve this paper.

(Received February 18, 2019)

REFERENCES

[1] A. Y. Aleksandrov, G. D. Hu, and A. P. Zhabko: Delay-independent stability conditions for some classes of nonlinear systems. IEEE Trans. Automat. Control *59* (2014), 2209–2214. DOI:10.1109/tac.2014.2299012

[2] A. Bellen and M. Zennaro: Numerical Methods for Delay Differential Equations. Oxford University Press, Oxford 2003. DOI:10.1093/acprof:oso/9780198506546.001.0001

[3] J. W. Brown and R. V. Churchill: Complex Variables and Applications. McGraw–Hill Companies, Inc. and China Machine Press, Beijing 2004.

[4] E. Fridman: Introduction to Time-Delay Systems: Analysis and Control. Birkhäuser, New York 2014. DOI:10.1007/978-3-319-09393-2

[5] J. K. Hale and S. M. Verduyn Lunel: Strong stabilization of neutral functional differential equations. IMA J. Math. Control Inform. *19* (2002), 5–23. DOI:10.1093/imamci/19.1_and_2.5

[6] Q. L. Han: Stability criteria for a class of linear neutral systems with time-varying discrete and distributed delays. IMA J. Math. Control Inform. *20* (2003), 371–386. DOI:10.1093/imamci/20.4.371

[7] G. D. Hu: Delay-dependent stability of Runge-Kutta methods for linear neutral systems with multiple delays. Kybernetika *54* (2018), 718–735. DOI:10.14736/kyb-2018-4-0718

[8] G. D. Hu and B. Cahlon: Estimations on numerically stable step-size for neutral delay differential systems with multiple delays. J. Compt. Appl. Math. *102* (1999), 221–234. DOI:10.1016/s0377-0427(98)00215-5

[9] G. D. Hu and M. Liu: Stability criteria of linear neutral systems with multiple delays. IEEE Trans. Automat. Control *52* (2007), 720–724. DOI:10.1109/tac.2007.894539

[10] C. Huang and S. Vandewalle: An analysis of delay-dependent stability for ordinary and partial differential equations with fixed and distributed delays. SIAM J. Scientific Computing *25* (2004), 1608–1632. DOI:10.1137/s1064827502409717

[11] L. W. Johnson, R. Dean Riess and J. T. Arnold: Introduction to Linear Algebra. Prentice-Hall, Englewood Cliffs 2000.

[12] E. I. Jury: Theory and Application of z-Transform Method. John Wiley and Sons, New York 1964.

[13] A. V. Kim and A. V. Ivanov: Multistep numerical methods for fuctional difefrential equations. Math. Comput. Simul. *45* (1998), 377–384. DOI:10.1016/s0378-4754(97)00117-1

[14] V. B. Kolmanovskii and A. Myshkis: Introduction to Theory and Applications of Functional Differential Equations. Kluwer Academic Publishers, Dordrecht 1999. DOI:10.1007/978-94-017-1965-0

[15] J. D. Lambert: Numerical Methods for Ordinary Differential Systems. John Wiley and Sons, New York 1999.

[16] P. Lancaster and M. Tismenetsky: The Theory of Matrices with Applications. Academic Press, Orlando 1985.

[17] S. Maset: Instability of Runge-Kutta methods when applied to linear systems of delay differential equations. Numer. Math. *90* (2002), 555–562. DOI:10.1007/s002110100266

[18] I. V. Medvedeva and A. P. Zhabko: Synthesis of Razumikhin and Lyapunov-Krasovskii approaches to stability analysis of time-delay systems. Automatica *51* (2015), 372–377. DOI:10.1016/j.automatica.2014.10.074

[19] H. Tian and J. Kuang: The stability of the θ-methods in numerical solution of delay differential equations with several delay terms. J. Comput. Appl. Math. *58* (1995), 171–181. DOI:10.1016/0377-0427(93)e0269-r

[20] T. Vyhlidal and P. Zitek: Modification of Mikhaylov criterion for neutral time-delay systems. IEEE Trans. Automat. Control *54* (2009), 2430–2435. DOI:10.1109/tac.2009.2029301

Guang-Da Hu, Department of Mathematics, Shanghai University, Shanghai, 200444. P. R. China.
e-mail: ghu@hit.edu.cn

Lizhen Shao, School of Automation and Electrical Engineering, University of Science and Technology Beijing, Beijing, 100083. P. R. China.
e-mail: lshao@ustb.edu.cn