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CONTINUOUS FEEDBACK STABILIZATION FOR A CLASS OF AFFINE STOCHASTIC NONLINEAR SYSTEMS

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We investigate the state feedback stabilization, in the sense of weak solution, of nonlinear stochastic systems when the drift is quadratic in the control and the diffusion term is affine in the control. Based on the generalised stochastic Lyapunov theorem, we derive the necessary conditions and the sufficient conditions, respectively, for the global asymptotic stabilization in probability by a continuous feedback explicitly computed. The interest of this work is that the existing control methods are inapplicable to a lot of systems contained in the class of stochastic systems considered in this paper.

Keywords: continuous state feedback, control stochastic nonlinear systems, global asymptotic stability in probability

Classification: 60H10, 93C10, 93D05, 93D15, 93E15

1. INTRODUCTION

In this paper, we consider a class of stochastic nonlinear systems described by

$$\mathrm{d}x = \Big(f_0(x) + uf_1(x) + u^2 f_2(x)\Big)\,\mathrm{d}t + \Big(g_0(x) + ug_1(x)\Big)\,\mathrm{d}\omega, \tag{1}$$

where $x \in \mathbb{R}^n$ is the state and $u \in \mathbb{R}$ is the control and ω is a standard $\mathbb{R}^m$-valued Wiener process defined on a probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ with Ω being a sample space, $\mathcal{F}$ a σ-algebra on Ω, $\mathcal{F}_t$ a filtration and P a probability measure. f_0, f_1, $f_2 : \mathbb{R}^n \to \mathbb{R}^n$ and g_0, $g_1 : \mathbb{R}^n \to \mathbb{R}^{n\times m}$ are continuous functions with $f_0(0) = g_0(0) = 0$.

Stochastic models have played an important role in many branches of manufacture and engineering applications. Since the works, such as Khasminskii [13], Kushner [15] and Mao [22] established a solid foundation for the stochastic stability theory, the design of stabilization controller has been investigated widely for various stochastic nonlinear systems. Several results on state-feedback stabilization and output-feedback stabilization for various classes of stochastic nonlinear systems have been achieved in [1, 5, 8, 16, 17, 32] and the references therein.

Most of the existing papers focus on the stabilization problems of nonlinear systems using some fundamental stochastic stability theories presented in [13] and [22], which were useful tools for the controller design of stochastic nonlinear systems. However, these

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theories require that the stochastic nonlinear systems satisfy the local Lipschitz condition or have a unique strong solution. This indeed impedes the application of the stability theory, because many practical stochastic systems do not satisfy the local Lipschitz condition, which is inherent or caused by a specified feedback controller. To relax this restriction, Li and Liu in [19] generalized the concepts and theorems of global stability to the stochastic nonlinear systems without the local Lipschitz condition or having more than one weak solution. Since then, many papers use the techniques developed in [19], one can cite [11], [27].

Stochastic control systems (1) are of interest since thier coefficients are only continuous, not necessary locally Lipschitz, and then can model many practical systems, such as, the stochastic models for a sequencing-batch reactor and for a chemostat proposed in [9], these models are affine in the control and have the coefficients involve the term $\sqrt{x(t)}$. Among other applications in which stochastic differential equations may be only continuous, we can cite stochastic financial and biological models in which, as mentioned in [26], the diffusion coefficients often contain the term $\sqrt{|x(t)|}$, or more general $|x(t)|^r$, $r \in (0,1)$, for more details, we refer the reader to [18, 23]. Therefore, the investigation of systems 1 is practically relevant.

For deterministic nonlinear control systems, many techniques for studying the stabilizability problem and for designing stabilizing feedback laws are known. Historically, one of the first significant results is because of Artstein [2] who introduced the notion of control Lyapunov function which gives a way to consider the choice of Lyapunov function and the design of control simultaneously. For affine nonlinear control systems, knowing a CLF, an explicit and simple proof of Artstein's Theorem [2] is given by Sontag in [25] and revisited later on by Lin and Sontag in [20].

In [6], Florchinger extends the Lin–Sontag's formula to the particular class of stochastic affine control systems

$$\mathrm{d}x = \Big(f_0(x) + uf_1(x)\Big)\,\mathrm{d}t + \Big(g_0(x)\Big)\,\mathrm{d}\omega. \tag{2}$$

For these systems, the associated infinitesimal generator $\mathcal{L}_u$ satisfies $\mathcal{L}_u V(x) = b(x)u + c(x)$, with known functions b and c. Knowing a stochastic control Lyapunov function (SCLF) V, the state-feedback control $u(x)$ defined in [20, 25] yields $\mathcal{L}_u V(x) < 0$, for all $x \in \mathbb{R}^n \setminus \{0\}$, which allows, as in [20, 25] for the deterministic case, to state in [6], by application of the stochastic versions of Lyapunov theorem [13], that the stochastic affine system (2) is globally asymptotically stabilizable in probability.

Later, Chabour and Oumoun in [3] and Daumail and Florchinger in [4] considered, respectively, the systems

$$\mathrm{d}x = \Big(f_0(x) + uf_1(x)\Big)\,\mathrm{d}t + \Big(g_0(x) + ug_1(x)\Big)\,\mathrm{d}\omega, \tag{3}$$

and the systems (1).

Contrary to system (2), for systems (3) and (1), in which everything is corrupted by a noise, the associated infinitesimal generator $\mathcal{L}_u$, applied to a Lyapunov function V, leads to $\mathcal{L}_u V(x) = a(x)u^2 + b(x)u + c(x)$, with known functions a, b and c. So, it appears that the Sontag feedback defined in [25, 20] is no more a stabilizing feedback for (3) and (1). In [3] and [4], explicit stabilizing feedbacks are, respectively, constructed for systems (3) and (1), provided that known SCLFs V satisfy the assumption $\mathcal{H}$: $a(x) < 0 \Longrightarrow b(x)^2 - 4a(x)c(x) < 0$.

More recently, Florchinger [7] gave an explicit stabilizer for system (1) under the same assumption ($\mathcal{H}$) with known stochastic $\alpha-$control Lyapunov function.

Note that all the results in the last four cited papers [3, 4, 6, 7] require, respectively, the coefficients of the systems (2), (3), (1) being continuously differentiable and the constructed feedbacks are, also, continuously differentiable to guarantee the local Lipschitz condition of the closed-loop system.

The aim of this paper is to relax this restriction, allowing the coefficients of the system (1) as well as the constructed feedback to be only continuous. Inspired by the deterministic case Maniar et al. [21] and based on a known SCLF, we first give a necessary condition for the stabilization in probability by a continuous feedback. After that, we present a sufficient stabilization condition which improve the stabilizability condition ($\mathcal{H}$). The main tool used in this paper is the generalised stochastic Lyapunov theorem proved by Li and Liu [19].

The remainder of this paper is organized as follows. Section 2 gives some preliminary knowledge on stochastic stability in the sense of weak solution. In section 3, we recall some definitions and preliminary results while section 4 gives the main results of this paper. In section 5 we provide a numerical example illustrating our results. Finally, section 6 draws some conclusions.

2. STOCHASTIC STABILITY OF WEAK SOLUTIONS

We first revisit some fundamental theory about the stochastic systems. Consider the following stochastic nonlinear system:

$$\mathrm{d}x = f(x)\,\mathrm{d}t + g(x)\,\mathrm{d}\omega, \quad x(0) = x_0 \in \mathbb{R}^n, \tag{4}$$

where $x \in \mathbb{R}^n$ is the system state; ω is an m-dimensional independent standard Wiener process. The functions $f : \mathbb{R}^n \to \mathbb{R}^n$ and $g : \mathbb{R}^n \to \mathbb{R}^{n\times m}$ are continuous and satisfy $f(0) = 0$ and $g(0) = 0$. Clearly, the origin is the equilibrium point of system (4).

As it is well known in stochastic differential equation theory (Khasminskii [13] and Mao [22]), in order to guarantee the existence and uniqueness of strong solutions of the stochastic systems (4), $f(.)$ and $g(.)$ are assumed to satisfy some definite conditions such as local Lipschitz. But, here, since both $f(.)$ and $g(.)$ are only continuous, not locally Lipschtiz, system (4) may not have the solution in the classical sense as in Khasminskii [13] and Mao [22]. However, the system always has weak solutions which are essentially different from the classical (or strong) solution since the former may not be unique and may be defined on a different probability spaces. The following definition gives the rigorous characterization of the weak solution of system (4), and for more details of this subject, we refer the reader to Ikeda and Watanabe [10], Klebaner [14], Li and Liu [19], Ondreját and Seidler [24], Zhang and Liu [31].

Definition 2.1. (Li and Liu [19]) If there exist a continuous adapted process $x(t)$ on a probability space $(\Omega^x, \mathcal{F}^x, P^x)$ with a filtration $\{\mathcal{F}_t\}_{t\geq t_0}$ satisfying the usual conditions, and an m-dimensional $\{\mathcal{F}_t\}$-adapted Brownian motion $\omega^x(t)$ with $P^x\{\omega^x(t_0) = 0\} = 1$, such that the initial condition $x(t_0)$ has the given distribution, and for all $t \in [t_0, \tau^x_{+\infty})$

$$x(t) = x(t_0) + \int_{t_0}^{t} f(x(s))\,\mathrm{d}s + \int_{t_0}^{t} g(x(s))\,\mathrm{d}\omega^x(s) \ \text{ a.s.},$$

then $x(t)$ is called a weak solution of system (4), where $\tau^x_{+\infty}$ is the explosion time of the weak solution $x(t)$, that is $\tau^x_{+\infty} = \lim_{\varepsilon\to+\infty}\inf\{t \geq t_0, \|x(t)\| \geq \varepsilon, \forall\varepsilon > 0\}$.

Now, we can introduce the notion of asymptotic stability in probability for the equilibrium solution of the stochastic differential equation (4) as follows.

Definition 2.2. (Li and Liu [19]) The zero solution of system (4) is said to be
(i) *globally stable in probability*: if for any $\varepsilon \in (0,1)$, there exists a class K function α such that for $\forall x_0 \in \mathbb{R}^n$, every weak solution $x(t)$ of system (4) satisfies

$$P^x\{\sup_{t\geq 0} \|x(t)\| < \alpha(\|x_0\|)\} \geq 1 - \varepsilon.$$

(ii) *globally asymptotically stable in probability*: if it is globally stable in probability and for $\forall x_0 \in \mathbb{R}^n$, every weak solution $x(t)$ of system (4) satisfies

$$P^x\{\lim_{t\to\infty} x(t) = 0\} = 1.$$

Definition 2.3. The infinitesimal generator associated with the stochastic differential equation (4), denoted by $\mathcal{L}$, is defined for any function Ψ in $C^2(\mathbb{R}^n)$ by

$$\mathcal{L}\Psi(x) = \nabla\Psi(x)f(x) + \frac{1}{2}\mathrm{Tr}\big(g(x)g(x)^\top\nabla^2\Psi(x)\big),$$

where Tr{.} represents the trace of the argument.

Definition 2.4. Let $\mathcal{V}$ be a neighborhood of the origin in $\mathbb{R}^n$. We say that a function $V \in C^2(\mathbb{R}^n, \mathbb{R}^+)$ is a stochastic Lyapunov function of system (4) on $\mathcal{V}$ if

(i) $V(0) = 0$ and $V(x) > 0$, $\forall x \in \mathcal{V} \setminus \{0\}$.

(ii) $\mathcal{L}V(x) < 0$, $\forall x \in \mathcal{V} \setminus \{0\}$.

(iii) If $\mathcal{V} = \mathbb{R}^n$, then V is proper, that is, $V(x) \to \infty$ as $\|x\| \to \infty$.

Lemma 2.5. (Li and Liu [19]) For system (4), suppose that there exist functions $V \in C^2(\mathbb{R}^n, \mathbb{R}^+)$, $\xi \in C^0(\mathbb{R}^n, \mathbb{R}^+)$ and class K_∞ functions α, β, such that

(i) $\alpha(\|x\|) \leq V(x) \leq \beta(\|x\|)$;

(ii) $\mathcal{L}V(x) \leq -\xi(x)$.

Then the zero solution of system (4) is globally stable in probability, and for $\forall x_0 \in \mathbb{R}^n$, every weak solution $x(t)$ of system (4) satisfies

$$P^x\left\{\lim_{t\to+\infty} \xi(x(t)) = 0\right\} = 1.$$

Particularly, if function ξ is positive definite, then the zero solution of system (4) is globally asymptotically stable in probability.

To conclude this section, we list a lemma which shows the connexion between a positive definite function and class $\mathcal{K}$ functions (see [12], Lemma 4.3).

Lemma 2.6. (Khalil [12]) Let $V : \mathbb{R}^n \to \mathbb{R}$ be a continuous positive definite function, then, there exit class $\mathcal{K}$ functions α and β, defined on $[0, \infty)$, such that

$$\alpha(\|x\|) \leq V(x) \leq \beta(\|x\|), \quad \forall x \in \mathbb{R}^n.$$

Moreover, if V is radially unbounded, then α and β can be chosen to belong to class $\mathcal{K}_\infty$.

3. PRELIMINARY RESULTS

For a C^2 function $V(x)$, the infinitesimal operator $\mathcal{L}_u V$ associated with the system (1) is defined as

$$\mathcal{L}_u V(x) = a_v(x)u^2 + b_v(x)u + c_v(x),$$

with

$$a_v(x) = \nabla V(x) f_2(x) + \frac{1}{2}\mathrm{Tr}\big(g_1(x) g_1(x)^\top \nabla^2 V(x)\big),$$

$$b_v(x) = \nabla V(x) f_1(x),$$

and

$$c_v(x) = \nabla V(x) f_0(x) + \frac{1}{2}\mathrm{Tr}\big(g_0(x) g_0(x)^\top \nabla^2 V(x)\big).$$

In addition, for each fixed $x \in \mathbb{R}^n$, let

$$\Delta_v(x) = b_v(x)^2 - 4a_v(x)c_v(x)$$

be the discriminant of the equation: $a_v(x)\lambda^2 + b_v(x)\lambda + c_v(x) = 0$, and

$$\lambda_{1,v}(x) = \frac{-b_v(x) - \sqrt{\Delta_v(x)}}{2a_v(x)} \quad , \quad \lambda_{2,v}(x) = \frac{-b_v(x) + \sqrt{\Delta_v(x)}}{2a_v(x)}$$

its roots when $\Delta_v(x) \geq 0$.

The following definitions come from [6].

Definition 3.1. (Florchinger [6]) A C^2 positive definite and proper function defined on $\mathbb{R}^n$ is a stochastic control Lyapunov function (SCLF) for the stochastic differential system (1) if

$$\inf_{u \in \mathbb{R}} \mathcal{L}_u V(x) = \inf_{u \in \mathbb{R}} \Big(a_v(x)u^2 + b_v(x)u + c_v(x)\Big) < 0, \quad \forall x \in \mathbb{R}^n \setminus \{0\}. \tag{5}$$

Remark 3.2. If V is a SCLF for system (1) then it is positive definite and proper, according to Lemma 2.6, there exit class $\mathcal{K}_\infty$ functions α and β satisfying the inequality (i) of Lemma 2.5, that is,

$$\alpha(\|x\|) \leq V(x) \leq \beta(\|x\|), \quad \forall x \in \mathbb{R}^n.$$

Definition 3.3. (Florchinger [6]) A function $V \in C^2(\mathbb{R}^n, \mathbb{R}^+)$ is said to satisfy the small control property with system (1), if for each $\varepsilon > 0$ there is $\delta > 0$ such that , if $x \neq 0$ satisfies $\|x\| < \delta$, then there is some u with $\|u\| < \varepsilon$ such that

$$\mathcal{L}_u V(x) = a_v(x)u^2 + b_v(x)u + c_v(x) < 0.$$

Remark 3.4. Note that if V is a SCLF for system (1) then:

(i) the functions $a_v(.)$, $b_v(.)$, $c_v(.)$ and $\Delta_v(.)$ are continuous over $\mathbb{R}^n$,

(ii) since V is positive definite, it has a minimum at 0, so $\nabla V(0) = 0$ and consequently $b_v(0) = 0$ and then $\Delta_v(0) = 0$ since $c_v(0) = 0$.

(iii) if $\Delta_v(x) > 0$, one has

$$\lambda_{1,v}(x) < \lambda_{2,v}(x), \quad \text{if} \quad a_v(x) > 0,$$

and

$$\lambda_{1,v}(x) > \lambda_{2,v}(x), \quad \text{if} \quad a_v(x) < 0.$$

Next, we recall the following result.

Proposition 3.5. (Chabour and Oumoun [3]) If V is a SCLF for the system (1), then we have:

1. $\big(\Delta_v(x) < 0\big) \Longrightarrow a_v(x) < 0,$
2. $\big(\Delta_v(x) = 0 \text{ and } a_v(x) \neq 0\big) \Longrightarrow a_v(x) < 0,$
3. $\big(\Delta_v(x) = 0 \text{ and } a_v(x) = 0\big) \Longrightarrow b_v(x) = 0 \text{ and } c_v(x) < 0.$

As far as possible, our objective is to define a state feedback law by explicit formulas in such a way that the resulting equilibrium solution of the closed-loop system is asymptotically stable in probability. To be more precise, according to Remark 3.2 and Lemma 2.5, if V is a known SCLF for the system (1), we shall define a feedback law u and a continuous definite function ξ such that

$$\mathcal{L}_u V(x) = a_v(x)u^2(x) + b_v(x)u(x) + c_v(x) \leq -\xi(x) < 0, \quad \forall x \in \mathbb{R}^n \setminus \{0\}.$$

The operator $\mathcal{L}_u V(x)$ regarded as a polynomial in $u(x)$, this last inequality leads to

$$u(x) \in \big]\lambda_{1,v}(x), \lambda_{2,v}(x)\big[, \quad \text{if} \quad a_v(x) > 0 \quad \text{and} \quad \Delta_v(x) > 0,$$

and

$$u(x) \notin \big[\lambda_{2,v}(x), \lambda_{1,v}(x)\big], \quad \text{if} \quad a_v(x) < 0 \quad \text{and} \quad \Delta_v(x) > 0.$$

In the next section, to prove main results of this paper, we use the following result.

Lemma 3.6. Let V be a SCLF for the system (1) and $x_0 \in \mathbb{R}^n \setminus \{0\}$ such that $a_v(x_0) = 0$, we have:

$$\lim_{\substack{x \to x_0 \\ a_v(x) > 0}} \lambda_{1,v}(x) = -\infty \quad \text{and} \quad \lim_{\substack{x \to x_0 \\ a_v(x) > 0}} \lambda_{2,v}(x) = -\frac{c_v(x_0)}{b_v(x_0)}, \quad \text{if} \quad b_v(x_0) > 0, \tag{6}$$

$$\lim_{\substack{x\to x_0\\ a_v(x)>0}} \lambda_{1,v}(x) = -\frac{c_v(x_0)}{b_v(x_0)} \quad\text{and}\quad \lim_{\substack{x\to x_0\\ a_v(x)>0}} \lambda_{2,v}(x) = +\infty, \quad\text{if}\quad b_v(x_0) < 0, \tag{7}$$

$$\lim_{\substack{x\to x_0\\ a_v(x)>0}} \lambda_{1,v}(x) = -\infty \quad\text{and}\quad \lim_{\substack{x\to x_0\\ a_v(x)>0}} \lambda_{2,v}(x) = +\infty, \quad\text{if}\quad b_v(x_0) = 0, \tag{8}$$

$$\lim_{\substack{x\to x_0\\ a_v(x)<0}} \lambda_{1,v}(x) = +\infty \quad\text{and}\quad \lim_{\substack{x\to x_0\\ a_v(x)<0}} \lambda_{2,v}(x) = -\frac{c_v(x_0)}{b_v(x_0)}, \quad\text{if}\quad b_v(x_0) > 0, \tag{9}$$

$$\lim_{\substack{x\to x_0\\ a_v(x)<0}} \lambda_{1,v}(x) = -\frac{c_v(x_0)}{b_v(x_0)} \quad\text{and}\quad \lim_{\substack{x\to x_0\\ a_v(x)<0}} \lambda_{2,v}(x) = -\infty, \quad\text{if}\quad b_v(x_0) < 0, \tag{10}$$

$$\lim_{\substack{x\to x_0\\ a_v(x)<0\\ b_v(x)<0}} \lambda_{1,v}(x) = -\infty \quad\text{and}\quad \lim_{\substack{x\to x_0\\ a_v(x)<0\\ b_v(x)<0}} \lambda_{2,v}(x) = -\infty, \quad\text{if}\quad b_v(x_0) = 0, \tag{11}$$

$$\lim_{\substack{x\to x_0\\ a_v(x)<0\\ b_v(x)>0}} \lambda_{1,v}(x) = +\infty \quad\text{and}\quad \lim_{\substack{x\to x_0\\ a_v(x)<0\\ b_v(x)>0}} \lambda_{2,v}(x) = +\infty, \quad\text{if}\quad b_v(x_0) = 0. \tag{12}$$

Proof. The proof is similar to Lemma 2.5 in [21] and is omitted here. □

4. MAIN RESULTS

The objective of this paper is to design a continuous feedback control law such that the resulting closed-loop system deduced from system (1) is globally asymptotically in probability.

4.1. Necessary condition for continuous feedback stabilization

Using Lemma 3.6, we can state the following necessary condition for the stabilization of system (1) by a continuous feedback.

Suppose there exist a SCLF V for system (1) and a connected open set $\mathcal{O}$ such that:

A1 $a_v(x) < 0$ and $\Delta_v(x) \geq 0$, for every $x \in \mathcal{O}$.

A2 There exist x_1 and x_2 in $\partial\mathcal{O}$ (the border of $\mathcal{O}$) such that $a_v(x_1) = a_v(x_2) = 0$.

A3 For any $\varepsilon > 0$, there exist $x \in B(x_1, \varepsilon) \cap \mathcal{O}$ and $y \in B(x_2, \varepsilon) \cap \mathcal{O}$ such that $b_v(x) \geq 0$ and $b_v(y) \leq 0$.

Remark 4.1. For reasons of continuity, if assumption **A3** holds, then $b_v(x_1) \geq 0$ and $b_v(x_2) \leq 0$.

In addition, if $b_v(x_1) = 0$ (respectively $b_v(x_2) = 0$), according to Proposition 3.5 we have $c_v(x_1) < 0$ (respectively $c_v(x_2) < 0$).

From assumption **A1** and definition of Δ_v, the inequality in assumption **A3** becomes strict, i. e., for any $\varepsilon > 0$, there exist $x \in B(x_1, \varepsilon) \cap \mathcal{O}$ such that $b_v(x) > 0$ (respectively there exist $y \in B(x_2, \varepsilon) \cap \mathcal{O}$ such that $b_v(y) < 0$).

Based on Lemma 3.6, we obtain the first main result in the following theorem. It provides a necessary condition for a stochastic control Lyapunov function to be a stochastic Lyapunov function.

Theorem 4.2. Let $\mathcal{V}$ be a neighborhood of the origin in $\mathbb{R}^n$ and assume that V is a SCLF for system (1). If there exists a connected open set $\mathcal{O} \subset \overline{\mathcal{O}} \subset \mathcal{V}$ such that assumptions **A1** − **A3** hold, then, system (1) is not asymptotically stabilizable in probability on $\mathcal{V}$ by a continuous feedback with V as the stochastic Lyapunov function.

Proof. First, from Remark 4.1, for any $\varepsilon > 0$, the sets $\{x \in \mathcal{O} | \|x - x_1\| < \varepsilon,\ b_v(x) > 0\}$ and $\{x \in \mathcal{O} | \|x - x_2\| < \varepsilon,\ b_v(x) < 0\}$ are not empty. Hence, from (10), (11) and (12)

$$\lim_{\substack{x \to x_1 \\ x \in \mathcal{O} \\ b_v(x) > 0}} \lambda_{1,v}(x) = +\infty \quad \text{and} \quad \lim_{\substack{x \to x_2 \\ x \in \mathcal{O} \\ b_v(x) < 0}} \lambda_{2,v}(x) = -\infty. \tag{13}$$

Now, assume that there exists a continuous feedback law u that asymptotically stabilizes in probability system (1) and for which the SCLF V is a stochastic Lyapunov function for the closed-loop system. Then, $\mathcal{L}_u V(x) = a_v(x) u^2(x) + b_v(x) u(x) + c_v(x) < 0$, for every $x \in \mathcal{O}$. This implies that either $u(x) < \lambda_{2,v}(x)$ or $\lambda_{1,v}(x) < u(x)$ for every $x \in \mathcal{O}$ (notice that $\lambda_{2,v}(x) \leq \lambda_{1,v}(x)$ for every $x \in \mathcal{O}$). Hence

- If $u(x) < \lambda_{2,v}(x)$ for every $x \in \mathcal{O}$ then, from (13) and since u is continuous, we have $\lim_{\substack{x \to x_2 \\ x \in \mathcal{O}}} u(x) \leq \lim_{\substack{x \to x_2 \\ x \in \mathcal{O} \\ b_v(x) < 0}} \lambda_{2,v}(x) = -\infty$.
- If $u(x) > \lambda_{1,v}(x)$ for every $x \in \mathcal{O}$ then, again from (13) and since u is continuous, we have $\lim_{\substack{x \to x_1 \\ x \in \mathcal{O}}} u(x) \geq \lim_{\substack{x \to x_1 \\ x \in \mathcal{O} \\ b_v(x) > 0}} \lambda_{1,v}(x) = +\infty$.

In both cases, we have a contradiction. □

4.2. Control design

In this section, we give universal formulas for the continuous stabilizers of system (1) via a known stochastic control Lyapunov function which not satisfies assumptions in Theorem 4.2.

First, we choose the two following continuous functions mapping $\mathbb{R}$ into itself

$$\varphi(x) = \begin{cases} 0 & \text{if } x \leq 0, \\ x & \text{if } x \geq 0, \end{cases} \qquad \text{and} \qquad \psi(x) = \begin{cases} x & \text{if } x \leq 0, \\ 0 & \text{if } x \geq 0. \end{cases}$$

We also need the following result.

Lemma 4.3. If V is a SCLF for system (1) which satisfies

$$\big(a_v(x) < 0 \ \text{ and } \ \Delta_v(x) \geq 0\big) \Longrightarrow b_v(x) \neq 0, \tag{14}$$

then the function

$$K(x) = \begin{cases} w(x) & \text{if } \Delta_v(x) > 0, \\ 0 & \text{if } \Delta_v(x) \leq 0, \end{cases} \tag{15}$$

where

$$w(x)=\begin{cases}\psi\Big(\frac{-b_v(x)+\sqrt{|\Delta_v(x)|}}{2a_v(x)}-\frac{\sqrt{|\Delta_v(x)|}}{2(1+a_v^2(x))}\Big) & \text{if } a_v(x)\neq 0 \text{ and } b_v(x)>0,\\ \varphi\Big(\frac{-b_v(x)-\sqrt{|\Delta_v(x)|}}{2a_v(x)}+\frac{\sqrt{|\Delta_v(x)|}}{2(1+a_v^2(x))}\Big) & \text{if } a_v(x)\neq 0 \text{ and } b_v(x)<0,\\ \psi\Big(-\frac{c(x)}{b(x)}-\frac{b(x)}{2}\Big) & \text{if } a(x)=0 \text{ and } b(x)>0,\\ \varphi\Big(-\frac{c(x)}{b(x)}-\frac{b(x)}{2}\Big) & \text{if } a(x)=0 \text{ and } b(x)<0,\\ 0 & \text{if } a(x)>0 \text{ and } b(x)=0\end{cases}$$

is continuous in $\mathbb{R}^n\setminus\{0\}$. Moreover, the function K is continuous at the origin if V satisfies the small control property.

Proof. Here, we follow the main lines of the proof of Theorem 3.9 from Maniar et al. [21]. From the continuity of φ, ψ, a_v and Δ_v, it is obvious that K is continuous on the open sets: $\{a_v(x)\neq 0,\Delta_v(x)>0,b_v(x)\neq 0\}$ and $\{a_v(x)<0,\Delta_v(x)<0\}$.

Since V is a SCLF, the set $\{a_v(x)>0,\Delta_v(x)\le 0\}$ is empty, and by assumption (14), the set $\{a_v(x)<0,\Delta_v(x)\ge 0,b_v(x)=0\}$ is also empty. Thus, it remains to study the continuity of K at the origin and on the sets:
$A_1=\{x\in\mathbb{R}^n\setminus\{0\}\ \mid\ \Delta_v(x)=0\}$,
$A_2=\{x\in\mathbb{R}^n\setminus\{0\}\ \mid\ a_v(x)>0,\Delta_v(x)>0,b_v(x)=0\}$,
$A_3=\{x\in\mathbb{R}^n\setminus\{0\}\ \mid\ a_v(x)=0,b_v(x)>0\}$,
$A_4=\{x\in\mathbb{R}^n\setminus\{0\}\ \mid\ a_v(x)=0,b_v(x)<0\}$.

1. If $x_0\in A_1$ then, $K(x_0)=0$. Note that $K(x)=0$ if $\Delta_v(x)\le 0$, so, from now on, we will focus on the case when $\Delta_v(x)>0$ and we will verify that K vanishes in some neighborhood of x_0. Since $\Delta_v(x_0)=0$, from Proposition 3.5, we either have $a_v(x_0)<0$ or $a_v(x_0)=b_v(x_0)=0$ and $c_v(x_0)<0$.

Case 1. If $a_v(x_0)<0$, then the functions $\lambda_{1,v}$ and $\lambda_{2,v}$ are continuous at x_0. Note that Δ_v and a_v are continuous and from assumption (14), $b_v(x_0)\neq 0$ since $\Delta_v(x_0)=0$. Hence, for x close enough to x_0 we have

- $\lambda_{1,v}(x)<\lambda_{1,v}(x_0)/2$ and $\frac{\sqrt{\Delta_v(x)}}{2(1+a_v^2(x))}<-\lambda_{1,v}(x_0)/2$, if $b_v(x_0)<0$,
- $\lambda_{2,v}(x)>\lambda_{2,v}(x_0)/2$ and $\frac{\sqrt{\Delta_v(x)}}{2(1+a_v^2(x))}<\lambda_{2,v}(x_0)/2$, if $b_v(x_0)>0$.

In both cases, from the definition of K, it follows that $K(x)=0$.

Case 2. In the case where $a_v(x_0)=b_v(x_0)=0$ and $c_v(x_0)<0$, again, from continuity, there exists a positive number l and a neighborhood $\mathcal{V}_{x_0}$ of x_0, such that if $x\in\mathcal{V}_{x_0}$ and $\Delta_v(x)>0$, we have $c_v(x)<0$, $\frac{|c_v(x)|}{|b_v(x)|}>l$, $\frac{\sqrt{\Delta_v(x)}}{2(1+a_v^2(x))}<l$ and $|b_v(x)|<l$. Moreover,

- $\lambda_{1,v}(x) < -l$ if $a_v(x) \neq 0$ and $b_v(x) < 0$ (cf. (8) and (11)),
- $\lambda_{2,v}(x) > l$ if $a_v(x) \neq 0$ and $b_v(x) > 0$ (cf. (8) and (12)).

It follows from the definition of K that in every cases, $K(x) = 0$. As a matter of fact, when $b_v(x) < 0$, it follows from the above inequalities that
if $a_v(x) \neq 0$, $K(x) = \varphi\big(\lambda_{1,v}(x) + \frac{\sqrt{\Delta_v(x)}}{2(1+a_v^2(x))}\big) = 0$, since $\lambda_{1,v}(x) < -l$ and $\frac{\sqrt{\Delta_v(x)}}{2(1+a_v^2(x))} < l$,
if $a_v(x) = 0$, $K(x) = \varphi\big(-\frac{c_v(x)}{b_v(x)} - \frac{b_v(x)}{2}\big) = 0$, since $-\frac{c_v(x)}{b_v(x)} < -l$ and $-\frac{b_v(x)}{2} < l$.

The cases when $b_v(x) > 0$ can be treated similary and the cases when $b_v(x) = 0$ are obvious, because, in this case, we deduce immediately from the definition of $\Delta_v(x)$ that $a_v(x) > 0$. So, from the definition of K, $K(x) = 0$.

Thus, K is identically equal to 0 in a neighborhood of x_0 and, therefore, is continuous at x_0.

2. If $x_0 \in A_2$, then $K(x_0) = 0$. Denote $\alpha(x) = -\frac{\sqrt{\Delta_v(x)}}{2a_v(x)} + \frac{\sqrt{\Delta_v(x)}}{2(1+a_v^2(x))}$. From the definition of K and the fact that $\alpha(x_0) < 0$, we obtain

$$\lim_{\substack{x\to x_0\\ a_v(x)>0\\ b_v(x)<0}} K(x) = \lim_{\substack{x\to x_0\\ a_v(x)>0\\ b_v(x)<0}} \varphi\Big(\lambda_{1,v}(x) + \frac{\sqrt{\Delta_v(x)}}{2(1+a_v^2(x))}\Big) = \varphi\Big(\alpha(x_0)\Big) = 0 = K(x_0),$$

and

$$\lim_{\substack{x\to x_0\\ a_v(x)>0\\ b_v(x)>0}} K(x) = \lim_{\substack{x\to x_0\\ a_v(x)>0\\ b_v(x)>0}} \psi\Big(\lambda_{2,v}(x) - \frac{\sqrt{\Delta_v(x)}}{2(1+a_v^2(x))}\Big) = \psi\Big(-\alpha(x_0)\Big) = 0 = K(x_0).$$

Also, if $b_v(x) = 0$ and $a_v(x) > 0$, then $K(x) = 0 = K(x_0)$. Thus K is continuous at $x_0 \in A_2$.

3. If $x_0 \in A_3$, for reasons of continuity, we have $b_v(x) > 0$ and $\Delta_v(x) > 0$ whenever x is close enough to x_0. So, continuity of b_v, c_v and ψ lead to

$$\lim_{\substack{x\to x_0\\ a_v(x)=0\\ b_v(x)>0}} K(x) = \lim_{\substack{x\to x_0\\ a_v(x)=0\\ b_v(x)>0}} \psi\Big(-\frac{c_v(x)}{b_v(x)} - \frac{b_v(x)}{2}\Big) = \psi\Big(-\frac{c_v(x_0)}{b_v(x_0)} - \frac{b_v(x_0)}{2}\Big) = K(x_0),$$

and, from (6) and (9) we have

$$\lim_{\substack{x\to x_0\\ a_v(x)\neq 0\\ b_v(x)>0}} K(x) = \lim_{\substack{x\to x_0\\ a_v(x)\neq 0\\ b_v(x)>0}} \psi\Big(\lambda_{2,v}(x) - \frac{\sqrt{\Delta_v(x)}}{2(1+a_v^2(x))}\Big) = \psi\Big(-\frac{c_v(x_0)}{b_v(x_0)} - \frac{b_v(x_0)}{2}\Big) = K(x_0).$$

It then follows that K is continuous at $x_0 \in A_3$.

4. If $x_0 \in A_4$, using the defintion of K and the assertions (7) and (10), this case is similar to the former one and is omitted here.

Thus K is continuous in $\mathbb{R}^n \setminus \{0\}$.

Finally, we wish to show that the function K is continuous at the origin. Note that $K(x) = 0 = K(0)$ when $\Delta_v(x) \leq 0$, so from now on, we consider $\Delta_v(x) > 0$. Assume that V satisfies the small control property and since Δ_v, a_v and b_v are continuous and $\Delta_v(0) = b_v(0) = 0$ see Remark 3.4, then, for every $\varepsilon > 0$, let $\varepsilon' = \varepsilon/2$ there exists $\delta > 0$ such that for every $x \neq 0$ satisfying $\|x\| < \delta$, there exists some w such that $|w| < \varepsilon'$ and

$$\mathcal{L}_w V(x) = a_v(x)w^2 + b_v(x)w + c_v < 0, \tag{16}$$

and

$$\frac{\sqrt{\Delta_v(x)}}{2(1 + a_v^2(x))} < \varepsilon' \quad \text{and} \quad |b_v(x)| < \varepsilon'. \tag{17}$$

The inequality (16) is equivalent to

$$\text{if} \quad a_v(x) > 0, \quad w \in]\lambda_{1,v}(x), \lambda_{2,v}(x)[, \tag{18}$$

$$\text{if} \quad a_v(x) < 0, \quad w \notin [\lambda_{2,v}(x), \lambda_{1,v}(x)], \tag{19}$$

$$\text{if} \quad a_v(x) = 0 \quad \text{and} \quad b_v(x) < 0, \quad w > -\frac{c_v(x)}{b_v(x)}, \tag{20}$$

$$\text{if} \quad a_v(x) = 0 \quad \text{and} \quad b_v(x) > 0, \quad w < -\frac{c_v(x)}{b_v(x)}. \tag{21}$$

Consider the three following cases according to the sign of $b_v(x)$.

- If $b_v(x) < 0$, we consider again three cases according to the sign of $a_v(x)$.

  (i) If $a_v(x) > 0$, then we have $K(x) = \varphi\big(\lambda_{1,v}(x) + \frac{\sqrt{\Delta_v(x)}}{2(1+a_v^2(x))}\big)$. From (18), we have $\lambda_{1,v}(x) < w < \varepsilon'$ which yields with (17), $\lambda_{1,v}(x) + \frac{\sqrt{\Delta_v(x)}}{2(1+a_v^2(x))} < 2\varepsilon' = \varepsilon$. Hence $|K(x)| = \varphi\big(\lambda_{1,v}(x) + \frac{\sqrt{\Delta_v(x)}}{2(1+a_v^2(x))}\big) < \varphi\big(\varepsilon\big) = \varepsilon$.

  (ii) If $a_v(x) < 0$, (19) is equivalent to $w < \lambda_{2,v}(x) < \lambda_{1,v}(x)$ or $\lambda_{2,v}(x) < \lambda_{1,v}(x) < w$. Since $a_v(x) < 0$ and $b_v(x) < 0$, we have $\lambda_{2,v}(x) < 0$ and $|\lambda_{1,v}(x)| < |\lambda_{2,v}(x)|$. So, we always have $\lambda_{1,v}(x) < |w| < \varepsilon'$. As above, we show again that $|K(x)| < \varepsilon$.

  iii) If $a_v(x) = 0$, in this case $K(x) = \varphi\big(- \frac{c_v(x)}{b_v(x)} - \frac{b_v(x)}{2}\big)$. From (20), we have $-\frac{c_v(x)}{b_v(x)} < w < \varepsilon'$ and the conclusion follows, as above, since $|b_v(x)| < \varepsilon'$.

- If $b_v(x) > 0$, this case can be treated similary as the former one and is omitted here.

- If $b_v(x) = 0$, from assumption (14), we have $a_v(x) \geq 0$ since $\Delta_v(x) > 0$. But $a_v(x) = 0$ would contradict $\Delta_v(x) > 0$ and so we have $a_v(x) > 0$. It then follows from the definition of K that $K(x) = 0$.

Thus, K is continuous at the origin and this complets the proof of Lemma 4.3. □

Now we study the asymptotic stability in probability of the stochastic system (1).

Theorem 4.4. If V is a SCLF which satisfies assumption (14) and the small control property, then the feedback law $u(x) = K(x)$ renders the stochastic system (1) globally asymptotically stable in probability.

Proof. From Lemma 4.3, since V satisfies the small control property, we deduce that u is continuous on $\mathbb{R}^n$. It follows that the coefficients of the closed-loop system deduced from the stochastic system (1) with the control u are continuous on $\mathbb{R}^n$, then so is the associated infinitesimal operator $\mathcal{L}_u V(x)$, since the fonctions a_v, b_v and c_v are continuous.

Next, we will verify that

$$\mathcal{L}_u V(x) = a_v(x)u^2(x) + b_v(x)u(x) + c_v(x) < 0, \quad \forall x \in \mathbb{R}^n \setminus \{0\},$$

according to the sign of $\Delta_v(x)$. To do this, we again follow the main lines of the proof of Theorem 3.9 from Maniar et al. [21].

1. If $\Delta_v(x) < 0$ then, $a_v(x) < 0$ according to Proposition 3.5, and the sign of $\mathcal{L}_u V(x)$ is that of $a_v(x)$, so $\mathcal{L}_u V(x) < 0$.
2. If $\Delta_v(x) = 0$, then $u(x) = 0$, and so, $\mathcal{L}_u V(x) = c_v(x) < 0$. As a matter of fact, from Proposition 3.5, we know that $a_v(x) = b_v(x) = 0$ and $c_v(x) < 0$ or $a_v(x) < 0$. If $a_v(x) < 0$, from assumption (14) we have $b_v(x) \neq 0$ and so, from defintion of Δ_v, we get $c_v(x) < 0$ since $\Delta_v(x) = 0$.
3. If $\Delta_v(x) > 0$, in this case, we consider three cases according to the sign of $b_v(x)$.

   i. If $b_v(x) < 0$, let $\beta(x) = \frac{\sqrt{\Delta_v(x)}}{2(1+a_v^2(x))}$ then, $u(x) = \varphi\big(\lambda_{1,v}(x) + \beta(x)\big)$. It is clear that $\lambda_{1,v}(x) < \lambda_{1,v}(x) + \beta(x) \leq \varphi\big(\lambda_{1,v}(x) + \beta(x)\big) = u(x)$. In case $a_v(x) < 0$, we have $\lambda_{2,v}(x) < \lambda_{1,v}(x) < u(x)$, and hence $\mathcal{L}_u V(x) < 0$.
   If $a_v(x) > 0$, we have $u(x) = \varphi\big(\lambda_{1,v}(x) + \beta(x)\big) \leq \max(0, \lambda_{1,v}(x) + \beta(x)) < \lambda_{2,v}(x)$. Indeed, $\lambda_{1,v}(x) + \beta(x) < \lambda_{1,v}(x) + \frac{\sqrt{\Delta_v(x)}}{2a_v(x)} = -\frac{\sqrt{b_v(x)}}{2a_v(x)} < \lambda_{2,v}(x)$, and since $a_v(x) > 0$ and $b_v(x) < 0$, we also have $0 < \lambda_{2,v}(x)$.
   Thus, $\lambda_{1,v}(x) < u(x) < \lambda_{2,v}(x)$ and then, $\mathcal{L}_u V(x) < 0$.
   If $a_v(x) = 0$, since $b_v(x) < 0$, we have

   $$u(x) = \varphi\Big(- \frac{c_v(x)}{b_v(x)} - \frac{b_v(x)}{2}\Big) \geq -\frac{c_v(x)}{b_v(x)} - \frac{b_v(x)}{2} > -\frac{c_v(x)}{b_v(x)},$$

   therefore, $\mathcal{L}_u V(x) = b_v(x)u(x) + c_v(x) < 0$.

   ii. If $b_v(x) > 0$, this case can be treated similary to the previous one.

   iii. If $b_v(x) = 0$, from assumption (14), we have $a_v(x) \geq 0$. But $a_v(x) = 0$ would contradict $\Delta_v(x) > 0$, so, $a_v(x) > 0$ which gives $c_v(x) < 0$, and therefore, $\mathcal{L}_u V(x) = c_v(x) < 0$, since $u(x) = 0$ according to the definition of u.

Hence, $\mathcal{L}_u V(x)$ is continuous and negative definite and since $V(x)$ is a SCLF for system (1), according to Lemma 2.6, there exit class $\mathcal{K}_\infty$ functions $\alpha(x)$ and $\beta(x)$ and class $\mathcal{K}$ function $\xi(x)$, defined on $\mathbb{R}^+$, such that

$$\alpha(\|x\|) \leq V(x) \leq \beta(\|x\|), \quad \forall x \in \mathbb{R}^n,$$

and

$$\mathcal{L}_u V(x) = a(x)u^2(x) + b(x)u(x) + c(x) \leq -\xi(\|x\|), \quad \forall x \in \mathbb{R}^n.$$

The conclusion follows with help of Lemma 2.5 and this completes the proof of Theorem 4.4. □

Remark 4.5. 1) In trying to build a continuous feedback u, the problem is when one has the following situation: $x_0 \in \mathbb{R}^n$ is such that, $a_v(x_0) < 0$, $\Delta_v(x_0) \geq 0$ and $b_v(x_0) = 0$. The global asymptotic stability require that $\mathcal{L}_u V(x_0) < 0$, two choices of $u(x_0)$ are possible: $u(x_0) < \lambda_{2,v}(x_0) \leq \lambda_{1,v}(x_0)$ or $\lambda_{2,v}(x_0) \leq \lambda_{1,v}(x_0) < u(x_0)$. In both cases, due to the continuity of u and according to Lemma 3.6, we can have $u(x_1)$ undefined for some $x_1 \in \mathbb{R}^n$. To avoid this situation, we use assumption (14).

2) If Florchinger [6] used a universal formula for the system (1) when $f_2 \equiv g_1 \equiv 0$ under a known SCLF, the problem of stabilization of system (1) has not yet a universal construction. Note that if V is a SCLF for the system (1), the sufficient condition: $a_v(x) < 0 \Rightarrow \Delta_v(x) < 0$, stated in [3, 4, 8], is more restrictive than the assumption (14) stated in this paper.

5. EXAMPLE

To illustrate the usefulness of the above result, let us give an example. Consider the following stochastic system

$$\mathrm{d}x = \begin{pmatrix} -x_1\cos(2x_1) + ux_1\sin^2(x_1) - u^2x_1\sin^2(x_1) \\ \\ -x_2\cos(2x_1) + ux_2\sin^2(x_2) - u^2x_2\sin^2(x_2) \end{pmatrix} \mathrm{d}t + \begin{pmatrix} ux_1\sin(x_1) \\ \\ ux_2\sin(x_2) \end{pmatrix} \mathrm{d}\omega. \tag{22}$$

Taking the following function V defined on $\mathbb{R}^2$ by

$$V(x) = \frac{x_1^2 + x_2^2}{2},$$

we have

$$\mathcal{L}_u V(x) = a_v(x)u^2(x) + b_v(x)u(x) + c_v(x),$$

with

$$a_v(x) = -\frac{1}{2}(x_1^2\sin^2(x_1) + x_2^2\sin^2(x_2)), \quad b_v(x) = x_1^2\sin^2(x_1) + x_2^2\sin^2(x_2),$$

and

$$c_v(x) = -(x_1^2 + x_2^2)\cos(2x_1).$$

It is easy to see that for every $x \in \mathbb{R}^2$, we have $a_v(x) \leq 0$, and if $x \neq 0$ is such that $a_v(x) = 0$ then, $b_v(x) = 0$ and $c_v(x) < 0$, so

$$\inf_{u\in\mathbb{R}} \mathcal{L}_u V(x) = \inf_{u\in\mathbb{R}} \Big(a_v(x)u^2 + b_v(x)u + c_v(x) \Big) < 0, \quad \forall x \in \mathbb{R}^2 \setminus \{0\}.$$

It then follows that V is a SCLF for system (22) which satisfies the small control property since $c_v(x) < 0$ for all $x \neq 0$ such that $x_1 \in]-\pi/4, \pi/4[$.

It can be seen that the stabilizability condition: $(a_v(x) < 0 \Longrightarrow \Delta_v(x) < 0)$ stated in Chabour and Oumoun [3] and Daumail and Florchinger [4] is not satisfied, since $a_v(\pi/4, \pi/4) < 0$ and $\Delta_v(\pi/4, \pi/4) > 0$.

However, for every $x \in \mathbb{R}^2 \setminus \{0\}$, we have $(a_v(x) < 0) \Rightarrow b_v(x) > 0$. Hence, assumption (14) is satisfied, and thus, according to Theorem 4.4, system (22) is globally asymptotically stable in probability with the feedback

$$u(x) = \begin{cases} \psi\Big(\frac{-b_v(x)+\sqrt{\Delta_v(x)}}{2a_v(x)} - \frac{\sqrt{\Delta_v(x)}}{2(1+a_v^2(x))} \Big) & \text{if } \ \Delta_v(x) > 0, \\ 0 & \text{if } \ \Delta_v(x) \leq 0. \end{cases}$$

6. CONCLUDING REMARKS

In this paper, the global state-feedback stabilization problem has been investigated for nonlinear stochastic systems when the drift is quadratic in the control and the diffusion term is affine in the control. In the sense of weak solution, under the assumption that a stochastic control Lyapunov function is known, we gave a necessary condition for the stabilization by a continuous state-feedback. Moreover, under appropriate condition, we designed state-feedback that ensures the zero solution of the closed-loop system is globally asymptotically stable in probability.

Since the studied system is with less restriction, this result can be applied to many practical models which are only continuous.

The following two problems are interesting for further investigation:

1. If the method can be used for the stabilization of deterministic nonlinear systems such as offshore structures, see [28, 29, 30]

2. The construction of a stabilizing controls in probability of multi-input nonlinear stochastic systems.

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