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CONSENSUS OF A MULTI-AGENT SYSTEMS WITH HETEROGENEOUS DELAYS

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The paper presents an algorithm for the solution of the consensus problem of a linear multi-agent system composed of identical agents. The control of the agents is delayed, however, these delays are, in general, not equal in all agents. The control algorithm design is based on the H_∞-control, the results are formulated by means of linear matrix inequalities. The dimension of the resulting convex optimization problem is proportional to the dimension of one agent only but does not depend on the number of agents, hence this problem is computationally tractable. It is shown that heterogeneity of the delays in the control loop can cause a steady error in the synchronization. Magnitude of this error is estimated. The results are illustrated by two examples.

Keywords: multi-agent system, time delay system, robust control, LMI

Classification: 93A14, 93B36

1. INTRODUCTION

1.1. State of the art

Consensus problem or leader-following problem of multi-agent systems gained strong attention in the recent times. For a survey of the pioneering works, see e. g. [14].

Control of multi-agent systems is implemented with communication networks. This, however, brings several issues. The data are transmitted in packets, delays in their delivery can occur. Also, the signals are sampled and quantized before their values are transmitted. Hence the need for control algorithms that are capable of dealing with these issues.

To deal with time delays in the multi-agent system control, the so-called descriptor approach, introduced in e. g. [7] or [6], was adopted. Its advantage is that no restriction on the derivative of the time delay is posed. In particular, it is applicable to sampled systems. This approach uses a Lyapunov-Krasovskii functional in connection with inequalities based on the well-known Jensen inequality.

From the above considerations follows that algorithms for achieving consensus in presence of time delay is needed. Consensus of a second-order multi-agent system with

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a constant delay is studied in [14] or [10]. Complex networks composed of more general agents are investigated in [11] for both continuous and discrete-time cases. The leader-following problem of multi-agent systems with constant delays is solved in [8], the approach based on the Razumikhin functional (for its definition, see e.g [20]) is used. A solution of this problem via the Lyapunov-Krasovskii functional is described in [9]. Consensus of a multi-agent system composed of nonlinear agents with delayed input is the problem solved in [19]; the delay can be time-varying. In all these papers, the assumption of equal delays for all agents throughout the network is made. Requiring this is apparently a rather restrictive and unnatural, however, it significantly simplifies analysis. One can also mention application of event-triggering mechanism for synchronization of multi-agent systems, e. g. in [13], where also sensor saturation is taken into account. A similar problem for discrete-time systems, namely event-triggered synchronization of complex networks, is studied in [12].

Let us also note that a solution of an analogous problem – stabilization of a large-scale system with communication delays – is presented in [2, 3] for the case of a linear system and in [21] for a nonlinear system.

Control of multi-agent systems where the delays are heterogeneous, that means, not equal in every agent throughout the network, is a much more difficult problem than the problem of synchronization of agents with equal delays. First, as demonstrated in [22], precise synchronization cannot be, in general, achieved. It was shown that a synchronization error appears, its norm does not decrease to zero with time. Nevertheless, the maximal value is bounded. This bound can be estimated using methods developed in the H_∞-control. Other papers dealing with systems with heterogeneous delays are [16, 15] where also jointly connected topologies are investigated, in [18, 5], the leader-following problem is solved. The authors of [17] study this problem for time varying agents with stable dynamics admitting quadratic Lyapunov function A similar assumption – that the agents are stable – is made in [25]. In [24], this problem is tackled from perspective of autonomous vehicles. The problem of synchronization of a sequence of integrators is studied in [1] while the paper [23] deals with synchronization of a chain of robots with delayed communication.

1.2. Purpose and outline of the paper

The purpose of this paper is to present an H_∞-based control design for a multi-agent system with heterogeneous delays. It will be shown that this situation, in contrast to the case when the delays of all agents are equal, may lead to a steady error in the consensus. It has a similar effect as disturbance acting upon the multi-agent system. It will be shown how the error caused by this non-homogeneity can be estimated. As far as we know, the problem of estimating steady synchronization error caused by non-equal time delays of the control signal has not been treated before.

The paper is organized as follows: in the second section, basic facts from the graph theory and from the multi-agent system theory are repeated. Third section introduces the disagreement dynamics. The core of the paper is the fourth section where the main results are derived. Fifth section contains examples.

The preliminary results were presented in the conference paper [22]. This article can be thus regarded as an expanded version of that paper.

1.3. Notation used in the paper

- The k-dimensional identity matrix is denoted by I_k; the zero matrix is denoted by 0, its dimension will be always clear from the context.
- The symbol $\|.\|$ means the quadratic norm.
- If a is a matrix, then a^T denotes the transpose of a.
- For functions of time, the time argument t is omitted: $f(t)$ is abbreviated as f. However, if the argument is different from t, it is written.
- The subscript denotes the time delay: $f(t-\tau) = f_\tau(t) = f_\tau$.
- For a matrix P, the inequality $P > 0$ means that the matrix P is symmetric positive definite.
- For symmetric matrices, the blocks below the diagonal are replaced by an asterisk:

$$\begin{pmatrix} a & b \\ b^T & c \end{pmatrix} = \begin{pmatrix} a & b \\ * & c \end{pmatrix}.$$

- If a, b are matrices, then

$$\mathrm{diag}(a,b) = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}.$$

- The symbol $\otimes$ denotes the Kronecker product.

For the reader's convenience, we repeat the most useful properties of the Kronecker product here.

- For any matrices A, B, C, D with compatible dimensions, $(A \otimes C)(B \otimes D) = AB \otimes CD$,
- $(A \otimes B)^T = A^T \otimes B^T$.

2. PRELIMINARIES

Only the notions most important for this paper are presented in this section. For more details, see e. g. [4].

2.1. Graph theory

An important tool for analysis of multi-agent systems is the graph theory, hence some facts of it are repeated here. Let N be a positive integer, $\mathcal{V} = \{1, \ldots, N\}$, $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$. The graph $\mathcal{G}$ is defined as the pair $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, the set $\mathcal{V}$ is called the *set of vertices*, the set $\mathcal{E}$ is the *set of edges*. The meaning is as follows: $(i, j) \in \mathcal{E}$ if and only if there is an edge from the node j to the node i. In the context of multi-agent systems, this means that the state of the jth agent is used to compute the control of the ith agent.

Assumption 2.1. The interconnection matrix satifies the following:

1. the graph $\mathcal{G}$ is not oriented: if $(i,j) \in \mathcal{E}$ then $(j,i) \in \mathcal{E}$,
2. for any $i \in \mathcal{V}$ holds $(i,i) \notin \mathcal{E}$,
3. the graph $\mathcal{G}$ is connected.

For the graph $\mathcal{G}$ defined above and satisfying 1) and 2) we define the *adjacency matrix* $E \in \mathbb{R}^{N\times N}$ as

$$\begin{aligned} E_{ij} =& 1 \text{ if } (i,j) \in \mathcal{E}, \\ E_{ij} =& 0 \text{ elsewhere.} \end{aligned}$$

From the previous assumptions about the graph $\mathcal{G}$ follows that matrix E is symmetric and $E_{ii} = 0$ for all $i = 1, \dots, N$.

With help of the adjacency matrix, one can define also the *Laplacian matrix* $L \in \mathbb{R}^{N\times N}$ as

$$\begin{aligned} L_{ij} =& -E_{ij} \text{ if } i \neq j, \\ L_{ii} =& \sum_{j=1}^{N} E_{ij} \text{ elsewhere.} \end{aligned}$$

The Laplacian matrix is symmetric positive definite [14]. Let

$$a = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \in \mathbb{R}^{N\times 1}, \quad r = \frac{1}{\sqrt{\|a\|}} a.$$

Symmetry of the Laplacian matrix L implies existence of an orthogonal matrix U and a diagonal matrix D so that

$$U^T L U = D. \tag{1}$$

As noted in [4], 0 is the eigenvalue of Laplacian matrix L corresponding to the eigenvector a. This eigenvalue is simple. Then, matrix D can be written as

$$D = \operatorname{diag}\Big(0, d_1, \dots, d_{N-1}\Big), \tag{2}$$

with $d_1 > 0, \dots, d_{N-1} > 0$ being its eigenvalues. Without loss of generality, we can assume the eigenvalues of matrix L are ordered so that

$$d_{i-1} \leq d_i, \; i = 2, \dots, N-1. \tag{3}$$

2.2. Multi-agent system

The multi-agent system considered in this paper is a system composed of N identical systems (agents). The ith agent ($i = 1, \dots, N$) is described by the equation

$$\dot{x}_i(t) = Ax_i(t) + Bu_i(t),\ y_i = Cx_i \tag{4}$$

where for all $i = 1, \dots, N$ holds $x_i : [0, \infty) \to \mathbb{R}^n$, $u_i : [0, \infty) \to \mathbb{R}^m$, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$.

Time delays occur in the control loop. It is assumed these time delays are uniformly bounded.

Assumption 2.2. There exists a positive constant $\bar{\tau}$ such that the time delays $\tau_i : [0, \infty) \to [0, \bar{\tau}]$. Moreover, the functions τ_i are measurable.

Let $\bar{x} = \frac{1}{N}\sum_{i=1}^{N} x_i$. The goal is to achieve consensus of the agents. This means, the aim is to satisfy

$$0 = \lim_{t\to\infty} \sum_{i=1}^{N} \|x_i - \bar{x}\|. \tag{5}$$

However, as will be shown, this goal is too ambitious since heterogeneity in the delays for different agents prevent us from achieving equality (5). Rather, a relaxed condition can be satisfied: one can find a constant $\mathbf{c} > 0$ (dependent on the time delays $\tau_1, \dots, \tau_N$) such that

$$\lim_{t\to\infty} \sum_{i=1}^{N} \|x_i - \bar{x}\| \le \mathbf{c}. \tag{6}$$

An estimate of this constant will be presented in the sequel.

In particular, the goal is find a control gain $K \in \mathbb{R}^{m\times n}$ so that (6) is achieved if for $i = 1, \dots, N$, the control u_i is given by

$$u_i = \sum_{i=1}^{N} E_{ij} K(x_{i,\tau_j} - x_{j,\tau_j}). \tag{7}$$

3. DISAGREEMENT DYNAMICS

Denote as $v \in \mathbb{R}^{nN}$ the following vector

$$v = \begin{pmatrix} \int_{t-\tau_1}^{t} \dot{x}_1(s)\,\mathrm{d}s \\ \vdots \\ \int_{t-\tau_N}^{t} \dot{x}_N(s)\,\mathrm{d}s \end{pmatrix}. \tag{8}$$

Then one can express the differential equation governing the multi-agent system as

$$\dot{x} = (I_N \otimes A)x + (L \otimes BK)x - (L \otimes BK)v + w. \tag{9}$$

The *disagreement vector* plays an important role in the solution of the consensus problem. This vector is denoted by ξ and is defined by

$$\xi = x - a \otimes \bar{x}. \tag{10}$$

Moreover, the following two vectors will be useful:

$$\omega_1 = \begin{pmatrix} \int_{t-\tau_1}^{t} \dot{\xi}_1(s)\,\mathrm{d}s \\ \vdots \\ \int_{t-\tau_N}^{t} \dot{\xi}_N(s)\,\mathrm{d}s \end{pmatrix}, \quad \omega_2 = \begin{pmatrix} \int_{t-\tau_1}^{t} \dot{\bar{x}}(s)\,\mathrm{d}s \\ \vdots \\ \int_{t-\tau_N}^{t} \dot{\bar{x}}(s)\,\mathrm{d}s \end{pmatrix} \tag{11}$$

The following result will be useful. Denote also $\bar{w} = a \otimes \frac{1}{N}\sum_{i=1}^{N} w_i$.

Lemma 3.1. Consider the system (9). Then

$$\dot{\bar{x}} = A\bar{x} + \bar{w}. \tag{12}$$

P r o o f . Let $\bar{a} = \frac{1}{N}a^T \otimes I_n$. Then $\bar{x} = \bar{a}x$ and also

$$a \otimes \dot{\bar{x}} = a \otimes \bar{a}\dot{x} = a \otimes \bar{a}\Big((I_N \otimes A)x + (L \otimes BK)x - (L \otimes BK)v + w\Big). \tag{13}$$

However, symmetry of L together with the above mentioned fact that a is an eigenvector of matrix L yields $aL = 0$, moreover equality $a \otimes \bar{a}(I_N \otimes A)x = a \otimes A\bar{x}$, hence (13) turns into

$$a \otimes \dot{\bar{x}} = a \otimes A\bar{x} + \bar{w}. \tag{14}$$

This nN-tuple of differential equations is a set composed of N equations (12). □

Corollary 3.2. The disagreement dynamics can be reformulated as

$$\dot{\xi} = (I_N \otimes A)\xi + (L \otimes BK)(\xi - \omega_1 - \omega_2) + w - \bar{w}. \tag{15}$$

P r o o f . First, note that $v = \omega_1 + \omega_2$. Then, subtracting (12) from (9) yields the result. □

The disagreement dynamics is described by a system of Nn differential equations which are "interconnected". Thus, another transformation of the system (15) is defined so that the transformed system is split into N autonomous equations. This means, a transformation is sought such that matrix L is replaced by a diagonal matrix. Using the orthogonal matrix U from (1) this transformation reads

$$\zeta' = (U^T \otimes I_n)\xi. \tag{16}$$

Define also $\eta_1 = -(U^T \otimes I_n)\omega_1$, $\eta_2 = -(U^T \otimes I_n)\omega_2$, $\nu = (U^T \otimes I_n)(w - \bar{w})$. Transformation (16) converts the system (15) into the form

$$\dot{\zeta}' = (I_N \otimes A)\zeta' + (D \otimes BK)(\zeta' + \eta_1 + \eta_2 + \nu). \tag{17}$$

Hence system (17) splits into N autonomous n-dimensional systems where the first n-tuple is identically equal to zero (see [14], Theorem 1).

Hence if the last $N-1$ components of the vector ζ converge to zero then consensus of the multi-agent system is achieved.

For $i = 1,\dots,N$, define also vector functions $\zeta_i : [0,\infty) \to \mathbb{R}^n$, $\eta_{1,i} : [0,\infty) \to \mathbb{R}^n$, $\eta_{2,i} : [0,\infty) \to \mathbb{R}^n$ and $\nu : [0,\infty) \to \mathbb{R}^n$ as

$$\bar{\zeta} = \begin{pmatrix} \zeta_1 \\ \vdots, \\ \zeta_N \end{pmatrix}, \quad \eta_1 = \begin{pmatrix} \eta_{1,1} \\ \vdots, \\ \eta_{1,N} \end{pmatrix}, \quad \eta_2 = \begin{pmatrix} \eta_{2,1} \\ \vdots, \\ \eta_{2,N} \end{pmatrix}, \quad \nu = \begin{pmatrix} \nu_1 \\ \vdots, \\ \nu_N \end{pmatrix}.$$

Dynamics of the ith autonomous systems is described by

$$\dot{\zeta}_i = A\zeta_i + d_i BK(\zeta_i + \eta_{1,i} + \eta_{2,i} + \nu_i). \tag{18}$$

Define also ζ by

$$\zeta = \begin{pmatrix} \zeta_2 \\ \vdots \\ \zeta_N \end{pmatrix}.$$

The following result is due to [4] for delay-free systems where, naturally, $\eta_{i,j} = 0$.

Lemma 3.3. Consider system (9). Assume systems (18) were obtained by transformation (16) and let $\bar{\tau} = 0$ and $\nu_i = 0$. Then stability of all systems (18) implies consensus of the multi-agent system (9).

As will be shown in the next section, if $\eta_{2,j} \neq 0$ or $\nu_i \neq 0$, the disagreement vector does not, in general, converge to zero. However, the norm of the disagreement vector is bounded.

4. H_∞ CONSENSUS PROBLEM

To formulate the main result of this paper. Assume first matrices $Q_1, Q_2, S \in \mathbb{R}^{n\times n}$ and $Y \in \mathbb{R}^{m\times n}$ are given. Then one can define functions $\sigma_{11}, \sigma_{12} : \mathbb{R} \to \mathbb{R}^{n\times n}$, $\sigma_{13}, \sigma_{23} : \mathbb{R} \to \mathbb{R}^{n\times m}$ and matrices σ_{16}, σ_{33} as

$$\begin{aligned}
\sigma_{11}(d) &= AQ_2 + Q_2^T A^T + d(BY + Y^T B^T), \\
\sigma_{12}(d) &= Q_1 - Q_2 + \varepsilon Q_2^T A^T + dY^T B^T, \\
\sigma_{13}(d) &= \bar{\tau} dBY, \\
\sigma_{16} &= Q_2^T C^T, \\
\sigma_{22} &= -\varepsilon(Q_2 + Q_2^T - \bar{\tau} S).
\end{aligned}$$

Using these functions, let us define matrix-valued function $\Sigma : \mathbb{R} \to \mathbb{R}^{(4n+m+p)\times(4n+m+p)}$ by

$$\Sigma(d) = \begin{pmatrix} \sigma_{11}(d) & \sigma_{12}(d) & \sigma_{13}(d) & I_n & 0 & \sigma_{16} \\ * & \sigma_{22} & \varepsilon\sigma_{13}(d) & 0 & I_n & 0 \\ * & * & \sigma_{33} & 0 & 0 & 0 \\ * & * & * & -\sigma I_n & 0 & 0 \\ * & * & * & * & -\frac{\gamma}{\varepsilon} I_n & 0 \\ * & * & * & * & * & -I_p \end{pmatrix}$$

Now we can formulate the main result:

Theorem 4.1. Consider the multi-agent system (9) satisfying Assumption (2.1). Assume also the minimal nonzero eigenvalue of the Laplacian matrix is equal to d_1 and maximal eigenvalue of the Laplacian matrix equals d_{N-1}. Let there exist $n \times n$-dimensional matrices $Q_1 > 0$, Q_2 nonsingular, $S > 0$, a $m \times n$-dimensional matrix Y and scalars $\gamma > 0$, $\varepsilon > 0$ such that

$$\Sigma(d_1) < 0,\ \Sigma(d_{N-1}) < 0 \tag{19}$$

holds. Then

1. if
$$w_1 = \cdots = w_N,\ \tau_1 = \cdots = \tau_N \ \text{ for every } t \geq 0 \tag{20}$$
then relation (5) holds, hence consensus is achieved;

2. if condition (20) is not satisfied, then there exists a constant $c > 0$ such that
$$\limsup_{t\to\infty} \sum_{i=1}^{N} \|x_i - \bar{x}\| \leq c(\|w_i - \bar{w}\| + \|\omega_2\|). \tag{21}$$

Outline of the proof: the proof is divided into several lemmas presented in the sequel. First, in Lemma 4.3, conditions (formulated by means of matrix inequalities) for synchronization of the original multi-agent system are derived using the descriptor approach. However, these conditions are not time-invariant and contain multiples of variables, hence LMI solvers are not applicable to find solution of these matrix inequalities. Moreover, the dimension of these matrix inequalities is proportional to the number of agents. In Lemmas 4.5 – 4.7, a set of linear matrix inequalities is derived such that its solvability is equivalent to existence of a solution of the previous set of nonlinear matrix inequalities. Nevertheless, the problem has still dimension proportional to N. Finally, conditions formulated by means of LMIs with dimension independent of N are derived in Lemma 4.8, these LMIs are equivalent to those derived in Lemma 4.7.

Remark 4.2. Note that, since $\Sigma(d) \in \mathbb{R}^{6n\times 6n}$, the dimension of LMIs constituting the condition (19) is $12n$, hence it is independent of the number of agents. However, the interconnection topology, and thus the number of agents, determines the values of the minimal and maximal eigenvalues.

The first lemma concerning H_∞-stability of the disagreement dynamics (15) will be presented in the sequel. The result is derived using the so-called descriptor approach for time-delay systems as introduced in e. g. [7].

The following definitions will be useful. Define matrix F_d by $F_d = \mathrm{diag}(d_1, \dots, d_{N-1})$ and let $C = I_n$.

Lemma 4.3. Assume there exist matrices P_1, P_2, R, all in $\mathbb{R}^{n\times n}$, $P_1 > 0$, $R > 0$, a matrix $K \in \mathbb{R}^{m\times n}$ and real constants $\gamma > 0$ and $\varepsilon > 0$ so that, with

$$\Gamma = \begin{pmatrix} \gamma_{11} & \gamma_{12} & \gamma_{13} \\ * & \gamma_{22} & \gamma_{23} \\ * & * & \gamma_{33} \end{pmatrix} \tag{22}$$

where

$$\begin{aligned}
\gamma_{11} =& (I_N \otimes P_2^T)\Big((I_N \otimes A) + (F_d \otimes BK)\Big) + \Big((I_N \otimes A) + (F_d \otimes BK)\Big)^T (I_N \otimes P_2) \\
&+ \frac{1}{\gamma}(I_N \otimes P_2^T)(I_N \otimes P_2) + (I_N \otimes C^T)(I_N \otimes C), \\
\gamma_{12} =& I_N \otimes (P_1 - P_2^T) + \varepsilon(I_N \otimes A + F_d \otimes BK)^T (I_N \otimes P_2), \\
\gamma_{13} =& \bar{\tau}(I_N \otimes P_2^T)(F_d \otimes BK), \\
\gamma_{22} =& -\varepsilon I_N \otimes (P_2 + P_2^T - \bar{\tau}R) + \frac{\varepsilon}{\gamma}(I_N \otimes P_2^T)(I_N \otimes P_2), \\
\gamma_{23} =& \varepsilon\gamma_{13}, \\
\gamma_{33} =& -\bar{\tau} I_N \otimes R
\end{aligned}$$

satisfies

$$\Gamma < 0, \tag{23}$$

then there exists a constant $k > 0$ such that

$$\lim_{t\to\infty} \|\zeta(t)\| \le k\sqrt{\gamma(1+\varepsilon)}\|\eta_2 + \nu\|. \tag{24}$$

Proof. The descriptor approach, see [6] or [7] for details, is used to prove this Lemma.

Let us first denote $z = \eta_2 + \nu$ and introduce the following Lyapunov-Krasovskii functional $V(\zeta)$ by

$$\begin{aligned}
V(\zeta) =& V_1(\zeta) + V_2(\zeta), \\
V_1(\zeta) =& \frac{1}{2}\zeta^T (I_N \otimes P_1)\zeta, \\
V_2(\zeta) =& \int_{-\bar{\tau}}^{0} \int_{t+s}^{t} \dot{\zeta}^T(\sigma)(I_N \otimes R)\dot{\zeta}(\sigma)\,\mathrm{d}\sigma\mathrm{d}s.
\end{aligned}$$

Note that the derivative of V_1 obeys the inequality

$$\begin{aligned}
&\dot{V}_1 + \zeta^T(I_N \otimes C^T)(I_N \otimes C)\zeta - y^T y \\
=&\dot{\zeta}^T(I_N \otimes P_1)\zeta + \left(\zeta^T(I_N \otimes P_2^T) + \varepsilon\dot{\zeta}^T(I_N \otimes P_2^T)\right) \\
&\times \left(-\dot{\zeta} + (I_N \otimes A)\zeta + (F_d \otimes BK)\zeta + \bar{\tau}(F_d \otimes BK)\eta_1 + (F_d \otimes BK)z\right) \\
&+ \zeta^T(I_N \otimes C^T)(I_N \otimes C)\zeta - y^T y.
\end{aligned}$$

Now consider the functional V_2. Due to Lemma A.1 from the Appendix, one has

$$\dot{V}_2 \le -\bar{\tau}\eta_1^T(I_N \otimes R)\eta_1 + \bar{\tau}\dot{\zeta}^T(I_N \otimes R)\dot{\zeta}. \tag{25}$$

Moreover, since $\gamma > 0$, one can write

$$\begin{aligned}
&\zeta^T(I_N \otimes P_2^T)(F_D \otimes BK)z \\
\le&\zeta^T\frac{1}{\gamma}(I_N \otimes P_2^T)(I_N \otimes P_2)\zeta + z^T\gamma(F_d \otimes BK)^T(F_d \otimes BK)z \\
\le&\zeta^T\frac{1}{\gamma}(I_N \otimes P_2^T)(I_N \otimes P_2)\zeta + \gamma\|BK\|^2 d_{N-1}^2\|z\|^2.
\end{aligned} \tag{26}$$

Analogously, one has

$$\dot{\zeta}^T(I_N \otimes P_2^T)(F_D \otimes BK)z \le \dot{\zeta}^T\frac{1}{\gamma}(I_N \otimes P_2^T)(I_N \otimes P_2)\dot{\zeta} + \gamma\|BK\|^2 d_{N-1}^2\|z\|^2. \tag{27}$$

These inequalities yield

$$\begin{aligned}
&\dot{V} + \zeta^T(I_N \otimes C^T)(I_N \otimes C)\zeta - y^T y \\
\le&\dot{\zeta}^T(I_N \otimes P_1)\zeta + \left(\zeta^T(I_N \otimes P_2^T) + \varepsilon\dot{\zeta}^T(I_N \otimes P_2^T)\right) \\
&\times \left(-\dot{\zeta} + (I_N \otimes A)\zeta + (F_d \otimes BK)\zeta + \bar{\tau}(F_d \otimes BK)\bar{\eta} + (F_d \otimes BK)z\right) \\
&+ \zeta^T(I_N \otimes C^T)(I_N \otimes C)\zeta - y^T y - \bar{\tau}\eta_1^T(I_N \otimes R)\eta_1 + \bar{\tau}\dot{\zeta}^T(I_N \otimes R)\dot{\zeta} \\
&+ \zeta^T\frac{1}{\gamma}(I_N \otimes P_2^T)(I_N \otimes P_2)\zeta + \dot{\zeta}^T\frac{\varepsilon}{\gamma}(I_N \otimes P_2^T)(I_N \otimes P_2)\dot{\zeta} + \gamma(1+\varepsilon)\|BK\|^2 d_{N-1}^2\|z\|^2 \\
&+ \zeta^T(I_N \otimes C^T)(I_N \otimes C)\zeta - y^T y.
\end{aligned} \tag{28}$$

With help of the matrix Γ, the previous inequality can be reformulated as

$$\dot{V} = (\zeta^T, \dot{\zeta}^T, \bar{\eta}^T)\Gamma \begin{pmatrix} \zeta \\ \dot{\zeta} \\ \bar{\eta} \end{pmatrix} - y^T y + \gamma(1+\varepsilon)\|BK\|^2 d_{N-1}^2\|z\|^2. \tag{29}$$

First, note that in absence of the distrubance z, the system is stabilized.

Let $T > 0$. If the disturbance is present, one can integrate Ineq. (29) from 0 to T. Hence

$$V(T) = \int_0^T (\zeta^T(s), \dot{\zeta}^T(s), \eta_1^T(s))\Gamma \begin{pmatrix} \zeta(s) \\ \dot{\zeta}(s) \\ \eta_1(s) \end{pmatrix} - y^T(s)y(s) + \gamma(1+\varepsilon)\|BK\|^2 d_{N-1}^2\|z(s)\|^2 \, \mathrm{d}s. \tag{30}$$

Note that the first term is nonpositive and $V(T) \geq 0$, thus

$$\int_0^T y^T(s)y(s)\,\mathrm{d}s \leq \int_0^T \gamma(1+\varepsilon)\|BK\|^2 d_{N-1}^2 \|z(s)\|^2\,\mathrm{d}s, \tag{31}$$

the claim is thus proved. □

Corollary 4.4. Let assumptions of Lemma are satisfied. Then

$$\lim_{t\to\infty} \|\xi(t)\| \leq c(\|\eta_2\| + \|\omega_2 + w - \bar{w}\|). \tag{32}$$

P r o o f. It is a consequence of definitions of vectors ζ and of matrix U. □

The previously obtained result cannot be directly used for computation as matrix Γ contains several multiples of variables. These will be removed in the following steps. First, the terms containing $\frac{1}{\gamma}(I_N \otimes P_2^T)(I_N \otimes P_2)$ are treated. Then, matrix Γ from (22) is replaced by

$$\Gamma' = \begin{pmatrix} \gamma'_{11} & \gamma_{12} & \gamma_{13} & \gamma'_{14} & 0 \\ * & \gamma'_{22} & \gamma_{23} & 0 & \gamma'_{25} \\ * & * & \gamma_{33} & 0 & 0 \\ * & * & * & -\gamma I_n & 0 \\ * & * & * & * & -\frac{\gamma}{\varepsilon} I_n \end{pmatrix} \tag{33}$$

with

$$\begin{aligned} \gamma'_{11} =& (I_N \otimes P_2^T)\Big((I_N \otimes A) + (F_d \otimes BK)\Big) + \Big((I_N \otimes A) + (F_d \otimes BK)\Big)^T (I_N \otimes P_2) \\ &+ (I_N \otimes C^T)(I_N \otimes C), \\ \gamma'_{14} =& I_N \otimes P_2^T, \\ \gamma'_{22} =& -\varepsilon I_N \otimes (P_2 + P_2^T - \bar{\tau}R), \\ \gamma'_{25} =& I_N \otimes P_2^T. \end{aligned}$$

Lemma 4.5. Let matrices P_1, P_2, R, K and constants γ, ε be as in Lemma 4.3. Then (23) holds if and only if

$$\Gamma' < 0. \tag{34}$$

P r o o f. Application of the Schur complement twice: on the element γ_{11} and γ_{22}. □

Matrix Γ' still contains multiples of variables, namely of P_2 and K. However, as matrix P_2 is nonsingular, one can introduce matrices

$$\begin{aligned} Q_2 =& P_2^{-1}, \\ Q_1 =& P_2^{-T} P_1 P_2^{-1}, \\ S =& P_2^{-T} R P_2^{-1}, \\ Y =& K P_2^{-1}. \end{aligned}$$

Define also matrix Γ'' by

$$\Gamma'' = \begin{pmatrix} \gamma''_{11} & \gamma''_{12} & \gamma''_{13} & \gamma''_{14} & 0 \\ * & \gamma''_{22} & \gamma''_{23} & 0 & \gamma''_{25} \\ * & * & \gamma''_{33} & 0 & 0 \\ * & * & * & -\gamma I_{nN} & 0 \\ * & * & * & * & -\frac{\gamma}{\varepsilon} I_{nN} \end{pmatrix}$$

where the elements of Γ'' are defined as

$$\begin{aligned} \gamma''_{11} =& (I_N \otimes A)(I_N \otimes Q_2) + (F_d \otimes BY) + (I_N \otimes Q_2^T)(I_N \otimes A^T) + (F_d \otimes Y^T B^T) \\ &+ (I_N \otimes Q_2^T C^T)(I_N \otimes CQ_2), \\ \gamma''_{12} =& I_N \otimes (Q_1 - Q_2) + \varepsilon(I_N \otimes Q_2^T A^T) + (F_D \otimes Y^Y B^T), \\ \gamma''_{13} =& \bar{\tau}(F_d \otimes BY), \\ \gamma''_{14} =& I_{nN}, \\ \gamma''_{22} =& -\varepsilon(Q_2 + Q_2^T - \bar{\tau} S), \\ \gamma''_{23} =& \varepsilon \gamma''_{13}, \\ \gamma''_{25} =& I_{nN}, \\ \gamma''_{33} =& -\bar{\tau}(I_N \otimes S). \end{aligned}$$

With these matrices, the following lemma can be proved:

Lemma 4.6. Let matrices P_1, P_2, R, K and constants γ, ε be as in Lemma 4.3. Let matrices Q_1, Q_2, S and Y are defined as above. Then (34) holds if and only if

$$\Gamma'' < 0. \tag{35}$$

Proof. Multiplication of matrix Γ'' by $\mathrm{diag}(I_N \otimes Q_2^T, I_N \otimes Q_2^T, I_N \otimes Q_2^T, I_{Nn}, I_{Nn})$ from the left and by $\mathrm{diag}(I_N \otimes Q_2, I_N \otimes Q_2, I_N \otimes Q_2, I_{Nn}, I_{Nn})$ yields the result. □

As the last transformation, we remove the multiple of $Q_2^T C^T C Q_2$ using the Schur complement. Define also matrix Γ''' by

$$\Gamma''' = \begin{pmatrix} \gamma''_{11} & \gamma''_{12} & \gamma''_{13} & \gamma''_{14} & 0 & \gamma'''_{16} \\ * & \gamma''_{22} & \gamma''_{23} & 0 & \gamma''_{25} & 0 \\ * & * & \gamma''_{33} & 0 & 0 & 0 \\ * & * & * & -\gamma I_{nN} & 0 & 0 \\ * & * & * & * & -\frac{\gamma}{\varepsilon} I_{nN} & 0 \\ * & * & * & * & * & -I_{pN} \end{pmatrix}$$

We thus obtain

Lemma 4.7. Let assumptions of Lemma 4.6 hold. Then (35) is valid if and only if

$$\Gamma''' < 0. \tag{36}$$

Inequality (36) is a LMI. Its solution could be used by an LMI solver, however, the size of this problem poses a limit to practical applicability of this method. This is since the dimension of matrix is proportional to nN. In the following text, a LMI problem is derived whose dimension is proportional to n only and whose solution implies validity of Ineq. (36).

Lemma 4.8. Let there exist $n \times n$-dimensional matrices Q_1, Q_2, S such that $Q_1 > 0$, $S > 0$, Q_2 is nonsingular, a $m \times n$-dimensional matrix Y and positive scalars γ, ε such that inequalities

$$\Sigma(d_1) < 0, \ \Sigma(d_{N-1}) < 0. \tag{37}$$

Then (36) holds.

Proof. First, note that $\sigma_{11}(d)$, $\sigma_{12}(d)$, $\sigma_{13}(d)$, $\sigma_{23}(d)$ are convex functions of d. Since eigenvalues of matrix D are supposed to be ordered as in (3), inequalities (37) imply

$$\mathrm{diag}\Big(\Sigma(d_1), \ldots, \Sigma(d_{N-1})\Big) < 0. \tag{38}$$

On the other hand, observe that there exists a permutation matrix Π such that

$$\Pi\Gamma'''\Pi^T = \mathrm{diag}\Big(\Sigma(d_1), \ldots, \Sigma(d_{N-1})\Big). \tag{39}$$

□

Proof of Theorem 4.1. Condition (19) guarantees that assumptions of Lemma 4.8 are satisfied. Thus, using Lemma 4.7, Lemma 4.6 and Lemma 4.5 implies that assumptions of 4.3 are satisfied.

Ad 1) Condition (20) implies $\omega_2 = a \otimes \int_{t-\tau_1}^{t} \dot{\bar{x}}(s)\,\mathrm{d}s$, hence

$$(L \otimes BK)\omega_2 = (L \otimes BK)(a \otimes \int_{t-\tau_1}^{t} \dot{\bar{x}}(s)\,\mathrm{d}s) = La \otimes BK \int_{t-\tau_1}^{t} \dot{\bar{x}}(s)\,\mathrm{d}s = 0 \tag{40}$$

since $La = A$. Thus $\eta_2 = 0$. Moreover, the first set of equations in (20) implies $\nu = 0$. Ineq. (24) holds with 0 on the right-hand side. Consensus is thus achieved for $t \to \infty$.

Ad 2) This is a direct consequence of Lemma 4.3. □

Remark 4.9. Differences in the time delays in agents cause the same effect as disturbances. In the example section is shown that this phenomenon cannot be circumvented. On the other hands, if the disturbance is equal in each agents, consensus is achieved. However, in this case, the disturbance influences the average dynamics.

5. EXAMPLES

5.1. Example 1

This example illustrates the fact that the consensus of agents with heterogeneous delays cannot be in some cases achieved. In other words, the results obtained in the previous sections are in some sense "optimal" – guaranteeing consensus in case of heterogeneous delays would require to impose additional requirements.

Assume a multi-agent system composed of three agents, every agent is a harmonic oscillator:

$$\begin{aligned} \dot{x}_{1,i} =& x_{2,i}, \\ \dot{x}_{2,i} =& -x_{1,i} + u_i, \\ i =& 1, 2, 3. \end{aligned}$$

Let $q > 0$ be a constant. Define the control inputs u_i as

$$\begin{aligned} u_1 =& x_{2,2}(t - \frac{\pi}{q}) - x_{2,1}(t), \\ u_2 =& x_{2,1}(t) - 2x_{2,2}(t - \frac{\pi}{q}) + x_{2,3}(t - \frac{2\pi}{q}), \\ u_3 =& x_{2,2}(t - \frac{\pi}{q}) - x_{2,3}(t - \frac{2\pi}{q}). \end{aligned}$$

In other words, $\tau_1 = 0$, $\tau_2 = \frac{\pi}{q}$, $\tau_3 = \frac{2\pi}{q}$.

Figure 1 shows the norm of the disagreement vector for this system if $q = 15$ and initial conditions

$$\begin{aligned} (x_{1,1}(0), x_{2,1}(0)) =& (-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}), \\ (x_{1,2}(0), x_{2,2}(0)) =& (0, 1), \\ (x_{1,3}(0), x_{2,3}(0)) =& (\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}). \end{aligned}$$

It is evident (and can be verified by an easy and simple computation) that the disagreement vector does not converge to zero.

On the other hand, if the example is changed so that $\tau_1 = \tau_2 = \tau_3 = \frac{2\pi}{q}$ (all other quantities remain unchanged), the consensus of this multi-agent system is achieved. This is illustrated by Figure 2.

5.2. Example 2

A network of 10 agents

$$\begin{aligned} \dot{x}_{1,i} =& x_{2,i}, \\ \dot{x}_{2,i} =& -x_{1,i} + u_i \end{aligned}$$

is considered. The agents are connected in a circular manner as seen in Figure 3.

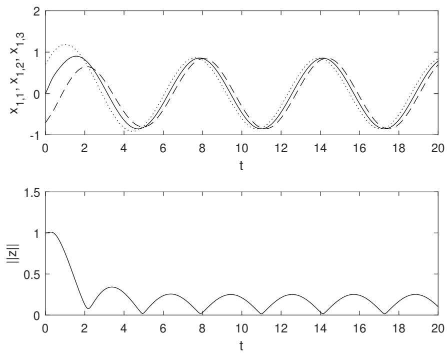

Fig. 1. Disagreement vector for heterogeneous delays.

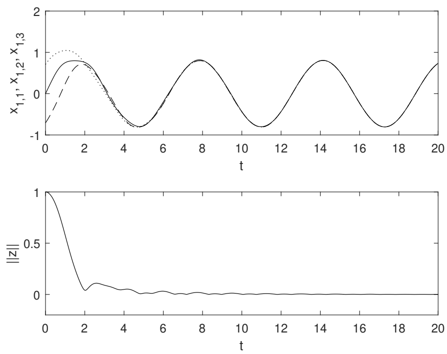

Fig. 2. Disagreement vector for homogeneous delays.

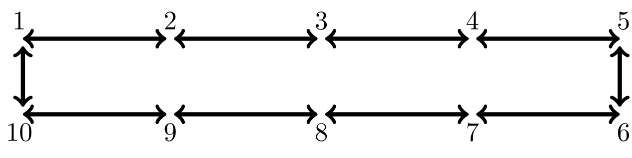

Fig. 3. Connection of agents.

Then, $d_1 = 0.38$, $d_{N-1} = 5.73$. It is assumed $\bar{\tau} = 0.2s$, minimal time delay was 0.

The algorithm presented in this paper yields $K = (-0.4219, -0.3719)$. The disturbance attenuation constant $\gamma = 5.52$.

Figure 4 illustrates the first component of the state of the first (solid bold line), fourth (solid line), seventh (dashed line) and tenth (dotted line) agents.

The norm of the disagreement vector of this multi-agent system is shown in Figure 5. Even here, the norm of the disagreement vector does not converge to zero.

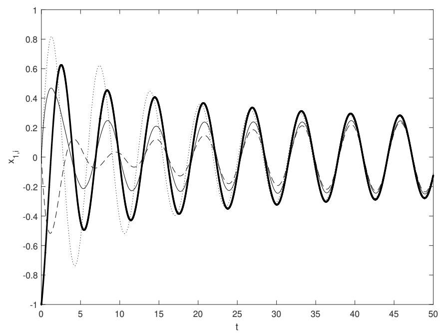

Fig. 4. State of the multi-agent system.

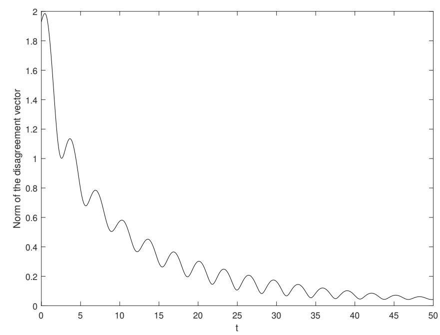

Fig. 5. Disagreement vector for heterogeneous delays.

6. CONCLUSIONS

The consensus problem for the multi-agent system composed of agents with heterogeneous time delays was presented. It was shown that, in case the delays in the agents are not equal, the disagreement vector may not, in general, converge to zero. However, the norm of the disagreement vector is estimated. This is achieved by means of the H_∞ control. The results were derived for the case when the interconnection of the agents was described by an undirected graph. In future, attention will be paid to the directed interconnection topology as well as to the case of jointly connected switching topology. Moreover, the case of different delays in different interconnections (not only different agents) will be treated. Also, event-triggered control of this kind of systems will be proposed, again, complemented by deriving of an estimate of the norm of the synchronization error.

A. TECHNICAL LEMMA

Lemma A.1. For any matrix $R \in \mathbb{R}^{n\times n}, R > 0$. Then

$$\frac{d}{dt}\int_{-\bar\tau}^{0}\int_{t+s}^{t}\dot\zeta^T(\sigma)(I_N\otimes R)\dot\zeta(\sigma)\,\mathrm{d}\sigma\,\mathrm{d}s \le -\frac{1}{\bar\tau}\eta_1^T(I_N\otimes R)\eta_1 + \bar\tau\dot\zeta^T(I_N\otimes R)\dot\zeta. \tag{41}$$

P r o o f. Note that for any orthogonal matrix W holds

$$(W\otimes I_n)(I\otimes R)(W^T\otimes I_n) = I\otimes R. \tag{42}$$

Ineq. (42) implies

$$\eta_1^T(I_N\otimes R)\eta_1 = \eta_1^T(U\otimes I_n)(I_N\otimes R)(U^T\otimes I_n)\eta_1 = \omega_1^T(I_N\otimes R)\omega_1, \tag{43}$$

$$\dot\xi^T(I\otimes R)\dot\xi = \dot\xi^T(U\otimes I_n)(I_N\otimes R)(U^T\otimes I_n)\dot\xi = \dot\zeta^T(I\otimes R)\dot\zeta. \tag{44}$$

Both inequalities $(43, 44)$ in connection with the Jensen inequality imply

$$\begin{aligned}\bar\tau\eta_1^T(I_N\otimes R)\eta_1 &= \bar\tau\omega_1^T(I_N\otimes R)\omega_1\\ &\le \begin{pmatrix}\frac{1}{\tau_1}\int_{t-\tau_1}^{t}\dot\xi_1(s)\,\mathrm{d}s\\ \vdots\\ \frac{1}{\tau_N}\int_{t-\tau_N}^{t}\dot\xi_N(s)\,\mathrm{d}s\end{pmatrix}^T (I_N\otimes R)\begin{pmatrix}\frac{1}{\tau_1}\int_{t-\tau_1}^{t}\dot\xi_1(s)\,\mathrm{d}s\\ \vdots\\ \frac{1}{\tau_N}\int_{t-\tau_N}^{t}\dot\xi_N(s)\,\mathrm{d}s\end{pmatrix}\\ &\le \sum_{i=1}^{N}\dot\xi_i^T(s)R\dot\xi_i(s)\,\mathrm{d}s\\ &\le \int_{t-\bar\tau}^{t}\dot\xi^T(s)(I_N\otimes R)\dot\xi(s)\,\mathrm{d}s = \int_{t-\bar\tau}^{t}\dot\zeta^T(s)(I_N\otimes R)\dot\zeta(s)\,\mathrm{d}s.\end{aligned} \tag{45}$$

This inequality yields

$$\begin{aligned}-\bar\tau\eta_1^T(I_N\otimes R)\eta_1 + \bar\tau\dot\zeta^T(I\otimes R)\dot\zeta &\ge -\int_{t-\bar\tau}\dot\zeta^T(s)(I\otimes R)\dot\zeta(s)\,\mathrm{d}s + \bar\tau\dot\zeta^T(I\otimes R)\dot\zeta\\ &= \frac{d}{dt}\int_{-\bar\tau}^{0}\int_{t+s}^{t}\dot\zeta^T(\sigma)(I_N\otimes R)\dot\zeta(\sigma)\,\mathrm{d}\sigma\,\mathrm{d}s\end{aligned} \tag{46}$$

which proves the lemma. □

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