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MODELLING AND OPTIMAL CONTROL OF NETWORKED SYSTEMS WITH STOCHASTIC COMMUNICATION PROTOCOLS

Chaoqun Zhu, Bin Yang and Xiang Zhu

This paper is concerned with the finite and infinite horizon optimal control issue for a class of networked control systems with stochastic communication protocols. Due to the limitation of networked bandwidth, only the limited number of sensors and actuators are allowed to get access to network mediums according to stochastic access protocols. A discrete-time Markov chain with a known transition probability matrix is employed to describe the scheduling behaviors of the stochastic access protocols, and the networked systems are modeled as a Markov jump system based on the augmenting technique. In such a framework, both the approaches of stochastic analysis and dynamic programming are utilized to derive the optimal control sequences satisfying the quadratic performance index. Moreover, the optimal controller gains are characterized by solving the solutions to coupled algebraic Riccati equations. Finally, a numerical example is provided to demonstrate the correctness and effectiveness of the proposed results.

Keywords: networked control systems, optimal control, stochastic communication protocol, markov chain

Classification: 93C05, 93E20

1. INTRODUCTION

Modelling, analysis and synthesis of networked control systems (NCSs) have recently attracted considerable research attention in the control community [1, 2, 3, 4]. One important feature of NCSs is that the controllers exchange information with the sensors and actuators over shared communication networks instead of the conventional point -to-point communication. The utilization of communication networks provides many advantages, such as lower cost, high reliability, reduced system wiring, flexible architecture, as well as simpler installation and maintenance, they have been found successful applications in a wide range of areas including environmental monitoring, process control, intelligent transportation, power systems, and many manufacturing plants. However, the introduction of network in control systems also brings some challenging issues, such as transmission delays, packet dropouts, quantization effects, etc. which are potential sources of poor performance and instability and have been primarily highlighted in the

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literature, see, e. g., [5, 6, 7, 8, 9, 10, 11] and references therein. For most underlying investigations on NCSs, a common assumption is that all the sensors and actuators could simultaneously access the network and communicate with the controller to exchange information during one sampling period. In many practical situations, however, it is quite unrealistic to implement such a communication scheme due to the fact that simultaneous multiple access over a shared network would result in unavoidable data collisions. In such a situation, in order to protect the information exchanging from network collisions and meet certain performance requirements, the control synthesis of NCSs involves not only stabilizing controller design but also efficiently utilizing access protocols to orchestrate the access order of network nodes.

Motivated by this fact, the control synthesis and access scheduling problem of NCSs with various communication protocols has been intensively investigated over the past decades. The communication protocols that have been explored to date could be generally classified into three categories, namely, static communication protocol [12, 13, 14, 15, 16], dynamic communication protocol [17, 18, 19, 20, 21], and stochastic communication protocol [22, 23, 24, 25, 26, 27, 28]. Static access protocol allocates the bandwidth of a network and determines which sensors and actuators could exchange data with controller based on a certain rule by off-line means. For instance, in [12] the Round-Robin (RR) protocol is applied to the sampled data control systems and the direct Lyapunov Krasovskii approach is employed to analyse the stability condition. The authors of [13] utilize the static periodic access protocol to describe the medium access constraints of NCSs and address the communication and control co-design problem with further proving a lower bound for the static communication sequences that preserved reachability and observability. Subsequently, the co-design method is extended to the NCSs with access constraints and network-induced delay in [15]. Generally speaking, static communication protocols are quite simple and easy to be implemented, but they are not adaptable to the situations with disturbances or uncertainties. An alternative to static access protocol is to let the medium access of sensors and/or actuators be determined based on the current information of the controlled systems, which is so-called dynamic communication protocol. The dynamic communication protocol has the ability to attenuate the disturbances quickly, and guarantee the stability of close-loop systems in complicated network environment. For example, in [17], the Try-Once-Discard (TOD) protocol is introduced for the first time to deal with the packets scheduling and the maximum allowable transfer interval is then obtained. [18] and [19] address the integrated design of controller and dynamic scheduling policy for the sensors/actuators of networked systems via a single-packet and multiple-packet transmission policy, respectively. The framework given in [19] is particularly interesting, as it is the first time that the control synthesis and scheduling problem are solved without decoupling them. Recently, the authors of [21] investigate the ultimate boundedness control problem for a class of networked nonlinear systems subject to the TOD protocol scheduling and uniform quantization effects.

Stochastic communication protocol serves as a widely used model in industrial control networks, such as the carrier-sense multiple access (CSMA) protocol of Ethernet and ALOHA protocol of local area network, and therefore much attention has been paid to networked systems with stochastic communication protocol. The property of this kind of protocol is that network nodes can obtain networked access rights randomly for data transmission. In general, two popular stochastic models are used to describe the stochastic communication protocol, namely, the Markovian jump model as well as the binary Bernoulli distribution model. Several results have been published regarding these stochastic access protocols. See, for example, [22] presents the stability analysis and controller design for the networked systems with random communication protocols and networked delay. The optimal controller design problem for networked systems with stochastic access constraints and packet dropouts is discussed in [24], However, this paper only considers the access constraints at the controller-to-actuator channel in the finite horizon, and the results are quite conservative. [25] is concerned with the H_∞ control problem for a class of linear time-varying networked systems with stochastic communication protocol. More recently, an output feedback controller is studied for the networked control systems with both stochastic access protocol, which is described by a vector-valued Markov process, and time-varying delays in [26]. Although the aforementioned literature have made some progress in the networked systems with stochastic communication protocol, there are still some interesting problems that deserve further research. So far, much attention has been focused on how to deal with the controller-to-actuator access constraints, while another important type of access constraints, sensors-to-controller access constraints, has not been fully investigated. Furthermore, most of the available results concerning networked systems with stochastic access protocol are confined to the stabilizing control or estimation problems, while the control performance issues are rarely addressed, which is equally important in many engineering applications.

In response to the aforementioned discussion, we aim to explore the finite and infinite horizon optimal control problem whose sensors and actuators access the medium subject to stochastic access protocols. More specifically, the objective of this paper is to achieve an optimal controller for the NCSs with stochastic communication protocol so as to guarantee the mean-square exponential stable performance of the systems in the finite and infinite horizon. The main contributions of this paper can be summarized as follows. (1) The comprehensive integration of the optimal controller design and the stochastic access protocols is investigated for the NCSs. (2) Medium access constraints characterized by stochastic communication protocol have been allowed to occur, simultaneously, in both the backward and the forward network channels in the framework of optimal controller design. (3) The calculation procedure of optimal state-feedback gains is presented by solving stabilizing solutions of coupled algebraic Riccati equations (CARE).

The rest of this paper is organized as follows. In the next Section we show how a networked system subject to stochastic access protocols could be modeled as a Markov jump system. In Section 3, we solve the optimal control problem of networked systems in the finite and infinite time horizon. The effectiveness of the proposed results is provided by an illustrative example in Section 4. Finally, Section 5 concludes the paper and discusses future research directions.

Notation. The notation used throughout the paper is fairly standard. R^n and $R^{n\times m}$ denote, respectively, the n dimensional Euclidean space and set of all $n \times m$ real matrices. A matrix $P > 0\,(P \geq 0)$ means that it is real symmetric and positive definite (semi-definite), and I_n is an $n \times n$ identity matrix. The superscripts $'T'$ and $'-1'$ stand for matrix transposition and matrix inverse, respectively. $\mathrm{tr}\{X\}$ denotes the trace of matrix X. $\{X_i\}_1^N$ means the set of matrices $X_1, X_2, \cdots, X_N$, and $\lambda_{\max}(X)(\lambda_{\min}(X))$ denotes the maximum (minimum) eigenvalue of a real symmetric matrix X. $\mathrm{E}[\zeta]$ and $\mathrm{E}[\zeta\,|v]$, respectively, stand expectation of the stochastic variable ζ and expectation of ζ conditional on v, and $\Pr\{\nu\}$ means the occurrence probability of the event ν. $diag\{\rho_1, \cdots, \rho_n\}$ stands for a diagonal matrix with the indicated elements on the diagonal and zeros elsewhere. C_a^b is the combinatorial number that b elements are selected from a total of a elements. $\mathrm{mod}(c, d)$ represents the remainder of the integer c to the integer d, and $\lfloor e \rfloor$ represents the largest integer not greater than real number e. Define the collections $\mathrm{T} \triangleq \{0, 1, \cdots, T-1\}$, $\mathrm{N}_m \triangleq \{1, 2, \cdots, N_m\}$, $\mathrm{N}_n \triangleq \{1, 2, \cdots, N_n\}$, $\mathrm{N}_r \triangleq \{1, 2, \cdots, N_m \times N_n\}$, where T, N_m, N_n and N_r are positive integers.

2. PROBLEM FORMULATION

In this section, we present the model of NCSs subject to stochastic communication protocol, whose structure is depicted in Fig. 1. Let us consider the linear time-invariant discrete-time plant described by

$$x(k+1) = Ax(k) + Bu(k) + D\omega(k), \tag{1}$$

where $A \in R^{n\times n}$, $B \in R^{n\times m}$, $D \in R^{n\times n}$, $x = [x_1 \cdots x_n]^T \in R^n$ is the plant state, which is directly measured by n sensors, with initial condition $x(0) = x_0$, and $u = [u_1 \cdots u_m]^T \in R^m$ is the control input actually executed by the actuators. $x_l(k)$ represents the measured state by the sensor l, and $u_i(k)$ denotes the control signal received by the actuator i. $\omega(k) \in R^n$ is the disturbance input, which satisfies $\omega(k) \sim N(0, W_\omega)$.

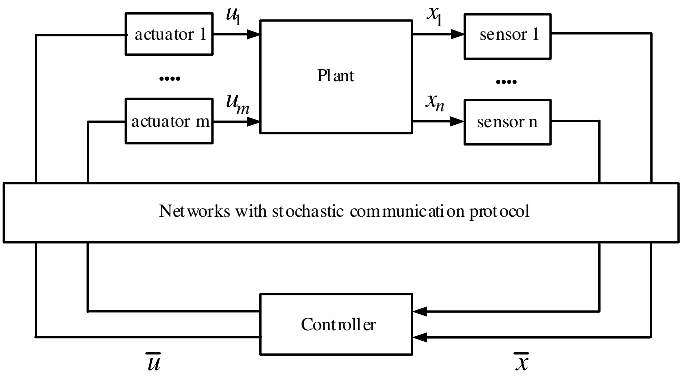

Fig. 1. The structure of the NCSs.

As shown in Fig. 1, there are n sensors and m actuators in the NCSs. For the bandwidth limitation, the communication medium imposes an upper bound, respectively, $1 \le q < n$ on the number of sensors and $1 \le p < m$ on the number of actuators which may communicate simultaneously with the controller at any time instant k. Another

q sensors and p actuators will be assigned the medium in the next sampling period. Therefore, there are total $N_n = C_n^q$ possible medium-access status for the sensors, and $N_m = C_m^p$ possible medium-access status for the actuators. For analyzing conveniently, only the medium access constrains are considered for the networked systems, while other communication limitations, such as transmission delays, packet dropouts, and quantization errors, are small enough and can be neglected in the controller design. In order to cope with the effect of access constraints, we construct the following compensation scheme: the zero-order holders are used at the receiving side of network medium, i. e., when the actuators or sensors fail to access the medium, the values stored in zero-order holders are fed into the plant or controller. Furthermore, without loss of generality, we also assume that the plant′s outputs and control signals are transmitted in separate packets to the controller and actuators via multiple communication channels.

2.1. Stochastic Communication Protocol

In what follows, we will introduce the stochastic communication protocol, which is governed by a discrete-time Markov chain. For the actuators′ side, let the binary valued function $\delta_i(k)$ denote medium-access status of the actuator i at time instant k, where $i = 1, \cdots, m$. $\delta_i(k) = 1$ means the actuator i is accessing the channel, and the ith control vector $\bar{u}_i(k)$ will be available to the actuator, i. e., $u_i(k) = \bar{u}_i(k) + E_i\nu(k)$, where $\nu(k) \in R^\nu$ is the network-induced disturbance, satisfying $\nu(k) \sim N(0, W_\nu)$, and $E_i \in R^{1\times\nu}$ is known constant matrix. Otherwise, $\delta_i(k) = 0$ means the control input packet is not transmitted and the actuator i will remain the last available value by used zero-order holder, i. e., $u_i(k) = u_i(k-1)$. Let $\delta(k) = [\delta_1(k), \cdots, \delta_m(k)]^T$ denote the stochastic protocol sequence that assigns medium access to the m actuators at time k. Defining

$$M_\delta(k) \triangleq diag\{\delta(k)\},\ E \triangleq \begin{bmatrix} E_1^T & E_2^T & \cdots & E_m^T \end{bmatrix}^T,$$

then we have

$$u(k) = M_\delta(k)(\bar{u}(k) + E\nu(k)) + \bar{M}_\delta(k)u(k-1), \tag{2}$$

where $\bar{M}_\delta(k) = I_m - M_\delta(k)$, and $\bar{u}(k)$ is the control signal actually generated by the controller.

According to the CSMA communication protocol [29], we assume that $M_\delta(k)$ can be modeled by a Markov process taking the matrix values in a finite set $\mathrm{M}_\delta = \{M_\delta^1, \ldots, M_\delta^{N_m}\}$ with the following conditional probability:

$$\Pr\{M_\delta(k) = M_\delta^i\} = \lambda_i(k),\ \Pr\left\{M_\delta(k+1) = M_\delta^j \,\middle|\, M_\delta(k) = M_\delta^i\right\} = \lambda_{ij}, \tag{3}$$

where $\lambda_{ij} > 0$ represents the transition probability from mode i to mode j, satisfying $\sum_{j=1}^{N_m} \lambda_{ij} = 1$, $i, j \in \mathrm{N}_m$. $\lambda_i(0)$ is the initial probability of mode i, while $\lambda_i(k)$ is the probability of mode i at time k, and $\lambda(k) = [\lambda_1(k) \cdots\ \lambda_{N_m}(k)\ ]^T$. For simplicity, let $\{\tau(k); k \in \mathrm{Z}\}$ be the indicator of the Markov process, satisfying $\tau(0) = \tau_0$ and $\tau(k) \in \mathrm{N}_m$. $\tau(k) = i$ if $M_\delta(k) = M_\delta^i$. According to (2), it is obtained

$$u(k) = M_\delta^{\tau(k)}(k)(\bar{u}(k) + E\nu(k)) + \bar{M}_\delta^{\tau(k)}(k)u(k-1), \tag{4}$$

where $\bar{M}_\delta^{\tau(k)}(k) = I_m - M_\delta^{\tau(k)}(k)$. Indeed, (4) reflects that the values in $u(k)$ corresponding to the ith group actuators are updated at time k, while the others remain the same values.

In the same manner, the medium-access status of the sensor l is denoted by another binary valued function $\sigma_l(k)$, where $l = 1, \cdots, n$. $\sigma_l(k) = 1$ indicates that the sensor l is accessing the medium, and the state information is updated, i. e., $\bar{x}_l(k) = x_l(k) + F_l\nu(k)$, where $F_l \in R^{1\times\nu}$ is known constant matrix, and $\bar{x}_l(k)$ denotes the lth received state data by the controller. Otherwise, $\sigma_l(k) = 0$ means that the current lth measured signal is not transmitted and the controller will use the state data of previous sampling period, i. e., $\bar{x}_l(k) = \bar{x}_l(k-1)$. Let $\sigma(k) = [\sigma_1(k), \cdots, \sigma_n(k)]^T$ denote the stochastic protocol sequence that assigns channel access to the n sensors at time k. Define

$$M_\sigma(k) \triangleq diag\{\sigma(k)\},\ F \triangleq \begin{bmatrix} F_1^T & F_2^T & \cdots & F_n^T \end{bmatrix}^T.$$

It is also assumed that $M_\sigma(k)$ is generated by another Markov process that takes the matrix values in finite set $\mathrm{M}_\sigma = \left\{M_\sigma^1, \cdots, M_\sigma^{N_n}\right\}$ with the following conditional probability:

$$\Pr\left\{M_\sigma(k) = M_\sigma^l\right\} = \pi_l(k),\ \Pr\left\{M_\sigma(k+1) = M_\sigma^h \,\middle|\, M_\sigma(k) = M_\sigma^l\right\} = \pi_{lh}, \tag{5}$$

where $\pi_{lh} > 0$, satisfying $\sum_{h=1}^{N_n} \pi_{lh} = 1$, $l, h \in \mathrm{N}_n$. Let $\{\theta(k); k \in \mathrm{Z}\}$ with $\theta(0) = \theta_0$, $\theta(k) \in \mathrm{N}_n$, denote the Markov process. We thus have

$$\bar{x}(k) = M_\sigma^{\theta(k)}(k)\left(x(k) + F\nu(k)\right) + \bar{M}_\sigma^{\theta(k)}(k)\,\bar{x}(k-1), \tag{6}$$

where $\bar{M}_\sigma^{\theta(k)}(k) = I_n - M_\sigma^{\theta(k)}(k)$.

2.2. System Modelling

The networked systems model can be attained by substituting (4) and (6) into (1), and introducing an augmented state variable

$$\tilde{x}(k) \triangleq \begin{bmatrix} x^T(k) & \bar{x}^T(k-1) & u^T(k-1) \end{bmatrix}^T.$$

This results in the discrete-time model

$$\tilde{x}(k+1) = A_{\tau(k),\theta(k)}\tilde{x}(k) + B_{\tau(k)}\bar{u}(k) + D_{\tau(k),\theta(k)}\xi(k), \tag{7}$$

where

$$A_{\tau(k),\theta(k)} = \begin{bmatrix} A & 0 & B\bar{M}_\delta^{\tau(k)}(k) \\ M_\sigma^{\theta(k)}(k) & \bar{M}_\sigma^{\theta(k)}(k) & 0 \\ 0 & 0 & \bar{M}_\delta^{\tau(k)}(k) \end{bmatrix},\ B_{\tau(k)} = \begin{bmatrix} BM_\delta^{\tau(k)}(k) \\ 0 \\ M_\delta^{\tau(k)}(k) \end{bmatrix},$$

$$D_{\tau(k),\theta(k)} = \begin{bmatrix} D & BM_\delta^{\tau(k)}(k)E \\ 0 & M_\sigma^{\theta(k)}(k)F \\ 0 & M_\delta^{\tau(k)}(k)E \end{bmatrix},$$

$$\xi(k) = \begin{bmatrix} \omega^T(k) & \nu^T(k) \end{bmatrix}^T, \ \tilde{x}(0) = \tilde{x}_0 = \begin{bmatrix} x_0^T \ \mathbf{0}^{1\times n} \ \mathbf{0}^{1\times m} \end{bmatrix}^T,$$

where $\xi(k) \sim N(0, W)$, satisfying $W = diag\{W_\omega, W_\nu\}$. For the simplicity of analysis and design, we utilize the method of [30] to reformulate the system (7) by mapping two stochastic processes $\tau(k)$ and $\theta(k)$ to one Markov chain $\{r(k); k \in \mathrm{Z}\}$, $r(0) = r_0$, $r(k) \in \mathrm{N}_r$. Define

$$r(k) = \Phi(\tau(k), \theta(k)) \triangleq \tau(k) + m(\theta(k) - 1). \tag{8}$$

Furthermore, if $r(k)$ is given, we can obtain the values of $\tau(k)$ and $\theta(k)$ in terms of functions $\phi_\tau(r(k))$ and $\phi_\theta(r(k))$, which are defined as following:

$$\begin{cases} \tau(k) = \phi_\tau(r(k)) \triangleq \left\lfloor \frac{r(k)-1}{m} \right\rfloor + 1 \\ \theta(k) = \phi_\theta(r(k)) \triangleq \mathrm{mod}(r(k) - 1, m) + 1. \end{cases} \tag{9}$$

According to (8), the transition probability matrix γ_{sw} of Markov chain $r(k)$ can be derived by the following conditional probability:

$$\begin{aligned} \gamma_{sw} &= \Pr\{r(k+1) = w \,|r(k) = s\} \\ &= \Pr\{\tau(k+1) = \phi_\tau(w) \,|\tau(k) = \phi_\tau(s)\} \Pr\{\theta(k+1) = \phi_\theta(w) \,|\sigma(k) = \phi_\theta(s)\} \\ &= \lambda_{\phi_\tau(w)\phi_\tau(s)} \pi_{\phi_\theta(w)\phi_\theta(s)}, \end{aligned}$$

where $s, w \in \mathrm{N}_r$, $\lambda_{\phi_\tau(w)\phi_\tau(s)}$ and $\pi_{\phi_\theta(w)\phi_\theta(s)}$ are given by (3) and (5). Considering (8) and (9), the augmented system (7) can be rewritten to an equivalent form as follows:

$$\tilde{x}(k+1) = \tilde{A}_{r(k)} \tilde{x}(k) + \tilde{B}_{r(k)} \bar{u}(k) + \tilde{D}_{r(k)} \xi(k), \tag{10}$$

where

$$\tilde{A}_{r(k)} = A_{\phi_\tau(r(k))\phi_\theta(r(k))}, \ \tilde{B}_{r(k)} = B_{\phi_\theta(r(k))}, \ \tilde{D}_{r(k)} = D_{\phi_\tau(r(k))\phi_\theta(r(k))}.$$

We assume that the current modes of the Markov process $\tau(k)$ and $\theta(k)$ are available at each time instant k. This assumption is critical, since it allows us to avoid the more difficult and generally unsolved 'dual control' problem. Let I_k denote the information sets available to the controller at time k. Then we have

$$\mathrm{I}_k = \{\tilde{x}(t), r(t) \,|t = 0, 1, \cdots, k\}.$$

It is clear that $\mathrm{I}_k \subset \mathrm{I}_{k+1} \subset \mathrm{I}_T$.

3. MAIN RESULTS

By employing dynamic programming approach, a finite-horizon optimal controller will be developed to guarantee the asymptotic stability of the system (10), and minimize the following quadratic cost function, namely:

$$J\left(\tilde{x}_0, r_0, \bar{u}\right) = \mathrm{E}\left[\tilde{x}^T\left(T\right)Q\left(T\right)\tilde{x}\left(T\right) + \sum_{k=0}^{T-1}\left(\tilde{x}^T\left(k\right)Q\left(k\right)\tilde{x}\left(k\right) + \bar{u}^T\left(k\right)R\left(k\right)\bar{u}\left(k\right)\right)|\tilde{x}_0, r_0, \bar{u}\right], \tag{11}$$

where $Q(T) \geq 0$, $Q(k) \geq 0$, $R(k) > 0$ are given symmetric matrices. The minimal is denoted by $J^*\left(\tilde{x}_0, r_0\right)$.

Theorem 1. Consider the system (10), and the quadratic cost function (11). The optimal controller is given by

$$\bar{u}\left(k\right) = K_{r(k)}\left(k\right)\tilde{x}\left(k\right), \tag{12}$$

where

$$K_{r(k)}\left(k\right) = -\left(R\left(k\right) + \tilde{B}^T_{r(k)}\bar{P}_{r(k)}\left(k+1\right)\tilde{B}_{r(k)}\right)^{-1}\tilde{B}^T_{r(k)}\bar{P}_{r(k)}\left(k+1\right)\tilde{A}_{r(k)}, \tag{13}$$

and the optimal cost is

$$J^*\left(\tilde{x}_0, r_0\right) = \sum_{i=1}^{N_r}\left[\mathrm{tr}\left(\gamma_i\left(0\right)\tilde{x}_0 P_i\left(0\right)\tilde{x}_0^T\right) + \sum_{k=0}^{T-1}\gamma_i\left(k\right)\mathrm{tr}\left(\tilde{D}_{r(k)}W\tilde{D}^T_{r(k)}\bar{P}_i\left(k+1\right)\right)\right]. \tag{14}$$

In the above, the matrix $\bar{P}_{r(k)}\left(k+1\right)$ obeys the following backward CRDE

$$\begin{aligned} P_{r(k)}\left(k\right) &= Q\left(k\right) + \tilde{A}^T_{r(k)}\bar{P}_{r(k)}\left(k+1\right)\tilde{A}_{r(k)} \\ &\quad - \tilde{A}^T_{r(k)}\bar{P}_{r(k)}\left(k+1\right)\tilde{B}_{r(k)}\left(R\left(k\right) + \tilde{B}^T_{r(k)}\bar{P}_{r(k)}\left(k+1\right)\tilde{B}_{r(k)}\right)^{-1} \\ &\quad \cdot \tilde{B}^T_{r(k)}\bar{P}_{r(k)}\left(k+1\right)\tilde{A}_{r(k)}, \end{aligned} \tag{15}$$

where $\bar{P}_{r(k)}\left(k+1\right) = \sum_{j=1}^{N_r}\pi_{r(k)j}P_j\left(k+1\right)$, $P_{r(T)}\left(T\right) = Q\left(T\right)$, $r\left(k\right) \in \mathrm{N}_r$, $k = T-1$, $\cdots, 1, 0$.

Proof. According to the quadratic cost function (11), we can obtain the Bellman functional as follows

$$\begin{aligned} S\left(\tilde{x}\left(k\right), r\left(k\right), k\right) &= \min_{\bar{u}(k)}\left\{\mathrm{E}\left[\tilde{x}^T\left(k\right)Q\left(k\right)\tilde{x}\left(k\right) + \bar{u}^T\left(k\right)R\left(k\right)\bar{u}\left(k\right)\right.\right. \\ &\quad \left.\left. + S\left(\tilde{x}\left(k+1\right), r\left(k+1\right), k+1\right)|\mathrm{I}_k\right]\right\}, \end{aligned} \tag{16}$$

for $k = T-1, \cdots, 1, 0$, such that $J^*(\tilde{x}_0, r_0) = S(\tilde{x}(0), r(0), 0)$. According to the characteristics of Bellman functional (16), its solution should be quadratic, i. e.,

$$S(\tilde{x}(k), r(k), k) = \mathrm{E}\left[\tilde{x}^T(k) P_{r(k)}(k) \tilde{x}(k) | \mathrm{I}_k\right] + \alpha(k), \tag{17}$$

where $k = T, \cdots, 1, 0$, $P_{r(k)}(k)$ is the undetermined symmetric and positive definite matrix, $r(k) \in \mathrm{N}_r$, and $\alpha(k)$ is the undetermined scalar function. Based on (17), one has the Bellman functional equation for $k+1$ as follows

$$\begin{aligned} & S(\tilde{x}(k+1), r(k+1), k+1) \\ & = \mathrm{E}\left[\tilde{x}^T(k+1) P_{r(k+1)}(k+1) \tilde{x}(k+1) | \mathrm{I}_{k+1}\right] + \alpha(k+1). \end{aligned} \tag{18}$$

Taking the conditional mathematical expectation on (18), we can obtain

$$\begin{aligned} & \mathrm{E}\left[S(\tilde{x}(k+1), r(k+1), k+1) | \mathrm{I}_k\right] \\ & = \mathrm{E}\left[\mathrm{E}\left[\tilde{x}^T(k+1) P_{r(k+1)}(k+1) \tilde{x}(k+1) | \mathrm{I}_{k+1}\right] + \alpha(k+1) \,\middle|\, \mathrm{I}_k\right]. \end{aligned} \tag{19}$$

By applying the smoothing property of the conditional expectations [31], one has

$$\begin{aligned} & \mathrm{E}\left[\tilde{x}^T(k+1) P_{r(k+1)}(k+1) \tilde{x}(k+1) | \mathrm{I}_k\right] \\ & = \mathrm{E}\left[\mathrm{E}\left[\tilde{x}^T(k+1) P_{r(k+1)}(k+1) \tilde{x}(k+1) | \mathrm{I}_{k+1}\right] \,\middle|\, \mathrm{I}_k\right]. \end{aligned} \tag{20}$$

According to the definition of the mathematical expectation, for any measurable functions g and f, the following equation holds

$$\mathrm{E}\left[g(\omega(k)) f(r(k+1)) | \mathrm{I}_k\right] = \mathrm{E}\left[g(\omega(k)) | \mathrm{I}_k\right] \sum_{j=1}^{N_r} \gamma_{r(k)j} f(j). \tag{21}$$

Considering (10) and (21), and substituting (20) into (19) leads to

$$\begin{aligned} & \mathrm{E}\left[S(\tilde{x}(k+1), r(k+1), k+1) | \mathrm{I}_k\right] \\ & = \mathrm{E}\left[\tilde{x}^T(k+1) P_{r(k+1)}(k+1) \tilde{x}(k+1) | \mathrm{I}_k\right] + \mathrm{E}\left[\alpha(k+1) | \mathrm{I}_k\right] \\ & = \mathrm{E}\Big[\tilde{x}^T(k) \tilde{A}^T_{r(k)} P_{r(k+1)}(k+1) \tilde{A}_{r(k)} \tilde{x}(k) \\ & \quad + 2\bar{u}^T(k) \tilde{B}^T_{r(k)} P_{r(k+1)}(k+1) \tilde{A}_{r(k)} \tilde{x}(k) \\ & \quad + 2\tilde{x}^T(k) A^T_{r(k)} P_{r(k+1)}(k+1) \tilde{D}_{r(k)} \xi(k) \\ & \quad + \bar{u}^T(k) \tilde{B}^T_{r(k)}(k) P_{r(k+1)}(k+1) \tilde{B}_{r(k)}(k) \bar{u}(k) \\ & \quad + 2\bar{u}^T(k) \tilde{B}^T_{r(k)}(k) P_{r(k+1)}(k+1) \tilde{D}_{r(k)}(k) \xi(k) + \xi^T(k) \tilde{D}^T_{r(k)} \\ & \quad \cdot P_{r(k+1)}(k+1) \tilde{D}_{r(k)}(k) \xi(k) | \mathrm{I}_k\Big] + \mathrm{E}\left[\alpha(k+1) | \mathrm{I}_k\right] \\ & = \tilde{x}^T(k) \tilde{A}^T_{r(k)} \bar{P}_{r(k)}(k+1) \tilde{A}_{r(k)} \tilde{x}(k) \\ & \quad + \bar{u}^T(k) \tilde{B}^T_{r(k)}(k) P_{r(k+1)}(k+1) \tilde{B}_{r(k)}(k) \bar{u}(k) \end{aligned}$$

$$
\begin{aligned}
&+2\bar{u}^T(k)\tilde{B}^T_{r(k)}\bar{P}_{r(k)}(k+1)\tilde{A}_{r(k)}\tilde{x}(k)+\alpha(k+1)\\
&+2\mathrm{tr}\left\{\tilde{D}_{r(k)}\mathrm{E}\left[\xi(k)x^T(k)|\mathrm{I}_k\right]A^T_{r(k)}\bar{P}_{r(k)}(k+1)\right\}\\
&+2\mathrm{tr}\left\{\tilde{D}_{r(k)}\mathrm{E}\left[\xi(k)\bar{u}^T(k)|\mathrm{I}_k\right]\tilde{B}^T_{r(k)}(k)\bar{P}_{r(k)}(k+1)\right\}\\
&+\mathrm{tr}\left\{\tilde{D}_{r(k)}E\left[\xi(k)\xi^T(k)|I_k\right]\tilde{D}^T_{r(k)}(k)\bar{P}_{r(k)}(k+1)\right\}.
\end{aligned}
\tag{22}
$$

Noting that $\xi(k)$ is uncorrelated random variables with $x(k)$ and $\bar{u}(k)$, and satisfies the statistical characteristics $\xi(k)\sim N(0,W)$, one has

$$\mathrm{E}\left[\xi(k)x^T(k)|\mathrm{I}_k\right]=0, \tag{23}$$

$$\mathrm{E}\left[\xi(k)\bar{u}^T(k)|\mathrm{I}_k\right]=0, \tag{24}$$

$$\mathrm{E}\left[\xi(k)\xi^T(k)|\mathrm{I}_k\right]=W. \tag{25}$$

Considering (23), (24) and (25), we can obtain

$$\mathrm{tr}\left\{\tilde{D}_{r(k)}\mathrm{E}\left[\xi(k)x^T(k)|\mathrm{I}_k\right]A^T_{r(k)}\bar{P}_{r(k)}(k+1)\right\}=0, \tag{26}$$

$$\mathrm{tr}\left\{\tilde{D}_{r(k)}\mathrm{E}\left[\xi(k)\bar{u}^T(k)|\mathrm{I}_k\right]\tilde{B}^T_{r(k)}\bar{P}_{r(k)}(k+1)\right\}=0, \tag{27}$$

$$
\begin{aligned}
&\mathrm{tr}\left\{\tilde{D}_{r(k)}\mathrm{E}\left[\xi(k)\xi^T(k)|\mathrm{I}_k\right]\tilde{D}^T_{r(k)}(k)\bar{P}_{r(k)}(k+1)\right\}\\
&=\mathrm{tr}\left\{\tilde{D}_{r(k)}W\tilde{D}^T_{r(k)}(k)\bar{P}_{r(k)}(k+1)\right\}\\
&=\sum\nolimits_{i=1}^{N_r}\gamma_i(k)\,\mathrm{tr}\left\{\tilde{D}_{r(k)}W\tilde{D}^T_{r(k)}(k)\bar{P}_i(k+1)\right\}.
\end{aligned}
\tag{28}
$$

Substituting (26), (27), and (28) into (22) yields

$$
\begin{aligned}
&\mathrm{E}\left[S(\tilde{x}(k+1),r(k+1),k+1)|\mathrm{I}_k\right]\\
&=\tilde{x}^T(k)\tilde{A}^T_{r(k)}\bar{P}_{r(k)}(k+1)\tilde{A}_{r(k)}\tilde{x}(k)\\
&\quad+\bar{u}^T(k)\tilde{B}^T_{r(k)}(k)P_{r(k+1)}(k+1)\tilde{B}_{r(k)}(k)\bar{u}(k)\\
&\quad+2\bar{u}^T(k)\tilde{B}^T_{r(k)}\bar{P}_{r(k)}(k+1)\tilde{A}_{r(k)}\tilde{x}(k)\\
&\quad+\sum\nolimits_{i=1}^{N_r}\gamma_i(k)\,\mathrm{tr}\left\{\tilde{D}_{r(k)}W\tilde{D}^T_{r(k)}(k)\bar{P}_i(k+1)\right\}+\alpha(k+1).
\end{aligned}
\tag{29}
$$

Substituting (29) into the Bellman functional (16) leads to

$$
\begin{aligned}
&S(\tilde{x}(k),r(k),k)\\
&=\min_{\bar{u}(k)}\left\{\tilde{x}^T(k)\left(Q(k)+\tilde{A}^T_{r(k)}\bar{P}_{r(k)}(k+1)\tilde{A}_{r(k)}\right)\tilde{x}(k)\right.\\
&\quad+\bar{u}^T(k)\left(R(k)+\tilde{B}^T_{r(k)}(k)P_{r(k+1)}(k+1)\tilde{B}_{r(k)}(k)\right)\bar{u}(k)
\end{aligned}
$$

$$+ 2\bar{u}^T(k)\tilde{B}^T_{r(k)}\bar{P}_{r(k)}(k+1)\tilde{A}_{r(k)}\tilde{x}(k)$$
$$+ \sum_{i=1}^{N_r}\gamma_i(k)\,\mathrm{tr}\left\{\tilde{D}_{r(k)}W\tilde{D}^T_{r(k)}(k)\bar{P}_i(k+1)\right\} + \alpha(k+1)\Bigg\}. \tag{30}$$

With symmetric matrix $\bar{P}_{r(k)}(k+1)$, it can be easily computed that

$$\partial S(\tilde{x}(k), r(k), k)/\partial\bar{u}(k)$$
$$= 2\left(R(k) + \tilde{B}^T_{r(k)}\bar{P}_{r(k)}(k+1)\tilde{B}_{r(k)}\right)\bar{u}(k)$$
$$+ 2\tilde{B}^T_{r(k)}\bar{P}_{r(k)}(k+1)\tilde{A}_{r(k)}\tilde{x}(k).$$

Since $\bar{u}(k)$ is unconstrained, its optimal value must satisfy

$$\partial S(\tilde{x}(k), r(k), k)/\partial\bar{u}(k) = 0,$$

then

$$\bar{u}(k) = -\left(R(k) + \tilde{B}^T_{r(k)}\bar{P}_{r(k)}(k+1)\tilde{B}_{r(k)}\right)^{-1}\tilde{B}^T_{r(k)}\bar{P}_{r(k)}(k+1)\tilde{A}_{r(k)}\tilde{x}(k). \tag{31}$$

Because of $R(k) > 0$, it is clear that

$$R(k) + \tilde{B}^T_{r(k)}\bar{P}_{r(k)}(k+1)\tilde{B}_{r(k)} > 0.$$

Then, (13) follows directly from (31). Substituting (31) into (30), one has

$$S(\tilde{x}(k), r(k), k)$$
$$=\mathrm{E}\Big[\tilde{x}^T(k)\Big(Q(k) + \tilde{A}^T_{r(k)}\bar{P}_{r(k)}(k+1)\tilde{A}_{r(k)} - \tilde{A}^T_{r(k)}\bar{P}_{r(k)}(k+1)$$
$$\cdot\tilde{B}_{r(k)}\left(R(k) + \tilde{B}^T_{r(k)}\bar{P}_{r(k)}(k+1)\tilde{B}_{r(k)}\right)^{-1}\tilde{B}^T_{r(k)}\bar{P}_{r(k)}(k+1)\tilde{A}_{r(k)}\tilde{x}(k)|\mathrm{I}_k\Big]$$
$$+\sum_{i=1}^{N_r}\gamma_i(k)\,\mathrm{tr}\left\{\tilde{D}_{r(k)}W\tilde{D}^T_{r(k)}(k)\bar{P}_i(k+1)\right\} + \alpha(k+1). \tag{32}$$

Comparing (32) with (17), one has

$$P_{r(k)}(k)$$
$$= Q(k) + \tilde{A}^T_{r(k)}\bar{P}_{r(k)}(k+1)\tilde{A}_{r(k)} - \tilde{A}^T_{r(k)}\bar{P}_{r(k)}(k+1)\tilde{B}_{r(k)}$$
$$\cdot\left(R(k) + \tilde{B}^T_{r(k)}\bar{P}_{r(k)}(k+1)\tilde{B}_{r(k)}\right)^{-1}\tilde{B}^T_{r(k)}\bar{P}_{r(k)}(k+1)\tilde{A}_{r(k)},$$
$$\alpha(k) = \sum_{i=1}^{N_r}\gamma_i(k)\,\mathrm{tr}\left\{\tilde{D}_{r(k)}W\tilde{D}^T_{r(k)}(k)\bar{P}_i(k+1)\right\} + \alpha(k+1)$$
$$= \sum_{t=k}^{T-1}\sum_{i=1}^{N_r}\gamma_i(t)\,\mathrm{tr}\left(\tilde{D}_{r(k)}W\tilde{D}^T_{r(k)}\bar{P}_i(t+1)\right),$$

with final value conditions

$$P_{r(T)}(T) = Q(T),\ \alpha(T) = 0,$$

where $r(k) \in \mathrm{N}_r$. Based on $J^*(\tilde{x}_0, r_0) = S(\tilde{x}(0), r(0), 0)$, we can obtain the optimal cost

$$\begin{aligned} & J^*(\tilde{x}_0, r_0) \\ & = \mathrm{E}\left[\tilde{x}^T(0) P_{r(0)}(0) \tilde{x}(0) | \mathrm{I}_0\right] + \alpha(0) \\ & = \sum_{i=1}^{N_r} \left[\mathrm{tr}\left(\gamma_i(0) \tilde{x}_0 P_i(0) \tilde{x}_0^T\right) + \sum_{k=0}^{T-1} \gamma_i(k) \,\mathrm{tr}\left(\tilde{D}_{r(k)} W \tilde{D}_{r(k)}^T \bar{P}_i(k+1)\right)\right]. \end{aligned}$$

The proof is completed. □

It is necessary to redefine the quadratic cost function for designing optimal controller in the infinite horizon. We define the long run average cost function as

$$J_{av}(\tilde{x}_0, r_0, \bar{u}) = \lim_{T\to\infty} \frac{1}{T} \left\{\mathrm{E}\left[\sum_{k=0}^{T-1} \tilde{x}^T(k) Q \tilde{x}(k) + \bar{u}^T(k) R \bar{u}(k)\right]\right\}, \tag{33}$$

and the optimal cost is denoted by

$$J_{av}^*(\tilde{x}_0, r_0) = \inf_{\bar{u}(k)} J_{av}(\tilde{x}_0, r_0, \bar{u}), \tag{34}$$

where $Q \geq 0$, $R > 0$ are given symmetric matrices.

In what follows, we will focus our attention on achieving a static state feedback $\{K_i\}_{i=1}^{N_r}$ to drive the system (10) asymptotically stable and minimize the cost function (33), which is summarized in the following theorem. Before proceeding further, several definitions are given for the CARE as follow [32, 33].

Definition 1. Let all parameters needed in the definition are given by the system (10). The CARE is given by the following collection of matrix equations

$$P_i = Q + \tilde{A}_i^T \bar{P}_i \tilde{A}_i - \tilde{A}_i^T \bar{P}_i \tilde{B}_i \left(R + \tilde{B}_i^T \bar{P}_i \tilde{B}_i\right)^{-1} \tilde{B}_i^T \bar{P}_i \tilde{A}_i, \tag{35}$$

and its set is defined as follow.

$$M = \left\{P_i | P_i = Q + \tilde{A}_i^T \bar{P}_i \tilde{A}_i - \tilde{A}_i^T \bar{P}_i \tilde{B}_i \left(R + \tilde{B}_i^T \bar{P}_i \tilde{B}_i\right)^{-1} \tilde{B}_i^T \bar{P}_i \tilde{A}_i\right\},$$

where $P_i = P_i^T > 0$, $\bar{P}_i = \sum_{j=1}^{N_r} \gamma_{ij} P_j$, $i = 1, 2, \cdots, N_r$.

Definition 2. Let $\{P_i\}_{i=1}^{N_r}$ be a solution of the CARE (35). If the feedback law

$$K_i = -\left(R + \tilde{B}_i^T \bar{P}_i \tilde{B}_i\right)^{-1} \tilde{B}_i^T \bar{P}_i \tilde{A}_i, \; i = 1, 2, \cdots, N_r, \tag{36}$$

stabilize the system (10), we say that $\{P_i\}_{i=1}^{N_r}$ is the stabilizing solution of the CARE (35).

Definition 3. We say that $\{P_i^m\}_{i=1}^{N_r}$ is the maximal solution of the CARE (35), if it satisfies (35) and $P_i^m \geq P_i$, for arbitrary $\{P_i\}_{i=1}^{N_r}$, $i = 1, 2, \cdots, N_r$.

Theorem 2. Given the system (10), if there exists stabilizing solution for the CARE (35), the optimal control law that minimizes the cost function (33) is given by

$$\bar{u}(k) = K_{r(k)}\tilde{x}(k), \tag{37}$$

and the optimal cost is

$$J_{av}^{*}(\tilde{x}_0, r_0) = \sum\nolimits_{i=1}^{N_r} \gamma_i \mathrm{tr}\left(\tilde{D}_i W \tilde{D}_i^T \bar{P}_i\right), \tag{38}$$

where $r(k) \in \mathrm{N}_r$, $\bar{P}_i$ and K_i, respectively, are given by (35) and (36).

Proof. Let $\tilde{x}(k)$ be the sequence generated by the system (10) when we apply the control input (37). Because $\{P_i\}_{i=1}^{N_r}$ is a stabilizing solution of the CARE (35),we have that

$$\lim_{k\to\infty} = \mathrm{E}\left[\|\tilde{x}(k)\|^2\right] = 0$$

is satisfied, that is, the control input (37) could render the system (10) asymptotically stable. The probability distribution $\gamma(k) = [\gamma_1(k) \ \cdots \ \gamma_{N_r}(k)]^T$ converges to a unique invariant vector $\gamma = [\gamma_1 \ \cdots \ \gamma_{N_r}]^T$ as $k \to \infty$. Considering (14) in Theorem 1 and combining the quadratic cost function (33) and (34), and the optimal control input $\bar{u}(k)$, $k \in \mathrm{Z}$, we have that

$$\begin{aligned} J_{av}^{*}(\tilde{x}_0, r_0) &= \lim_{T\to\infty} \tfrac{1}{T}\left\{\inf_{\bar{u}(k)} \sum\nolimits_{k=0}^{T-1} \mathrm{E}\left[\sum\nolimits_{k=0}^{T-1} \tilde{x}^T(k) Q\tilde{x}(k) + \bar{u}^T(k) R\bar{u}(k)\right]\right\} \\ &= \sum\nolimits_{i=1}^{N_r}\left\{\lim_{T\to\infty}\tfrac{1}{T}\left[\sum\nolimits_{k=0}^{T-1}\gamma_i(k)\,\mathrm{tr}\left(\tilde{D}_{r(k)} W \tilde{D}_{r(k)}^T \bar{P}_i\right)\right]\right\} \\ &= \sum\nolimits_{i=1}^{N_r}\gamma_i \mathrm{tr}\left(\tilde{D}_i W \tilde{D}_i^T \bar{P}_i\right). \end{aligned}$$

The proof is completed. □

Although the design method of optimal controller in infinite horizon is presented by Theorem 2, it involves solving the CARE (35) to achieve the optimal feedback gains $\{K_i\}_{i=1}^{N_r}$. The following lemmas will help us in deriving the stabilizing solution of CARE (35) [33].

Lemma 1. Suppose that system (10) is stabilizable. Then there exists maximal solution $\{P_i^m\}_{i=1}^{N_r}$ satisfying (35) such that $P_i^m \geq P_i$ for all $\{P_i\}_{i=1}^{N_r} \in M$, if and only if there exists a solution $\left\{\hat{P}_i\right\}_{i=1}^{N_r}$ for the following convex programming problem. Moreover, the solution $\hat{P}_i = P_i^m$, $i = 1, 2, \cdots, N_r$.

$$\begin{cases} \max \mathrm{tr}\left(\sum_{i=1}^{N} P_i\right), \\ s.t. \ \ \text{(i)} \ \begin{bmatrix} -P_i + \tilde{A}_i^T \bar{P}_i \tilde{A}_i + Q & \tilde{A}_i^T \bar{P}_i \tilde{B}_i \\ \tilde{B}_i^T \bar{P}_i \tilde{A}_i & R + \tilde{B}_i^T \bar{P}_i \tilde{B}_i \end{bmatrix} \geq 0, \\ \qquad \text{(ii)} \ R + \tilde{B}_i^T \bar{P}_i \tilde{B}_i > 0. \end{cases} \tag{39}$$

Lemma 2. Suppose that there exists a stabilizing solution for the CARE (35), then the maximal solution $P_i^m \geq P_i$, $i = 1, 2, \cdots, N_r$, of the CARE (35), also is its stabilizing solution.

Lemma 1 and Lemma 2 provide a sufficient condition for the existence of stabilizing solutions for the CARE (35). By means of these lemmas, we can summarize the seeking steps of stabilizing solutions as follow.

Step1. Solve (39), and the maximal solution of the CARE (35) can be obtained.

Step2. Verify whether the maximal solution stabilizes the system (10). If it does, then the attained maximal solution is also the stabilizing solution of the CARE (35).

On the basis of preliminary results, we are now in a position to discuss the exponentially mean-square stability of the system (10). To state the subsequent results, we introduce the following definition [34].

Definition 4. A system in the form of (10) is said to be exponentially mean-square stable if for any initial condition (x_0, θ_0) with $\xi(k) = 0$, there exists constants $\alpha > 0$ and $0 < \beta < 1$ such that,

$$\mathrm{E}\left[\|\tilde{x}(k)\|^2 | \tilde{x}_0, r_0\right] \leq \alpha \beta^k \|\tilde{x}_0\|^2, \ \forall k \geq 0.$$

Theorem 3. The optimal control law (37) renders the system (10) to be exponentially mean-square stable, if the disturbance input $\xi(k)$ satisfies $\xi(k) = 0$.

Proof. It follows from (35) that $P_{r(k)} > 0$, $r(k) \in \mathrm{N}_r$. Let us consider the candidate Lyapunov function

$$V(k) = \tilde{x}^T(k) P_{r(k)}(k) \tilde{x}(k),$$

which is positive definite when $\tilde{x}(k) \neq 0$. We evaluate the first forward difference of $V(k)$ along the trajectories of the system (10) using the optimal control sequence $\bar{u}(k) = K_{r(k)}(k) \tilde{x}(k)$, then we have

$$\begin{aligned}
&\mathrm{E}[V(k+1) | \mathrm{I}_k] - V(k) \\
&= \mathrm{E}\left[\tilde{x}^T(k+1) P_{r(k+1)}(k+1) \tilde{x}(k+1) | \mathrm{I}_k\right] - \tilde{x}^T(k) P_{r(k)} \tilde{x}(k) \\
&= \mathrm{E}\left[\left(\tilde{A}_{r(k)}(k) \tilde{x}(k) + \tilde{B}_{r(k)} \bar{u}(k)\right)^T P_{r(k+1)}(k+1) \left(\tilde{A}_{r(k)}(k) \tilde{x}(k) + \tilde{B}_{r(k)} \bar{u}(k)\right) | \mathrm{I}_k\right] \\
&\quad - \tilde{x}^T(k) P_{r(k)}(k) \tilde{x}(k) \\
&= \tilde{x}^T(k) \left[\left(\tilde{A}_{r(k)}(k) \tilde{x}(k) + \tilde{B}_{r(k)} K_{r(k)}(k)\right)^T \bar{P}_{r(k)}(k+1) \right. \\
&\quad \left. \cdot \left(\tilde{A}_{r(k)}(k) \tilde{x}(k) + \tilde{B}_{r(k)} K_{r(k)}(k)\right) - P_{r(k)}(k)\right] \tilde{x}(k) \\
&= -\tilde{x}^T(k) \left(K_{r(k)}^T R K_{r(k)} + Q\right) \tilde{x}(k).
\end{aligned} \tag{40}$$

Let

$$L_{r(k)} = K_{r(k)}^T R K_{r(k)} + Q,$$

and hence

$$\begin{aligned} \mathrm{E}\left[V(k+1)\,|\mathrm{I}_k\right] &= V(k) - \tilde{x}^T(k)\, L_{r(k)}\tilde{x}(k) \\ &\le \left(1 - \lambda_{\min}\left(L_{r(k)}\right)\lambda_{\max}^{-1}\left(P_{r(k)}\right)\right) V(k) \\ &< \left(1 - \mu\sigma^{-1}\right) V(k) = \beta V(k), \end{aligned} \tag{41}$$

where $0 < \mu < \lambda_{\min}\left(F_{r(k)}\right)$, $\sigma > \lambda_{\max}\left(P_{r(k)}\right)$. It is obvious that $\mu < \sigma$, $0 < \beta < 1$. Similar to (41), one has

$$\mathrm{E}\left[V(k)\,|\mathrm{I}_{k-1}\right] < \beta V(k-1). \tag{42}$$

Based on (21), it can be obtained that

$$\begin{aligned} \mathrm{E}\left[V(k)\,|\mathrm{I}_{k-2}\right] &= \mathrm{E}\left[\mathrm{E}\left[V(k)\,|\mathrm{I}_{k-2}\right]|\mathrm{I}_{k-2}\right] \\ &< \beta\mathrm{E}\left[V(k-1)\,|\mathrm{I}_{k-2}\right] < \beta^2 V(k-2). \end{aligned} \tag{43}$$

By applying (43) recursively, we finally obtain

$$\mathrm{E}\left[V(k)\,|\mathrm{I}_0\right] < \beta^k V(0).$$

Therefore,

$$\mathrm{E}\left[\tilde{x}^T(k)\, P_{r(k)}\tilde{x}(k)\,|\tilde{x}_0, r_0\right] < \beta^k \tilde{x}_0^T P_{r_0}\tilde{x}_0,$$

which yields

$$\begin{aligned} \mathrm{E}\left[\tilde{x}^T(k)\,\tilde{x}(k)\,|\tilde{x}_0, r_0\right] &< \beta^k\lambda_{\min}^{-1}\left(P_{r(k)}\right)\tilde{x}_0^T P_{r_0}\tilde{x}_0 \\ &\le \lambda_{\max}\left(P_{r_0}\right)\lambda_{\min}^{-1}\left(P_{r(k)}\right)\beta^k\tilde{x}_0^T\tilde{x}_0 \\ &= \alpha\beta^k\tilde{x}_0^T\tilde{x}_0, \end{aligned}$$

where $\alpha = \lambda_{\max}\left(P_{r_0}\right)\lambda_{\min}^{-1}\left(P_{r(k)}\right)$. Therefore, the system (10) is exponentially mean-square stable according to Definition 1.
The proof is completed. □

4. NUMERICAL EXAMPLE

In what follows, we will show the effectiveness of the presented method by a numerical example. For this purpose, we consider a discrete-time plant in the form of (1) with the following parameters:

$$A = \begin{bmatrix} 0.1600 & -1.2005 \\ -1.1042 & -0.8890 \end{bmatrix},\ B = \begin{bmatrix} -1.5350 & 1.8918 \\ -1.2902 & -1.6869 \end{bmatrix},\ D = \begin{bmatrix} 0.1000 & 0.0000 \\ 0.0000 & 0.1000 \end{bmatrix},$$

$$E = \begin{bmatrix} 0.1000 & 0.0000 \\ 0.0000 & 0.1000 \end{bmatrix},\ F = \begin{bmatrix} 0.1000 & 0.0000 \\ 0.0000 & 0.1000 \end{bmatrix}.$$

The eigenvalues of A are 0.9007 and -1.6297, thus the above system is unstable. Our objective is to derive an optimal state feedback controller to render the system (10) exponentially mean-square stability and minimize the quadratic cost function (33). In the underlying networked systems, we consider the worst-case scenario, i. e., only one actuator and one sensor can access to medium at any time. Then, the actuators and sensors access sequences set are

$$\{M_\delta^1, M_\delta^2\} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\},\ \{M_\sigma^1, M_\sigma^2\} = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right\}.$$

In the simulation, we assume the following transition matrices for the Markov chains in (3) and (5), respectively,

$$\lambda = \begin{bmatrix} 0.7775 & 0.2225 \\ 0.2253 & 0.7747 \end{bmatrix},\ \pi = \begin{bmatrix} 0.6975 & 0.3025 \\ 0.3150 & 0.6850 \end{bmatrix}.$$

The stochastic communication sequences generated for the actuators and sensors are shown in Fig. 2 and Fig. 3, where $'1'$ and $'2'$ in the y-axis denote the working modes. We take the initial state x_0 as $x(0) = \begin{bmatrix} 7 & -6 \end{bmatrix}^T$, and the disturbance input terms in (1) and (2), respectively, are $\omega(\cdot) \sim N(0, 0.02I_{4\times 4})$ and $v(\cdot) \sim N(0, 0.02I_{2\times 2})$. We formulate the optimal control problem with the state weighting matrix $Q = I_4$, and the control weighting matrix $R = I_4$. The maximum solution P_i^m of the CARE (35) can be obtained by solving (39) in terms of linear matrix inequalities. Applying Theorem 2, the corresponding optimal control gains, which minimize the quadratic cost function (33), can be obtained as follow.

$$K_1 = \begin{bmatrix} -0.5893 & -1.0326 & -0.0063 & 0 & 0 & 0.1042 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$K_2 = \begin{bmatrix} -0.4992 & -0.9350 & 0 & 0.0014 & 0 & 0.4542 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$K_3 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ -0.6806 & -0.9800 & -0.0005 & 0 & -0.7571 & 0 \end{bmatrix},$$

$$K_4 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ -0.3314 & -0.4209 & 0 & 0.0039 & -0.2527 & 0 \end{bmatrix}.$$

It can be verified that the maximum solution P_i^m obtained by Lemma 1 could stabilize the system (10). Therefore, the maximum solution is also the stabilizing solution of the CARE (35). The corresponding state response of the closed-loop NCS is depicted in Fig. 4. The simulation result has confirmed that the designed optimal controller performs very well.

Control scheme	State index	Control index	The quadratic performance index
Our results	0.3294e+003	0.0136e+003	0.3430e+003
The method of [35]	0.9416e+003	0.1859e+003	1.1275e+003

Tab. 1. Comparison of quadratic performance index.

For comparison, the method from [35] are also simulated under the same conditions. The state trajectories are depicted in Fig. 5, and Tab. 1 lists the summary of performance index for the presented scheme in our paper as well as that of [35]. It can be found from Table 1 that the scheme in this paper not only guarantee the networked system to be exponentially mean-square stable, but also have better control performance than the method of [35].

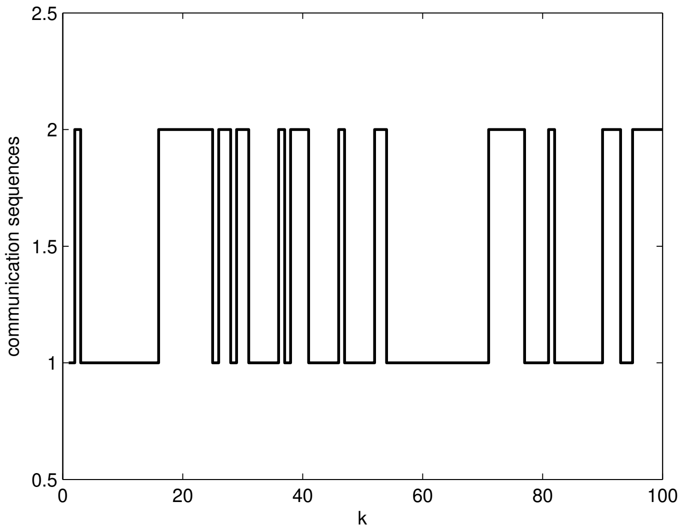

Fig. 2. Stochastic communication sequence of actuators.

5. CONCLUSIONS

In this paper, the finite and infinite horizon optimal control problems have been investigated for a class of networked systems with stochastic communication protocols. Owing to limited networked bandwidth, only a group of the measured and control signals are allowed to gain access to network medium according to stochastic access protocols in each sampling period. When sensors and actuators are not assigned to communicate with the controller, the state and control data of previous sampling period are supplied

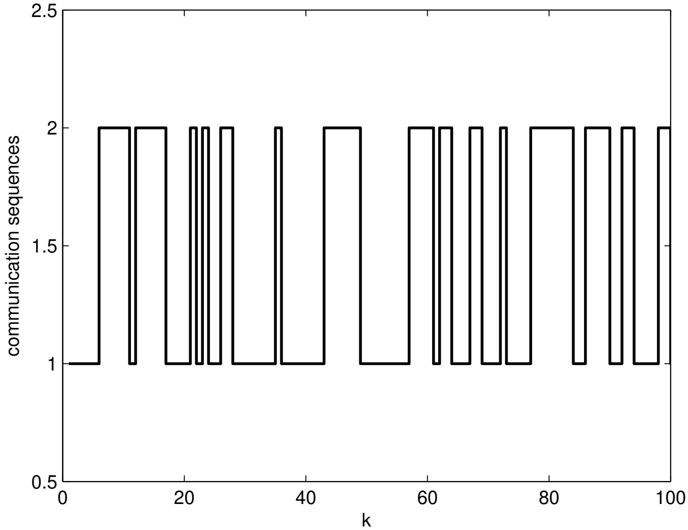

Fig. 3. Stochastic communication sequence of sensors.

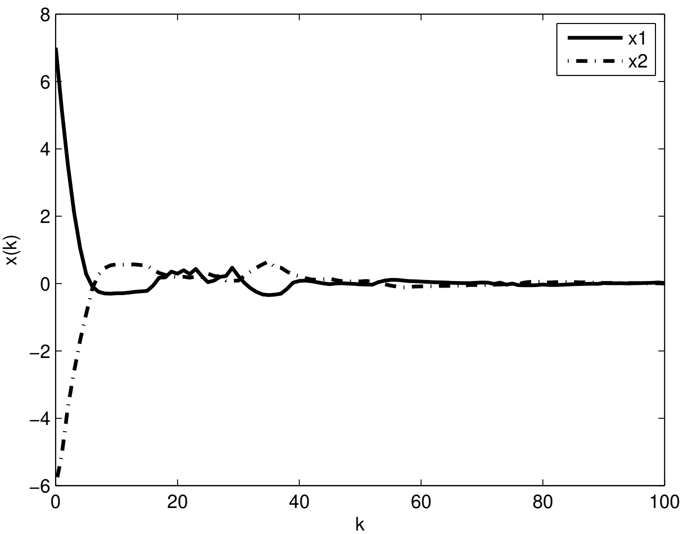

Fig. 4. State trajectories.

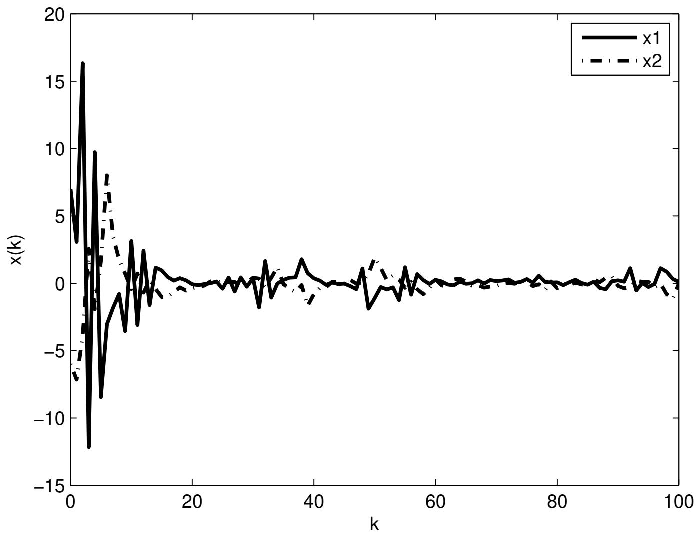

Fig. 5. State trajectories of [35].

by the zero-order holders. A discrete-time Markov chain with known transition probability matrix has been introduced to model the scheduling behaviors of the stochastic access protocol, and the NCSs are modeled as a Markov jump system based on lifting technique. The finite and infinite horizon optimal controllers are designed such that the exponentially mean-square stable performance of networked systems is guaranteed. Furthermore, the procedure for calculating optimal control gains has been given by solving the solutions of a family of CARE subject to the stochastic communication protocol. Finally, an illustrative example has been provided to demonstrate to verify the correctness and effectiveness of the proposed scheme.

This paper has investigated the case where the state is measured directly by the sensors. If the output of networked systems is available instead of state, we can still apply the present method by obtaining a state estimation first, and then accomplishing an optimal controller based on the estimated state. Another interesting issue worthy of investigation is the scenario where the medium-access status of the nodes is governed by random access protocols with simultaneous consideration of network induced delays, packet dropouts, and measurement quantization, etc.

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REFERENCES

[1] J. P. Hespanha, P. Naghshtabrizi, and Y. Xu: A survey of recent results in networked control systems. Proc. IEEE *95* (2007), 138–162. DOI:10.1109/JPROC.2006.887288

[2] X.-M. Zhang, Q.-L. Han, X. Ge, D. Ding, L. Ding, D. Yue, and C. Peng: Networked control systems: A survey of trends and techniques. IEEE/CAA J. Automat. Sinica (2019). DOI:10.1109/JAS.2019.1911651

[3] D. Ding, Q.-L. Han, Z. Wang, and X. Ge: A survey on model-based distributed control and filtering for industrial cyber-physical systems. IEEE Trans. Industr. Inform. *15* (2019), 2483–2499. DOI:10.1109/TII.2019.2905295

[4] L. Ding, Q.-L. Han, and X.-M. Zhang: Distributed secondary control for active power sharing and frequency regulation in islanded microgrids using an event-triggered communication mechanism. IEEE Trans. Industr. Inform. *15* (2019), 3910–3922. DOI:10.1109/TII.2018.2884494

[5] S. Shu and F. Lin: Deterministic networked control of discrete event systems with nondeterministic communication delays. IEEE Trans. Automat. Control *62* (2017), 190–205. DOI:10.1109/TAC.2016.2553959

[6] X. Ge, Q.-L. Han, X.-M. Zhang, L. Ding, and F. Yang: Distributed event-triggered estimation over sensor networks: A survey. IEEE Trans. Cybernet. (2019). DOI:10.1109/TCYB.2019.2917179

[7] H. Song, S.-C. Chen and, Y. Yam: Sliding mode control for discrete-time systems with markovian packet dropouts. IEEE Trans. Cybernet. *47* (2017), 3669–3679. DOI:10.1109/TCYB.2016.2577340

[8] K. Okano, M. Wakaiki, G. Yang, and J. P. Hespanha: Stabilization of networked control systems under clock offsets and quantization. IEEE Trans. Automat. Control *63* (2018), 1708–1723. DOI:10.1109/TAC.2017.2753938

[9] L. Ding, Q.-L. Han, X. Ge, and X.-M. Zhang: An overview of recent advances in event-triggered consensus of multi-agent systems. IEEE Trans. Cybernet. *48* (2018), 1110–1123. DOI:10.1109/TCYB.2017.2771560

[10] Q.-L. Han, Y. Liu, and F. Yang: Optimal communication network-based H_∞ quantized control with packet dropouts for a class of discrete-time neural networks with distributed time delay. IEEE Trans. Neural Networks Learning Syst. *27* (2016), 426–434. DOI:10.1109/TNNLS.2015.2411290

[11] B. Xue, N. Li, S. Li, and Q. Zhu: Moving horizon scheduling for networked control systems with communication constraints. IEEE Trans. Industr. Electron. *60* (2013), 3318–3327. DOI:10.1109/TIE.2012.2200219

[12] K. Liu, E. Fridman, and L. Hetel: Stability and L_2-gain analysis of networked control systems under round-robin scheduling: A time-delay approach. Systems Control Lett. *61* (2012), 666–675. DOI:10.1016/j.sysconle.2012.03.002

[13] L. Zhang and D. Hristu-Varsakelis: Communication and control co-design for networked control systems. Automatica *42* (2006), 953–958. DOI:10.1016/j.automatica.2006.01.022

[14] S. Longo, G. Herrmann and P. Barber: Robust scheduling of sampled-data networked control systems. IEEE Trans. Control Systems Technol. *20* (2012), 1613–1621. DOI:10.1109/TCST.2011.2170172

[15] D. Hristu-Varsakelis and L. Zhang: LQG control of networked control systems with access constraints and delays. Int. J. Control *81* (2008), 1266–1280. DOI:10.1080/00207170701697742

[16] W.-A. Zhang, L. Yu, and G. Feng: Optimal linear estimation for networked systems with communication constraints. Automatica *47* (2011), 1992–2000. DOI:10.1016/j.automatica.2011.05.020

[17] G. C. Walsh, H. Ye, and L. G. Bushnell: Stability analysis of networked control systems. IEEE Trans. Control Systems Technol. *10* (2002), 438–446. DOI:10.1109/87.998034

[18] G. Guo and H. Jin: A switching system approach to actuator assignment with limited channels. Int. J. Robust Nonlinear Control *20* (2010), 140–1426. DOI:10.1002/rnc.1522

[19] G. Guo: A switching system approach to sensor and actuator assignment for stabilization via limited multi-packet transmitting channels. Int. J. Control *84* (2011), 78–93. DOI:10.1080/00207179.2010.540715

[20] W.-A. Zhang, L. Yu, and G. Feng: Stabilization of linear discrete-time networked control systems via protocol and controller co-design. Int. J. Robust Nonlinear Control *25* (2015), 3072–3085. DOI:10.1002/rnc.3248

[21] L. Zou, Z. Wang, Q.-L. Han, and D. Zhou: Ultimate boundedness control for networked systems with try-once-discard protocol and uniform quantization effects. IEEE Trans. Automat. Control *62* (2017), 6582–6588. DOI:10.1109/TAC.2017.2713353

[22] G. Guo, Z. Lu, and Q.-L. Han: Control with Markov sensors/actuators assignment. IEEE Trans. Automat. Control *57* (2012), 1799–1804. DOI:10.1109/TAC.2011.2176393

[23] L. Zou, Z. Wang, Q.-L. Han, and D. Zhou: Recursive filtering for time-varying systems with random access protocol. IEEE Trans. Automat. Control *64* (2019), 720–727. DOI:10.1109/TAC.2018.2833154

[24] C. Zhu, G. Guo, B. Yang, and Z. Wang: Networked optimal control with random medium access protocol and packet dropouts. Math. Problems Engrg. *2015* (2015), 1–11. DOI:10.1155/2015/105416

[25] L. Zou, Z. Wang, and H. Gao: Observer-based H_∞ control of networked systems with stochastic communication protocol: The finite-horizon case. Automatica *63* (2016), 366–373. DOI:10.1016/ j.automatica.2015.10.045

[26] J. Zhang, C. Peng, M.-R. Fei, and Y.-C. Tian: Output feedback control of networked systems with a stochastic communication protocol. J. Franklin Inst. *354* (2017), 3838–3853. DOI:10.1016/j.jfranklin.2016.02.009

[27] D. Ding, Z. Wang, and Q.-L. Han: Neural-network-based output-feedback control with stochastic communication protocols. Automatica *106* (2019), 221–229. DOI:10.1016/j.automatica. 2019.04.025

[28] Y. Long, S. Liu, and L. Xie: Stochastic channel allocation for networked control systems. IEEE Control Systems Lett. *1* (2017), 176–181. DOI:10.1109/LCSYS.2017.2712100

[29] M. C. F. Donkers, W. P. M. H. Heemels, D. Bernardini, A. Bemporad, and V. Shneer: Stability analysis of stochastic networked control systems. Automatica *48* (2012), 917–925. DOI:10.1016/j.automatica.2012.02.029

[30] Y. Xu, H. Su, Y.-J. Pan, Z.-G. Wu, and W. Xu: Stability analysis of network control systems with round-robin scheduling and packet dropouts. J. Franklin Inst. *350* (2013), 2013–2027. DOI:10.1016/j.jfranklin.2013.05.024

[31] E. Yaz: Control of randomly varying systems with prescribed degree of stability. IEEE Trans. Automat. Control *33* (1988), 407–410. DOI:10.1109/9.192203

[32] K. D. Running, and N. C. Martins: Optimal preview control of Markovian jump linear systems. IEEE Trans. Automat. Control *54* (2009), 2260–2266. DOI:10.1109/TAC.2009.2026860

[33] O. L. V. Costa, and R. P. Marques: Maximal and stabilizing hermitian solutions for discrete-time coupled algebraic Riccati equations. Math. Control Signals Syst. *12* (1999), 167–195. DOI:10.1007/PL00009849

[34] F. Yang, Z. Wang, Y. S. Hung, and M. Gani: H_∞ control for networked systems with random communication delays. IEEE Trans. Automat. Control *51* (2006), 511–518. DOI:10.1109/TAC.2005.864207

[35] G. Guo and Z. Lu: Markov actuator assignment for networked control systems. Euroup. J. Control *18* (2012), 323–330. DOI:10.3166/EJC.18.323-330

Chaoqun Zhu, Corresponding author, College of Electrical and Information Engineering, Lanzhou University of Technology, LanZhou 730050. P. R. China.
e-mail: chaoqunzhu@yeah.net

Bin Yang, College of Electrical and Information Engineering, Lanzhou University of Technology, LanZhou 730050. P. R. China.
e-mail: yangbin0@yeah.net

Xiang Zhu, College of Electrical and Information Engineering, Lanzhou University of Technology, LanZhou 730050. P. R. China.
e-mail: zhuxiang@lut.cn