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CONSTRUCTION METHODS FOR UNI-NULLNORMS AND NULL-UNINORMS ON BOUNDED LATTICE

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In this paper, two construction methods have been proposed for uni-nullnorms on any bounded lattices. The difference between these two construction methods and the difference from the existing construction methods have been demonstrated and supported by an example. Moreover, the relationship between our construction methods and the existing construction methods for uninorms and nullnorms on bounded lattices are investigated. The charactertics of null-uninorms on bounded lattice L are given and a contruction method is presented.

Keywords: uni-nullnorm, bounded lattices, uni-nullnorms, null-norms, uninorms, null-norms, t-norms

Classification: 03E72, 03B52, 03G10

1. INTRODUCTION

Aggregation functions basicly are functions satisfying the monotonicity and boundary conditions. Because they have many applications (e. g., probability, statistics, artificial intelligence, operational research, economy and finance, pattern recognition and image processing, data fusion, multicriteria decision aid, automated reasoning, etc.), they are studied both in theory and in practice. Triangular norms and conorms (shortly t-norms and t-conorms, respectively), uninorms, nullnorms, uni-nullnorms and null-uninorms are special aggregation functions as well. In this respect, it is quite clear that there are many studies on these aggregation operators and that they have been working intensively([5, 6, 15, 21]).

Uninorms are special aggregation operators and they generalize t-norms and t-conorms since their neutral elements lie anywhere in the lattice. They were introduced by Yager and Rybalov [24], their structure was studied by Fodor et al. [9] and its properties and structures were investigated by many other researchers [3, 8, 11, 13].

Nullnorms, as well as uninorms, are the combination of t-norms and t-conorms. The story of these functions began with the works of Mas et al. [17] and Calvo et al. [4]. It is observed that nullnorms and t-operators are equivalent [18]. Some of their properties have been studied by researchers [7, 12, 14]

2-uninorms, which are comprised of two uninorms, are defined by Akella using the ordinal sum construction [1]. The uni-nullnorms, a special case of 2-uninorms, were

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presented and studied by Sun et al. [19, 20]. In [19], Sun et al. have investigated uni-nullnorms and null-uninorms on unit real interval $[0,1]$ and they pose the characteristics of uni-nullnorms and null-uninorms on $[0,1]$. In [20], they characterized uni-nullnorms with continuous Archimedean underlying t-norms and t-conorms. In [22], distributivity and conditional distributivity properties for uni-nullnorms are researched and in [23], they propose two construction methods for uni-nullnorms and a construction method for idempotent uni-nullnorms. In this study, we present two construction methods for uni-nullnorms, which we observed to be more general than the construction methods in [19], on any bounded lattices. We have demonstrated the relationship of our construction methods with the existing construction methods for uni-nullnorms, uninorms and nullnorms. Moreover, based on the facts $V(0,a)$ is either 0 or a for the uni-nullnorm F having a 2-neutral element $\{e,1\}_a$ on $[0,1]$ and $V(0,a) \leq a$ for the uni-nullnorm V having a 2-neutral element $\{e,1\}_a$ on any bounded lattice L, we have proved with an example that there may be an uni-nullnorm V having a 2-neutral element $\{e,1\}_a$ on a bounded lattice L such that $0 < V(0,a) < a$. The characteristic properties of null-uninorms on bounded lattices are investigated and an example is given to prove that there may be a null-uninorm F on a bounded lattice L having a 2-neutral element $\{0,e\}_a$ with a being an annihilator over $[0,e]$ such that $1 > F(1,a) > a$ (recall that for null-uninorm on chain $F(1,a) = a$ or $F(1,a) = 1$) The paper is organized as follows. In Section 2, we recall some basic information we need them for the construction of uni-nullnorms. In Section 3, we give two construction methods for uni-nullnorms and compare these methods with existing construction methods for uni-nullnorms, uninorms and nullnorms under some restrictions. The some concluding remarks are in Section 4.

2. NOTATIONS, DEFINITIONS AND A REVIEW OF PREVIOUS RESULTS

In this section, we recall some basic notions and results.

Definition 2.1. (Birkhoff [2]) A lattice $(L, \leq)$ is bounded lattice if L has the top element 1 and bottom element 0, that is, there exist two elements $1, 0 \in L$ such that $0 \leq x \leq 1$ for all $x \in L$.

Definition 2.2. (Birkhoff [2]) Let $(L, \leq, 0, 1)$ be a bounded lattice. The elements x and y are called comparable if $x \leq y$ or $y \leq x$. Otherwise, x and y are called incomparable and the notation $x||y$ is used for such elements.

Let $t \in L$. The set of elements which are incomparable with t is denoted by I_t^*, that is, $I_t^* = \{x \in L \ : \ x||t\}$.

Definition 2.3. (Birkhoff [2]) Let $(L, \leq, 0, 1)$ be a bounded lattice and $a, b \in L$ with $a \leq b$. The sublattice $[a,b]$ is defined as

$$[a,b] = \{x \in L| \quad a \leq x \leq b\}.$$

Similarly, $(a,b] = \{x \in L| \quad a < x \leq b\}$, $[a,b) = \{x \in L| \quad a \leq x < b\}$ and $(a,b) = \{x \in L| \quad a < x < b\}$ can be defined.

Definition 2.4. (Klement et al. [16]) Let $(L, \leq, 0, 1)$ be a bounded lattice. An operation T (S) on a bounded lattice L is called a triangular norm (triangular conorm) if it is commutative, associative, increasing with respect to the both variables and has a neutral element 1 (0).

Definition 2.5. (Karaçal et al. [11]) Let $(L, \leq, 0, 1)$ be a bounded lattice. A binary operation $U : L^2 \to L$ is called a uninorm if it has a neutral element $e \in L$ such that $U(e, x) = x$ for all $x \in L$ and satisfies the commutativity, associativity, monotonicity with respect to both variables.

Definition 2.6. (Ertuğrul [7]) Let $(L, \leq, 0, 1)$ be a bounded lattice. A binary operation $V : L^2 \to L$ is called a nullnorm if it has a zero element $a \in L$ such that $V(x, 0) = x$ for all $x \leq a$ and $V(x, 1) = x$ for all $x \geq a$, and satisfies the commutativity, associativity, monotonicity with respect to both variables.

Definition 2.7. (Akella [1]) Let $(L, \leq, 0, 1)$ be a bounded lattice and $e_1, a, e_2 \in L$ with $e_1 \leq a \leq e_2$. Let V be a binary operator on L. Then $\{e_1, e_2\}_a$ with $0 \leq e_1 \leq a \leq e_2 \leq 1$ is called a 2-neutral element of V if $V(e_1, x) = x$ for all $x \leq a$ and $V(e_2, x) = x$ for all $x \geq a$ where $0 < a < 1$ and $e_1 \in [0, a]$, $e_2 \in [a, 1]$.

Definition 2.8. (Akella [1]) Let $(L, \leq, 0, 1)$ be a bounded lattice and $e_1, a, e_2 \in L$ with $e_1 \leq a \leq e_2$. A commutative, associative, increasing binary operator on L having a 2-neutral element $\{e_1, e_2\}_a$ is called a 2-uninorm on L.

Definition 2.9. (Sun et al. [19]) Let $(L, \leq, 0, 1)$ be a bounded lattice and $e, a \in L$ with $e \leq a$. A uni-nullnorm is a 2-uninorm having a 2-neutral element $\{e, 1\}_a$ with a being an annihilator over $[e, 1]$.

Definition 2.10. (Sun et al. [19]) Let $(L, \leq, 0, 1)$ be a bounded lattice and $e, a \in L$ with $e \leq a$. A null-uninorm is a 2-uninorm having a 2-neutral element $\{0, e\}_a$ with a being an annihilator over $[0, e]$.

Definition 2.11. (Grabisch et al. [10]) Let U and V be two aggregation operators on L. U is called smaller than V if for any elements $x, y \in L$, $U(x, y) \leq V(x, y)$.

In this sense, the smallest and greatest t-norms on a bounded lattice L are given respectively as follows:

$$T_D(x, y) = \begin{cases} y & x = 1, \\ x & y = 1, \\ 0 & \text{otherwise} \end{cases} \quad \text{and } T_\wedge(x, y) = x \wedge y.$$

Similarly, the smallest t-conorm $S_\vee$ and greatest t-conorm S_D on a bounded lattice L,

$$S_D(x, y) = \begin{cases} y & x = 0, \\ x & y = 0, \\ 1 & \text{otherwise} \end{cases} \quad \text{and } S_\vee(x, y) = x \vee y$$

are given.

We denote by I_e for the set of elements which are incomparable with e but comparable with a, in which case such an element should be smaller than the element a and we denote by I_a for the set of elements which are in comparable with a but comparable with e,in which case such an element should be greater than the element e,that is, $I_e = \{x \in L \ : \ x||e \text{ and } x \nparallel a\}$ and $I_a = \{x \in L \ : \ x||a \text{ and } x \nparallel e\}$. By $I_{e,a}$, we denote the set of elements which are incomparable with e and a, that is, $I_{e,a} = \{x \in L \ : \ x||e \text{ and } x||a\}$.

Proposition 2.12. (Wang et al. [23]) Let $(L, \leq, 0, 1)$ be a bounded lattice, $e, a \in L$ with $e \leq a$. F be a uni-nullnorm with 2-neutral element $\{e, 1\}_a$.

(i) If $x \in I_e$, then $x < a$.

(ii) If $x \in I_a$, then $x > e$.

The following two propositions can be obtained from Definition 2.9 directly.

Proposition 2.13. (Wang et al. [23]) Let $(L, \leq, 0, 1)$ be a bounded lattice, $e, a \in L$ with $e \leq a$. F be a uni-nullnorm with 2-neutral element $\{e, 1\}_a$.

(i) $U^* = F|_{[0,a]^2} : [0, a]^2 \to [0, a]$ is a uninorm on $[0, a]$.

(ii) $V^* = F|_{[e,1]^2} : [e, 1]^2 \to [e, 1]$ is a nullnorm on $[e, 1]$.

(iii) $T_1^* = F|_{[0,e]^2} : [0, e]^2 \to [0, e]$ is a t-norm on $[0, e]$.

(iv) $S^* = F|_{[e,a]^2} : [e, a]^2 \to [e, a]$ is a t-conorm on $[e, a]$.

(v) $T_2^* = F|_{[a,1]^2} : [a, 1]^2 \to [a, 1]$ is a t-norm on $[a, 1]$.

Proposition 2.14. (Wang et al. [23]) Let $e, a \in L \setminus \{0, 1\}$ and F be a uni-nullnorm with the neutral element e and the zero element a on L. Then

(i) $F(x, y) \leq x$ for $(x, y) \in [0, a] \times [0, e] \cup I_e \times [0, e] \cup [a, 1] \times L$.

(ii) $F(x, y) \leq y$ for $(x, y) \in [0, e] \times [0, a] \cup [0, e] \times I_e \cup L \times [a, 1]$.

(iii) $F(x, y) \geq x$ for $(x, y) \in [0, a] \times [e, 1] \cup [0, a] \times I_a \cup I_e \times I_a \cup I_e \times [e, 1]$.

(iv) $F(x, y) \geq y$ for $(x, y) \in [e, 1] \times [0, a] \cup [e, 1] \times I_e \cup I_a \times I_e \cup I_a \times [0, a]$.

(v) $0 \leq F(x, y) \leq x \wedge y$ for $(x, y) \in [0, e]^2$.

(vi) $x \vee y \leq F(x, y) \leq a$ for $(x, y) \in [e, a]^2$.

(vii) $a \leq F(x, y) \leq x \wedge y$ for $(x, y) \in [a, 1]^2$.

(viii) $F(x, y) = a$ for $(x, y) \in [e, a] \times [a, 1] \cup [a, 1] \times [e, a]$.

(ix) $x \wedge y \leq F(x, y) \leq x \vee y$ for $(x, y) \in [0, e) \times (e, a] \cup (e, a] \times [0, e)$.

(x) $F(x, y) \geq (x \wedge a) \vee (y \wedge a)$ for $(x, y) \in [e, a] \times I_a \cup I_a \times [e, a] \cup I_a \times I_a$.

(xi) $F(x, y) \leq (x \vee a) \wedge (y \vee a)$ for $(x, y) \in [a, 1] \times I_a \cup I_a \times [a, 1] \cup I_a \times I_a$.

(xii) $F(x, y) \leq a$ for $(x, y) \in [0, a] \times L \cup L \times [0, a] \cup I_e \times L \cup L \times I_e$.

(xiii) $F(x, y) \geq a$ for $(x, y) \in [a, 1] \times [e, 1] \cup [e, 1] \times [a, 1] \cup I_a \times [a, 1] \cup [a, 1] \times I_a$.

In [23], the existence of uni-nullnorms on any bounded lattice is proved in the following Theorem 2.15.

Theorem 2.15. (Wang et al. [23]) Let L be a bounded lattice $e, a \in L \setminus \{0,1\}$ such that $0 \leq e \leq a \leq 1$ and $0 < a < 1$. If T_1 is a t-norm on $[0,e]^2$, S is a t-conorm on $[e,a]^2$ and T_2 is a t-norm on $[a,1]^2$, then the following functions $F_\wedge : L^2 \to L$ and $F_\vee : L^2 \to L$ are uni-nullnorms with a 2-neutral element $\{e,1\}_a$ on L.

$$F_\wedge(x,y) = \begin{cases} S(x,y) & (x,y) \in [e,a]^2, \\ T_2(x,y) & (x,y) \in [a,1]^2, \\ x \wedge y & (x,y) \in [0,e) \times [0,a) \cup [e,a) \times [0,e) \cup [0,e) \times I_e \cup I_e \times [0,e) \cup I_e \times I_e, \\ y & (x,y) \in [e,a) \times I_e, \\ x & (x,y) \in I_e \times [e,a), \\ a & \text{otherwise,} \end{cases} \tag{1}$$

and

$$F_\vee(x,y) = \begin{cases} T_1(x,y) & (x,y) \in [0,e]^2, \\ T_2(x,y) & (x,y) \in [a,1]^2, \\ x \vee y & (x,y) \in [0,a] \times (e,a] \cup (e,a] \times [0,e] \cup (e,a] \times I_e \cup I_e \times (e,a] \cup I_e \times I_e, \\ y & (x,y) \in [0,e) \times I_e, \\ x & (x,y) \in I_e \times [0,e), \\ a & \text{otherwise.} \end{cases} \tag{2}$$

3. CONSTRUCTION OF UNI-NULLNORMS ON BOUNDED LATTICES

In this section, we give two construction methods for uni-nullnorms on a bounded lattices.

Theorem 3.1. Let $(L, \leq, 0, 1)$ be a bounded lattice and $e, a \in L \setminus \{0,1\}$ with $e \leq a$. If T_1 is a t-norm on $[0,e]^2$, S is a t-conorm on $[e,a]^2$ such that $S(x,y) < a$ for all $x, y < a$ and T_2 is a t-norm on $[a,1]^2$, then the following functions $F_{T_1} : L^2 \to L$ and $F_S : L^2 \to L$ are uni-nullnorms with a 2-neutral element $\{e,1\}_a$ on L.

$$F_{T_1}(x,y) = \begin{cases} T_1(x,y) & (x,y) \in [0,e)^2, \\ S(x,y) & (x,y) \in [e,a)^2, \\ T_2(x,y) & (x,y) \in [a,1]^2, \\ T_1(x \wedge e, y \wedge e) & (x,y) \in [0,e) \times I_e \cup I_e \times [0,e) \cup I_e \times I_e, \\ x \wedge y & (x,y) \in [0,e) \times [e,a) \cup [e,a) \times [0,e), \\ y & (x,y) \in [e,a) \times I_e, \\ x & (x,y) \in I_e \times [e,a), \\ a & \text{otherwise,} \end{cases} \tag{3}$$

and

$$F_S(x,y)=\begin{cases} T_1(x,y) & (x,y)\in[0,e]^2,\\ S(x,y) & (x,y)\in[e,a)^2,\\ T_2(x,y) & (x,y)\in[a,1]^2,\\ S(x\vee e,y\vee e) & (x,y)\in[e,a)\times I_e\cup I_e\times[e,a)\cup I_e\times I_e,\\ x\vee y & (x,y)\in[0,e]\times(e,a)\cup(e,a)\times[0,e],\\ y & (x,y)\in[0,e)\times I_e,\\ x & (x,y)\in I_e\times[0,e),\\ a & \text{otherwise.}\end{cases} \tag{4}$$

P r o o f.

It is obvious that F_{T_1} satisfies the commutativity, $F_{T_1}(x,e)=x$ for all $x\in[0,a]$ and $F_{T_1}(x,a)=a$ for all $x\in[e,1]$. Furthermore, $F_{T_1}(x,a)=a$ for all $x\in L$.

(i) *Monotonicity*: Let us show that for every elements $x,y\in L$ with $x\le y$, $F_{T_1}(x,z)\le F_{T_1}(y,z)$ for all $z\in L$. If the elements $x,y\in L$ are simultaneously elements of $[0,e]$ or $[e,a]$ or $[a,1]$ or I_e or I_a or $I_{e,a}$, the proof is clear. Similarly, if $z\in I_a$ or $z\in I_{e,a}$, the inequality holds. The proof is split into all the other remain possible cases.

1. Let $x\in[0,e)$.

1.1. $y\in[e,a)$,

1.1.1. If $z\in[0,e)$, then $F_{T_1}(x,z)=T_1(x,z)\le x\wedge z\le y\wedge z=F_{T_1}(y,z)$.

1.1.2. If $z\in[e,a)$, then $F_{T_1}(x,z)=x\wedge z\le z\le y\vee z\le S(y,z)=F_{T_1}(y,z)$.

1.1.3. If $z\in[a,1]$, then $F_{T_1}(x,z)=a=F_{T_1}(y,z)$.

1.1.4. If $z\in I_e$, then $F_{T_1}(x,z)=T_1(x\wedge e,z\wedge e)\le z\wedge e\le z=F_{T_1}(y,z)$.

1.2. $y\in[a,1]$,

1.2.1. If $z\in[0,e)$, then $F_{T_1}(x,z)=T_1(x,z)\le a=F_{T_1}(y,z)$.

1.2.2. If $z\in[e,a)$, then $F_{T_1}(x,z)=x\wedge z\le z\le a=F_{T_1}(y,z)$.

1.2.3. If $z\in[a,1]$, then $F_{T_1}(x,z)=a\le T_2(x,y)=F_{T_1}(y,z)$.

1.2.4. If $z\in I_e$, then $F_{T_1}(x,z)=T_1(x\wedge e,z\wedge e)\le z\wedge e\le a=F_{T_1}(y,z)$.

1.3. $y\in I_e$,

1.3.1. If $z\in[0,e)$, then $F_{T_1}(x,z)=T_1(x,z)=T_1(x\wedge e,z\wedge e)\le T_1(y\wedge e,z\wedge e)=F_{T_1}(y,z)$.

1.3.2. If $z\in[e,a)$, then $F_{T_1}(x,z)=x\wedge z\le x=F_{T_1}(y,z)$.

1.3.3. If $z\in[a,1]$, then $F_{T_1}(x,z)=a=F_{T_1}(y,z)$.

1.3.4. If $z\in I_e$, then $F_{T_1}(x,z)=T_1(x\wedge e,z\wedge e)\le T_1(y\wedge e,z\wedge e)=F_{T_1}(y,z)$.

1.4. $y\in I_a$ or $y\in I_e\cap I_a$,

In this case, $F_{T_1}(y,z)=a$. Therefore, it must be showed that $F_{T_1}(x,z)\le a$.

1.4.1. If $z\in[0,e)$, then $F_{T_1}(x,z)=T_1(x,z)\le a$.

1.4.2. If $z\in[e,a)$, then $F_{T_1}(x,z)=x\wedge z\le a$.

1.4.3. If $z\in[a,1]$, then $F_{T_1}(x,z)=a\le a$.

1.4.4. If $z \in I_e$, then $F_{T_1}(x,z) = T_1(x \wedge e, z \wedge e) \leq a$.

2. Let $x \in [e,a)$.

2.1. $y \in [a,1]$,

2.1.1. If $z \in [0,e)$, then $F_{T_1}(x,z) = x \wedge z \leq a = F_{T_1}(y,z)$.

2.1.2. If $z \in [e,a)$, then $F_{T_1}(x,z) = S(x,z) \leq a = F_{T_1}(y,z)$.

2.1.3. If $z \in [a,1]$, then $F_{T_1}(x,z) = a \leq T_2(x,y) = F_{T_1}(y,z)$.

2.1.4. If $z \in I_e$, then $F_{T_1}(x,z) = z \leq a = F_{T_1}(y,z)$.

2.2. $y \in I_a$,

In this case, $F_{T_1}(y,z) = a$. Therefore, it must be showed that $F_{T_1}(x,z) \leq a$.

2.2.1. If $z \in [0,e)$, then $F_{T_1}(x,z) = x \wedge z \leq a$.

2.2.2. If $z \in [e,a)$, then $F_{T_1}(x,z) = S(x,z) \leq a$.

2.2.3. If $z \in [a,1]$, then $F_{T_1}(x,z) = a \leq a$.

2.2.4. If $z \in I_e$, then $F_{T_1}(x,z) = z \leq a$.

3. $x \in I_e$,

3.1. $y \in [e,a)$,

3.1.1. If $z \in [0,e)$, then $F_{T_1}(x,z) = T_1(x \wedge e, z \wedge e) = T_1(x \wedge e, z) \leq x \wedge z \wedge e = x \wedge z \leq y \wedge z = F_{T_1}(y,z)$.

3.1.2. If $z \in [e,a)$, then $F_{T_1}(x,z) = x \leq y \leq S(y,z) = F_{T_1}(y,z)$.

3.1.3. If $z \in [a,1]$, then $F_{T_1}(x,z) = a = F_{T_1}(y,z)$.

3.1.4. If $z \in I_e$, then $F_{T_1}(x,z) = T_1(x \wedge e, z \wedge e) \leq x \wedge z \wedge e \leq z = F_{T_1}(y,z)$.

3.2. $y \in [a,1]$,

3.2.1. If $z \in [0,e)$, then $F_{T_1}(x,z) = T_1(x \wedge e, z \wedge e) = T_1(x \wedge e, z) \leq x \wedge z \wedge e \leq a = F_{T_1}(y,z)$.

3.2.2. If $z \in [e,a)$, then $F_{T_1}(x,z) = x \leq a = F_{T_1}(y,z)$.

3.2.3. If $z \in [a,1]$, then $F_{T_1}(x,z) = a \leq T_2(y,z) = F_{T_1}(y,z)$.

3.2.4. If $z \in I_e$, then $F_{T_1}(x,z) = T_1(x \wedge e, z \wedge e) \leq x \wedge z \wedge e \leq e \leq a = F_{T_1}(y,z)$.

3.3. $y \in I_a$ or $y \in I_e \cap I_a$,

In this case, $F_{T_1}(y,z) = a$. Therefore, it must be showed that $F_{T_1}(x,z) \leq a$.

3.3.1. If $z \in [0,e)$, then $F_{T_1}(x,z) = T_1(x \wedge e, z \wedge e) \leq a$.

3.3.2. If $z \in [e,a)$, then $F_{T_1}(x,z) = x \leq a$.

3.3.3. If $z \in [a,1]$, then $F_{T_1}(x,z) = a$.

3.3.4. If $z \in I_e$, then $F_{T_1}(x,z) = T_1(x \wedge e, z \wedge e) \leq a$.

4. $x \in I_a$.

4.1. $y \in (a,1]$,

4.1.1. If $z \in [0,a] \cup I_e$, then $F_{T_1}(x,z) = a = F_{T_1}(y,z)$.

5. $x \in I_{e,a}$.

5.1. $y \in (a,1]$,

5.1.1. If $z \in [0,a] \cup I_e$, then $F_{T_1}(x,z) = a = F_{T_1}(y,z)$.

5.1.2. If $z \in (a,1]$, then $F_{T_1}(x,z) = a \leq T_2(y,z) = F_{T_1}(y,z)$.

(ii) *Associativity*: We present that $F_{T_1}(x, F_{T_1}(y,z)) = F_{T_1}(F_{T_1}(x,y),z)$ for any $x,y,z \in L$. If one of the elements x,y and z is in I_a or $I_{e,a}$, it is clear that the equality is always satisfied. The proof is split into all remain possible cases by considering the situation of the elements x,y,z.

1. Let $x \in [0,e)$.

1.1. $y \in [0,e)$,

1.1.1. If $z \in [0,e)$, then $F_{T_1}(x,F_{T_1}(y,z)) = F_{T_1}(x,T_1(y,z)) = T_1(x,T_1(y,z)) = T_1(F_{T_1}(x,y),z) = F_{T_1}(F_{T_1}(x,y),z)$.

1.1.2. If $z \in [e,a)$, then $F_{T_1}(x,F_{T_1}(y,z)) = F_{T_1}(x,y \wedge z) = F_{T_1}(x,y) = T_1(x,y) = T_1(x,y) \wedge z = F_{T_1}(T_1(x,y),z) = F_{T_1}(F_{T_1}(x,y),z)$.

1.1.3. If $z \in [a,1]$, then $F_{T_1}(x,F_{T_1}(y,z)) = F_{T_1}(x,a) = a = F_{T_1}(T_1(x,y),z) = F_{T_1}(F_{T_1}(x,y),z)$.

1.1.4. If $z \in I_e$, then $F_{T_1}(x,F_{T_1}(y,z)) = F_{T_1}(x,T_1(y \wedge e, z \wedge e)) = F_{T_1}(x,T_1(y,z \wedge e)) = T_1(x,T_1(y,z \wedge e)) = T_1(T_1(x,y),z \wedge e) = T_1(T_1(x,y) \wedge e, z \wedge e) = F_{T_1}(T_1(x,y),z) = F_{T_1}(F_{T_1}(x,y),z)$.

1.2. $y \in [e,a)$,

1.2.1. If $z \in [0,e)$, then $F_{T_1}(x,F_{T_1}(y,z)) = F_{T_1}(x,y \wedge z) = F_{T_1}(x,z) = T_1(x,z) = F_{T_1}(x,z) = F_{T_1}(x \wedge y,z) = F_{T_1}(F_{T_1}(x,y),z)$

1.2.2. If $z \in [e,a)$, then $F_{T_1}(x,F_{T_1}(y,z)) = F_{T_1}(x,S(y,z)) = x \wedge S(y,z) = x = x \wedge z = F_{T_1}(x,z) = F_{T_1}(x \wedge y,z) = F_{T_1}(F_{T_1}(x,y),z)$.

1.2.3. If $z \in [a,1]$, then $F_{T_1}(x,F_{T_1}(y,z)) = F_{T_1}(x,a) = a = F_{T_1}(x,z) = F_{T_1}(x \wedge y,z) = F_{T_1}(F_{T_1}(x,y),z)$.

1.2.4. If $z \in I_e$, then $F_{T_1}(x,F_{T_1}(y,z)) = F_{T_1}(x,z) = T_1(x \wedge e, z \wedge e) = F_{T_1}(x,z) = F_{T_1}(x \wedge y,z) = F_{T_1}(F_{T_1}(x,y),z)$.

1.3. $y \in [a,1]$,

1.3.1. If $z \in [0,a] \cup I_e$, then $F_{T_1}(x,F_{T_1}(y,z)) = F_{T_1}(x,a) = a = F_{T_1}(a,z) = F_{T_1}(F_{T_1}(x,y),z)$.

1.3.2. If $z \in [a,1]$, then $F_{T_1}(x,F_{T_1}(y,z)) = F_{T_1}(x,T_2(y,z)) = a = F_{T_1}(a,z) = F_{T_1}(F_{T_1}(x,y),z)$.

1.4. $y \in I_e$,

1.4.1. If $z \in [0,e)$, then $F_{T_1}(x,F_{T_1}(y,z)) = F_{T_1}(x,T_1(y \wedge e, z \wedge e)) = T_1(x,T_1(y \wedge e, z \wedge e)) = T_1(T_1(x,y \wedge e),z \wedge e) = F_{T_1}(T_1(x,y \wedge e),z) = F_{T_1}(T_1(x \wedge e, y \wedge e),z) = F_{T_1}(F_{T_1}(x,y),z)$.

1.4.2. If $z \in [e,a)$, then $F_{T_1}(x,F_{T_1}(y,z)) = F_{T_1}(x,y) = T_1(x \wedge e, y \wedge e) = T_1(x,y \wedge e) = T_1(x,y \wedge e) \wedge z = F_{T_1}(T_1(x,y \wedge e),z) = F_{T_1}(T_1(x \wedge e,y \wedge e),z) = F_{T_1}(F_{T_1}(x,y),z)$.

1.4.3. If $z \in [a,1]$, then $F_{T_1}(x,F_{T_1}(y,z)) = F_{T_1}(x,a) = a = F_{T_1}(T_1(x,y \wedge e),z) = F_{T_1}(T_1(x \wedge e,y \wedge e),z) = F_{T_1}(F_{T_1}(x,y),z)$.

1.4.4. If $z \in I_e$, then $F_{T_1}(x, F_{T_1}(y,z)) = F_{T_1}(x, T_1(y \wedge e, z \wedge e)) = T_1(x, T_1(y \wedge e, z \wedge e)) = T_1(T_1(x, y \wedge e) \wedge e, z \wedge e) = F_{T_1}(T_1(x, y \wedge e), z) = F_{T_1}(T_1(x \wedge e, y \wedge e), z) = F_{T_1}(F_{T_1}(x,y),z)$.

2. $x \in [e,a)$.

2.1. $y \in [0,e)$,

2.1.1. If $z \in [0,e)$, then $F_{T_1}(x, F_{T_1}(y,z)) = F_{T_1}(x, T_1(y,z)) = x \wedge T_1(y,z) = T_1(y,z) = F_{T_1}(y,z) = F_{T_1}(x \wedge y, z) = F_{T_1}(F_{T_1}(x,y),z)$.

2.1.2. If $z \in [e,a)$, then $F_{T_1}(x, F_{T_1}(y,z)) = F_{T_1}(x, y \wedge z) = F_{T_1}(x,y) = x \wedge y = y = y \wedge z = F_{T_1}(y,z) = F_{T_1}(x \wedge y, z) = F_{T_1}(F_{T_1}(x,y),z)$.

2.1.3. If $z \in [a,1]$, then $F_{T_1}(x, F_{T_1}(y,z)) = F_{T_1}(x,a) = a = F_{T_1}(y,z) = F_{T_1}(x \wedge y, z) = F_{T_1}(F_{T_1}(x,y),z)$.

2.1.4. If $z \in I_e$, then $F_{T_1}(x, F_{T_1}(y,z)) = F_{T_1}(x, T_1(y \wedge e, z \wedge e)) = x \wedge T_1(y \wedge e, z \wedge e) = T_1(y \wedge e, z \wedge e) = F_{T_1}(y,z) = F_{T_1}(x \wedge y, z) = F_{T_1}(F_{T_1}(x,y),z)$.

2.2. $y \in [e,a)$,

2.2.1. If $z \in [0,e)$, then $F_{T_1}(x, F_{T_1}(y,z)) = F_{T_1}(x, y \wedge z) = F_{T_1}(x,z) = z = F_{T_1}(S(x,y),z) = F_{T_1}(F_{T_1}(x,y),z)$.

2.2.2. If $z \in [e,a)$, then $F_{T_1}(x, F_{T_1}(y,z)) = F_{T_1}(x, S(y,z)) = S(x, S(y,z)) = S(S(x,y),z) = F_{T_1}(S(x,y),z) = F_{T_1}(F_{T_1}(x,y),z)$.

2.2.3. If $z \in [a,1]$, then $F_{T_1}(x, F_{T_1}(y,z)) = F_{T_1}(x,a) = a = F_{T_1}(S(x,y),z) = F_{T_1}(F_{T_1}(x,y),z)$.

2.2.4. If $z \in I_e$, then $F_{T_1}(x, F_{T_1}(y,z)) = F_{T_1}(x,z) = z = F_{T_1}(S(x,y),z) = F_{T_1}(F_{T_1}(x,y),z)$.

2.3. $y \in [a,1]$,

2.3.1. If $z \in [0,a) \cup I_e$, then $F_{T_1}(x, F_{T_1}(y,z)) = F_{T_1}(x,a) = a = F_{T_1}(a,z) = F_{T_1}(F_{T_1}(x,y),z)$.

2.3.2. If $z \in [a,1]$, then $F_{T_1}(x, F_{T_1}(y,z)) = F_{T_1}(x, T_2(y,z)) = a = F_{T_1}(a,z) = F_{T_1}(F_{T_1}(x,y),z)$.

2.4. $y \in I_e$,

2.4.1. If $z \in [0,e)$, then $F_{T_1}(x, F_{T_1}(y,z)) = F_{T_1}(x, T_1(y \wedge e, z)) = x \wedge T_1(y \wedge e, z) = T_1(y \wedge e, z) = F_{T_1}(y,z) = F_{T_1}(F_{T_1}(x,y),z)$.

2.4.2. If $z \in [e,a)$, then $F_{T_1}(x, F_{T_1}(y,z)) = F_{T_1}(x,y) = y = F_{T_1}(y,z) = F_{T_1}(F_{T_1}(x,y),z)$.

2.4.3. If $z \in [a,1]$, then $F_{T_1}(x, F_{T_1}(y,z)) = a = F_{T_1}(y,z) = F_{T_1}(F_{T_1}(x,y),z)$.

2.4.4. If $z \in I_e$, then $F_{T_1}(x, F_{T_1}(y,z)) = F_{T_1}(x, T_1(y \wedge e, z \wedge e)) = T_1(y \wedge e, z \wedge e) = F_{T_1}(y,z) = F_{T_1}(F_{T_1}(x,y),z)$.

3. $x \in [a,1]$.

3.1. $y \in [0,e)$,

3.1.1. If $z \in [0,a) \cup I_e$, then $F_{T_1}(x, F_{T_1}(y,z)) = F_{T_1}(x, T_1(y,z)) = a = F_{T_1}(a,z) = F_{T_1}(F_{T_1}(x,y),z)$.

3.1.2. If $z \in [a,1]$, then $F_{T_1}(x, F_{T_1}(y,z)) = F_{T_1}(x,a) = a = F_{T_1}(a,z) = F_{T_1}(F_{T_1}(x,y),z)$.

3.2. $y \in [e,a)$,

3.2.1. If $z \in [0,e)$, then $F_{T_1}(x, F_{T_1}(y,z)) = F_{T_1}(x, y \wedge z) = F_{T_1}(x,z) = a = F_{T_1}(a,z) = F_{T_1}(F_{T_1}(x,y),z)$.

3.2.2. If $z \in [e,a)$, then $F_{T_1}(x, F_{T_1}(y,z)) = F_{T_1}(x, S(y,z)) = a = F_{T_1}(a,z) = F_{T_1}(F_{T_1}(x,y),z)$.

3.2.3. If $z \in [a,1]$, then $F_{T_1}(x, F_{T_1}(y,z)) = F_{T_1}(x,a) = a = F_{T_1}(a,z) = F_{T_1}(F_{T_1}(x,y),z)$.

3.2.4. If $z \in I_e$, then $F_{T_1}(x, F_{T_1}(y,z)) = F_{T_1}(x,z) = a = F_{T_1}(a,z) = F_{T_1}(F_{T_1}(x,y),z)$.

3.3. $y \in [a,1]$,

3.3.1. If $z \in [0,a) \cup I_e$, then $F_{T_1}(x, F_{T_1}(y,z)) = F_{T_1}(x,a) = a = F_{T_1}(T_2(x,y),z) = F_{T_1}(F_{T_1}(x,y),z)$.

3.3.2. If $z \in [a,1]$, then $F_{T_1}(x, F_{T_1}(y,z)) = F_{T_1}(x, T_2(y,z)) = T_2(x, T_2(y,z)) = T_2(T_2(x,y),z) = F_{T_1}(T_2(x,y),z) = F_{T_1}(F_{T_1}(x,y),z)$.

3.4. $y \in I_e$,

3.4.1. If $z \in [0,e) \cup I_e$, then $F_{T_1}(x, F_{T_1}(y,z)) = F_{T_1}(x, T_1(y,z)) = a = F_{T_1}(a,z) = F_{T_1}(F_{T_1}(x,y),z)$.

3.4.2. If $z \in [e,a)$, then $F_{T_1}(x, F_{T_1}(y,z)) = F_{T_1}(x,y) = a = F_{T_1}(a,z) = F_{T_1}(F_{T_1}(x,y),z)$.

3.4.3. If $z \in [a,1]$, then $F_{T_1}(x, F_{T_1}(y,z)) = F_{T_1}(x,a) = a = F_{T_1}(a,z) = F_{T_1}(F_{T_1}(x,y),z)$.

4. $x \in I_e$.

4.1. $y \in [0,e)$,

4.1.1. If $z \in [0,e)$, then $F_{T_1}(x, F_{T_1}(y,z)) = F_{T_1}(x, T_1(y,z)) = T_1(x \wedge e, T_1(y,z) \wedge e) = T_1(x \wedge e, T_1(y,z)) = T_1(T_1(x \wedge e, y), z) = F_{T_1}(T_1(x \wedge e, y), z) = F_{T_1}(T_1(x \wedge e, y \wedge e), z) = F_{T_1}(F_{T_1}(x,y),z)$.

4.1.2. If $z \in [e,a)$, then $F_{T_1}(x, F_{T_1}(y,z)) = F_{T_1}(x, y \wedge z) = F_{T_1}(x,y) = T_1(x \wedge e, y \wedge e) = T_1(x \wedge e, y) = T_1(x \wedge e, y) \wedge z = F_{T_1}(T_1(x \wedge e, y \wedge e), z) = F_{T_1}(F_{T_1}(x,y),z)$.

4.1.3. If $z \in [a,1]$, then $F_{T_1}(x, F_{T_1}(y,z)) = F_{T_1}(x,a) = a = F_{T_1}(T_1(x \wedge e, y), z) = F_{T_1}(T_1(x \wedge e, y \wedge e), z) = F_{T_1}(F_{T_1}(x,y),z)$.

4.1.4. If $z \in I_e$, then $F_{T_1}(x, F_{T_1}(y,z)) = F_{T_1}(x, T_1(y \wedge e, z \wedge e)) = F_{T_1}(x, T_1(y, z \wedge e)) = T_1(x \wedge e, T_1(y, z \wedge e)) = T_1(T_1(x \wedge e, y), z \wedge e) = F_{T_1}(T_1(x \wedge e, y), z) = F_{T_1}(T_1(x \wedge e, y \wedge e), z) = F_{T_1}(F_{T_1}(x,y),z)$.

4.2. $y \in [e,a)$,

4.2.1. If $z \in [0,e) \cup I_e$, then $F_{T_1}(x, F_{T_1}(y,z)) = F_{T_1}(x,z) = F_{T_1}(x,z) = F_{T_1}(F_{T_1}(x,y),z)$.

4.2.2. If $z \in [e,a)$, then $F_{T_1}(x, F_{T_1}(y,z)) = F_{T_1}(x, S(y,z)) = x = F_{T_1}(x,z) = F_{T_1}(F_{T_1}(x,y),z)$.

4.2.3. If $z \in [a,1]$, then $F_{T_1}(x, F_{T_1}(y,z)) = F_{T_1}(x,a) = a = F_{T_1}(x,z) = F_{T_1}(F_{T_1}(x,y),z)$.

4.3. $y \in [a,1]$,

4.3.1. If $z \in L$, then $F_{T_1}(x, F_{T_1}(y,z)) = a = F_{T_1}(a,z) = F_{T_1}(F_{T_1}(x,y),z)$.

4.4. $y \in I_e$,

4.4.1. If $z \in [0,e)$, then $F_{T_1}(x, F_{T_1}(y,z)) = F_{T_1}(x, T_1(y \wedge e, z \wedge e)) = F_{T_1}(x, T_1(y \wedge e, z)) = T_1(x \wedge e, T_1(y \wedge e, z)) = T_1(T_1(x \wedge e, y \wedge e), z) = F_{T_1}(T_1(x \wedge e, y \wedge e), z) = F_{T_1}(F_{T_1}(x,y),z)$.

4.4.2. If $z \in [e, a)$, then $F_{T_1}(x, F_{T_1}(y, z)) = F_{T_1}(x, y) = T_1(x \wedge e, y \wedge e)) = F_{T_1}(T_1(x \wedge e, y \wedge e), z) = F_{T_1}(F_{T_1}(x, y), z)$.

4.4.3. If $z \in [a, 1]$, then $F_{T_1}(x, F_{T_1}(y, z)) = F_{T_1}(x, a) = a = F_{T_1}(T_1(x \wedge e, y \wedge e), z) = F_{T_1}(F_{T_1}(x, y), z)$.

4.4.4. If $z \in I_e$, then $F_{T_1}(x, F_{T_1}(y, z)) = F_{T_1}(x, T_1(y \wedge e, z \wedge e)) = T_1(x \wedge e, T_1(y \wedge e, z \wedge e)) = F_{T_1}(T_1(x \wedge e, y \wedge e), z) = F_{T_1}(F_{T_1}(x, y), z)$.

Therefore, we obtain that F_{T_1} is a uni-nullnorm on L with the neutral element e and the zero element a. Similarly, we can prove that F_S is a uni-nullnorm on L with the neutral element e and the zero element a. □

Remark 3.2. Let $(L, 0, 1, \leq)$ be a bounded lattice, $e, a \in L \setminus \{0, 1\}$, T_1 be a t-norm on $[0, e]$, S be a t-conorm on $[e, a]$ and T_2 be a t-norm on $[a, 1]$. Consider the uni-nullnorms $F_\wedge$, $F_\vee$, F_{T_1} and F_S on bounded lattice L considering Theorems 2.15 and 3.1.

(i) If $T_1 = \wedge$, then $F_\wedge = F_{T_1}$. Although, the uni-nullnorm F_{T_1} given in Theorem 3.1 is much more general than the uni-nullnorm $F_\wedge$ in Theorem 2.15 when $T_1 \neq \wedge$ in formula (3) (see Example 3.4).

(ii) If $S = \vee$, then $F_\vee = F_S$. Although, the uni-nullnorm F_S given in Theorem 3.1 is much more general than the uni-nullnorm $F_\vee$ in Theorem 2.15 when $S \neq \vee$ in formula (3) (see Example 3.4).

(iii) If we take $e = 0$ in formulas (1), (2), (3) and (4), then the formula (3) is coincident to formula (1) and produce a nullnorm on bounded lattice L and formula (4) is coincided to (2) and produce a nullnorm too on bounded lattice L.

(iv) If $a = 1$, then the formula (1) is coincided to construction method given in Proposition 1 in [3] and the formula (2) is coincided to Proposition 2 in [3].

Remark 3.3. From Theorem 3.1, the structure of two construction methods of a uni-nullnorm on the bounded lattice L can be visualized, see Figures 1 and 2.

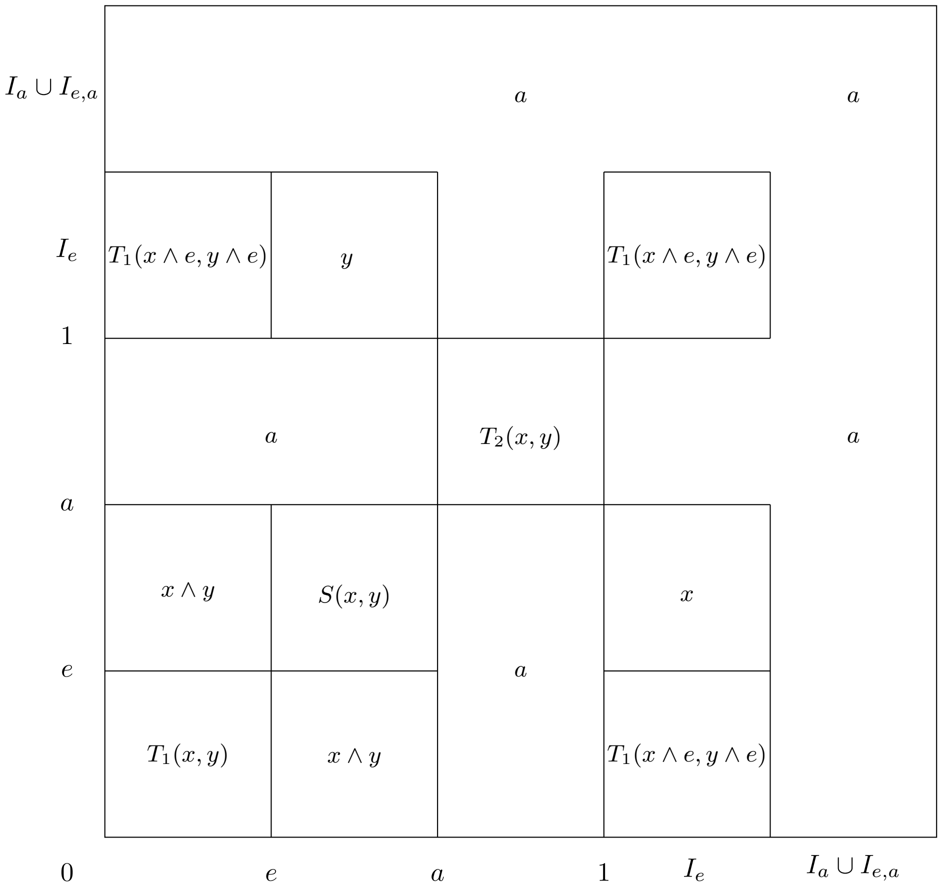

Fig. 1. The representation of the uni-nullnorm F_{T_1} obtained from formula (3).

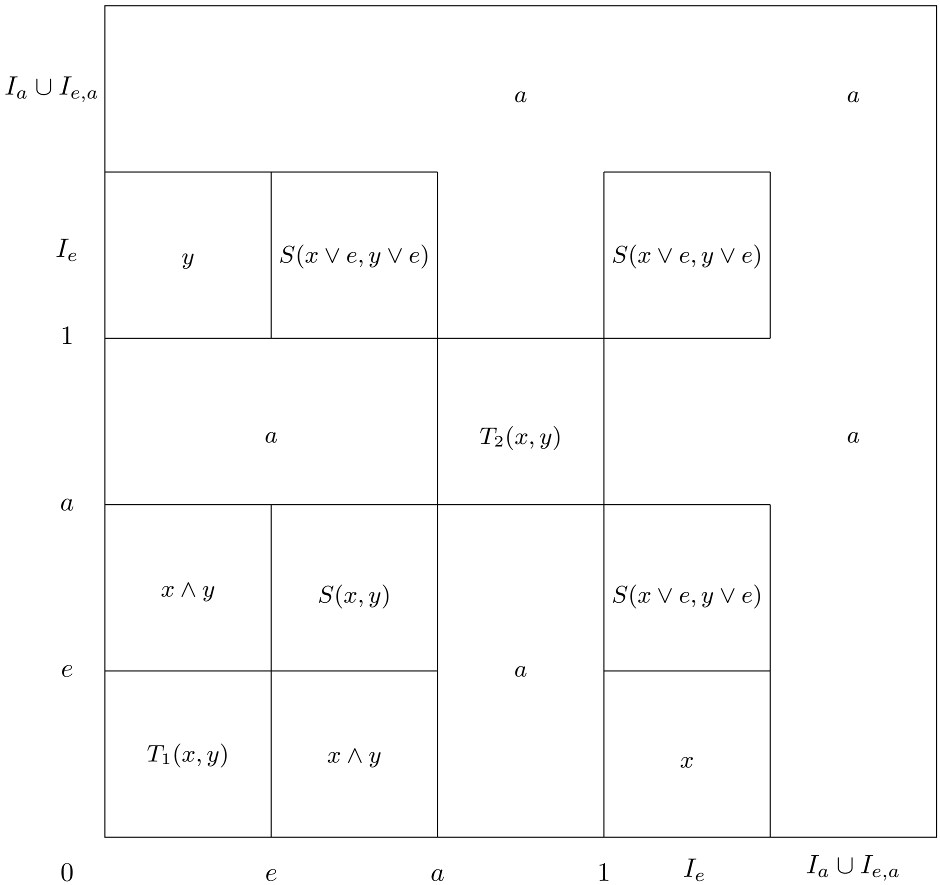

Fig. 2. The representation of the uni-nullnorm F_S ontained from formula (4).

Example 3.4. Consider the lattice L whose Hasse diagram is as in Figure 3.

Let us take the t-norm $T_2 = T_\wedge$ on $[a,1]$, the t-norm $T_1 = T_D$ on $[0,e]$ and the t-conorm S on $[e,a]$ as following.

S	e	i	h	a
e	e	i	h	a
i	i	i	a	a
h	h	a	a	a
a	a	a	a	a

Tab. 1. The t-conorm S on $[e,a]$.

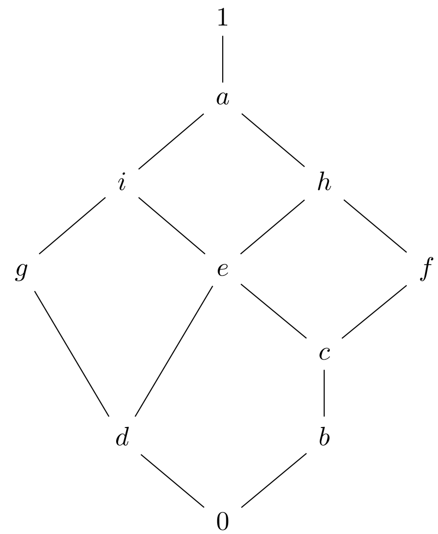

Fig. 3. Lattice diagram of L

The uni-nullnorms $F_\wedge, F_\vee, F_{T_1}$ and F_S obtained from the formulas (1), (2), (3) and (4) are respectively as in Tables 2, 3, 4 and 5.

$F_\wedge$	0	b	c	d	e	f	g	h	i	a	1
0	0	0	0	0	0	0	0	0	0	a	a
b	0	b	b	0	b	b	0	b	b	a	a
c	0	b	c	0	c	c	0	c	c	a	a
d	0	0	0	d	d	0	d	d	d	a	a
e	0	b	c	d	e	f	g	h	i	a	a
f	0	b	c	0	f	f	0	f	f	a	a
g	0	0	0	d	g	0	g	g	g	a	a
h	0	b	c	d	h	f	g	a	a	a	a
i	0	b	c	d	i	f	g	a	i	a	a
a	a	a	a	a	a	a	a	a	a	a	a
1	a	a	a	a	a	a	a	a	a	a	1

Tab. 2. The uni-nullnorm $F_\wedge$ induced by the formula (1) in Theorem 2.15.

$F_\vee$	0	b	c	d	e	f	g	h	i	a	1
0	0	0	0	0	0	f	g	h	i	a	a
b	0	0	0	0	b	f	g	h	i	a	a
c	0	0	0	0	c	f	g	h	i	a	a
d	0	0	0	0	d	f	g	h	i	a	a
e	0	b	c	d	e	f	g	h	i	a	a
f	f	f	f	f	f	f	a	a	a	a	a
g	g	g	g	g	g	a	g	a	a	a	a
h	h	h	h	h	h	a	a	h	a	a	a
i	i	i	i	i	i	a	a	a	i	a	a
a	a	a	a	a	a	a	a	a	a	a	a
1	a	a	a	a	a	a	a	a	a	a	1

Tab. 3. The uni-nullnorm $F_\vee$ induced by the formula (2) in Theorem 2.15.

F_{T_D}	0	b	c	d	e	f	g	h	i	a	1
0	0	0	0	0	0	0	0	0	0	a	a
b	0	0	0	0	b	0	0	b	b	a	a
c	0	0	0	0	c	0	0	c	c	a	a
d	0	0	0	0	d	0	0	d	d	a	a
e	0	b	c	d	e	f	g	h	i	a	a
f	0	0	0	0	f	0	0	f	f	a	a
g	0	0	0	0	g	0	0	g	g	a	a
h	0	b	c	d	h	f	g	a	a	a	a
i	0	b	c	d	i	f	g	a	i	a	a
a	a	a	a	a	a	a	a	a	a	a	a
1	a	a	a	a	a	a	a	a	a	a	1

Tab. 4. The uni-nullnorm F_{T_1} induced by the formula (3) in Theorem 3.1.

F_S	0	b	c	d	e	f	g	h	i	a	1
0	0	0	0	0	0	f	g	h	i	a	a
b	0	0	0	0	b	f	g	h	i	a	a
c	0	0	0	0	c	f	g	h	i	a	a
d	0	0	0	0	d	f	g	h	i	a	a
e	0	b	c	d	e	h	i	h	i	a	a
f	f	f	f	f	h	a	a	a	a	a	a
g	g	g	g	g	i	a	i	a	i	a	a
h	h	h	h	h	h	a	a	a	a	a	a
i	i	i	i	i	i	a	i	a	i	a	a
a	a	a	a	a	a	a	a	a	a	a	a
1	a	a	a	a	a	a	a	a	a	a	1

Tab. 5. The uni-nullnorm F_S induced by the formula (4) in Theorem 3.1.

By comparing Table 2 and Table 4, one can observe that the uni-nullnorm $F_\wedge$ does not coincide with the uni-nullnorm F_{T_D} (since $T_D \neq \wedge$ in this lattice). Similarly, one can observe by comparing Table 3 and Table 5 that the uni-nullnorm $F_\vee$ does not coincide with the uni-nullnorm F_S (since $S \neq \vee$ in this lattice).

Remark 3.5. Let $(L, \leq, 0, 1)$ be a bounded lattice, $e, a \in L$ with $e \leq a$ and F be a uni-nullnorm with 2-neutral element $\{e, 1\}_a$. We know that $F(0, a) = 0$ or $F(0, a) = a$ if we take $L = [0, 1]$. In general, $F(0, a) \leq a$ on any bounded lattice L. A natural question arises: Is there a uni-nullnorm F such that $F(0, a) = b$ for an element b satisfying $0 < b < a$? The following example shows that the claim may be true.

Example 3.6. Consider the lattice L whose Hasse diagram is in Figure 4.

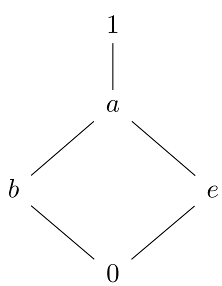

Fig. 4. Lattice diagram of L.

Take the uni-nullnorm F on L as in Table 6

F	0	e	a	b	1
0	0	0	b	b	b
e	0	e	a	b	a
a	b	a	a	b	a
b	b	b	b	b	b
1	b	a	a	b	1

Tab. 6. The uni-nullnorm F on L.

When checking the Table 6, it has seen that $F(0, a) = b < a$. Therefore, this example indicates that $F(0, a)$ may be different from 0 or a for uni-nullnorm F defined on bounded lattice L different from the real unit interval $[0, 1]$.

4. CONSTRUCTION OF NULL-UNINORM ON BOUNDED LATTICES

In this section, after giving a proposition that characterizes the general characteristics of null-uninorm on the bounded lattices, two construction methods for null-uninorm are given on the bounded lattices.

Proposition 4.1. Let $(L, \leq, 0, 1)$ be a bounded lattice, $e, a \in L$ with $a \leq e$ and F be a null-uninorm having a 2-neutral element $\{0, e\}_a$ with a being an annihilator over $[0, e]$. Then,

(i) $x \leq F(x, y)$ for $(x, y) \in [a, 1] \times [e, 1] \cup I_e \times [e, 1] \cup [0, a] \times L$.

(ii) $y \leq F(x, y)$ for $(x, y) \in [e, 1] \times [a, 1] \cup [e, 1] \times I_e \cup L \times [0, a]$.

(iii) $x \leq F(x, y)$ for $(x, y) \in [a, 1] \times [0, e] \cup [a, 1] \times I_a \cup I_e \times I_a \cup I_e \cup [0, e]$.

(iv) $y \leq F(x, y)$ for $(x, y) \in [0, e] \times [a, 1] \cup [0, e] \times I_e \cup I_a \times I_e \cup I_a \cup [a, 1]$.

(v) $x \vee y \leq F(x, y) \leq 1$ for $(x, y) \in [e, 1]^2$.

(vi) $a \leq F(x, y) \leq x \wedge y$ for $(x, y) \in [a, e]^2$.

(vii) $x \vee y \leq F(x, y) \leq a$ for $(x, y) \in [0, a]^2$.

(viii) $F(x, y) = a$ for $(x, y) \in [a, 1] \times [0, a] \cup [0, a] \times [a, 1]$.

(ix) $x \wedge y \leq F(x, y) \leq x \vee y$ for $(x, y) \in (e, 1] \times [a, e) \cup [a, e) \times (e, 1]$.

(x) $F(x, y) \geq (x \wedge a) \vee (y \wedge a)$ for $(x, y) \in [0, a] \times I_a \cup I_a \times [0, a] \cup I_a \times I_a$.

(xi) $F(x, y) \leq (x \vee a) \wedge (y \vee a)$ for $(x, y) \in [a, e] \times I_a \cup I_a \times [a, e] \cup I_a \times I_a$.

(xii) $F(x, y) \geq a$ for $(x, y) \in [a, 1] \times L \cup L \times [a, 1] \cup I_e \times L \cup L \times I_e$.

(xiii) $F(x, y) \leq a$ for $(x, y) \in [0, a] \times [0, e] \cup [0, e] \times [0, a] \cup I_a \times [0, a] \cup [0, a] \times I_a$.

Proposition 4.2. Let $(L, \leq, 0, 1)$ be a bounded lattice, $e, a \in L$ with $a \leq e$ and F be a null-uninorm having a 2-neutral element $\{0, e\}_a$ with a being an annihilator over $[0, e]$. Then,

(i) $V^* = F|_{[0,e]^2} : [0, e]^2 \to [0, e]$ is a nullnorm on $[0, e]$.

(ii) $U^* = F|_{[a,1]^2} : [a, 1]^2 \to [a, 1]$ is a uninorm on $[a, 1]$.

(iii) $S_1^* = F|_{[0,a]^2} : [0, a]^2 \to [0, a]$ is a t-conorm on $[0, a]$.

(iv) $T^* = F|_{[a,e]^2} : [a, e]^2 \to [a, e]$ is a t-norm on $[a, e]$.

(v) $S_2^* = F|_{[e,1]^2} : [e, 1]^2 \to [e, 1]$ is a t-conorm on $[e, 1]$.

The above propositions can be obtained immediately from Definition 2.10.

Theorem 4.3. Let $a, e \in L \setminus \{0, 1\}$ with $a \leq e$. If S_2 is a t-conorm on $[0, a]^2$, T is a t-norm on $[a, e]^2$ such that $T(x, y) > a$ for all $x, y > a$ and S_1 is a t-conorm on $[e, 1]^2$, then the following function $F_T : L^2 \to L$ is null-uninorm a 2-neutral element $\{0, e\}_a$ on L.

$$F_T(x,y) = \begin{cases} S_2(x,y) & (x,y) \in [0,a]^2, \\ T(x,y) & (x,y) \in [a,e]^2, \\ S_1(x,y) & (x,y) \in [e,1]^2, \\ S_1(x \vee e, y \vee e) & (x,y) \in (e,1] \times I_e \cup I_e \times (e,1] \cup I_e \times I_e, \\ x \vee y & (x,y) \in (e,1] \times (a,e] \cup (a,e] \times (e,1], \\ y & (x,y) \in (a,e] \times I_e, \\ x & (x,y) \in I_e \times (a,e], \\ a & \text{otherwise}. \end{cases} \tag{5}$$

The proof of Theorem 4.3 is omitted since it could be done in a way similar to Theorem 3.1.

Remark 4.4. From Theorem 4.3, the structure of construction method of a null-uninorm on the bounded lattice L can be visualized, see Figure 5.

Remark 4.5. Let $(L, \leq, 0, 1)$ be a bounded lattice, $e, a \in L$ with $a \leq e$ and F be a null-uninorm with a 2-neutral element $\{0, e\}_a$. We know that $F(1, a) = 1$ or $F(1, a) = a$ if we take $L = [0, 1]$. In general $F(1, a) \geq a$ on any bounded lattice L. Similarly, a natural question arises: Is there a null-uninorm F such that $F(1, a) = b$ for an element b satisfying $a < b < 1$? The following example shows that the claim may be true.

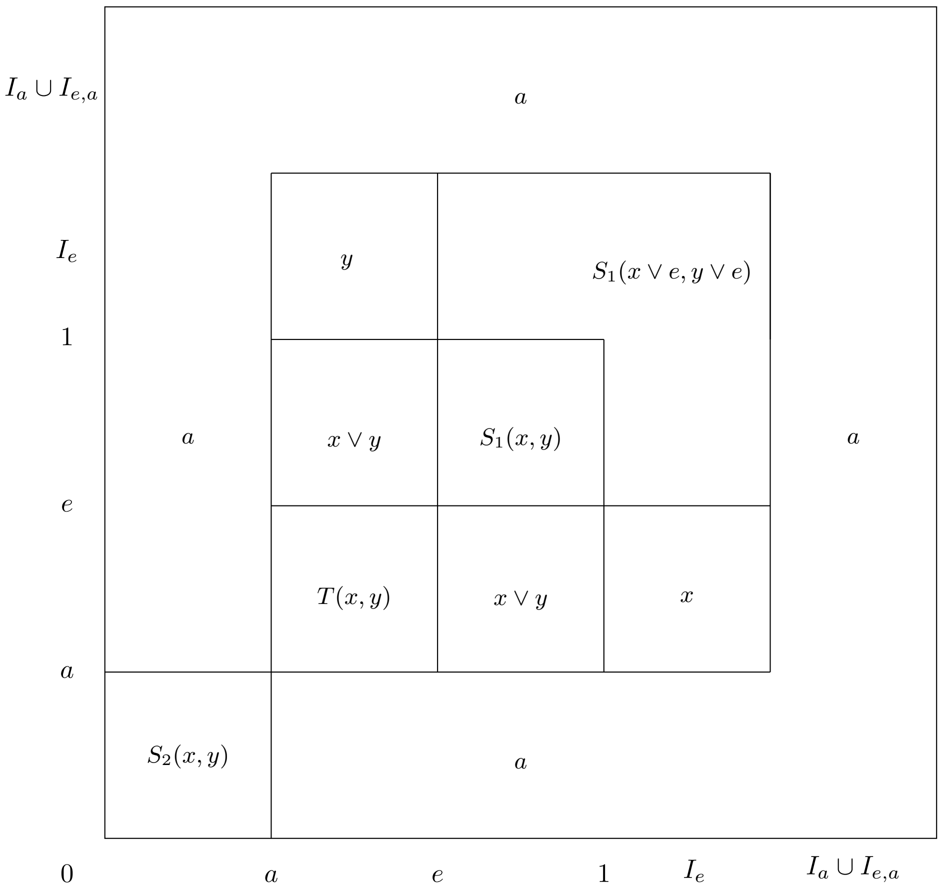

Fig. 5. The representation of the null-uninorm F_T obtained from formula (5).

Example 4.6. Consider the lattice L whose Hasse diagram is as in Figure 6.

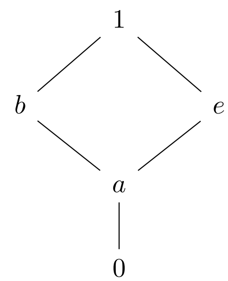

Fig. 6. Lattice diagram of L

Take the null-uninorm F on L as in Table 7

F	0	e	a	b	1
0	0	a	a	b	b
e	a	e	a	b	1
a	a	a	a	b	b
b	b	b	b	b	b
1	b	1	b	b	1

Tab. 7. The null-uninorm F on L.

When checking the Table 7, it has seen that $F(1,a) = b > a$. Therefore, this example indicates that $F(1,a)$ may be different from 1 and a for null-uninorm F defined on bounded lattice L different from the real unit interval $[0,1]$.

5. CONCLUDING REMARKS

After the definition and characteristics of the uni-nullnorms and null-uninorms on $[0,1]$ were given by Sun et al. [19], a study was conducted by Wang et al. on the methods of constructing uni-nullnorm [23] and moreover, these functions have been studied by researchers from different perspectives [20, 22]. Uni-nullnorm functions are also studied on bounded lattices such as other logical operators (t-norms, t-conorms, uninorms, nullnorms) which are defined and studied on the unit real interval at first and then studied on the bounded lattices. In this paper, two construction methods are proposed and investigated their relations with other construction methods for uni-nullnorms, nullnorms and uninorms. Characteristic properties of nulll-uninorms on bounded lattice are introduced and a construction method for null-uninorm is given.

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