RANDOM FIELDS AND RANDOM SAMPLING

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We study the limiting distribution of the maximum value of a stationary bivariate real random field satisfying suitable weak mixing conditions. In the first part, when the double dimensions of the random samples have a geometric growing pattern, a max-semistable distribution is obtained. In the second part, considering the random field sampled at double random times, a mixture distribution is established for the limiting distribution of the maximum.

Keywords: stationary random fields, max-semistable laws, random double sample size

Classification: 60G60, 60G70

1. INTRODUCTION

Extreme values of stationary and non stationary random fields have been studied by many authors in the last two decades. The usual extremal properties and concepts, well known for real sequences of random variables, were extended showing a great commitment between the classical results and the necessary inspiration to deal with random fields. The characterization of the limiting distribution of the extremes of a random field, studied in the scenarios of asymptotic independence, local dependence, exceedance point process, clustering of high values, asymptotic location, etc, can be seen in Leadbetter and Rootzén [13], Ferreira and Pereira [6, 7], Pereira and Ferreira [16, 17], among many others. Gaussian random fields have received special attention, which stand out due to its valuable properties, as we can see for instance in Pereira [15], Tan [20, 21] and Harshova et al. [11]. Almost sure convergence of maxima of random fields has also been an interesting topic developed in many works, which can be checked in Tan and Wang [22] and Pereira and Tan [18] and references therein.

In this paper we study the limit in distribution of the maximum of a stationary bivariate real random field, sampled at double random times. Starting by considering a deterministic sample size, the aim of Section 2 is to establish this limit under the context of max-semistability. The results are simultaneously extensions of the ones of Temido and Canto e Castro [23] and Choi [4], concerning stationary real sequences and stationary real random fields, respectively. In Section 3, we extend the results of Freitas et al. [8], where a random sample size of a stationary real sequence is considered.

In the remaining part of this introduction, we provide the required background support led by the class of max-semistable laws.

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A distribution function (d.f.) G is max-semistable if there are reals $r > 1$, $a > 0$ and b such that $G(x) = G^r(ax + b)$. The class of these d.f.'s - MSS class - coincides with the class of all possible limits in

$$F^{k_n}(a_n x + b_n) \to G(x),\ n \to +\infty,$$

where F is a d.f., $\{a_n > 0\}$ and $\{b_n\}$ are real sequences and $\{k_n\}$ is an integer sequence satisfying $k_{n+1} \geq k_n \geq 1$ and $\frac{k_{n+1}}{k_n} \to r \geq 1$ $(r < \infty)$, $n \to +\infty$. This means that MSS is the class of all the possible limits in distribution of the maximum, under linear normalization, of k_n independent and identically distributed random variables.

Max-semistable laws were introduced in Extreme Values Theory in the works of Pancheva [14] and Grinevich [10]. The three families of real max-semistable d.f.'s are defined by:

$$\begin{aligned}
\Phi_{\alpha,\nu}(x) &= \exp\{-(x-t)^{-\alpha}\nu(\log(x-t))\}\,\mathbb{I}_{]t,+\infty[}(x),\\
\Psi_{\alpha,\nu}(x) &= \exp\{-(t-x)^{\alpha}\nu(\log(t-x))\}\,\mathbb{I}_{]-\infty,t[}(x) + \mathbb{I}_{[t,+\infty[}(x) \text{ and}\\
\Lambda_\nu(x) &= \exp\{-\exp(-\beta x)\nu(x)\},
\end{aligned}$$

where $\alpha = |\log\ r/\log\ a|$, $\beta = \log\ r/b$, $t \in \mathbb{R}$ and ν are positive, bounded and periodic suitable functions.

The MSS class perspectives a significant increase in applications of Extreme Value Theory, since it contains discontinuous d.f.'s as well as a wide set of multimodal d.f.'s. Statistical inference in max-semistable models (estimation of parameters and an useful test) was studied mainly in Canto e Castro and Dias [1] and Canto e Castro et al. [2, 3].

Temido and Canto e Castro [23] consider stationary sequences under an asymptotic independence condition that is an adaptation of Leadbetter's $D(u_n)$ condition. They proved that the class of possible limits in distribution of the maximum of the first k_n random variables, linearly normalized, coincides with the MSS class.

Also in the context of max-semistability for stationary real sequences, Freitas et al. [8] establishes the limit in distribution of the maximum M_{T_n}, under linear normalization, where $\{T_n\}$ is a sequence of positive integer random variables satisfying $T_n/k_n \to T$ in probability, for some strictly positive random variable T. An additional mixing condition was also introduced in order to deal with the weak convergence with mixing property. Then $a_n^{-1}(M_{T_n} - b_n)$ converges in distribution to the mixture defined by $E(G^T)$, where G stands for the limit in distribution of $a_n^{-1}(M_{k_n} - b_n)$ stated by Temido and Canto e Castro [23].

2. RANDOM FIELDS AND MAX-SEMISTABILITY

Let us consider a bivariate stationary real random field, $\{X_{n,m}\}$, with (n, m) in $\mathbb{N}^2$. Let $\{k_n\}$ and $\{k^*_m\}$ be two positive integer sequences satisfying

$$\frac{k_{n+1}}{k_n} \to r \geq 1,\ n \to +\infty, \quad \text{and} \quad \frac{k^*_{m+1}}{k^*_m} \to r^* \geq 1,\ m \to +\infty. \tag{1}$$

We use the usual notation $M(I) := \max\{X_{i,j}, (i,j) \in I\}$ for $I \subseteq]0,a]\times]0,b]$ and $M_{a,b} := M(]0,a]\times]0,b])$, for any $a, b \in \mathbb{R}^+$.

The following dependence restriction is, at the same time, a generalization of the condition $D_{k_n}(u_n)$ provided in Temido and Canto e Castro [23] and an adaptation of $\mathbf{\Delta}(u_n)$ introduced in Pereira and Ferreira [17], this one being an extension of $\Delta(u_n)$ presented in Choi [4].

Definition 2.1. Let $\{u_{n,m}\}$ be a real sequence and $\{r_n\}$, $\{r_m^*\}$, $\{\ell_n\}$ and $\{\ell_m^*\}$ positive integer sequences such that

$$r_n \to +\infty, \qquad r_n = o_n(k_n), \qquad \ell_n = o_n(r_n), \quad n \to +\infty, \tag{2}$$

and

$$r_m^* \to +\infty, \qquad r_m^* = o_m(k_m^*), \qquad \ell_m^* = o_m(r_m^*), \quad m \to +\infty. \tag{3}$$

The random field $\{X_{n,m}\}$ satisfies the condition $D_{k_n,k_m^*}(u_{n,m})$ if

i) for each pair of rectangles $I_1 =]a_1, b_1] \times]c_1, d_1]$ and $I_2 =]a_2, b_2] \times]c_1, d_1]$ such that $a_2 - b_1 > \ell_n$ and $b_2 - a_2 < r_n$, we have

$$|P(M(I_1) \le u_{n,m}, M(I_2) \le u_{n,m}) - P(M(I_1) \le u_{n,m})P(M(I_2) \le u_{n,m})| \le \alpha_{n,m}$$

with $\frac{k_n}{r_n}\alpha_{n,m} \to 0$, $n, m \to +\infty$;

ii) for each pair of rectangles $I_1^* =]a_1^*, b_1^*] \times]c_1^*, d_1^*]$ and $I_2^* =]a_1^*, b_1^*] \times]c_2^*, d_2^*]$ such that $c_2^* - d_1^* > \ell_m^*$ and $d_2^* - c_2^* < r_m^*$, we have

$$|P(M(I_1^*) \le u_{n,m}, M(I_2^*) \le u_{n,m}) - P(M(I_1^*) \le u_{n,m})P(M(I_2^*) \le u_{n,m})| \le \alpha_{n,m}^*$$

with $\frac{k_m^*}{r_m^*}\frac{k_n}{r_n}\alpha_{n,m}^* \to 0$, $n, m \to +\infty$.

The next lemma establishes the asymptotic independence of linearized maxima over disjoint rectangles for stationary random fields under the condition $D_{k_n,k_m^*}(u_{n,m})$, as well as estimates of $P(M_{k_n,k_m^*} \le u_{n,m})$. More precisely, this lemma is a generalization of Lemma 4.4.1 of Choi [4] and Lemmas 2 and 3 of Temido and Canto e Castro [23]. As a consequence we deduce an extremal type theorem for stationary random fields, where a max-semistable limit law is obtained. In what follows, we present the construction of those disjoint rectangles.

Let us consider two nondecreasing sequences of positive integers $\{s_n\}$ and $\{s_m^*\}$ defined by $r_n = \lfloor k_n/s_n \rfloor$ and $r_m^* = \lfloor k_m^*/s_m^* \rfloor$ satisfying $\frac{s_n \ell_n}{k_n} \to 0$, $n \to +\infty$, and $\frac{s_m^* \ell_m^*}{k_m^*} \to 0$, $m \to +\infty$.

Consider the disjoint rectangles $E_j :=](j-1)r_n, jr_n] \times]0, s_m^* r_m^*]$, $j = 1, ..., s_n$, and $E_j^* :=]0, r_n] \times](j-1)r_m^*, jr_m^*]$, $j = 1, ..., s_m^*$, and observe that $]0, s_n r_n] \times\ ]0, s_m^* r_m^*] := \bigcup_{j=1}^{s_n} E_j$ and $E_1 =]0, r_n] \times]0, s_m^* r_m^*] := \bigcup_{j=1}^{s_m^*} E_j^*$. Take also the separated rectangles $I_j :=](j-1)r_n, jr_n - \ell_n] \times]0, s_m^* r_m^*]$, $j = 1, ..., s_n$, and $I_j^* :=]0, r_n] \times](j-1)r_m^*, jr_m^* - \ell_m^*]$, $j = 1, ..., s_m^*$.

Indeed, we split the rectangle $]0, s_n r_n] \times]0, s_m^* r_m^*]$ into s_n rectangles ℓ_n-separated, as well as $E_1 =]0, r_n] \times]0, s_m^* r_m^*]$ into s_m^* rectangles ℓ_m^*-separated.

Lemma 2.2. Let $\{X_{n,m}\}$ be a bivariate stationary real random field. Consider two sequences of positive integers, $\{k_n\}$ and $\{k_m^*\}$, satisfying (1) and the sequences $\{s_n\}$, $\{r_n\}$, $\{s_m^*\}$ and $\{r_m^*\}$ as before. If $k_n k_m^*(1-F(u_{n,m})) < +\infty$ and $D_{k_n,k_m^*}(u_{n,m})$ holds, then

1.
$$\left| P\left(M\left(\bigcup_{j=1}^{s_n} I_j \right) \le u_{n,m} \right) - \prod_{j=1}^{s_n} P\left(M(I_j) \le u_{n,m}\right) \right| \le (s_n - 1)\alpha_{n,m} \quad (4)$$

and

$$\left| P\left(M\left(\bigcup_{j=1}^{s_m^*} I_j^* \right) \le u_{n,m} \right) - \prod_{j=1}^{s_m^*} P\left(M(I_j^*) \le u_{n,m}\right) \right| \le (s_m^* - 1)\alpha_{n,m}^*. \quad (5)$$

2.
$$P(M_{k_n,k_m^*} \le u_{n,m}) - P^{s_n s_m^*}(M_{r_n,r_m^*} \le u_{n,m}) \to 0, \quad n,m \to +\infty. \quad (6)$$

3. (a) If $r > 1$ and $r^* > 1$,

$$P(M_{k_n,k_m^*} \le u_{n,m}) - P^{rr^*}(M_{k_{n-1},k_{m-1}^*} \le u_{n,m}) \to 0,\ n,m \to +\infty; \quad (7)$$

(b) If $r = 1$ and $r^* = 1$, $\forall\, h_1, h_2 \in \mathbb{R}^+$, there are positive integer sequences $\{p_n\}$ and $\{p_m^*\}$ such that $k_{p_n} \sim k_n/h_1$, $n \to +\infty$, and $k_{p_m^*}^* \sim k_m^*/h_2$, $m \to +\infty$, and

$$P(M_{k_n,k_m^*} \le u_{n,m}) - P^{h_1 h_2}(M_{k_{p_n},k_{p_m^*}^*} \le u_{n,m}) \to 0, \quad n,m \to +\infty; \quad (8)$$

(c) If $r = 1$ and $r^* > 1$, $\forall h \in \mathbb{R}^+$, there is a positive integer sequence $\{p_n\}$ such that $k_{p_n} \sim k_n/h$, $n \to +\infty$, and

$$P(M_{k_n,k_m^*} \le u_{n,m}) - P^{hr^*}(M_{k_{p_n},k_{m-1}^*} \le u_{n,m}) \to 0, \quad n,m \to +\infty; \quad (9)$$

(d) If $r > 1$ and $r^* = 1$, $\forall h \in \mathbb{R}^+$, there is a positive integer sequence $\{p_m^*\}$ such that $k_{p_m^*} \sim k_m^*/h$, $m \to +\infty$, and

$$P(M_{k_n,k_m^*} \le u_{n,m}) - P^{rh}(M_{k_{n-1},k_{p_m^*}^*} \le u_{n,m}) \to 0, \quad n,m \to +\infty. \quad (10)$$

Proof. We omit the proof of 1. due to its similitude with the proof of Lemma 3.2.2 in Leadbetter et al. [12], in spite of the approach of random fields. In the following we present the appropriate main arguments to separated rectangles, which are generalizations of the ones presented in Temido and Canto e Castro [23] for separated intervals.

2. Let us observe that

$$
\begin{aligned}
&|P(M_{k_n,k^*_m} \le u_{n,m}) - P^{s_n s^*_m}(M_{r_n,r^*_m} \le u_{n,m})| \\
&\quad \le |P(M_{k_n,k^*_m} \le u_{n,m}) - P(M_{s_n r_n, s^*_m r^*_m} \le u_{n,m})| \qquad (11)\\
&\qquad +|P(M_{s_n r_n, s^*_m r^*_m} \le u_{n,m}) - P^{s_n}(M(E_1) \le u_{n,m})| \qquad (12)\\
&\qquad +|P^{s_n}(M(E_1) \le u_{n,m}) - P^{s_n s^*_m}(M_{r_n,r^*_m} \le u_{n,m})|. \qquad (13)
\end{aligned}
$$

Due to the fact that $r_n s_n \le k_n$ and $r^*_m s^*_m \le k^*_m$, taking $R^{(1)}_{n,m} :=]s_n r_n, k_n] \times\,]0, s^*_m r^*_m]$ and $R^{(2)}_{n,m} :=]0, k_n] \times]s^*_m r^*_m, k^*_m]$, we prove that (11) is bounded by

$$
\begin{aligned}
&P(M(R^{(1)}_{n,m}) > u_{n,m}) + P(M(R^{(2)}_{n,m}) > u_{n,m}) \\
&\quad \le k_n \left(1 - \frac{s_n r_n}{k_n}\right) \frac{s^*_m r^*_m}{k^*_m} k^*_m \overline{F}(u_{n,m}) + k^*_m \left(1 - \frac{s^*_m r^*_m}{k^*_m}\right) k_n \overline{F}(u_{n,m}) \to 0, \quad n,m \to +\infty,
\end{aligned}
$$

where $\overline{F}(x) := 1 - F(x)$. We also deduce that

$$
\begin{aligned}
0 &\le P(M(\cup_{j=1}^{s_n} I_j) \le u_{n,m}) - P(M(\cup_{j=1}^{s_n} E_j) \le u_{n,m}) \\
&\le \frac{s_n \ell_n}{k_n} \frac{s^*_m r^*_m}{k^*_m} k_n k^*_m \overline{F}(u_{n,m}) \to 0, \quad n,m \to +\infty, \qquad (14)
\end{aligned}
$$

where $M(\cup_{j=1}^{s_n} E_j) = M_{s_n r_n, s^*_m r^*_m}$.

Moreover, due to (4) and using the stationarity of $\{X_{n,m}\}$, we get

$$
\left|P(M(\cup_{j=1}^{s_n} I_j) \le u_{n,m}) - P^{s_n}(M(I_1) \le u_{n,m})\right| \le s_n \alpha_{n,m} \to 0,\ n,m \to +\infty. \qquad (15)
$$

In addition, due to the well known inequality $\left|\prod_{i=1}^{k} a_i - \prod_{i=1}^{k} b_i\right| \le \sum_{i=1}^{k} |a_i - b_i|$ for $a_i, b_i \in [0,1]$, we obtain

$$
\begin{aligned}
P^{s_n}(M(I_1) \le u_{n,m}) - P^{s_n}(M(E_1) \le u_{n,m}) &\le s_n \left[P(M(I_1) \le u_{n,m}) - P(M(E_1) \le u_{n,m})\right] \\
&\le s_n P(M(]r_n - \ell_n, r_n] \times]0, s^*_m r^*_m]) > u_{n,m}) \\
&= \frac{s_n \ell_n}{k_n} \frac{s^*_m r^*_m}{k^*_m} k_n k^*_m \overline{F}(u_{n,m}) \to 0, \qquad (16)
\end{aligned}
$$

as $n, m \to +\infty$. Then (14), (15) and (16) prove that (12) is asymptotically zero. Similarly, considering the vertical scheme for $E_1^*, ..., E_{s_m^*}^*$, we obtain

$$|P^{s_n}(M(E_1) \leq u_{n,m}) - P^{s_n s_m^*}(M(E_1^*) \leq u_{n,m})|$$

$$\leq s_n |P(M(\cup_{j=1}^{s_m^*} E_j^*) \leq u_{n,m}) - P(M(\cup_{j=1}^{s_m^*} I_j^*) \leq u_{n,m})|$$

$$+ s_n |P(M(\cup_{j=1}^{s_m^*} I_j^*) \leq u_{n,m}) - P^{s_m^*}(M(I_1^*) \leq u_{n,m})|$$

$$+ s_n |P^{s_m^*}(M(I_1^*) \leq u_{n,m}) - P^{s_m^*}(M(E_1^*) \leq u_{n,m})|$$

$$\leq 2 s_n s_m^* \ell_m^* \overline{F}(u_{n,m}) + s_n s_m^* \alpha_{n,m}^* \to 0, \ n, m \to +\infty.$$

The proof of 2 is concluded.

In order to prove 3.(a) we proceed with the rectangle $]0, k_{n-1}] \times]0, k_{m-1}^*]$ and consider the sequences $\beta_n = \lfloor \frac{s_n}{r} \rfloor$, $q_n = \lfloor \frac{k_{n-1} r}{s_n} \rfloor$, $\beta_m^* = \lfloor \frac{s_m^*}{r^*} \rfloor$ and $q_m^* = \lfloor \frac{k_{m-1}^* r^*}{s_m^*} \rfloor$ for which holds: $\frac{q_n}{r_n} \to 1$, $\frac{q_m^*}{r_m^*} \to 1$, $\frac{k_{n-1}}{\beta_n} \to +\infty$, $\frac{k_{m-1}^*}{\beta_m^*} \to +\infty$, $\beta_n \alpha_{n,m} \to 0$, $\beta_m^* \alpha_{n,m}^* \to 0$, $\frac{\ell_n \beta_n}{k_{n-1}} \to 0$, $\frac{\ell_m^* \beta_m^*}{k_{m-1}^*} \to 0$, when $n, m \to +\infty$.

Since the condition $D_{k_{n-1}, k_{m-1}^*}(u_{n,m})$ holds as well, *mutatis mutandis*, we prove that

$$P(M_{k_{n-1}, k_{m-1}^*} \leq u_{n,m}) - P^{\beta_n \beta_m^*}(M_{q_n, q_m^*} \leq u_{n,m}) \to 0, \ n, m \to +\infty. \tag{17}$$

Furthermore

$$|P(M_{k_n, k_m^*} \leq u_{n,m}) - P^{r r^*}(M_{k_{n-1}, k_{m-1}^*} \leq u_{n,m})|$$

$$\leq |P(M_{k_n, k_m^*} \leq u_{n,m}) - P^{s_n s_m^*}(M_{r_n, r_m^*} \leq u_{n,m})| \tag{18}$$

$$+ |P^{s_n s_m^*}(M_{r_n, r_m^*} \leq u_{n,m}) - P^{s_n s_m^*}(M_{q_n, q_m^*} \leq u_{n,m})| \tag{19}$$

$$+ |P^{s_n s_m^*}(M_{q_n, q_m^*} \leq u_{n,m}) - P^{\beta_n r \beta_m^* r^*}(M_{q_n, q_m^*} \leq u_{n,m})| \tag{20}$$

$$+ |P^{\beta_n r \beta_m^* r^*}(M_{q_n, q_m^*} \leq u_{n,m}) - P^{r r^*}(M_{k_{n-1}, k_{m-1}^*} \leq u_{n,m})| \tag{21}$$

Here the difference (18) is asymptotically zero due to (6). Likewise (21) tends to zero, when $n, m \to +\infty$, because $|x^r - y^r| \leq r|x - y|$ for all $x, y \in [0, 1]$. Assuming without loss of generality that $q_n \leq r_n$ and $q_m^* \leq r_m^*$ and taking $R_{n,m}^{(3)} :=]q_n, r_n] \times]0, q_m^*]$ and $R_{n,m}^{(4)} :=]0, r_n] \times]q_m^*, r_m^*]$, we prove that (19) is bounded by

$$s_n s_m^* \{P(M(R_{n,m}^{(3)}) > u_{n,m}) + P(M(R_{n,m}^{(4)}) > u_{n,m})\}$$

$$\leq s_n s_m^* \{(r_n - q_n) q_m^* + (r_m^* - q_m^*) r_n\} \overline{F}(u_{n,m}) \to 0, \ n, m \to +\infty.$$

In what concerns (20), we use the inequality $0 \leq u^x - u^y \leq u^x (\log u)(x - y)$ valid for $0 < x \leq y < \infty$ and $u \in]0, 1[$ (Lagrange's Theorem) and the fact that $\beta_n \beta_m^* \leq \frac{s_n}{r} \frac{s_m^*}{r^*}$.

Then we establish that (20) does not exceed

$$\begin{aligned}&P^{\beta_n r\beta^*_m r^*}(M_{q_n,q^*_m}\le u_{n,m})\log P(M_{q_n,q^*_m}\le u_{n,m})\times(\beta_n r\beta_m r^*-s_ns^*_m)\\&\quad=P^{\beta_n r\beta^*_m r^*}(M_{q_n,q^*_m}\le u_{n,m})\log P^{\beta_n r\beta^*_m r^*}(M_{q_n,q^*_m}\le u_{n,m})(1-\frac{s_ns^*_m}{\beta_n r\beta_m r^*})\\&\quad\to 0,\ n,m\to+\infty,\end{aligned}$$

because $x\log(x)$ is a bounded real function on $]0,1[$.

The proof of (7) is concluded. In order to prove (8), we start by observing that for all $(h_1,h_2)\in\mathbb{R}^+\times\mathbb{R}^+$, with $\lfloor\frac{k_n}{h_1}\rfloor$, $\lfloor\frac{k^*_m}{h_2}\rfloor$, $\lfloor\frac{s_n}{h_1}\rfloor$ and $\lfloor\frac{s^*_m}{h_2}\rfloor$ instead of k_n, k^*_m, s_n and s^*_m, from (6) we deduce

$$P(M_{\lfloor\frac{k_n}{h_1}\rfloor,\lfloor\frac{k^*_m}{h_2}\rfloor}\le u_{n,m})-P^{\lfloor\frac{s_n}{h_1}\rfloor\lfloor\frac{s^*_m}{h_2}\rfloor}(M_{r_n,r^*_m}\le u_{n,m})\to 0,\ n,m\to+\infty.$$

On the other hand, since $r=1$, for any $h_1>0$, there exists a positive integer sequence $p_n=p_n(h_1)$ such that $k_{p_n-1}<k_n/h_1\le k_{p_n}$. Likewise for k^*_m. Then, we easily obtain

$$P(M_{k_{p_n},k^*_{p^*_m}}\le u_{n,m})-P^{\frac{s_ns^*_m}{h_1h_2}}(M_{r_n,r^*_m}\le u_{n,m})\to 0,\ n,m\to+\infty.$$

This last limit, jointly with (6), gives us (8).

If $r=1$ and $r^*>1$, for any $h>0$, we obtain similarly

$$P(M_{\lfloor\frac{k_n}{h}\rfloor,\lfloor\frac{k^*_m}{r^*}\rfloor}\le u_{n,m})-P^{\lfloor\frac{s_n}{h}\rfloor\lfloor\frac{s^*_m}{r^*}\rfloor}(M_{r_n,r^*_m}\le u_{n,m})\to 0,\ n,m\to+\infty,$$

and consequently

$$P^{hr^*}(M_{k_{p_n},k^*_{m-1}}\le u_{n,m})-P^{s_ns^*_m}(M_{r_n,r^*_m}\le u_{n,m})\to 0,\ n,m\to+\infty,$$

as well as (6).

The same happens for $r>1$ and $r^*=1$. The proof of the lemma is concluded. □

The next theorem, the main result of this section, establishes that the limit in distribution of the sequence of linearized maxima $a^{-1}_{n,m}(M_{k_n,k^*_m}-b_{n,m})$ is a max-semistable random variable. This extremal type theorem is the expected generalization of Theorem 2 of Temido and Canto e Castro [23], when the approach of random fields is considered.

Theorem 2.3. Let $\{X_{n,m}\}$ be a stationary bivariate random field. Suppose that there are positive integer sequences $\{k_n\}$ and $\{k^*_m\}$ satisfying (1) and real sequences $\{a_{n,m}>0\}$ and $\{b_{n,m}\}$ such that $k_nk^*_m(1-F(a_{n,m}x+b_{n,m}))<+\infty$. If $\{X_{n,m}\}$ satisfies $D_{k_n,k^*_m}(a_{n,m}x+b_{n,m})$ and there exists a non degenerate d.f. G, such that

$$P(M_{k_n,k^*_m}\le a_{n,m}x+b_{n,m})\ \to G(x),\ n,m\to+\infty,\tag{22}$$

then G is max-semistable.

P r o o f. Suppose that $r > 1$ and $r^* > 1$. By assumption we have

$$P(M_{k_{n-1},k^*_{m-1}} \leq u_{n-1,m-1}) \to G(x),\ n,m \to +\infty,$$

and, due to the last result, we get

$$P(M_{k_{n-1},k^*_{m-1}} \leq u_{n,m}) \to G^{1/rr^*}(x),\ n,m \to +\infty.$$

Applying Khintchine's Convergence Theorem we prove that there are real constants $A > 0$ and B such that $G(x) = G^{rr^*}(Ax + B)$. Then G is max-semistable.

If $r = 1$ and $r^* > 1$, from (22) and (9) we get, respectively

$$P(M_{k_{p_n},k^*_{m-1}} \leq u_{p_n,m-1}) \to G(x),\ n,m \to +\infty,$$

and, for all $h > 0$,

$$P(M_{k_{p_n},k^*_{m-1}} \leq u_{n,m}) \to G^{1/hr^*}(x),\ n,m \to +\infty.$$

Once again, Khintchine's Convergence Theorem gives us $G^{h^*}(Ax+B) = G(x)$, $\forall h^* > 0$, $(h^* = hr^*)$. In this case G is max-stable. The cases $r = r^* = 1$ and $r > r^* = 1$ are similar. □

This section is finalized with the following useful proposition.

Proposition 2.4. Let $\{X_{n,m}\}$ be a bivariate stationary random field and $\{k_n\}$, $\{k^*_m\}$ positive integer sequences satisfying (1). Suppose that $\{a_{n,m} > 0\}$ and $\{b_{n,m}\}$ are real sequences such that $k_n k^*_m(1 - F(a_{n,m}x + b_{n,m})) < +\infty$ and

$$P(M_{k_n,k^*_m} \leq a_{n,m}x + b_{n,m}) \to G(x),\ n,m \to +\infty, \tag{23}$$

with G non degenerated. If $\{X_{n,m}\}$ satisfies the condition $D_{k_n,k^*_m}(a_{n,m}x + b_{n,m})$, for every x in the support of G, then

$$P(M_{\lfloor \lambda k_n \rfloor,\lfloor \lambda^* k^*_m \rfloor} \leq a_{n,m}x + b_{n,m}) \to G^{\lambda\lambda^*}(x),\ n,m \to +\infty, \tag{24}$$

for any positive reals λ and λ^*.

P r o o f. Assume first that $r > 1$ and $r^* > 1$. Lemma 1 in Freitas et al. [8] states that for $\lambda > 1$ there are two positive numbers $j := j(\lambda) = \min\{i : \lambda k_n < k_{n+i}\}$ and $n_0 := n_0(\lambda)$ such that $\lambda k_n < k_{n+j}$, for $n > n_0$. When $\lambda^* > 1$ we obtain similarly $\lambda^* k^*_m < k^*_{m+j^*}$, for $j^* := j^*(\lambda^*) = \min\{i : \lambda k^*_m < k^*_{m+i}\}$ and $m > m_0$. Then, by the assumption that $D_{k_{n+j},k^*_{m+j^*}}(u_{n+j,m+j^*})$ holds, we deduce the validity of $D_{k_n,k^*_m}(u_{n+j,m+j^*})$ and $D_{\lfloor \lambda k_n \rfloor,\lfloor \lambda^* k^*_m \rfloor}(u_{n+j,m+j^*})$. Observe that if $\lambda \leq 1$ and $\lambda^* \leq 1$ all these conditions are trivially valid for any j, $j^* \geq 0$. Otherwise if $\lambda \leq 1$ and $\lambda^* > 1$ or $\lambda > 1$ and $\lambda^* \leq 1$ it is enough to merge these arguments to establish the validity of these two conditions. So, taking into account the limit (22) and the steps used in the proof of Lemma 2.2 (see also the proof of Proposition 1. in Freitas et al. [8], we obtain

$$P^{\lambda\lambda^*}(M_{k_n,k^*_m} \leq u_{n+j,m+j^*}) - P(M_{\lfloor \lambda k_n \rfloor,\lfloor \lambda^* k^*_m \rfloor} \leq u_{n+j,m+j^*}) \to 0,\ n,m \to +\infty. \tag{25}$$

Now, in the proof of (7), replace r and r^* by r^j and $(r^*)^{j^*}$, respectively, in the definition of β_n, q_n, β_m^* and q_m^*. Then with a suitable change in (17) − (21) we get

$$P(M_{k_{n+j},k^*_{m+j^*}} \leq u_{n+j,m+j^*}) - P^{r^j(r^*)^{j^*}}(M_{k_n,k^*_m} \leq u_{n+j,m+j^*}) \to 0,\ n,m \to +\infty,$$

which implies

$$P(M_{k_n,k^*_m} \leq u_{n+j,m+j^*}) \to G^{r^{-j}(r^*)^{-j^*}}(x),\ n,m \to +\infty.$$

However, $G^{r^{-j}(r^*)^{-j^*}}(x) = G(Ax+B)$, where $A := A(j,j^*)$ and $B := B(j,j^*)$ are obtained from Khintchine's Convergence Theorem.

Combining this last limit with (25) and using again Khintchine's Convergence Theorem we obtain (24).

Assume now $r = r^* = 1$. For any λ there is a positive integer sequence $p_n := p_n(\lambda)$ such that $k_{p_n-1} \leq \lfloor \lambda k_n \rfloor \leq k_{p_n}$ and then $\lfloor \lambda k_n \rfloor \sim k_{p_n}$, $n \to +\infty$. The same happens with $\lfloor \lambda^* k^*_m \rfloor \sim k^*_{p^*_m}$, $m \to +\infty$. As well as we obtained (8) we achieve now (24). Moreover, if $r = 1$ and $r^* > 1$ consider p_n and j^* as before and observe that $D_{k_{p_n},k^*_{m+j^*}}(u_{p_n,m+j^*})$ implies the validity of $D_{\lfloor \lambda k_n \rfloor, \lfloor \lambda^* k^*_m \rfloor}(u_{p_n,m+j^*})$ and $D_{k_n,k^*_m}(u_{p_n,m+j^*})$.

Starting in (7) we obtain

$$P(M_{k_{p_n},k^*_{m+j^*}} \leq u_{p_n,m+j^*}) - P^{\lambda (r^*)^{j^*}}(M_{k_n,k^*_m} \leq u_{p_n,m+j^*}) \to 0,\ n,m \to +\infty,$$

and

$$P(M_{\lfloor \lambda k_n \rfloor, \lfloor \lambda^* k^*_m \rfloor} \leq u_{p_n,m+j^*}) - P^{\lambda\lambda^*}(M_{k_n,k^*_m} \leq u_{p_n,m+j^*}) \to 0,\ n,m \to +\infty.$$

Once again Khintchine's Convergence Theorem gives us the existence of A and B such that

$$P(M_{\lfloor \lambda k_n \rfloor, \lfloor \lambda^* k^*_m \rfloor} \leq u_{p_n,m+j^*}) \to G^{\lambda\lambda^*}(Ax+B),\ n,m \to +\infty,$$

as well as the limit (24). The proof is concluded. □

3. RANDOM SAMPLE SIZE

When a sample with random dimension is considered, in order to establish the limit in distribution of maxima, sums, maxima of sums and many other statistics, the weak convergence with mixing property in Rényi's sense has been an useful result. Namely, following Rényi [19] a sequence of real random variables $\{Z_n\}$, specified on a common probability space $(\Omega, \mathcal{A}, P)$, is weakly convergent with mixing property if there exists a d.f. F such that

$$P(\{Z_n \leq x\} \cap B) \to F(x)P(B),\ n \to +\infty, \tag{26}$$

for any event $B \in \mathcal{A}$ and any x in the set of continuity points of F, say $\mathcal{C}(F)$. Obviously the weak convergence with mixing property implies the weak convergence. Rényi [19] proved a criterion for this type of convergence according to which (26) holds if and only if

$$P(Z_n \le x, Z_k \le x) \to F(x)P(Z_k \le x),\ n \to +\infty, \tag{27}$$

for any positive integer k and any x in $\mathcal{C}(F)$. The proof of this equivalence can be seen in Galambos [9] and can be easily extended to higher dimensions.

In order to apply this criterion to the sequence of maxima $a_{n,m}^{-1}(M_{\lfloor\lambda k_n\rfloor,\lfloor\lambda^* k_m^*\rfloor} - b_{n,m})$, for any positive λ and λ^*, we need to introduce the following additional mixing condition $D_{k_n,k_m^*}^{(q,q^*)}(v, u_{n,m})$, which extends the condition $D_{m,k_n}(v, u_n)$ of Freitas et al. [8], being this one inspired by condition $\underline{\Delta}(u_n)$ of Ferreira [5].

Definition 3.1. Let $\{k_n\}$ and $\{k_m^*\}$ be positive integer sequences satisfying (1), $\{u_{n,m}\}$ a real sequence and $\{r_n\}$, $\{r_m^*\}$, $\{\ell_n\}$ and $\{\ell_m^*\}$ positive integer sequences satisfying (2) and (3). The bivariate random field $\{X_{n,m}\}$ satisfies $D_{k_n,k_m^*}^{(q,q^*)}(v, u_{n,m})$, for $q > 0$, $q^* > 0$ and $v > 0$, if for any integers $1 \le i_1 \le q < i_2 < k_n$ and $1 \le j_1 \le q^* < j_2 < k_m^*$, for which $i_2 - i_1 > \ell_n$, $j_2 - j_1 > \ell_m^*$, the rectangles $J_{i_1,j_1} :=]0, i_1]\times]0, j_1]$ and $J'_{i_2,j_2} :=]i_2, k_n]\times]j_2, k_m^*]$ are such that

$$\left|P(M(J_{i_1,j_1}) \le v, M(J'_{i_2,j_2}) \le u_{n,m}) - P(M(J_{i_1,j_1}) \le v)P(M(J'_{i_2,j_2}) \le u_{n,m})\right| \le \alpha'_{n,m},$$

with $\alpha'_{n,m} \to 0$, $n, m \to +\infty$.

The counterpart of Rényi's property is now deduced.

Lemma 3.2. Let $\{X_{n,m}\}$ be a bivariate stationary random field and $\{k_n\}$ and $\{k_m^*\}$ be positive integer sequences satisfying (1). Suppose that $u_{n,m} = a_{n,m}x + b_{n,m}$ is a real sequence such that $\{k_n k_m^*(1 - F(u_{n,m}))\}_{n,m}$ is bounded, the convergence (23) holds and the condition $D_{k_n,k_m^*}(u_{n,m})$ is satisfied, for any x in the support of G. Assume that $D_{\lfloor\lambda k_n\rfloor,\lfloor\lambda^* k_m^*\rfloor}^{(\lambda k_q,\lambda^* k_{q^*}^*)}(u_{q,q^*}, u_{n,m})$ is satisfied for any fixed and positive reals λ and λ^* and for any positive integers q and q^*. Then

$$P(M_{\lfloor\lambda k_n\rfloor,\lfloor\lambda^* k_m^*\rfloor} \le u_{n,m} | M_{\lfloor\lambda k_q\rfloor,\lfloor\lambda^* k_{q^*}^*\rfloor} \le u_{q,q^*}) \to G^{\lambda\lambda^*}(x),\ n, m \to +\infty,$$

and

$$P(M_{\lfloor\lambda k_n\rfloor,\lfloor\lambda^* k_m^*\rfloor} \le u_{n,m} | B) \to G^{\lambda\lambda^*}(x),\ n, m \to +\infty, \tag{28}$$

for any event B with positive probability.

Proof. For each rectangle $]0, \lfloor\lambda k_n\rfloor]\times]0, \lfloor\lambda^* k_m^*\rfloor]$, define the disjoint sub-rectangles $J_1 :=]\lfloor\lambda k_q\rfloor + \ell_n, \lfloor\lambda k_n\rfloor]\times]0, \lfloor\lambda^* k_{q^*}^*\rfloor + \ell_m^*]$, $J_2 :=]0, \lfloor\lambda k_q\rfloor + \ell_n]\times]\lfloor\lambda^* k_{q^*}^*\rfloor + \ell_m^*, \lfloor\lambda^* k_m^*\rfloor]$ and $J_3 := J_{\lfloor\lambda k_q\rfloor+\ell_n,\lfloor\lambda^* k_{q^*}^*\rfloor+\ell_m^*}$. For simplicity, write $J' := J'_{\lfloor\lambda k_q\rfloor+\ell_n,\lfloor\lambda^* k_{q^*}^*\rfloor+\ell_m^*}$.

We first prove that

$$\begin{aligned} &P(M_{\lfloor\lambda k_n\rfloor,\lfloor\lambda^* k_m^*\rfloor} \le u_{n,m}, M_{\lfloor\lambda k_q\rfloor,\lfloor\lambda^* k_{q^*}^*\rfloor} \le u_{q,q^*}) \\ &\quad = P(M(J') \le u_{n,m}, M(J_{\lfloor\lambda k_q\rfloor,\lfloor\lambda^* k_{q^*}^*\rfloor}) \le u_{q,q^*}) + o_{n,m}(1). \end{aligned} \tag{29}$$

In fact, we have

$$P(M_{\lfloor\lambda k_n\rfloor,\lfloor\lambda^* k^*_m\rfloor} \le u_{n,m}, M_{\lfloor\lambda k_q\rfloor,\lfloor\lambda^* k^*_{q^*}\rfloor} \le u_{q,q^*})$$
$$= P(M(J') \le u_{n,m}, M(J_1 \cup J_2) \le u_{n,m}, M(J_{\lfloor\lambda k_q\rfloor,\lfloor\lambda^* k^*_{q^*}\rfloor}) \le u_{q,q^*}, M(J_3) \le u_{n,m})$$
$$= P(M(J') \le u_{n,m}, M(J_1 \cup J_2) \le u_{n,m}, M(J_{\lfloor\lambda k_q\rfloor,\lfloor\lambda^* k^*_{q^*}\rfloor}) \le u_{q,q^*})$$
$$-P(M(J') \le u_{n,m}, M(J_1 \cup J_2) \le u_{n,m}, M(J_{\lfloor\lambda k_q\rfloor,\lfloor\lambda^* k^*_{q^*}\rfloor}) \le u_{q,q^*}, M(J_3) > u_{n,m}),$$

where the last probability does not exceed

$$P(M(J_3) > u_{n,m}, M(J_{\lfloor\lambda k_q\rfloor,\lfloor\lambda^* k^*_{q^*}\rfloor}) \le u_{q,q^*})$$
$$\le P\left(\left\{\bigcup_{i=\lfloor\lambda k_q\rfloor+1}^{\lfloor\lambda k_q\rfloor+\ell_n}\ \bigcup_{j=1}^{\lfloor\lambda^* k^*_{q^*}\rfloor+\ell^*_m}\{X_{i,j} > u_{n,m}\}\right\}\bigcup\left\{\bigcup_{i=1}^{\lfloor\lambda k_q\rfloor}\ \bigcup_{j=\lfloor\lambda^* k^*_{q^*}\rfloor+1}^{\lfloor\lambda^* k^*_{q^*}\rfloor+\ell^*_m}\{X_{i,j} > u_{n,m}\}\right\}\right)$$
$$\le (\ell_n(\lfloor\lambda^* k^*_{q^*}\rfloor + \ell^*_m) + \ell^*_m\lfloor\lambda k_q\rfloor)\overline{F}(u_{n,m})$$
$$= \left(\frac{\ell_n}{k_n}\frac{\lfloor\lambda^* k^*_{q^*}\rfloor + \ell^*_m}{k^*_m} + \frac{\lfloor\lambda k_q\rfloor}{k_n}\frac{\ell^*_m}{k^*_m}\right) k_n k^*_m \overline{F}(u_{n,m}) \to 0,\ n,m \to +\infty.$$

On the other hand

$$P(M(J') \le u_{n,m}, M(J_1 \cup J_2) \le u_{n,m}, M(J_{\lfloor\lambda k_q\rfloor,\lfloor\lambda^* k^*_{q^*}\rfloor}) \le u_{q,q^*})$$
$$= P(M(J') \le u_{n,m}, M(J_{\lfloor\lambda k_q\rfloor,\lfloor\lambda^* k^*_{q^*}\rfloor}) \le u_{q,q^*})$$
$$-P(M(J') \le u_{n,m}, M(J_1 \cup J_2) > u_{n,m}, M(J_{\lfloor\lambda k_q\rfloor,\lfloor\lambda^* k^*_{q^*}\rfloor}) \le u_{q,q^*})$$

where

$$P(M(J') \le u_{n,m}, M(J_1 \cup J_2) > u_{n,m}, M(J_{\lfloor\lambda k_q\rfloor,\lfloor\lambda^* k^*_{q^*}\rfloor}) \le u_{q,q^*})$$
$$\le P(M(J_1 \cup J_2) > u_{n,m})$$
$$\le P(M(J_1) > u_{n,m}) + P(M(J_2) > u_{n,m})$$
$$\le \left(\frac{\lfloor\lambda k_n\rfloor - \lfloor\lambda k_q\rfloor - \ell_n}{k_n}\frac{\lfloor\lambda^* k^*_{q^*}\rfloor + \ell^*_m}{k^*_m} + \right.$$
$$\left. + \frac{\lfloor\lambda k_q\rfloor + \ell_n}{k_n}\frac{\lfloor\lambda^* k^*_m\rfloor - \lfloor\lambda^* k^*_{q^*}\rfloor - \ell^*_m}{k^*_m}\right) k_n k^*_m \overline{F}(u_{n,m}) \to 0,\ n,m \to +\infty.$$

Henceforth, (29) is proved. Now, due to the fact that $D^{(\lambda k_q,\lambda^* k^*_{q^*})}_{\lfloor\lambda k_n\rfloor,\lfloor\lambda^* k^*_m\rfloor}(u_{q,q^*}, u_{n,m})$ holds, we also get

$$P(M(J') \le u_{n,m}, M(J_{\lfloor\lambda k_q\rfloor,\lfloor\lambda^* k^*_{q^*}\rfloor}) \le u_{q,q^*})$$
$$= P(M(J') \le u_{n,m})P(M(J_{\lfloor\lambda k_q\rfloor,\lfloor\lambda^* k^*_{q^*}\rfloor}) \le u_{q,q^*}) + o_{n,m}(1). \tag{30}$$

So it remains to prove that

$$P(M_{\lfloor\lambda k_n\rfloor,\lfloor\lambda^* k_m^*\rfloor} \leq u_{n,m}) = P(M(J') \leq u_{n,m}) + o_{n,m}(1). \tag{31}$$

Indeed

$$\begin{aligned}P(M_{\lfloor\lambda k_n\rfloor,\lfloor\lambda^* k_m^*\rfloor} \leq u_{n,m}) &= P(M(J') \leq u_{n,m}, M(J_1 \cup J_2 \cup J_3) \leq u_{n,m}) \\ &= P(M(J') \leq u_{n,m})) - P(M(J') \leq u_{n,m}, M(J_1 \cup J_2 \cup J_3) > u_{n,m}),\end{aligned}$$

where, using the previous arguments, we deduce

$$\begin{aligned}&P(M(J') \leq u_{n,m}, M(J_1 \cup J_2 \cup J_3) > u_{n,m}) \\ &\quad\leq P(M(J_1 \cup J_2 \cup J_3) > u_{n,m}) \\ &\quad\leq \left(\frac{\lfloor\lambda k_n\rfloor}{k_n}\frac{\lfloor\lambda^* k_{q^*}^*\rfloor + \ell_m^*}{k_m^*} + \frac{\lfloor\lambda k_q\rfloor + \ell_n}{k_n}\frac{\lfloor\lambda^* k_m^*\rfloor - \lfloor\lambda^* k_{q^*}^*\rfloor - \ell_m^*}{k_m^*}\right) k_n k_m^* \overline{F}(u_{n,m}) \\ &\quad\to 0,\ n, m \to +\infty.\end{aligned}$$

Due to (29), (30) and (31) we conclude that

$$\begin{aligned}&P(M_{\lfloor\lambda k_n\rfloor,\lfloor\lambda^* k_m^*\rfloor} \leq u_{n,m}, M_{\lfloor\lambda k_q\rfloor,\lfloor\lambda^* k_{q^*}^*\rfloor} \leq u_{q,q^*}) \\ &\quad= P(M_{\lfloor\lambda k_n\rfloor,\lfloor\lambda^* k_m^*\rfloor} \leq u_{n,m}) P(M_{\lfloor\lambda k_q\rfloor,\lfloor\lambda^* k_{q^*}^*\rfloor} \leq u_{q,q^*}) + o_{n,m}(1).\end{aligned}$$

Taking into account the limit (24) and using Rényi's property, we are able to deduce (28). □

In what follows $\{T_n\}$ and $\{T_m^*\}$ are sequences of positive integer random variables satisfying

$$T_n/k_n \to T,\ n \to +\infty,\quad T_m^*/k_m^* \to T^*,\ m \to +\infty \text{ in probability,} \tag{32}$$

for some strictly positive random variables T and T^*.

The main result of this paper, Theorem 3.4, establishes the limit in distribution of the linearized maxima $a_{n,m}^{-1}(M_{T_n,T_m^*} - b_{n,m})$. For its proof we need the following lemma that presents the asymptotic closeness between $a_{n,m}^{-1} M_{T_n,T_m^*}$ and $a_{n,m}^{-1} M_{\lfloor k_n T\rfloor,\lfloor k_m^* T^*\rfloor}$.

Lemma 3.3. Let $\{X_{n,m}\}$ be a bivariate stationary real random field under the conditions of Lemma 3.2. Let $\{T_n\}$ and $\{T_m^*\}$ be sequences of positive integer random variables satisfying (32). Then

$$P(|(M_{T_n,T_m^*} - M_{\lfloor k_n T\rfloor,\lfloor k_m^* T^*\rfloor})| > \varepsilon a_{n,m}) \to 0,\ n, m \to +\infty,$$

for all $\varepsilon > 0$.

P r o o f. Due to the fact that T and T^* are strictly positive, for any $\varepsilon_1, \varepsilon_1^* > 0$ there exist $x_0 > 0$ and $x_0^* > 0$ such that $P(T \le x_0) \le \varepsilon_1$ and $P(T^* \le x_0^*) \le \varepsilon_1^*$. Thus, for any $a > 0$ and $b > 0$, we have

$$P\left(\left|\frac{T_n}{k_n} - T\right| > aT\right) \le P\left(\left|\frac{T_n}{k_n} - T\right| > ax_0\right) + \varepsilon_1 < \varepsilon_2$$

and

$$P\left(\left|\frac{T_m^*}{k_m^*} - T^*\right| > bT^*\right) \le P\left(\left|\frac{T_m^*}{k_m^*} - T^*\right| > bx_0^*\right) + \varepsilon_1^* < \varepsilon_2^*.$$

On the other hand, for any $\varepsilon_3 > 0$ we may choose d_0, d_1, d_0^* and d_1^* such that

$$P((T, T^*) \in D) \ge 1 - \varepsilon_3, \tag{33}$$

with $D := [d_0, d_1] \times [d_0^*, d_1^*]$. As a consequence, for any $c, c^* > 0$, we have

$$\begin{aligned} &P(|M_{T_n,T_m^*} - M_{\lfloor k_nT\rfloor,\lfloor k_m^*T^*\rfloor}| > \varepsilon a_{n,m}) \\ &\le P\Bigg(|M_{T_n,T_m^*} - M_{\lfloor k_nT\rfloor,\lfloor k_m^*T^*\rfloor}| > \varepsilon a_{n,m}, \left|\frac{T_n}{k_n} - T\right| \le cT, \\ &\left|\frac{T_m^*}{k_m^*} - T^*\right| \le c^*T^*, (T, T^*) \in D\Bigg) + \varepsilon_2 + \varepsilon_2^* + \varepsilon_3. \end{aligned} \tag{34}$$

Consider that the inequalities $\left|\frac{T_n}{k_n} - T\right| \le cT$ and $\left|\frac{T_n^*}{k_m^*} - T^*\right| \le c^*T^*$ hold. Then $k_nT(1-c) \le T_n \le k_nT(1+c)$ and $k_m^*T^*(1-c^*) \le T_m^* \le k_m^*T^*(1+c^*)$. Then, assuming that $|M_{T_n,T_m^*} - M_{\lfloor k_nT\rfloor,\lfloor k_m^*T^*\rfloor}| > \varepsilon a_{n,m}$, we have the following cases:

i) if $T_n > k_nT$ and $T_m^* > k_m^*T^*$ then $M_{T_n,T_m^*} > M_{\lfloor k_nT\rfloor,\lfloor k_m^*T^*\rfloor}$, and therefore there is at least one $(i,j) \in]k_nT, T_n] \times]0, T_m^*] \bigcup]0, k_nT] \times]k_m^*T^*, T_m^*]$ such that $X_{i,j} > M_{\lfloor k_nT\rfloor,\lfloor k_m^*T^*\rfloor} + \varepsilon a_{n,m}$;

ii) if $T_n > k_nT$ and $T_m^* \le k_m^*T^*$, it can occur:

a) $M_{T_n,T_m^*} > M_{\lfloor k_nT\rfloor,\lfloor k_m^*T^*\rfloor}$ and then there is at least one $(i,j) \in]k_nT, T_n] \times]0, T_m^*]$ such that $X_{i,j} > M_{\lfloor k_nT\rfloor,\lfloor k_m^*T^*\rfloor} + \varepsilon a_{n,m}$;

b) $M_{\lfloor k_nT\rfloor,\lfloor k_m^*T^*\rfloor} > M_{T_n,T_m^*}$ and then there is at least one $(i,j) \in]0, k_nT] \times]T_m^*, k_m^*T^*]$ such that $X_{i,j} > M_{T_n,T_m^*} + \varepsilon a_{n,m}$;

iii) if $T_n \le k_nT$ and $T_m^* > k_m^*T^*$, it can occur:

a) $M_{T_n,T_m^*} > M_{\lfloor k_nT\rfloor,\lfloor k_m^*T^*\rfloor}$ and then there is at least one $(i,j) \in]0, T_n] \times]k_m^*T^*, T_m^*]$ such that $X_{i,j} > M_{\lfloor k_nT\rfloor,\lfloor k_m^*T^*\rfloor} + \varepsilon a_{n,m}$;

b) $M_{\lfloor k_nT\rfloor,\lfloor k_m^*T^*\rfloor} > M_{T_n,T_m^*}$ and then there is at least one $(i,j) \in]T_n, k_nT] \times]0, k_m^*T^*]$ such that $X_{i,j} > M_{T_n,T_m^*} + \varepsilon a_{n,m}$;

iv) if $T_n \le k_nT$ and $T_m^* \le k_m^*T^*$ then $M_{\lfloor k_nT\rfloor,\lfloor k_m^*T^*\rfloor} > M_{T_n,T_m^*}$, and therefore there is at least one $(i,j) \in]T_n, k_nT] \times]0, k_m^*T^*] \cup]0, T_n] \times]T_m^*, k_m^*T^*]$ such that $X_{i,j} > M_{T_n,T_m^*} + \varepsilon a_{n,m}$.

Take into consideration that, when $\left|\frac{T_n}{k_n} - T\right| \leq cT$ holds, both T_n and k_nT are between $k_nT(1-c)$ and $k_nT(1+c)$ and similarly for T_m^* and $k_m^*T^*$. Then we can enlarge suitably the underlying rectangles. Furthermore since $M_{\lfloor k_nT(1-c)\rfloor,\lfloor k_m^*T^*(1-c^*)\rfloor}$ is less or equal than all the other maxima, by (34), we obtain

$$
\begin{aligned}
&P(|M_{T_n,T_m^*} - M_{\lfloor k_nT\rfloor,\lfloor k_m^*T^*\rfloor}| > \varepsilon a_{n,m}) \\
&\quad\leq \sum_{i=1}^{\lfloor k_nT(1+c)\rfloor} \sum_{j=\lfloor k_m^*T^*(1-c^*)\rfloor+1}^{\lfloor k_m^*T^*(1+c^*)\rfloor} 4P\left(X_{i,j} \geq M_{\lfloor k_nT(1-c)\rfloor,\lfloor k_m^*T^*(1-c^*)\rfloor} + \varepsilon a_{n,m}, (T,T^*) \in D\right) \\
&\qquad+ \sum_{i=\lfloor k_nT(1-c)\rfloor+1}^{\lfloor k_nT(1+c)\rfloor} \sum_{j=1}^{\lfloor k_m^*T^*(1+c^*)\rfloor} 4P\left(X_{i,j} \geq M_{\lfloor k_nT(1-c)\rfloor,\lfloor k_m^*T^*(1-c^*)\rfloor} + \varepsilon a_{n,m}, (T,T^*) \in D\right) \\
&\qquad+\varepsilon_2 + \varepsilon_2^* + \varepsilon_3.
\end{aligned}
$$

Meanwhile, due to Proposition 2.4, there is $\widetilde{\varepsilon} > 0$ and $x_1 := x_1(\widetilde{\varepsilon})$ such that

$$P(M_{\lfloor k_nd_0(1-c)\rfloor,\lfloor k_m^*d_0^*(1-c^*)\rfloor} \leq a_{n,m}x_1 + b_{n,m}) \leq \widetilde{\varepsilon}.$$

Then,

$$
\begin{aligned}
&P(|M_{T_n,T_m^*} - M_{\lfloor k_nT\rfloor,\lfloor k_m^*T^*\rfloor}| > \varepsilon a_{n,m}) \\
&\quad\leq 4 \sum_{i=1}^{\lfloor k_nT(1+c)\rfloor} \sum_{j=\lfloor k_m^*T^*(1-c^*)\rfloor+1}^{\lfloor k_m^*T^*(1+c^*)\rfloor} P\left(\frac{X_{i,j} - b_{n,m}}{a_{n,m}} \geq \frac{M_{\lfloor k_nd_0(1-c)\rfloor,\lfloor k_m^*d_0^*(1-c^*)\rfloor} - b_{n,m}}{a_{n,m}} + \varepsilon,\right. \\
&\qquad\left.\frac{M_{\lfloor k_nd_0(1-c)\rfloor,\lfloor k_m^*d_0^*(1-c^*)\rfloor} - b_{n,m}}{a_{n,m}} > x_1, (T,T^*) \in D\right) \\
&\qquad+ 4 \sum_{i=\lfloor k_nT(1-c)\rfloor+1}^{\lfloor k_nT(1+c)\rfloor} \sum_{j=1}^{\lfloor k_m^*T^*(1+c^*)\rfloor} P\left(\frac{X_{i,j} - b_{n,m}}{a_{n,m}} \geq \frac{M_{\lfloor k_nd_0(1-c)\rfloor,\lfloor k_m^*d_0^*(1-c^*)\rfloor} - b_{n,m}}{a_{n,m}} + \varepsilon,\right. \\
&\qquad\left.\frac{M_{\lfloor k_nd_0(1-c)\rfloor,\lfloor k_m^*d_0^*(1-c^*)\rfloor} - b_{n,m}}{a_{n,m}} > x_1, (T,T^*) \in D\right) + \varepsilon_4 \\
&\quad\leq 4 \sum_{i=1}^{\lfloor k_nT(1+c)\rfloor} \sum_{j=\lfloor k_m^*T^*(1-c^*)\rfloor+1}^{\lfloor k_m^*T^*(1+c^*)\rfloor} P\left(\frac{X_{i,j} - b_{n,m}}{a_{n,m}} \geq x_1 + \varepsilon, (T,T^*) \in D\right) \\
&\qquad+ 4 \sum_{i=\lfloor k_nT(1-c)\rfloor+1}^{\lfloor k_nT(1+c)\rfloor} \sum_{j=1}^{\lfloor k_m^*T^*(1+c^*)\rfloor} P\left(\frac{X_{i,j} - b_{n,m}}{a_{n,m}} \geq x_1 + \varepsilon, (T,T^*) \in D\right) + \varepsilon_4 \\
&\quad\leq (8k_nd_1(1+c)k_m^*d_1^*c^* + 8k_nd_1ck_m^*d_1^*(1+c^*))\overline{F}(a_{n,m}(x_1+\varepsilon) + b_{n,m}) + \varepsilon_4,
\end{aligned}
$$

where $\varepsilon_4 = \widetilde{\varepsilon} + \varepsilon_2 + \varepsilon_2^* + \varepsilon_3$. By establishing that $c, c^* \to 0$ and $\varepsilon_4 \to 0$, we obtain the desired result.

□

Theorem 3.4. Let $\{X_{n,m}\}$ be a bivariate stationary random field. Let $\{k_n\}$ and $\{k_m^*\}$ be positive integer sequences satisfying (1) and suppose that there are real sequences $\{a_{n,m} > 0\}$ and $\{b_{n,m}\}$, with $u_{n,m} = a_{n,m}x + b_{n,m}$, such that

$$P(M_{k_n,k_m^*} \leq u_{n,m}) \to G(x),\ n,m \to +\infty,$$

with G non degenerate. Let $\{T_n\}$ and $\{T_m^*\}$ be sequences of positive integer random variables satisfying (32). If $\{X_{n,m}\}$ satisfies the conditions $D_{k_n,k_m^*}(u_{n,m})$ and $D_{k_n,k_m^*}^{(\lambda k_q,\lambda^* k_s^*)}(u_{q,s}, u_{n,m})$ for any (λ, λ^*) in the support of (T, T^*), $S_{(T,T^*)}$, then

$$P(M_{T_n,T_m^*} \leq u_{n,m}) \to \int_{S_{(T,T^*)}} G^{\lambda\lambda^*}(x)\,\mathrm{d}F_{(T,T^*)}(\lambda, \lambda^*),\ n,m \to +\infty.$$

Proof. For each $z \in \mathbb{N}$ and $\mathbf{h} := (h, h^*) \in \mathbb{N}_0^2$ define the disjoint rectangles $A_{\mathbf{h}} =]h2^{-z}, (h+1)2^{-z}] \times]h^*2^{-z}, (h^*+1)2^{-z}]$. Let $d_h := h2^{-z}$, $d_h^* := h^*2^{-z}$ and $\varepsilon > 0$. We have

$$\begin{aligned} P(M_{T_n,T_m^*} \leq u_{n,m}) &= P(M_{T_n,T_m^*} \leq u_{n,m}, |M_{T_n,T_m^*} - M_{\lfloor k_n T\rfloor,\lfloor k_m^* T^*\rfloor}| \leq \varepsilon a_{n,m}) \\ &+P(M_{T_n,T_m^*} \leq u_{n,m}, |M_{T_n,T_m^*} - M_{\lfloor k_n T\rfloor,\lfloor k_m^* T^*\rfloor}| > \varepsilon a_{n,m}). \end{aligned} \tag{35}$$

Considering Lemma 3.3, we prove that the last term in (35) tends to zero as $n, m \to +\infty$.

We also obtain an upper bound for the second term in (35). In fact

$$\begin{aligned} &P\left(M_{T_n,T_m^*} \leq u_{n,m}, |M_{T_n,T_m^*} - M_{\lfloor k_n T\rfloor,\lfloor k_m^* T^*\rfloor}| \leq \varepsilon a_{n,m}\right) \\ &\leq P\left(M_{\lfloor k_n T\rfloor,\lfloor k_m^* T^*\rfloor} - \varepsilon a_{n,m} \leq u_{n,m}, |M_{T_n,T_m^*} - M_{\lfloor k_n T\rfloor,\lfloor k_m^* T^*\rfloor}| \leq \varepsilon a_{n,m}\right) \\ &= \sum_{\mathbf{h}: A_{\mathbf{h}} \cap S_{(T,T^*)} \neq \emptyset} P\left(M_{\lfloor k_n T\rfloor,\lfloor k_m^* T^*\rfloor} - \varepsilon a_{n,m} \leq u_{n,m},\right. \\ &\qquad\qquad \left.|M_{T_n,T_m^*} - M_{\lfloor k_n T\rfloor,\lfloor k_m^* T^*\rfloor}| \leq \varepsilon a_{n,m}, (T, T^*) \in A_{\mathbf{h}}\right) \\ &\leq \sum_{\mathbf{h}: A_{\mathbf{h}} \cap S_{(T,T^*)} \neq \emptyset} P(M_{\lfloor k_n d_h\rfloor,\lfloor k_m^* d_{h^*}\rfloor} \leq u_{n,m} + \varepsilon a_{n,m} | (T, T^*) \in A_{\mathbf{h}}) P((T, T^*) \in A_{\mathbf{h}}) \\ &= \sum_{\mathbf{h}: A_{\mathbf{h}} \cap S_{(T,T^*)} \neq \emptyset} P(M_{\lfloor k_n d_h\rfloor,\lfloor k_m^* d_{h^*}\rfloor} \leq (x+\varepsilon)a_{n,m} + b_{n,m} | (T, T^*) \in A_{\mathbf{h}}) \times \\ &\qquad \times P((T, T^*) \in A_{\mathbf{h}}). \end{aligned}$$

Using (28) and the dominated convergence theorem we get for continuity points x of G

$$\limsup_{z\to+\infty}\lim_{n,m\to+\infty}\sum_{\mathbf{h}:A_{\mathbf{h}}\cap S_{(T,T^*)}\neq\emptyset}P(M_{\lfloor k_n d_h\rfloor,\lfloor k_m^* d_{h^*}\rfloor}\leq(x+\varepsilon)a_{n,m}+b_{n,m}|(T,T^*)\in A_{\mathbf{h}})\times$$
$$\times P((T,T^*)\in A_{\mathbf{h}})$$
$$=\limsup_{z\to+\infty}\sum_{\mathbf{h}:A_{\mathbf{h}}\cap S_{(T,T^*)}\neq\emptyset}G^{d_h d_{h^*}}(x+\varepsilon)P((T,T^*)\in A_{\mathbf{h}})$$
$$=\int_{S_{(T,T^*)}}G^{\lambda\lambda^*}(x+\varepsilon)\,\mathrm{d}F_{(T,T^*)}(\lambda,\lambda^*).$$

Similarly, we get a lower bound of the second term in (35). In fact

$$P(M_{T_n,T_m^*}\leq u_{n,m},|M_{T_n,T_m^*}-M_{\lfloor k_n T\rfloor,\lfloor k_m^* T^*\rfloor}|\leq\varepsilon a_{n,m})$$
$$\geq P(M_{\lfloor k_n T\rfloor,\lfloor k_m^* T^*\rfloor}+\varepsilon a_{n,m}\leq u_{n,m},|M_{T_n,T_m^*}-M_{\lfloor k_n T\rfloor,\lfloor k_m^* T^*\rfloor}|\leq\varepsilon a_{n,m})$$
$$=P(M_{\lfloor k_n T\rfloor,\lfloor k_m^* T^*\rfloor}+\varepsilon a_{n,m}\leq u_{n,m})$$
$$-P(M_{\lfloor k_n T\rfloor,\lfloor k_m^* T^*\rfloor}+\varepsilon a_{n,m}\leq u_{n,m},|M_{T_n,T_m^*}-M_{\lfloor k_n T\rfloor,\lfloor k_m^* T^*\rfloor}|>\varepsilon a_{n,m})$$
$$\geq P(M_{\lfloor k_n T\rfloor,\lfloor k_m^* T^*\rfloor}+\varepsilon a_{n,m}\leq u_{n,m})-P(|M_{T_n,T_m^*}-M_{\lfloor k_n T\rfloor,\lfloor k_m^* T^*\rfloor}|>\varepsilon a_{n,m})$$
$$=\sum_{\mathbf{h}:A_{\mathbf{h}}\cap S_{(T,T^*)}\neq\emptyset}P(M_{\lfloor k_n T\rfloor,\lfloor k_m^* T^*\rfloor}\leq u_{n,m}-\varepsilon a_{n,m}|(T,T^*)\in A_{\mathbf{h}})\times$$
$$\times P((T,T^*)\in A_{\mathbf{h}})-o_n(1)$$
$$=\sum_{\mathbf{h}:A_{\mathbf{h}}\cap S_{(T,T^*)}\neq\emptyset}P(M_{\lfloor k_n d_h\rfloor,\lfloor k_m^* d_{h^*}\rfloor}\leq(x-\varepsilon)a_{n,m}+b_{n,m}|(T,T^*)\in A_{\mathbf{h}})\times$$
$$\times P((T,T^*)\in A_{\mathbf{h}})-o_n(1).$$

Applying the same arguments as for the upper bound, we get for continuity points x of G,

$$\limsup_{z\to+\infty}\lim_{n,m\to+\infty}\sum_{\mathbf{h}:A_{\mathbf{h}}\cap S_{(T,T^*)}\neq\emptyset}P(M_{\lfloor k_n d_h\rfloor,\lfloor k_m^* d_{h^*}\rfloor}\leq(x-\varepsilon)a_{n,m}+b_{n,m}|(T,T^*)\in A_{\mathbf{h}})\times$$
$$\times P((D,D^*)\in A_{\mathbf{h}})$$
$$=\int_{S_{(T,T^*)}}G^{\lambda\lambda^*}(x-\varepsilon)\,\mathrm{d}F_{(T,T^*)}(\lambda,\lambda^*).$$

Combining the two bounds, for continuity points x of G, we have

$$\lim_{\varepsilon\to0}\int_{S_{(T,T^*)}}G^{\lambda\lambda^*}(x-\varepsilon)\,\mathrm{d}F_{(T,T^*)}(\lambda,\lambda^*)\leq\lim_{n,m\to+\infty}P(M_{T_n,T_m^*}\leq u_{n,m})$$
$$\leq\lim_{\varepsilon\to0}\int_{S_{(T,T^*)}}G^{\lambda\lambda^*}(x-\varepsilon)\,\mathrm{d}F_{(T,T^*)}(\lambda,\lambda^*).$$

The proof is concluded. □

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