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# OBSERVER BASED CONTROL FOR STRONG PRACTICAL STABILIZATION OF A CLASS OF UNCERTAIN TIME DELAY SYSTEMS

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In this paper, we address the strong practical stabilization problem for a class of uncertain time delay systems with a nominal part written in triangular form. We propose, firstly, a strong practical observer. Then, we show that strong practical stability of the closed loop system with a linear, parameter dependent, state feedback is achieved. Finally, a separation principle is established, that is, we implement the control law with estimate states given by the strong practical observer and we prove that the closed loop system is strong practical stable. With the help of a numerical example, effectiveness of the proposed approach is demonstrated.

*Keywords:* observer, exponential stability, strong practical stability, time delay, Lyapunov–Krasovskii

*Classification:* 93C10, 93D15

## 1. INTRODUCTION

The separation principle involves the design of a state observer and a state feedback stabilization controller independently. The problem of theory of output feedback stabilization of nonlinear systems using high gain observers has become the focal interest of researchers and it has received many attentions. In these studies, a high gain observer has been used by [2, 3] to provide a separation principle for the considered uncertain system. In this context, [4] proved a separation principle for nonlinear uncertain systems with nominal linear part.

Recently, adaptive observer and controller design for nonlinear time-delay systems have been very prominent and active research fields due to their importance in controlling engineering practice [20] and of real phenomena such as biological systems [27], chemistry [22], electrical [1], population dynamics models [30] and economic model [19]. Under a Lyapunov–Krasovskii functional, suitable choice, [24] derived a control scheme to design an adaptive control to stabilize the nonlinear time-delay systems. These stability findings obtained for delayed systems can be generally classified into two main types, namely delay independent [4, 5, 12, 31] and delay dependent [11, 15]. [16] has suggested the problem of observer for a class of nonlinear delay systems. Global and delay independent conditions are provided. Under a linear observer-based feedback, the

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problem global asymptotic stability for a triangular structure of time-delay systems, is reached by [21]. The problem of state and output feedback stabilization for nonlinear systems that are diffeomorphic has been considered in [17] and [18] with a constructive approach. The systems have triangular form with nonlinearities that appear on each component but there is a restrictive assumption on the nonlinearities. In [9] the problem of exponential stabilization for nonlinear uncertain systems with time-varying delay is proved by using the Lyapunov approach and solving linear matrix inequalities. Under Lipschitz condition and by constructing an appropriate Krasovskii functional and solving linear matrix inequalities [28], an adaptive observer for a class of nonlinear systems with time delays is introduced.

Much attention has been paid to solving the problem of exponential stability because it is an important index to obtain the convergence rates of prescribed time-delay systems. Therefore, exponential stability of systems with delays has been the interest of researchers and the subject of numerous papers and monographs [5, 9, 11, 19, 26, 29, 31]. Based on the Lyapunov method, [26] derived a linear matrix inequality. It proposes a robust exponential stability criterion which is delay dependent. In [10] sufficient conditions are provided to prove the practical stability for a class of nonlinear delay systems satisfying some relaxed triangular-type condition. According to the Lyapunov–Krasovskii functional, the problem of global exponential stability of a class of nonlinear time-delay systems written in triangular form that satisfies a linear growth condition is achieved by [5]. [29] derives sufficient conditions expressed in terms of linear matrix inequalities (LMIs) for exponential stability of linear time-delay systems. For constant time delay, the exponential convergence of the observer is achieved by [19, 31].

However, from a practical point of view, dynamics, measurement noises and other disturbances often prevent origin from being an equilibrium point of the uncertain system. So we can no longer expect to design a controller that guarantees the stability of the origin as an equilibrium point. Due to this reason, the property is referred to as a practical stability which is more suitable for nonlinear free-delay systems see [6, 7, 8] and for nonlinear systems with time-delay see [13, 14, 19, 31, 34]. Practically an exponential convergence of the observation for unknown time delay nonlinear system in triangular form has been proposed in [19]. In [31] a Lyapunov–Krasovskii function is chosen, and sufficient assumptions are provided for the purpose of proving the practical stability. The problem of global uniform practical exponential stability of general nonlinear non autonomous differential delay equations is proved in [13]. The authors of [34] present the practical stability of time delay system by constructing an appropriate Krasovskii functional and solving linear matrix inequalities. By the Lyapunov–Krasovskii theorem, the problem of global practical tracking for a class of uncertain nonlinear multiple time-delay systems by output feedback is derived in [23].

Now, we want to establish some comparisons with results that addressed issues similar to those addressed in this document. It is well known that the problem of stabilization nonlinear systems with time delay is generally more difficult than that of systems without delays. In [8] the authors consider a class of for free-delay nonlinear uncertain systems. However, the system considered in [8] is a particular case of the system considered in our paper. [15] consider a specific canonical form with time delays, which is uniformly observable. Current work is the natural extension of that one proposed [15] to a class of uncertain time delay systems with a nominal part written in a triangular form.

In this paper, motivated by [8] and [28] an observer-based for a class of nonlinear time delay systems is presented. This class of systems having a triangular structure. Constant delays for nonlinear parts of the system have been considered in which the nonlinear parts satisfy the Lipschitz condition while the uncertain part is bounded. Based on Lyapunov–Krasovskii functional, we investigate the problem of designing an observer-based output feedback controller in order to practically exponentially stabilize the closed-loop system. Uncertain bounded and sufficient conditions are given to insure the strong practical exponential stability of the proposed observer.

The paper is organized as follows: Section 2 presents the basic definitions, some preliminary results and the system description. The required assumptions and the statement of the main results as strong practical exponential stability for time varying nonlinear systems are provided in section 3. Section 4 illustrates the validity of our design method in the selected numerical example. Section 5 draws some conclusions based on the findings of the research. Finally, this paper is concluded.

## 2. SYSTEM DESCRIPTION AND BASIC RESULTS

In this paper, we investigate stability of nonlinear time delay systems. Consider the nonlinear differential equations:

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) + f(x(t), x(t-\tau), u(t)) + Bg(t, x(t), x(t-\tau)), & \forall t \geq 0, \\ y(t) = Cx(t), & \\ x(t) = \varphi(t), & \forall t \in [-\tau, 0], \end{cases} \tag{1}$$

where $\tau > 0$ denotes the time delay, $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}$ is the input of the system, $y(t) \in \mathbb{R}$ is the measured output and the initial condition is specified as a continuous function $\varphi \in \mathcal{C}$, where $\mathcal{C}$ denotes the Banach space of continuous functions mapping the interval $[-\tau, 0] \to \mathbb{R}^n$ equipped with the supremum-norm:

$$\| \varphi \|_\infty = \max_{s \in [-\tau, 0]} \| \varphi(s) \|$$

$\| \, \|$ being the Euclidean-norm. The matrices $A$, $B$ and $C$ are given by,

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \end{bmatrix},$$

and the perturbed term is

$$f(x(t), x(t-\tau), u(t)) = \begin{bmatrix} f_1(x_1(t), x_1(t-\tau), u(t)) \\ f_2(x_1(t), x_2(t), x_1(t-\tau), x_2(t-\tau), u(t)) \\ \vdots \\ f_n(x(t), x(t-\tau), u(t)) \end{bmatrix}.$$

We assume that the mappings $f_i : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$, $i = 1, \ldots, n$, are smooth with $f_i(0, 0, u(t)) = 0$ and $g : \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, is unknown and represent the parameter perturbations. For $\theta \in [-\tau, 0]$, we define the function segment $x_t$ by $x_t(\theta) = x(t + \theta)$. For $\varphi \in \mathcal{C}$, the solution of (1) that satisfies $x_0 = \varphi$ denote by $x(t, \varphi)$ or shortly $x(t)$. Finally, we denote the segment of this solution by $x_t(\varphi)$ or shortly $x_t$.

The main objective of this paper is the design of a nonlinear observer based controller that stabilizes system (1). We generalize the systems considered by [6] for free-delay systems. We shall explore a strong practical observer introduced in [33]. That is the error converges exponentially towards an arbitrarily small neighborhood of the origin. All these observers are parameter dependent systems.

For free delay systems, the problem of the conception of a strong practical observer is addressed in [33] for an uncertain system with nominal linear part and in [8] for an uncertain system with nonlinear nominal part. Inspired from [33] and [8], we introduce the following definition of a strong practical observer for the uncertain time delay system (1).

**Definition 2.1.** (Benabdallah et al. [8]) Consider a system that depends on a parameter $\varepsilon > 0$:

$$\begin{cases} \dot{x}(t) &= F_\varepsilon(t, x(t), x(t-\tau)), \\ y(t) &= h(x(t), x(t-\tau)), \\ x(t_0) &= \varphi(s), \end{cases} \tag{2}$$

where $x(t) \in \mathbb{R}^n$, $t \in \mathbb{R}_+$ and $F_\varepsilon : \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous in $t$ and locally Lipschitz in $x(t)$ and $x(t-\tau)$ uniformly with respect to $t$. We call the origin of system (2) is practically globally uniformly exponentially stable if there exists $\varepsilon^* > 0$ such that for all $\varepsilon < \varepsilon^*$ there exist positive scalars $K(\varepsilon)$, $\lambda(\varepsilon)$ and $\rho(\varepsilon)$, such that for all $t \geq t_0$ and $\varphi \in \mathcal{C}$, we have

$$\|x(t)\| \leq K(\varepsilon)\|\varphi\| \exp(-\lambda(\varepsilon)(t - t_0)) + \rho(\varepsilon)$$

with $\lim_{\varepsilon \to 0} \lambda(\varepsilon) = +\infty$ and $\lim_{\varepsilon \to 0} \rho(\varepsilon) = 0$.

**Definition 2.2.** A strong practical observer for system (2) is a family of auxiliary dynamic systems written as $\dot{\hat{x}} = G_\varepsilon(t, \hat{x}(t), \hat{x}(t-\tau), y(t))$ such that for all $t \geq 0$

$$\|\hat{x}(t) - x(t)\| \leq K(\varepsilon)e^{-\lambda(\varepsilon)t} \sup_{s \in [-\tau, 0]} \|\hat{x}(s) - x(s)\| + \rho(\varepsilon)$$

with $\lim\limits_{\varepsilon \to 0} \lambda(\varepsilon) = +\infty$ and $\lim\limits_{\varepsilon \to 0} \rho(\varepsilon) = 0$.

**Remark 2.3.** When $\rho(\varepsilon) = 0$, in this case the origin is an equilibrium point, then we point the classical definition of the exponential stability (see[5, 32]).

**Notation 1.** Throughout the paper, the notation $A^T$ means the transpose of $A$. For a real matrix $A$ we denote the minimal and maximal eigenvalue of a matrix $A$ by $\lambda_{\min}(A)$ and $\lambda_{\max}(A)$ respectively. We note $x^\tau$ the delayed state vector $x(t - \tau)$.

## 3. SEPARATION PRINCIPLE

To complete the description of the time-delay system, we suppose the following assumptions.

**Assumption 1.** The nonlinearity $f(x(t), x(t-\tau), u(t))$ is smooth, globally Lipschitz,to $x$ and $x(t-\tau)$, uniformly with respect to $u$ and well-defined for all $x(t) \in \mathbb{R}^n$ with $f(0,0,u)=0$.

**Assumption 2.** For all $t \geq 0$, the delay $\tau$ is known and constant.

**Assumption 3.** There exists a positive scalar $M$ such that $\|g(t, x(t), x(t-\tau))\| \leq M$.

**Remark 3.1.** In [25] time delays are considered only in the linear component of the system, whereas the present work deals delays nonlinear parts of the system.

**Remark 3.2.** The systems (1) generalizes the systems introduced by [21] for the case of an nonlinear system without disturbance.

### 3.1. Observer design

In this subsection, we are studying in designing an observer to estimate the states of the time-delay nonlinear system. Generally, the direct measurement of all states of a system is not achieved. It is this fact which has a great interest in studying the systems whose desired behavior makes the zeros of the state space exponentially stable or an approximation close to it. The main objective of the next subsection is to construct a state estimator to ensure the practical convergence of the stability of the resulting error system. Under the time delay constant and known, we present delay-independent conditions to ensure strong practical exponential convergence of the observation error. We will design an observer for system (1) under assumptions **A1**, **A2** and **A3**. We propose the following system:

$$\begin{cases} \dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) + f(\hat{x}(t), \hat{x}^\tau, u(t)) + L(\varepsilon)(C\hat{x}(t) - y(t)), \\ \hat{y}(t) = C\hat{x}(t), \end{cases} \tag{3}$$

where $L(\varepsilon) = [\frac{l_1}{\varepsilon}, \ldots, \frac{l_n}{\varepsilon^n}]^T$, with $\varepsilon > 0$ and $L = [l_1, \ldots, l_n]^T$ such that $A_L := A + LC$ is Hurwitz, $\hat{x}(s) = \hat{\phi}(s),\ -\tau \leq s \leq 0$ with $\hat{\phi} : [-\tau, 0] \to \mathbb{R}^n$ being any known continuous function. Let us now define $e = \hat{x} - x$ the observation error, which denotes the difference between the actual state and estimated states. The observation error, whose dynamics is

$$\dot{e} = (A + L(\varepsilon)C)e + f(\hat{x}, \hat{x}^\tau, u) - f(x, x^\tau, u) - Bg(t, x, x^\tau). \tag{4}$$

Let $P$ the symmetric positive definite which satisfies solution of the Lyapunov equation

$$A_L^T P + P A_L = -I. \tag{5}$$

**Theorem 3.3.** Consider the time-delay system (1) under assumptions **A1**, **A2** and **A3**. Then, the system (3) is a strong practical observer for system (1).

Proof. For $\varepsilon > 0$, let $D(\varepsilon) = diag[1, \varepsilon, \dots, \varepsilon^{n-1}]$. Let $\eta = D(\varepsilon)e$.
Using the fact that $A + L(\varepsilon)C = \frac{1}{\varepsilon}D(\varepsilon)^{-1}A_L D(\varepsilon)$, we get

$$\dot{\eta} = \frac{1}{\varepsilon}A_L\eta + D(\varepsilon)(f(\hat{x}, \hat{x}^\tau, u) - f(x, x^\tau u)) - D(\varepsilon)Bg(t, x, x^\tau). \tag{6}$$

Let us choose a Lyapunov–Krasovskii functional candidate as follows

$$V(\eta_t) = \eta^T P\eta + \int_{t-\tau}^{t} e^{\frac{\ln\frac{1}{\varepsilon}}{2\tau}(s-t)}\|\eta(s)\|^2 \mathrm{d}s. \tag{7}$$

The time derivative of (7) along the trajectories of (6) can be expressed as

$$\dot{V}(\eta_t) = 2\eta^T P\dot{\eta} + \|\eta\|^2 - \sqrt{\varepsilon}\|\eta^\tau\|^2 - \frac{\ln\frac{1}{\varepsilon}}{2\tau}(V(\eta_t) - \eta^T P\eta). \tag{8}$$

Making use of (5), one obtains for $\varepsilon < 1$

$$\begin{aligned}\dot{V}(\eta_t) + \tfrac{\ln\frac{1}{\varepsilon}}{2\tau}V(\eta_t) \quad &\le \quad -(\tfrac{1}{\varepsilon} - 1)\|\eta\|^2 + \tfrac{\ln\frac{1}{\varepsilon}}{2\tau}\eta^T P\eta - \sqrt{\varepsilon}\|\eta^\tau\|^2 \\ &\quad +2\eta^T PD(\varepsilon)(f(\hat{x}, \hat{x}^\tau, u) - f(x, x^\tau u)) - 2\eta^T PD(\varepsilon)Bg(t, x, x^\tau).\end{aligned}$$

Since $P$ satisfies the following inequality, for all $\eta \in \mathbb{R}^n$,

$$\lambda_{\min}(P)\|\eta\|^2 \le \eta^T P\eta \le \lambda_{\max}(P)\|\eta\|^2. \tag{9}$$

The following inequality holds thanks to **Assumption 1**

$$\begin{aligned}\|D(\varepsilon)(f(\hat{x}, \hat{x}^\tau, u) - f(x, x^\tau u))\| \quad &\le \quad k\|D(\varepsilon)(\hat{x} - x)\| + k\|D(\varepsilon)(\hat{x}^\tau - x^\tau)\| \\ &\le \quad k(\|\eta\| + \|\eta^\tau\|)\end{aligned} \tag{10}$$

where $k$ is a Lipschitz constant in (10).
So using Assumption 3, (10) and (9), one obtains

$$\begin{aligned}\dot{V}(\eta_t) + \tfrac{\ln\frac{1}{\varepsilon}}{2\tau}V(\eta_t) \quad &\le \quad -\left\{\tfrac{1}{\varepsilon} - 1\right\}\|\eta\|^2 + \tfrac{\ln\frac{1}{\varepsilon}}{2\tau}\eta^T P\eta + 2k\|P\|\|\eta\|^2 + 2k\|P\|\|\eta\|\|\eta^\tau\| \\ &\quad +2M\varepsilon^{n-1}\|P\|\|\eta\| - \sqrt{\varepsilon}\|\eta^\tau\|^2 \\ &\le \quad -\left\{\tfrac{1}{\varepsilon} - 1 - \lambda_{\max}(P)\tfrac{\ln\frac{1}{\varepsilon}}{2\tau} - 2k\|P\|\right\}\|\eta\|^2 + 2k\|P\|\|\eta\|\|\eta^\tau\| \\ &\quad +2M\varepsilon^{n-1}\|P\|\|\eta\| - \sqrt{\varepsilon}\|\eta^\tau\|^2\end{aligned}$$

Let $\mu = 2M\varepsilon^{n-1}\|P\|$. Using the fact that

$$\begin{aligned}\mu\|\eta\| \quad &\le \quad \tfrac{1}{4}\|\eta\|^2 + \mu^2 \\ 2k\|P\|\|\eta\|\|\eta^\tau\| \quad &\le \quad \tfrac{1}{\sqrt{\varepsilon}}k^2\|P\|^2\|\eta\|^2 + \sqrt{\varepsilon}\|\eta^\tau\|^2\end{aligned}$$

we deduce that

$$\dot{V}(\eta_t)+\frac{\ln\frac{1}{\varepsilon}}{2\tau}V(\eta_t)-\mu^2\leq-\left\{\frac{1}{\varepsilon}-\|P\|\frac{\ln\frac{1}{\varepsilon}}{2\tau}-\frac{5}{4}-2k\|P\|-\frac{1}{\sqrt{\varepsilon}}k^2\|P\|^2\right\}\|\eta\|^2. \quad (11)$$

Now, all we need to do is to choose $\varepsilon$ small enough such that

$$k(\varepsilon)=\frac{1}{\varepsilon}-\|P\|\frac{\ln\frac{1}{\varepsilon}}{2\tau}-\frac{5}{4}-2k\|P\|-\frac{1}{\sqrt{\varepsilon}}k^2\|P\|^2>0. \quad (12)$$

It is obvious that $k(\varepsilon)$ tends to $\infty$ as $\varepsilon$ tends to zero. This implies that there exists $\varepsilon_0\in]0,1[$ such that for all $0<\varepsilon<\varepsilon_0$ condition (12) is fulfilled.
From (11) and (12) we get

$$\dot{V}(\eta_t)+\frac{\ln\frac{1}{\varepsilon}}{2\tau}V(\eta_t)-\mu^2\leq 0. \quad (13)$$

Set $\lambda(\varepsilon)=\frac{\ln\frac{1}{\varepsilon}}{4\tau}$. Using the well-known Gronwall inequality, inequality (13) is equivalent to

$$V(\eta_t)\leq e^{-2\lambda(\varepsilon)t}V(\eta_0)+\mu^2\frac{2\tau}{\ln\frac{1}{\varepsilon}}. \quad (14)$$

Using (7) and (9), we have, on the one hand,

$$\begin{aligned} V(\eta_0) &\leq \lambda_{\max}(P)\|\eta_0\|^2+\int_{-\tau}^{0}e^{\frac{s}{2\tau}\ln\frac{1}{\varepsilon}}\|\eta(s)\|^2\mathrm{d}s \\ &\leq (\lambda_{\max}(P)+\tau)\sup_{s\in[-\tau,0]}\|\eta(s)\|^2, \end{aligned}$$

and on the other hand,

$$\lambda_{\min}(P)\|\eta(t)\|^2\leq V(\eta(t)).$$

We deduce that

$$\|\eta(t)\|\leq\sqrt{\frac{\|P\|+\tau}{\lambda_{\min}(P)}}e^{-\lambda(\varepsilon)t}\sup_{s\in[-\tau,0]}\|\eta(s)\|+\sqrt{\frac{2\mu^2\tau}{\lambda_{\min}(P)\ln\frac{1}{\varepsilon}}}. \quad (15)$$

Since $\varepsilon<1$, one has

$$\|\eta(t)\|\leq\|e\|\leq\frac{1}{\varepsilon^{n-1}}\|\eta(t)\|$$

and inequality (15) can be written as

$$\|e(t)\|\leq K(\varepsilon)e^{-\lambda(\varepsilon)t}\sup_{s\in[-\tau,0]}\|e(s)\|+\rho(\varepsilon)$$

where

$$\begin{aligned} K(\varepsilon) &= \tfrac{1}{\varepsilon^{n-1}}\sqrt{\tfrac{\|P\|+\tau}{\lambda_{\min}(P)}} \\ \rho(\varepsilon) &= 2M\|P\|\sqrt{\tfrac{2\tau}{\lambda_{\min}(P)\ln\frac{1}{\varepsilon}}} \\ \lambda(\varepsilon) &= \tfrac{\ln\frac{1}{\varepsilon}}{4\tau}, \end{aligned}$$

with $\lim_{\varepsilon\to 0}\rho(\varepsilon)=0$ and $\lim_{\varepsilon\to 0}\lambda(\varepsilon)=+\infty$. □

**Remark 3.4.** In [25, 28], the sufficient conditions which guarantee that the estimation error converges asymptotically towards zero are given in terms of a linear matrix inequality. The conditions given in Theorem 3.3 are quite different from the ones used in [25, 28]. Comparing with [25, 28], our results are less conservative and more convenient to use since they are independent of time delays and the results in [25, 28], are not practical for networks with large size. The free terms in Theorem, for example, $\varepsilon_0$ and $\varepsilon$, help in the observer design (3) for the system (1), but the linear matrix inequality in [25, 28] do not.

### 3.2. Global exponential stabilization by state feedback

In this subsection, using the Lyapunov–Krasovskii method, we establish a delay-independent condition for the strong practical exponential stabilization of the nonlinear system (1). The state feedback controller is given by

$$u = K(\varepsilon)x \tag{16}$$

where $K(\varepsilon) = [\frac{k_1}{\varepsilon^n}, \dots, \frac{k_n}{\varepsilon}]$ and $K = [k_1, \dots, k_n]$ such that $A_K := A + BK$ is Hurwitz. Let $S$ be the symmetric positive definite solution of the Lyapunov equation

$$A_K^T S + S A_K = -I. \tag{17}$$

**Theorem 3.5.** Consider the time-delay system (1) under assumptions **A1**, **A2** and **A3**. Then, the closed loop time-delay system (1) – (16) is strong globally practically exponentially stable.

Proof. The closed-loop dynamics are given by

$$\dot{x} = (A + BK(\varepsilon))x + f(x, x^\tau, u) + Bg(t, x, x^\tau). \tag{18}$$

For $\varepsilon > 0$, let $D(\varepsilon) = diag[1, \varepsilon, \dots, \varepsilon^{n-1}]$ and $\chi = D(\varepsilon)x$.
Using the fact that $A + BK(\varepsilon) = \frac{1}{\varepsilon} D(\varepsilon)^{-1} A_K D(\varepsilon)$, we get

$$\dot{\chi} = \frac{1}{\varepsilon} A_K \chi + D(\varepsilon) f(x, x^\tau, u) + D(\varepsilon) Bg(t, x, x^\tau). \tag{19}$$

Let the Lyapunov–Krasovskii

$$W(\chi_t) = \chi^T S \chi + \int_{t-\tau}^{t} e^{\frac{\ln \frac{1}{\varepsilon}}{2\tau}(s-t)} \|\chi(s)\|^2 \mathrm{d}s. \tag{20}$$

Taking the time derivative of (20) along the trajectories of (19), can be expressed as

$$\dot{W}(\chi_t) = 2\chi^T S \dot{\chi} + \|\chi\|^2 - \sqrt{\varepsilon}\|\chi^\tau\|^2 - \frac{\ln \frac{1}{\varepsilon}}{2\tau}(W(\chi_t) - \chi^T S \chi). \tag{21}$$

As in the proof of Theorem 3.3, we have,

$$\lambda_{\min}(S)\|\chi\|^2 \le \chi^T S \chi \le \lambda_{\max}(S)\|\chi\|^2. \tag{22}$$

Since $f(0,0,u)=0$, (10) implies that

$$\|D(\varepsilon)f(x,x^\tau u)\| \leq k(\|\chi\|+\|\chi^\tau\|). \tag{23}$$

So using Assumption 3, (23) and (22), one obtains

$$\begin{aligned}\dot{W}(\chi_t)+\tfrac{\ln\frac{1}{\varepsilon}}{2\tau}W(\chi_t) &\leq -\left\{\tfrac{1}{\varepsilon}-1\right\}\|\chi\|^2+\tfrac{\ln\frac{1}{\varepsilon}}{2\tau}\chi^TS\chi+2k\|S\|\|\eta\|^2+2k\|S\|\|\chi\|\|\chi^\tau\| \\ &\quad +2M\varepsilon^{n-1}\|S\|\|\chi\|-\sqrt{\varepsilon}\|\chi^\tau\|^2 \\ &\leq -\left\{\tfrac{1}{\varepsilon}-1-\|S\|\tfrac{\ln\frac{1}{\varepsilon}}{2\tau}-2k\|S\|\right\}\|\chi\|^2+2k\|S\|\|\chi\|\|\chi^\tau\| \\ &\quad +2M\varepsilon^{n-1}\|S\|\|\chi\|-\sqrt{\varepsilon}\|\chi^\tau\|^2.\end{aligned}$$

Let $\mu_1=2M\varepsilon^{n-1}\|S\|$. Using the fact that

$$\begin{aligned}\mu_1\|\chi\| &\leq \tfrac{1}{4}\|\chi\|^2+\mu_1^2 \\ 2k\|S\|\|\chi\|\|\chi^\tau\| &\leq \tfrac{1}{\sqrt{\varepsilon}}k^2\|S\|^2\|\chi\|^2+\sqrt{\varepsilon}\|\chi^\tau\|^2\end{aligned}$$

we deduce that

$$\dot{W}(\chi_t)+\frac{\ln\frac{1}{\varepsilon}}{2\tau}W(\chi_t)-\mu_1^2 \leq -\left\{\frac{1}{\varepsilon}-\|S\|\frac{\ln\frac{1}{\varepsilon}}{2\tau}-\frac{5}{4}-2k\|S\|-\frac{1}{\sqrt{\varepsilon}}k^2\|S\|^2\right\}\|\eta\|^2. \tag{24}$$

Now, all we need to do is to choose $\varepsilon$ small enough such that

$$k_1(\varepsilon)=\frac{1}{\varepsilon}-\|S\|\frac{\ln\frac{1}{\varepsilon}}{2\tau}-\frac{5}{4}-2k\|S\|-\frac{1}{\sqrt{\varepsilon}}k^2\|S\|^2>0. \tag{25}$$

It is obvious that $k_1(\varepsilon)$ tends to $\infty$ as $\varepsilon$ tends to zero. This implies that there exists $\varepsilon_0\in]0,1[$ such that for all $0<\varepsilon<\varepsilon_0$ condition (25) is fulfilled.
From (24) and (25) we get

$$\dot{W}(\chi_t)+\frac{\ln\frac{1}{\varepsilon}}{2\tau}W(\chi_t)-\mu_1^2 \leq 0. \tag{26}$$

Set $\lambda(\varepsilon)=\frac{\ln\frac{1}{\varepsilon}}{4\tau}$. Using the well-known Gronwall inequality, inequality (26) is equivalent to

$$W(\chi_t) \leq e^{-2\lambda(\varepsilon)t}W(\chi_0)+\mu_1^2\frac{2\tau}{\ln\frac{1}{\varepsilon}}. \tag{27}$$

Using (20) and (22), we have, on the one hand,

$$\begin{aligned}W(\chi_0) &\leq \lambda_{\max}(S)\|\chi_0\|^2+\int_{-\tau}^{0}e^{\frac{s}{2\tau}\ln\frac{1}{\varepsilon}}\|\chi(s)\|^2\mathrm{d}s \\ &\leq (\lambda_{\max}(S)+\tau)\sup_{s\in[-\tau,0]}\|\chi(s)\|^2,\end{aligned}$$

and on the other hand,

$$\lambda_{\min}(S)\|\chi(t)\|^2 \leq W(\chi(t)).$$

We deduce that

$$\|\chi(t)\| \leq \sqrt{\frac{\|S\|+\tau}{\lambda_{\min}(S)}} e^{-\lambda(\varepsilon)t} \sup_{s\in[-\tau,0]} \|\chi(s)\| + \sqrt{\frac{2\mu_1^2\tau}{\lambda_{\min}(S)\ln\frac{1}{\varepsilon}}}. \tag{28}$$

Since $\varepsilon < 1$, one has

$$\|\chi(t)\| \leq \|x\| \leq \frac{1}{\varepsilon^{n-1}}\|\chi(t)\|$$

and inequality (28) can then be expressed as follow

$$\|x(t)\| \leq K_1(\varepsilon)e^{-\lambda(\varepsilon)t} \sup_{s\in[-\tau,0]} \|x(s)\| + \rho_1(\varepsilon)$$

where

$$\begin{aligned} K_1(\varepsilon) &= \tfrac{1}{\varepsilon^{n-1}}\sqrt{\tfrac{\|S\|+\tau}{\lambda_{\min}(S)}} \\ \rho_1(\varepsilon) &= 2M\|S\|\sqrt{\tfrac{2\tau}{\lambda_{\min}(S)\ln\frac{1}{\varepsilon}}} \\ \lambda(\varepsilon) &= \tfrac{\ln\frac{1}{\varepsilon}}{4\tau}, \end{aligned}$$

with $\lim_{\varepsilon\to 0}\rho_1(\varepsilon) = 0$ and $e\lim_{\varepsilon\to 0}\lambda(\varepsilon) = +\infty$. □

### 3.3. Observer-based control stabilization

In this subsection, is devoted to the design of the observer-based controller. We implement the control law with estimate states. The observer-based controller is given by:

$$u = K(\varepsilon)\hat{x}, \tag{29}$$

where $\hat{x}$ is provided by the observer (3).

**Theorem 3.6.** Suppose that Assumptions 1−2−3 are satisfied. Then, there exists $\varepsilon_0 \in ]0,1[$ such that the origin of the closed loop time-delay system (1)−(29) is strong globally practically exponentially stable.

P r o o f . The closed loop system in the $(\chi,\eta)$ coordinates can be written as follows:

$$\begin{aligned} \dot{\chi} &= \tfrac{1}{\varepsilon}A_K\chi + \tfrac{1}{\varepsilon}BK\eta + D(\varepsilon)f(x,x^\tau,u) + D(\varepsilon)Bg(t,x,x^\tau), \\ \dot{\eta} &= \tfrac{1}{\varepsilon}A_L\eta + D(\varepsilon)(f(\hat{x},\hat{x}^\tau,u) - f(x,x^\tau,u)) - D(\varepsilon)Bg(t,x,x^\tau). \end{aligned} \tag{30}$$

Let

$$U(\eta_t,\chi_t) = \alpha V(\eta_t) + W(\chi_t).$$

From the proof of Theorems 3.3 and 3.5, we get

$$\dot{U}(\eta_t,\chi_t) + \tfrac{\ln\frac{1}{\varepsilon}}{2\tau}U(\eta_t,\chi_t) - \alpha\mu^2 - \mu_1^2 \;\leq\; -\alpha k(\varepsilon)\|\eta\|^2 - k_1(\varepsilon)\|\chi\|^2 + \tfrac{2}{\varepsilon}\|S\|\|K\|\|\eta\|\|\chi\|.$$

Now using the fact that for all $\theta > 0$,

$$2\|\chi\|\|\eta\| \leq \theta\|\chi\|^2 + \frac{1}{\theta}\|\eta\|^2.$$

Then, by taking $\theta = \frac{\varepsilon k_1(\varepsilon)}{2\|S\|\|K\|}$, we can write that

$$\dot{U}(\eta_t, \chi_t) + \frac{\ln\frac{1}{\varepsilon}}{2\tau}U(\eta_t, \chi_t) - \alpha\mu^2 - \mu_1^2 \leq -\alpha k(\varepsilon)\|\eta\|^2 - \frac{k_1(\varepsilon)}{2}\|\chi\|^2 + \frac{2}{\varepsilon^2 k_1(\varepsilon)}\|S\|^2\|K\|^2\|\eta\|^2.$$

Finally we select $\alpha$ such that

$$\alpha k(\varepsilon) - \frac{2}{\varepsilon^2 k_1(\varepsilon)}\|S\|^2\|K\|^2 > 0$$

to deduce that the origin of system (30) is globally practically exponentially stable. □

## 4. NUMERICAL EXAMPLE

In this section, as an illustration, we give an example of application of our observer to a biological Predator-Prey Interaction. We consider a system with two populations. Such model is described by the following system:

$$\begin{array}{lcl} \dot{x}_1 & = & x_2(t) + \frac{1}{2}x_1(t-\tau) + x_1(t)\cos u, \\ \dot{x}_2 & = & \frac{1}{2}x_2(t-\tau) + \frac{1}{2}x_2(t) + u + g(t), \\ y(t) & = & x_1(t) \end{array} \tag{31}$$

where, $x_1(t)$ denotes the prey population, $x_2(t)$ represents mature predators. $u$ denotes the change rate of the predators and is regarded as an known input. $g$ is the transition from homogeneous to diffusive predator movement.(i. e. the introduction of mobility limitation.) The constant delay in system (31) can be regarded as a gestation period or reaction time of the predators. In order to test the obtained solution the unknown exogenous disturbance is supposed as

$$g(t) = \begin{bmatrix} 7.5\cos(80t) \\ 10\sin(20t) \end{bmatrix}.$$

The initial conditions for the system are $x(0) = [-10, -20]^T$, and the initial conditions the observer have been given by $\hat{x}(0) = [10, 10]^T$. We have the Lipschitz constant defined in (10) equal to $\frac{1}{\sqrt{2}}$. Now, select $K = [-4 - 9]$ and $L = [-5 - 5]^T$ , $A_K$ and $A_L$ are Hurwitz. Using Matlab, the solutions of the Lyapunov equations (5) and (17) are given by

$$P = \begin{bmatrix} 0.1200 & 0.1000 \\ 0.1000 & 1.100 \end{bmatrix}$$

$$S = \begin{bmatrix} 1.1944 & -0.5000 \\ -0.5000 & 0.2778 \end{bmatrix}.$$

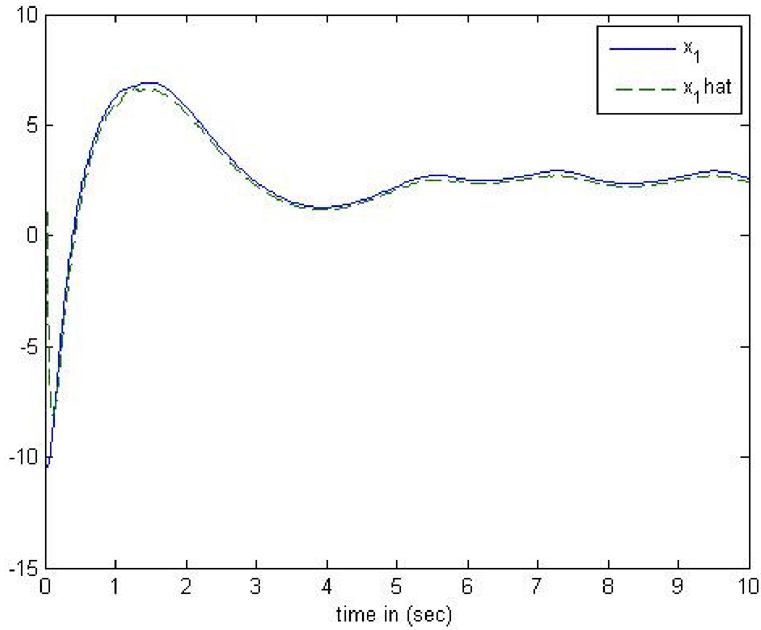


**Fig. 1.** The evolution of $x_1$ and its estimate $\hat{x}_1$ for $\tau = 1$ s.

So, $\|P\| = 1.1101$ and $\|S\| = 1.4144$. According to the strong practical stability enhanced in the proof of Theorem 3.3 and Theorem 3.5, it is clear that the observation error of the observer converges to a ball with a radius $r > 0$ depending on a parameter $\varepsilon$. The value of the observer-based control parameter $\varepsilon$ has been set to 0.01.
Fig. 1 and Fig. 2 shows that the estimated magnitudes converges practically to the real one, for a constant delay equal to $\tau = 1\ s$. It can be seen, from Fig. 3 and Fig. 4, that for a large delay $\tau = 10\ s$ and for different initial conditions the converges practically to the real one is still verified which means that are not dependent on the size of the delay, for all parameter $\varepsilon \leq 0.01$. Fig. 5 and Fig. 6, we show that for a delay $\tau = 1$ and parameter $\varepsilon = 10^{-4}$. Under different time delays, the above Figures 1-6 show the performance of the observer-controller are bound to on a parameter $\varepsilon$. By comparing the convergence radius $r$, we note that if the parameter $\varepsilon$ tends to 0 the radius of convergence is also.

**Remark 4.1.** It is obvious that our result is an extension observer synthesis for a class of nonlinear time delay systems for a similar class of free-delay systems inspired from [6, 8].

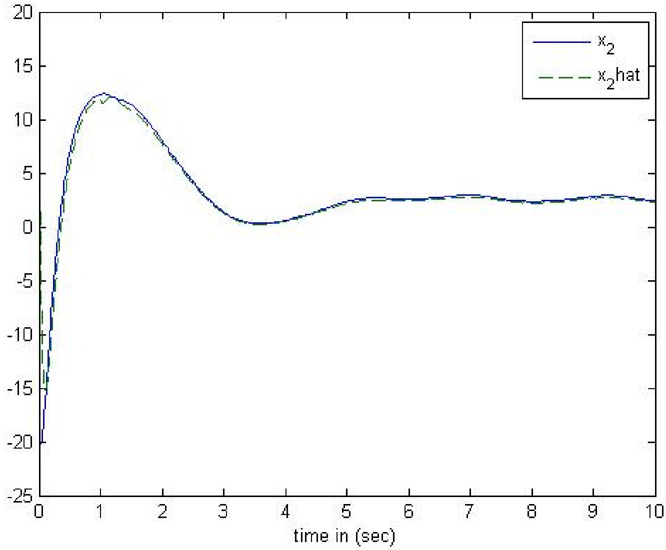


**Fig. 2.** The evolution of $x_2$ and its estimate $\hat{x}_2$ for $\tau = 1$ s.

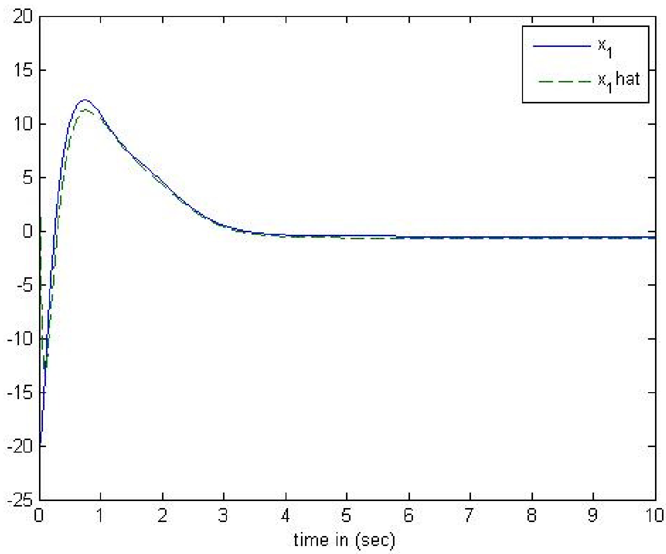


**Fig. 3.** The evolution of $x_1$ and its estimate $\hat{x}_1$ for $\tau = 10$ s.

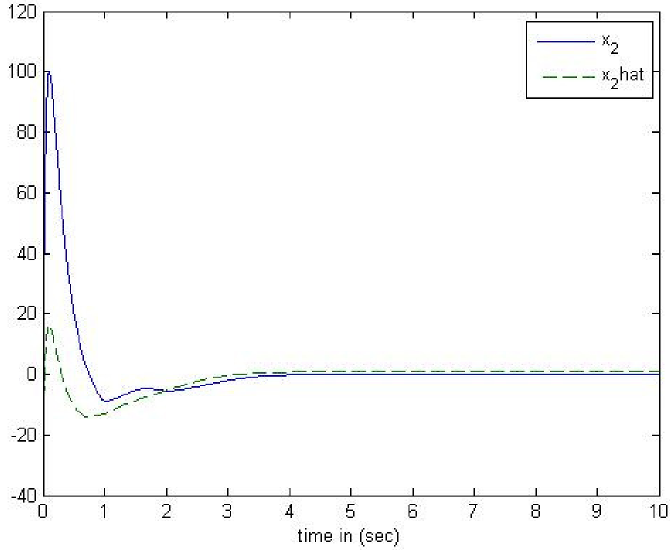


**Fig. 4.** The evolution of $x_2$ and its estimate $\hat{x}_2$ for $\tau = 10$ s.

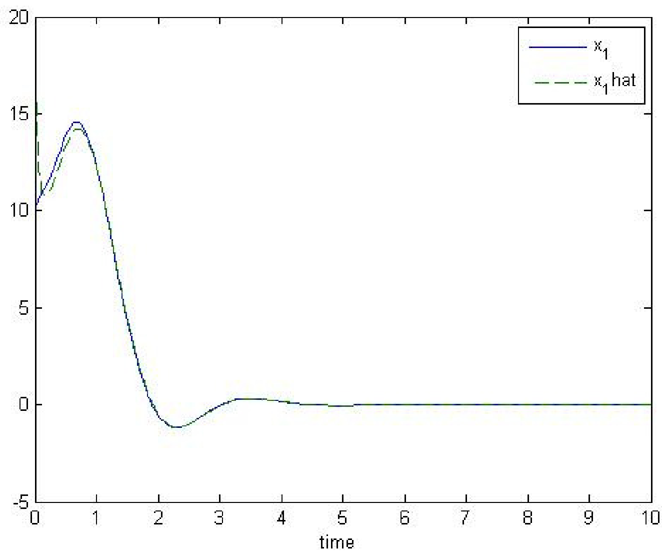


**Fig. 5.** The evolution of $x_1$ and its estimate $\hat{x}_1$ for $\varepsilon = 10^{-4}$, $\tau = 1$ s.

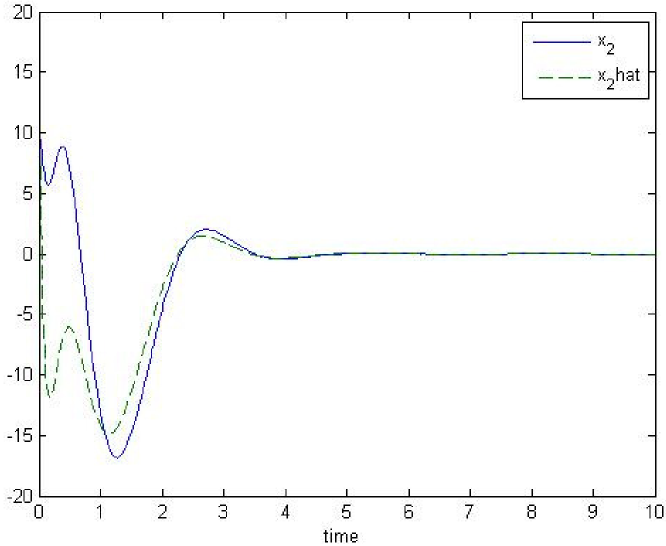


**Fig. 6.** The evolution of $x_2$ and its estimate $\hat{x}_2$ for $\varepsilon = 10^{-4}$, $\tau = 1$ s.

## 5. CONCLUSION

In this paper, we have proposed a separation principle for a class of nonlinear time-delay systems. The nonlinearities of this class of systems satisfy the Lipschitz condition while the uncertain term is bounded. Since the origin is not supposed to be an equilibrium point, using Lyapunov–Krasovskii functionals, we proposed a strong practical observer, a state feedback, and we proved that the observer based controller asserts strong practical stability of the closed loop system. Finally, simulation study has been undertaken to illustrate the theory.

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