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TRACKING CONTROL DESIGN FOR NONLINEAR POLYNOMIAL SYSTEMS VIA AUGMENTED ERROR SYSTEM APPROACH AND BLOCK PULSE FUNCTIONS TECHNIQUE

Bassem Iben Warrad, Mohamed Karim Bouafoura and Naceur Benhadj Braiek

In this paper, tracking control design for a class of nonlinear polynomial systems is investigated by augmented error system approach and block pulse functions technique. The proposed method is based on the projection of the close loop augmented system and the associated linear reference model that it should follow over a basis of block pulse functions. The main advantage of using this tool is that it allows to transform the analytical differential calculus into an algebraic one relatively easy to solve. The developments presented have led to the formulation of a linear system of algebraic equations depending only on parameters of the feedback control. Once the control gains are determined by solving the latter optimization problem in least square sense, the practical stability of the closed loop augmented system is checked through given conditions. A double inverted pendulums benchmark is used to validate the proposed tracking control method.

Keywords: tracking control, nonlinear polynomial systems, augmented error system approach, block pulse functions, practical stability

Classification: 93Cxx, 93Dxx

1. INTRODUCTION

Tracking control design for nonlinear systems is deemed as one of the most crucial topics in modern control theory [13, 15]. This is due to the difficulty of synthesizing a tracking control law with guaranteed performance and stability of the closed loop system. This is why problem related to the tracking control synthesis of nonlinear systems is intensively studied nowadays.

For handling this problem, there are various Lyapunov-based methods, that can provide an asymptotic tracking performance. They include control Lyapunov function [20], sliding mode control [6, 12], backstepping control [30, 31], trajectory linearization control

The main obstacles of those methods arise firstly when choosing a suitable Lyapunov function, which is in most cases of quadratic form and secondly from the difficulty to test algorithmically the obtained non negativity conditions. These researches could

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only handle some subclasses of nonlinear systems, namely nonlinear system satisfies the Lipschitz condition [20], nonlinear system described by a specific triangular form [6, 12], nonlinear system in strict-feedback form [30, 31], differentiable nonlinear system [22, 28] and nonlinear system described by a neural networks or fuzzy logic structure [5, 7, 8, 11, 19, 24, 25].

Moreover, to the best of our knowledge the time varying setpoint tracking control for a particular class of nonlinear systems, namely nonlinear polynomial system, is not yet developed until now. Recently, some works are devoted the tracking control problem for such system in the case of a step input [26, 27]. We recall that, the polynomial model is considered as an important class of nonlinear systems. In fact, this kind of models has the advantages to conserve the non-linearity, to extend the range of validity of the approximation around an operating point, and to describe the high order nonlinear systems [2, 16]. Since, any analytic dynamic system can be approached by a polynomial model using the tensoriel Kronecker product [4] and the development into the Taylor series expansion of vectorial functions, intervening in the system equations [3, 21]. Hence, the polynomial model can be considered as an unified form to describe analytic dynamic processes.

In order to overcome the drawbacks of the exiting tracking control methods based on the Lyapunov function approach and also to address the time varying setpoint tracking control problem for nonlinear polynomial system, it would be interesting to exploit the augmented error system approach jointly used with the development of numerical tools. In fact, this latter approach is usually used in the tracking control synthesis by augmenting the state vector with the integral of the output tracking error vector [29], which allows to reduce effectively the static errors. Besides, the main characteristic of the development of numerical tools is that it convert the differential or integral calculus into of solving a system of algebraic equations [9, 10].

Based on the above idea, a new algebraic approach for the synthesis of a polynomial state feedback control with integral actions for nonlinear polynomial system is presented in this paper. The whole proposed development uses the augmented error system approach to derive an equivalent augmented form of the closed loop system, and the block pulse functions as a tool of approximation as well as their operational matrices to transform the original control problem into a system of linear algebraic equations.

Among the existing piecewise constant basis functions, the block pulse functions set has a simple structure and can be implemented without too much effort. As explained in [9, 10], easy implementation, simple operations, short execution time and accurate solutions are the principal features of the block-pulse functions. Thus, the projection of the augmented state space model of the controlled system over a basis of block pulse functions as well the use of the latter tool properties, permits to convert nonlinear differential equations into linear algebraic ones depending only on parameters of the feedback regulator.

The main key of this work consists on equalizing the augmented state vector of the close loop system and the state vector of a chosen reference model, and thus the equalization of their projections over the basis of block pulse functions. A linear system of algebraic equations is then formulated and solved by the mean of least squares minimization. That lead to the control law parameters. New sufficient conditions are further derived to ensure the practical stability of the augmented close loop system.

The remainder of this paper is organized as follows. In section 2, we introduce the studied systems and we explains the main objective of the work. In section 3, the main results are presented. In section 4, a double inverted pendulums benchmark is used as an example to show the usefulness of the developed results. Finally, conclusion is given in section 5.

Notation: Throughout this paper, $\mathbb{R}^n$ denotes the n dimensional Euclidean, while $\mathbb{R}^{n\times m}$ refers to the set of all real matrices with n rows and m columns. I_n denotes the identity matrix of size $n\times n$. $0_{n\times m}$ denotes the zero matrix of size $n\times m$. A^T represents the transpose of the matrix A. The adopted vector norm is the Euclidean norm and the matrix norm is the corresponding induced norm.

2. SYSTEM DESCRIPTION AND CONTROL OBJECTIVE

In this paper, we focus on the control of nonlinear polynomial system (S), which is described by the following state equation:

$$(S)\begin{cases} \dot{x}(t) = \sum\limits_{i=1}^{q} A_i x^{[i]}(t) + \sum\limits_{j=1}^{s} G_j(u(t)\otimes x^{[j]}(t)) + Bu(t) \\ y(t) = Cx(t) \\ x(t) = x_0 \end{cases} \tag{1}$$

where $u(t)\in\mathbb{R}^m$ is the input vector, $x(t)\in\mathbb{R}^n$ is the state vector and $y(t)\in\mathbb{R}^p$ is the output vector.

In (1), $A_i\in\mathbb{R}^{n\times n^i}$, $G_j\in\mathbb{R}^{n\times mn^j}$, $B\in\mathbb{R}^{n\times m}$ and $x^{[i]}(t)$ is the i^{th} Kronecker power of the state vector $x(t)$, defined by:

$$\begin{cases} x^{[0]}(t) = 1 \\ x^{[i]}(t) = x^{[i-1]}(t)\otimes x(t) \quad \forall i > 1. \end{cases}$$

Assumption: The system (1) is locally controllable around x_0 and its n state components are all physically measurable.

Strategy of Control: For given reference input vector $y_c(t)\in\mathbb{R}^p$, the strategy of control is to design a polynomial state feedback controller with compensator gain:

$$u(t) = \bar{N}R(t) - \sum_{i=1}^{q} K_i x^{[i]}(t) \tag{2}$$

with

$$R(t) = \begin{bmatrix} e(t)^T & e^{(1)}(t)^T & \dots & e^{(w-2)}(t)^T & e^{(w-1)}(t)^T \end{bmatrix}^T \tag{3}$$

where

$$e(t) = \underbrace{\int_0^t \dots \int_0^t}_{w \text{ times}} (y_c(\sigma) - y(\sigma))\,\mathrm{d}\sigma^w \tag{4}$$

with w is a positive integer which depends on the type of reference input vector $y_c(t) \in \mathbb{R}^p$. In (2), $\bar{N} \in \mathbb{R}^{m\times wp}$, $K_i \in \mathbb{R}^{m\times n^i}$ and $e^{(h)}(t)$ is the hth derivative of $e(t)$.

Control objective: Using the augmented error system approach, we define the augmented state vector as follows:

$$X(t) = \begin{bmatrix} x(t)^T & R(t)^T \end{bmatrix}^T. \tag{5}$$

By taking into account that:

$$x(t) = \begin{bmatrix} I_n & 0_{n,wp} \end{bmatrix} X(t) = \Psi X(t) \quad \text{and} \quad x^{[i]}(t) = \Psi^{[i]} X^{[i]}(t) \tag{6}$$

then, the augmented system (S_a) is described by the following state equation:

$$(S_a) \begin{cases} \dot{X}(t) = \sum_{i=1}^{q} \bar{A}_i X^{[i]}(t) + \sum_{j=1}^{s} \bar{G}_j (u(t) \otimes (\Psi^{[j]} X^{[j]}(t))) + \bar{B}u(t) + \bar{E}y_c(t) \\ Y(t) = \bar{C}X(t) \\ X(0) = \begin{bmatrix} x_0{}^T & 0_{1,wp} \end{bmatrix}^T \end{cases} \tag{7}$$

with

$$\bar{A}_1 = \begin{bmatrix} A_1 & 0_{n,p} & 0_{n,p} & \cdots & \cdots & 0_{n,p} \\ 0_{p,n} & 0_{p,p} & I_p & 0_{p,p} & \cdots & 0_{p,p} \\ 0_{p,n} & 0_{p,p} & 0_{p,p} & I_p & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & 0_{p,p} \\ 0_{p,n} & 0_{p,p} & 0_{p,p} & \cdots & 0_{p,p} & I_p \\ -C & 0_{p,p} & 0_{p,p} & \cdots & \cdots & 0_{p,p} \end{bmatrix}, \quad \bar{B} = \begin{bmatrix} B \\ 0_{wp,m} \end{bmatrix}$$

$$\bar{C} = \begin{bmatrix} C & 0_{p,wp} \end{bmatrix}, \quad \bar{E} = \begin{bmatrix} 0_{n+(w-1)p,p} \\ I_p \end{bmatrix}$$

and for each $i \in \{2 \dots q\}$

$$\bar{A}_i = \begin{bmatrix} A_i \\ 0_{wp,n^i} \end{bmatrix} \Psi^{[i]}$$

and for each $j \in \{1 \dots s\}$

$$\bar{G}_j = \begin{bmatrix} G_j \\ 0_{wp,mn^j} \end{bmatrix}$$

The closed loop augmented system should reproduce sharply the dynamical behavior of a linear reference model (S_r) and therefore responds to desired performances. Such reference model is described by the following state equations:

$$(S_r) \begin{cases} \dot{X}_r(t) = EX_r(t) + Fy_c(t) \\ Y_r(t) = ZX_r(t) \\ X_r(0) = X_{0r} \end{cases} \tag{8}$$

where $X_r(t) \in \mathbb{R}^{n+wp}$ and $Y_r(t) \in \mathbb{R}^p$ are respectively the state and the output vectors of the reference model.

The above reference model is designed such as its output vector $Y_r(t)$ tracks perfectly the reference input vector $y_c(t)$.

3. MAIN RESULTS

3.1. Reference model construction

In order to meet the desired performance, while respecting a realistic sense of the control problem (2), we construct a parsimonious reference model as follows:
Firstly, we take the linear part of the original system (1), that is to say:

$$\begin{cases} \dot{x}_l(t) = A_1 x_l(t) + B u_l(t) \\ y_l(t) = C x_l(t) \\ x_l(0) = x_0. \end{cases} \tag{9}$$

The dynamic behavior of linear system (9) could be easily tuned as desired simply with a state feedback and compensator gain of the following form:

$$u_l(t) = \bar{N}_0 R_l(t) - K_0 x_l(t) \tag{10}$$

with

$$R_l(t) = \begin{bmatrix} e_l(t)^T & e_l^{(1)}(t)^T & \dots & e_l^{(w-2)}(t)^T & e_l^{(w-1)}(t)^T \end{bmatrix}^T \tag{11}$$

where

$$e_l(t) = \underbrace{\int_0^t \dots \int_0^t}_{w \text{ times}} (y_c(\sigma) - y_l(\sigma))\, \mathrm{d}\sigma^w. \tag{12}$$

Secondly, we define the state vector of reference model (8) as follows:

$$X_r(t) = \begin{bmatrix} x_l(t)^T & R_l(t)^T \end{bmatrix}^T \tag{13}$$

then, the control law (10) can be written as follows:

$$u_l(t) = -\begin{bmatrix} K_0 & -\bar{N}_0 \end{bmatrix} X_r(t) = -\bar{K}_0 X_r(t). \tag{14}$$

Assuming that pair $(\bar{A}_1, \bar{B})$ is controllable, then the parameters of reference model (8) are given by:

$$E = \bar{A}_1 - \bar{B}\bar{K}_0, \quad F = \bar{E}, \quad Z = \bar{C}, \quad X_{0r} = \begin{bmatrix} x_0 \\ 0_{wp,1} \end{bmatrix} \tag{15}$$

where matrix $\bar{K}_0 \in \mathbb{R}^{m \times (n+wp)}$ is synthesized by pole placement approach.

Remark 3.1.1. The value of w is chosen such that the output vector $Y_r(t)$ of the reference model (8) tracks perfectly the reference input vector $y_c(t)$.

Remark 3.1.2. The reference model (8) is strongly inspired from the original nonlinear system (1). Accordingly, by taking into account that the general idea of this work consists on equalizing the state vector of the augmented controlled system and the state vector of the reference model, then the synthesis of the control law (2) is reduced to a simple adjustment of the control law (10) relative to the reference model.

3.2. Proposed nonlinear polynomial system tracking control approach

The control law (2) can be written as follows:

$$u(t) = -\sum_{i=1}^{q} \bar{K}_i X^{[i]}(t) \tag{16}$$

with

$$\bar{K}_1 = \begin{bmatrix} K_1 & -\bar{N} \end{bmatrix} \quad and \quad \bar{K}_i = K_i \Psi^{[i]} \quad \forall i \in \{2 \dots q\}\,. \tag{17}$$

From relations (7) and (16), augmented state equation could be written as follows:

$$\dot{X}(t) = \sum_{i=1}^{q} (\bar{A}_i - \bar{B}\bar{K}_i) X^{[i]}(t) - \sum_{j=1}^{s}\sum_{i=1}^{q} \bar{G}_j((\bar{K}_i X^{[i]}(t)) \otimes (\Psi^{[j]} X^{[j]}(t))) + \bar{E} y_c(t). \tag{18}$$

The integration of equation (18) with respect to t over the interval $[0, T]$ leads to:

$$\begin{aligned} X(t) - X(0) &= \sum_{i=1}^{q} (\bar{A}_i - \bar{B}\bar{K}_i) \int_0^t X^{[i]}(\sigma)\,\mathrm{d}\sigma \\ &- \sum_{j=1}^{s}\sum_{i=1}^{q} \bar{G}_j(\bar{K}_i \otimes \Psi^{[j]}) \int_0^t (X^{[i]}(\sigma) \otimes X^{[j]}(\sigma)) d\sigma + \bar{E} \int_0^t y_c(\sigma)\,\mathrm{d}\sigma. \end{aligned} \tag{19}$$

We underline that the main idea consists on equalizing the state vector of the controlled augmented system (S_a) and the state vector of the reference model (S_r). That is to say $\forall t \in [0, T]$:

$$X(t) = X_r(t) = X_{1,rN} S_N(t) \tag{20}$$

where $X_{1,rN} \in \mathbb{R}^{(n+wp)\times N}$ denote state coefficients of the reference model, computing from the scalar product (A.3) given in appendix A.

The expansion of fixed reference input vector $y_c(t)$ over the same basis of the Block-pulse functions is given by:

$$y_c(t) = Y_{1,cN} S_N(t) \tag{21}$$

where $Y_{1,cN} \in \mathbb{R}^{p\times N}$ denote reference input coefficients, computing from the scalar product (A.3) given in appendix A.

Based on the operational matrix of product defined by equation (A.6) given in appendix A, the i^{th} Kronecker power of $X(t)$ could be also expanded over the same basis:

$$\begin{aligned} & X^{[2]}(t) = X(t) \otimes X(t) = ((X_{1,rN} S_N(t)) \otimes (X_{1,rN} S_N(t))) \\ & = (X_{1,rN} \otimes X_{1,rN})(S_N(t) \otimes S_N(t)) = (X_{1,rN} \otimes X_{1,rN}) M_N S_N(t) = X_{2,rN} S_N(t) \\ & \vdots \\ & X^{[q]}(t) = X^{[q-1]}(t) \otimes X(t) = ((X_{q-1,rN} S_N(t)) \otimes (X_{1,rN} S_N(t))) \\ & = (X_{q-1,rN} \otimes X_{1,rN})(S_N(t) \otimes S_N(t)) = (X_{q-1,rN} \otimes X_{1,rN}) M_N S_N(t) = X_{q,rN} S_N(t)\,. \end{aligned} \tag{22}$$

Furthermore, the Kronecker product terms $X^{[i]}(t) \otimes X^{[j]}(t)$ for each $i \in \{1 \dots q\}$ and $j \in \{1 \dots s\}$ can be also expanded as follows:

$$\begin{aligned} &X^{[i]}(t) \otimes X^{[j]}(t) = ((X_{i,rN} S_N\,(t)) \otimes (X_{j,rN} S_N\,(t))) = \\ &(X_{i,rN} \otimes X_{j,rN})\,(S_N\,(t) \otimes S_N\,(t)) = (X_{i,rN} \otimes X_{j,rN})\,M_N S_N\,(t)\,. \end{aligned} \tag{23}$$

Then, the expansion of equation (19) over the considered block pulse functions basis yields:

$$\begin{aligned} &X_{1,rN} S_N\,(t) - X_{0N} S_N\,(t) = \sum_{i=1}^{q} (\bar{A}_i - \bar{B}\bar{K}_i) X_{i,rN} \int_0^t S_N(\sigma)\,\mathrm{d}\sigma \\ &- \sum_{j=1}^{s} \sum_{i=1}^{q} \bar{G}_j (\bar{K}_i \otimes \Psi^{[j]})(X_{i,rN} \otimes X_{j,rN}) M_N \int_0^t S_N(\sigma)\,\mathrm{d}\sigma + \bar{E} Y_{1,cN} \int_0^t S_N(\sigma)\,\mathrm{d}\sigma \end{aligned} \tag{24}$$

where $X_{0N} = \begin{bmatrix} x_0 & 0_{n,1} & \dots & 0_{n,1} \\ 0_{wp,1} & 0_{wp,1} & \dots & 0_{wp,1} \end{bmatrix} \in \mathbb{R}^{(n+wp)\times N}$.

The use of the integration operational matrix P_N, defined by equation (A.4) given in appendix A, yields:

$$\begin{aligned} &X_{1,rN} S_N\,(t) - X_{0N} S_N\,(t) = \sum_{i=1}^{q} (\bar{A}_i - \bar{B}\bar{K}_i) X_{i,rN} P_N S_N(t) \\ &- \sum_{j=1}^{s} \sum_{i=1}^{q} \bar{G}_j (\bar{K}_i \otimes \Psi^{[j]})(X_{i,rN} \otimes X_{j,rN}) M_N P_N S_N(t) + \bar{E} Y_{1,cN} P_N S_N(t). \end{aligned} \tag{25}$$

Simplifying the vector $S_N\,(t)$ in both sides of relation (25), we obtain the following algebraic equations, in which unknowns are the control law parameters:

$$\begin{aligned} &X_{1,rN} - X_{0N} = \sum_{i=1}^{q} (\bar{A}_i - \bar{B}\bar{K}_i) X_{i,rN} P_N \\ &- \sum_{j=1}^{s} \sum_{i=1}^{q} \bar{G}_j (\bar{K}_i \otimes \Psi^{[j]})(X_{i,rN} \otimes X_{j,rN}) M_N P_N + \bar{E} Y_{1,cN} P_N. \end{aligned} \tag{26}$$

Using the *vec* operator (See Appendix B), it comes out:

$$\beta = -\sum_{i=1}^{q} \chi_i vec(\bar{K}_i) - \sum_{j=1}^{s} \sum_{i=1}^{q} \xi_{ij} vec(\bar{K}_i \otimes \Psi^{[j]}) \tag{27}$$

with

$$\beta = vec(X_{1,rN}) - vec(X_{0N}) - \sum_{i=1}^{q} (P_N^T \otimes \bar{A}_i) vec(X_{i,rN}) - (P_N^T \otimes \bar{E}) vec(Y_{1,cN})$$

where for each $i \in \{1 \dots q\}$

$$\chi_i = \left((X_{i,rN} P_N)^T \otimes \bar{B}\right)$$

and for each $i \in \{1 \dots q\}$ and $j \in \{1 \dots s\}$

$$\xi_{ij} = \left(((X_{i,rN} \otimes X_{j,rN}) M_N P_N)^T \otimes \bar{G}_j\right).$$

Based on the propriety of the *vec* operator (See Appendix C), relation (27) becomes:

$$\beta = -\sum_{i=1}^{q} \chi_i vec(\bar{K}_i) - \sum_{j=1}^{s}\sum_{i=1}^{q} \xi_{ij}\Pi_{m,(n+wp)^i}(\Psi^{[j]})vec(\bar{K}_i). \tag{28}$$

From relation (17), we can deduce that:

$$vec(\bar{K}_1) = \begin{bmatrix} I_{mn} \\ 0_{mwp,mn} \end{bmatrix} vec(K_1) - \begin{bmatrix} 0_{mn,mwp} \\ I_{mwp} \end{bmatrix} vec(\bar{N}) \tag{29}$$

and for each $i \in \{2 \dots q\}$:

$$vec(\bar{K}_i) = \left(\left(\Psi^{[i]}\right)^T \otimes I_m\right) vec(K_i) \tag{30}$$

then, relation (28) could be written as follows:

$$\beta = \alpha_1 vec(\bar{N}) - \sum_{i=1}^{q} \bar{\chi}_i vec(K_i) \tag{31}$$

with

$$\alpha_1 = \left(\chi_1 + \sum_{j=1}^{s} \xi_{1j}\Pi_{m,(n+wp)}(\Psi^{[j]})\right) \begin{bmatrix} 0_{mn,mwp} \\ I_{mwp} \end{bmatrix}$$

$$\bar{\chi}_1 = \left(\chi_1 + \sum_{j=1}^{s} \xi_{1j}\Pi_{m,(n+wp)}(\Psi^{[j]})\right) \begin{bmatrix} I_{mn} \\ 0_{mwp,mn} \end{bmatrix}$$

and for each $i \in \{2 \dots q\}$:

$$\bar{\chi}_i = \left(\chi_i + \sum_{j=1}^{s} \xi_{ij}\Pi_{m,(n+wp)^i}(\Psi^{[j]})\right) \left(\left(\Psi^{[i]}\right)^T \otimes I_m\right).$$

Let

$$\alpha_2 = -\begin{bmatrix} \bar{\chi}_1 & \bar{\chi}_2 & \dots & \bar{\chi}_q \end{bmatrix}$$

then, from (31), it comes out:

$$\beta = \alpha_1 vec(\bar{N}) + \alpha_2 \theta_K \tag{32}$$

with

$$\theta_K = \begin{bmatrix} vec(K_1) \\ \vdots \\ vec(K_q) \end{bmatrix}.$$

It would be interesting to formulate this problem under the following simple form:

$$\mathrm{H}\theta = \mathrm{S} \tag{33}$$

where constant matrix H and constant vector S are given by:

$$\mathrm{H} = \begin{bmatrix} \alpha_1 & \alpha_2 \end{bmatrix}, \quad \mathrm{S} = \beta$$

and vector of searched parameters is given by:

$$\theta = \begin{bmatrix} vec(\bar{N}) \\ \theta_K \end{bmatrix}.$$

Equation (33) is an overdetermined linear system of $(n+wp)N$ algebraic equations with respect to $\bar{n} = m\left(wp + \sum_{i=1}^{q} n^i\right)$ unknowns, which can be solved in the least square sense.

3.3. Stability analysis and attraction domain estimation

Once control parameters $\bar{N}$, K_i are determined by solving equation (33) for fixed reference input $y_c(t)$, we propose to analyze the stability of the close loop augmented system (18).

Now, let us define the following matrices, for each $i \in \{1 \dots q\}$ and $j \in \{1 \dots s\}$:

$$M_i = \bar{A}_i - \bar{B}\bar{K}_i \quad and \quad L_{ij} = \bar{G}_j \left(\bar{K}_i \otimes \Psi^{[j]}\right). \tag{34}$$

Definition 3.3.1. The system (18) is said to be practically stable, if there exist $0 < R_0 < r$, such that [17]:

$$\|X(0)\| < R_0 \quad \Rightarrow \quad \|X(t)\| < r, \quad \forall t \geq 0. \tag{35}$$

Theorem 3.3.2. The closed loop system (18) is practical stable if all eigenvalues of matrix M_1 have a strictly negative real part and if

$$\|X(0)\| < R_0 \tag{36}$$

where $R_0 > 0$, $\lambda_1 > 0$ and $\omega_1 < 0$ are given scalars satisfying:

$$\left\|e^{M_1 t}\right\| \leq \lambda_1 e^{\omega_1 t}, \forall t \geq 0 \tag{37}$$

and

$$R_0 = \frac{R_1}{\lambda_1} \tag{38}$$

with R_1 is the unique positive solution of the following equation:

$$\lambda_1 \left(\sum_{i=2}^{q} \|M_i\| R_1^{i-1} + \sum_{j=1}^{s} \sum_{i=1}^{q} \|L_{ij}\| R_1^{i+j-1} \right) + \omega_1 = 0. \tag{39}$$

P r o o f. We propose to prove the existence of a region of initial conditions ensuring the practical stability of closed loop system (18). This region is assumed to be a ball centred in the origin and of radius R_0, i. e.

$$\Im(0, R_0) = \{X(0) \in \mathbb{R}^{n+wp}, \|X(0)\| < R_0\}.$$

By taking into account the relation (34), then the closed loop system can be written from equation (18) as follows:

$$\dot{X}(t) - M_1 X(t) = \sum_{i=2}^{q} M_i X^{[i]}(t) - \sum_{j=1}^{s}\sum_{i=1}^{q} L_{ij}(X^{[i]}(t) \otimes X^{[j]}(t)) + \bar{E} y_c(t). \tag{40}$$

From the series definition which permits to have $M_1 e^{-M_1 t} = e^{-M_1 t} M_1$, we can write

$$\begin{aligned} &\tfrac{\mathrm{d}}{\mathrm{d}t}\left(e^{-M_1 t} X(t)\right) = -M_1 e^{-M_1 t} X(t) + e^{-M_1 t}\dot{X}(t) = e^{-M_1 t}\left(\dot{X}(t) - M_1 X(t)\right) \\ &= e^{-M_1 t}\left(\sum_{i=2}^{q} M_i X^{[i]}(t) - \sum_{j=1}^{s}\sum_{i=1}^{q} L_{ij}(X^{[i]}(t) \otimes X^{[j]}(t)) + \bar{E} y_c(t)\right) \end{aligned}$$

then, the integration of the last equation from 0 to t gives

$$\begin{aligned} X(t) &= e^{M_1 t} X(0) + \int_0^t e^{M_1(t-\sigma)}\left(\sum_{i=2}^{q} M_i X^{[i]}(\sigma) + \bar{E} y_c(\sigma)\right) \mathrm{d}\sigma \\ &- \int_0^t e^{M_1(t-\sigma)}\left(\sum_{j=1}^{s}\sum_{i=1}^{q} L_{ij}(X^{[i]}(\sigma) \otimes X^{[j]}(\sigma))\right) \mathrm{d}\sigma. \end{aligned} \tag{41}$$

Since, all eigenvalues of matrix M_1 have negative real part, then there exist two reals $\lambda_1 > 0$ and $\omega_1 < 0$ such that relation (37) holds on, then $X(t)$ can be bounded as:

$$\begin{aligned} e^{-\omega_1 t}\|X(t)\| &\le \lambda_1 \|X(0)\| + \lambda_1 \sum_{i=2}^{q} \|M_i\| \int_0^t e^{-\omega_1 \sigma}\left\|X^{[i]}(\sigma)\right\| \mathrm{d}\sigma \\ &+ \lambda_1 \left\|\bar{E}\right\| \int_0^t e^{-\omega_1 \sigma}\|y_c(\sigma)\| \mathrm{d}\sigma \\ &+ \lambda_1 \sum_{j=1}^{s}\sum_{i=1}^{q} \|L_{ij}\| \int_0^t e^{-\omega_1 \sigma}\left\|X^{[i]}(\sigma) \otimes X^{[j]}(\sigma)\right\| \mathrm{d}\sigma. \end{aligned} \tag{42}$$

Let us assume that:

$$\|X(t)\| < R. \tag{43}$$

Then using the following matrix norm property, for each $i \in \{2 \dots q\}$ and $j \in \{2 \dots s\}$

$$\left\|X^{[i]}(t)\right\| \le \left\|X^{[i-1]}(t)\right\| \|X(t)\| < R^{i-1}\|X(t)\|$$

$$\left\|X^{[i]}(t) \otimes X^{[j]}(t)\right\| \le \left\|X^{[i]}(t)\right\| \left\|X^{[j]}(t)\right\| < R^{i+j-1}\|X(t)\|.$$

This results in:

$$e^{-\omega_1 t}\|X(t)\| < \lambda_1 \|X(0)\| + f(R)\int_0^t e^{-\omega_1 \sigma}\|X(\sigma)\| \mathrm{d}\sigma + \bar{\eta}\int_0^t e^{-\omega_1 \sigma} \mathrm{d}\sigma \tag{44}$$

where

$$f(R) = \lambda_1\left(\sum_{i=2}^{q} \|M_i\| R^{i-1} + \sum_{j=1}^{s}\sum_{i=1}^{q} \|L_{ij}\| R^{i+j-1}\right)$$

and $\bar{\eta} = \lambda_1 \left\|\bar{E}\right\| \delta$, with $\delta > 0$ verifying $\|y_c(t)\| < \delta$.

Let us define $Q(t)$ as the right hand member of inequality (44):

$$Q(t) = \lambda_1 \|X(0)\| + f(R) \int_0^t e^{-\omega_1 \sigma} \|X(\sigma)\| \, \mathrm{d}\sigma + \bar{\eta} \int_0^t e^{-\omega_1 \sigma} \, \mathrm{d}\sigma. \tag{45}$$

The derivation of function $Q(t)$ leads to:

$$\frac{\mathrm{d}Q(t)}{\mathrm{d}t} = f(R) \, e^{-\omega_1 t} \|X(t)\| + \bar{\eta} e^{-\omega_1 t} \tag{46}$$

then the following inequality can be deduced from (45):

$$\frac{\mathrm{d}Q(t)}{\mathrm{d}t} < f(R) \, Q(t) + \bar{\eta} e^{-\omega_1 t}. \tag{47}$$

Using the Bellman Gronwall inequality in differential form (See Appendix D), the integration of the inequality (47) on the time interval $[0, t]$, leads:

$$Q(t) < e^{f(R)t} Q(0) + \frac{\bar{\eta}}{f(R)+\omega_1} \left(e^{f(R)t} - e^{-\omega_1 t} \right) \tag{48}$$

where

$$Q(0) = \lambda_1 \|X(0)\| .$$

Inequality (44) and expression (48) imply:

$$\|X(t)\| < \lambda_1 \|X(0)\| \, e^{(f(R)+\omega_1)t} + \frac{\bar{\eta}}{f(R)+\omega_1} \left(e^{(f(R)+\omega_1)t} - 1 \right). \tag{49}$$

Then for

$$f(R) + \omega_1 < 0 \tag{50}$$

the following inequality ensures the boundedness of solution $X(t)$:

$$\|X(t)\| < \lambda_1 \|X(0)\| - \frac{\bar{\eta}}{f(R)+\omega_1}. \tag{51}$$

Condition (50) is satisfied for $R < R_1$, where R_1 is the unique positive solution of the following equation:

$$f(R_1) + \omega_1 = 0. \tag{52}$$

Now, to ensure the hypothesis given by relation (43) for all t, it suffices to have:

$$\lambda_1 \|X(0)\| < R_1. \tag{53}$$

So, from inequality (51) and condition (53), it follows that:

$$\|X(0)\| < R_0 = \frac{R_1}{\lambda_1} \quad \Rightarrow \quad \|X(t)\| < r = \lambda_1 R_0 - \frac{\bar{\eta}}{f(R)+\omega_1}. \tag{54}$$

Hence, the closed loop system (18) is practical stable. □

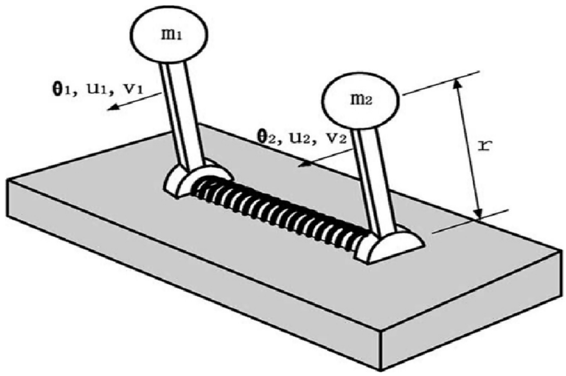

Fig. 1. Double-inverted pendulums connected by a spring.

Symbol	Value	Unit	Physical property
g	9.81	m/s^2	Gravity acceleration
r	1	m	Height of each of the two pendulums
k	2	N/m	Constant of the connecting torsional spring
m_1	2	kg	Mass at the end of first pendulum
m_2	2.5	kg	Mass at the end of second pendulum
J_1	2	$kg.m^2$	Moment of inertia for first pendulum
J_2	2.5	$kg.m^2$	Moment of inertia for second pendulum

Tab. 1. Physical parameters of the double inverted pendulums connected by a spring.

4. APPLICATION TO A DOUBLE-INVERTED PENDULUMS

In this section, the obtained results are simulated to verify the effectiveness of the proposed method. For this purpose, the tracking control problem is considered for a double inverted pendulums benchmark, represented by Figure 1.

The physical parameters of the double inverted pendulums connected by a spring are listed in Table 1.

4.1. System modeling

Each pendulum may be positioned by a torque input $u_i(t)$ for $i = 1, 2$, applied by a servomotor at its base. Denote $x_1(t) = \theta_1(t)$ (angular position), $x_2(t) = \dot{\theta}_1(t)$ (angular rate), $x_3(t) = \theta_2(t)$, $x_4(t) = \dot{\theta}_2(t)$. Thus, the inverted pendulums equation can be described as [32]:

$$\begin{cases} \dot{x}_1(t) = x_2(t) \\ \dot{x}_2(t) = \frac{m_1 gr}{J_1} sin(x_1(t)) - \frac{k}{J_1} x_1(t) + \frac{k}{J_1} x_3(t) + \left(\frac{1}{J_1} + f_1(x_1(t), x_3(t))\right) u_1(t) + \frac{v_1(t)}{J_1} \\ \dot{x}_3(t) = x_4(t) \\ \dot{x}_4(t) = \frac{m_2 gr}{J_2} sin(x_3(t)) - \frac{k}{J_2} x_3(t) + \frac{k}{J_2} x_1(t) + \left(\frac{1}{J_2} + f_2(x_1(t), x_3(t))\right) u_2(t) + \frac{v_2(t)}{J_2} \end{cases} \tag{55}$$

where $v_1(t)$ and $v_2(t)$ are the torque disturbance and $f_1(x_1, x_3)$ and $f_2(x_1, x_3)$ are nonlinearity associated respectively with each input channels $u_1(t)$ and $u_2(t)$. Therefore, the origin $x_0 = \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}^T$ is the equilibrium point of this system. The outputs system are $\theta_1(t)$ and $\theta_2(t)$.

In addition, we assumed that $v_1(t) = v_2(t) = 0$ and

$$f_1(x_1(t), x_3(t)) = f_2(x_1(t), x_3(t)) = -0.25x_1^2(t) + 0.25x_3^2(t).$$

4.2. Polynomial system description

The nonlinear system (55) can be developed into polynomial form by a Taylor series expansions, then we have:

$$\begin{aligned} &\dot{x}(t) = A_1 x(t) + A_2 x^{[2]}(t) + A_3 x^{[3]}(t) + G_1(u(t) \otimes x(t)) + G_2(u(t) \otimes x^{[2]}(t)) + Bu(t) \\ &y(t) = Cx(t) \end{aligned} \tag{56}$$

with:

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix}, A_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \left(\frac{m_1 gr}{J_1} - \frac{k}{J_1}\right) & 0 & \frac{k}{J_1} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{k}{J_2} & 0 & \left(\frac{m_2 gr}{J_2} - \frac{k}{J_2}\right) & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 0 \\ \frac{1}{J_1} & 0 \\ 0 & 0 \\ 0 & \frac{1}{J_2} \end{bmatrix}$$

$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$, $A_2(i,j) = 0$ for $\left(\mathrm{i} = 1 \dots 4; \mathrm{j} = 1 \dots 4^2\right)$; $A_3(2,1) = -\frac{m_1 gr}{6J_1}$; $A_3(4,43) = -\frac{m_2 gr}{6J_2}$; $A_3(i,j) = 0$ elsewhere $\left(\mathrm{i} = 1 \dots 4; \mathrm{j} = 1 \dots 4^3\right)$; $G_1(i,j) = 0$ for $(\mathrm{i} = 1 \dots 4; \mathrm{j} = 1 \dots 8)$ and $G_2(2,1) = -0.25$; $G_2(2,11) = 0.25$; $G_2(4,17) = -0.25$; $G_2(4,27) = 0.25$; $G_2(i,j) = 0$ elsewhere $(\mathrm{i} = 1 \dots 4; \mathrm{j} = 1 \dots 32)$.

4.3. Fidelity of the adopted polynomial system

Given $u_1(t) = -36x_1(t) - 12x_2(t)$ and $u_2(t) = -36x_3(t) - 12x_4(t)$, then the evolution of the state variables in closed loop for both nonlinear (55) and polynomial (56) systems, is shown in Figure 2.

It appears from the simulations that the polynomial modeling is very suitable and valid for this system. Indeed, we can't distinguish the real system behavior from the polynomial one.

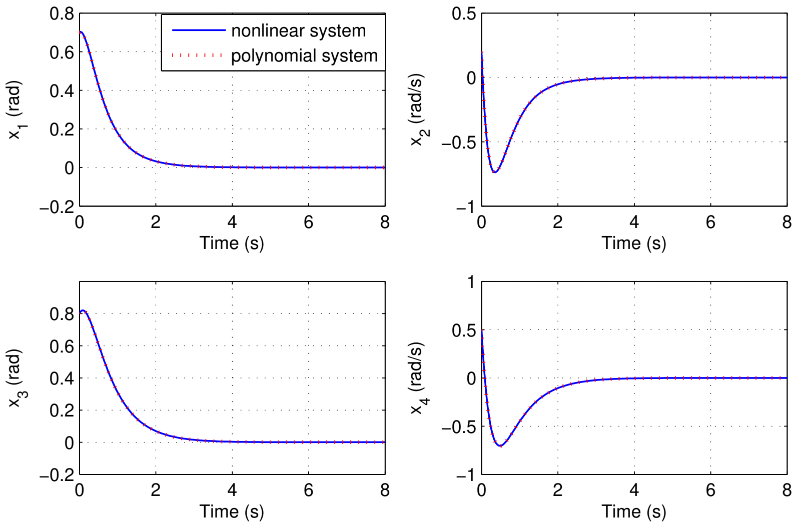

Fig. 2. Evolution of state variables for both real model and polynomial model.

4.4. Control specifications and considered reference model

In this application, we plan to control angular positions $\theta_1(t)$ and $\theta_2(t)$ of the inverted pendulums, which will be aimed to track respectively the reference signals $y_{c1}(t) = y_{c2}(t) = 0.5(sin(t) + sin(0.5t))$.

For $w = 3$ (Number of integrators) and by choosing the poles of the designed reference system (8) as $p_1 = -7.5$, $p_2 = -10$, $p_3 = -12.5$, $p_4 = -15$, $p_5 = -1.5$, $p_6 = -3$, $p_7 = -4$, $p_8 = -5$, $p_9 = -6$, $p_{10} = -7$, then the proposed method for reference model construction leads to the following parameters:

$$E = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -480.9 & -35.8 & -122.4 & -4.5 & 9518.9 & 6513.6 & 8825.3 & 4682.2 & 3019.3 & 1175.4 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -110.8 & -4 & -477.8 & -35.7 & 5864.3 & 9597.1 & 4218.6 & 8821.2 & 1060.9 & 3004 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$F = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}^T, \quad Z = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Note that the output vector $Y_r(t)$ of the above reference model tracks perfectly the reference input vector $y_c(t) = \begin{bmatrix} y_{c1}(t) & y_{c2}(t) \end{bmatrix}^T$.

4.5. Control parameters gains and simulation results

For $N = 2^6$ (Number of BPFs) and $T = 20s$, the implementation of the proposed tracking control approach leads to the following control gains:

$$\bar{N} = \begin{bmatrix} 19635 & 12528 & 18227 & 8861 & 6139 & 2267 \\ 15654 & 23161 & 11493 & 21226 & 2810 & 7380 \end{bmatrix} K_1 = \begin{bmatrix} 976 & 71.4 & 252.9 & 9.3 \\ 272.2 & 9.7 & 1228.2 & 89.9 \end{bmatrix}$$

$K_2(1,1) = 1.1705$, $K_2(2,1) = 1.7244$, $K_2(1,4) = -221.8346$, $K_2(2,4) = -318.2420$, $K_2(1,5) = 220.9817$, $K_2(2,5) = 316.9508$, $K_2(1,6) = 24.8810$, $K_2(2,6) = 37.0761$, $K_2(1,7) = -220.3052$, $K_2(2,7) = -315.9582$, $K_2(1,11) = -1.1675$, $K_2(2,11) = -1.7201$, $K_2(1,12) = 221.1558$, $K_2(2,12) = 317.2457$, $K_2(1,14) = -49.7165$, $K_2(2,14) = -74.0935$, $K_2(1,16) = 24.8339$, $K_2(2,16) = 37.0150$, $K_2(i,j) = 0$ elsewhere (i = 1 ... 2; j = 1 ... 16) $K_3(1,1) = 0.1892$, $K_3(2,1) = 5.3061$, $K_3(2,2) = -448.6026$, $K_3(1,4) = 280.1324$, $K_3(2,4) = 448.4058$, $K_3(1,5) = -280.3216$, $K_3(1,6) = 5.3174$, $K_3(2,6) = 5.9939$, $K_3(1,14) = -2.6195$, $K_3(1,22) = 15.9250$, $K_3(2,22) = 24.5114$, $K_3(1,23) = -3.9641$, $K_3(2,26) = -4.8170$, $K_3(1,27) = 279.6198$, $K_3(2,27) = 447.4656$, $K_3(2,29) = -2.1975$, $K_3(1,32) = -47.3243$, $K_3(2,32) = -72.8843$, $K_3(1,43) = -3.4629$, $K_3(2,43) = -9.4002$, $K_3(2,44) = -447.2738$, $K_3(1,47) = -279.4336$, $K_3(1,48) = 1.2793$, $K_3(2,63) = 1.0404$, $K_3(1,64) = 31.4045$, $K_3(2,64) = 48.3802$, $K_3\,(i,j) = 0$ elsewhere (i = 1 ... 2; j = 1 ... 64).

In Figure 3, the responses of controlled inverted pendulums system outputs using the obtained polynomial controller and the considered reference model are plotted.

It can be seen that the polynomial state feedback controller with triple action integral, applied to the considered system, permits to achieve her purpose.

From Figure 4, we can see that the tracking errors $e_1(t) = \theta_1(t) - y_{c1}(t)$ and $e_2(t) = \theta_2(t) - y_{c2}(t)$, converge to a small neighborhood of the origin, which confirm that the proposed control law computed using the developed approach, permits to have a good tracking performance.

The Figure 5 shows the variation of the control signals $u_1(t)$ and $u_2(t)$. We notice that control level is acceptable.

4.6. Practical stability test

The obtained gains verified that all eigenvalues of matrix M_1 have a strictly negative real part, and

$$\left\| e^{M_1 t} \right\| < 380 e^{-1.2t}$$

which corresponds to inequality (37) with $\lambda_1 = 380$ and $\omega_1 = -1.2$. Then the Theorem 3.3.2 provides an estimation of the radius of practical stability of the studied system $R_0 = 0.000002$.

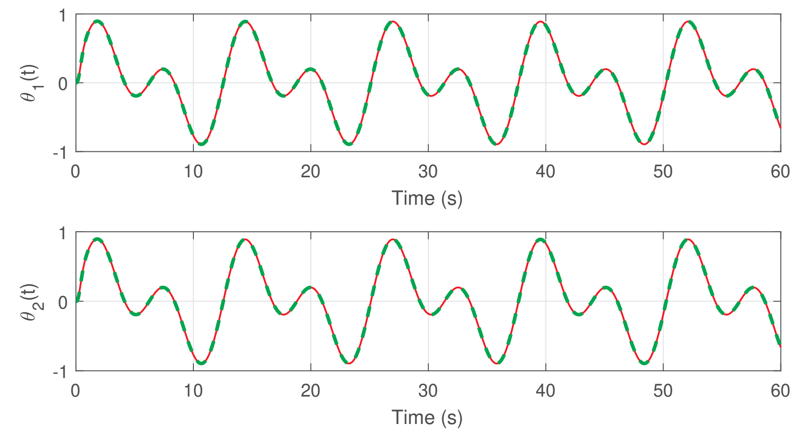

Fig. 3. Responses of the controlled inverted pendulums system outputs using the obtained polynomial controller −− and the considered reference model −.

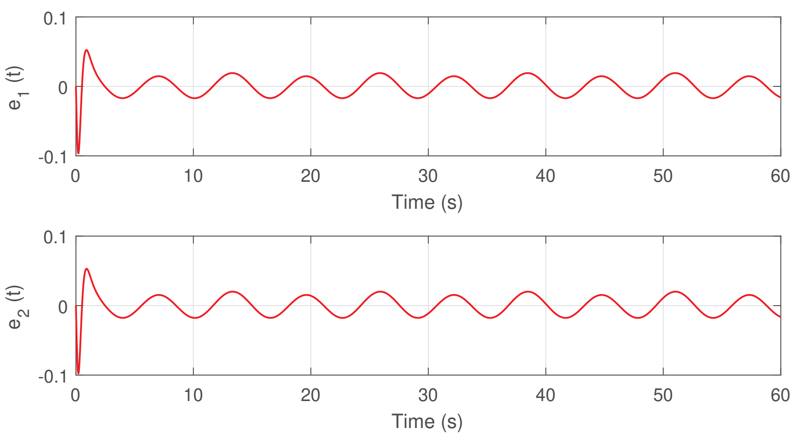

Fig. 4. Variation of the tracking errors $e_1(t)$ and $e_2(t)$.

5. CONCLUSION

An algebraic technique for tracking control design of nonlinear systems with polynomial vector fields has been presented in this paper. Firstly, the augmented error system approach is used to derive an equivalent augmented form of the closed-loop system.

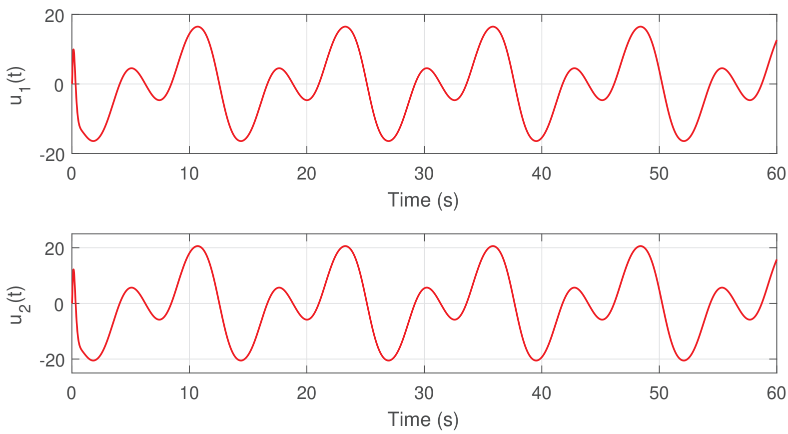

Fig. 5. Variation of the control signals $u_1(t)$ and $u_2(t)$.

Secondly, the proposed approach of control design is formulated as a linear equation of the controller parameters, which can be solved in the least square sense. Finally, the practical stability of the close loop augmented system is tested by means of simple conditions, where a domain of attraction is estimated. The effectiveness of the developed method is verified on a double inverted pendulums benchmark system.

APPENDICES

Appendix A

Block pulse functions (BPFs) have been used extensively in recent years as a basic set of functions in many fields of engineering [1, 14, 18, 23, 27].

N set of Block-Pulse Functions (BPFs) over the interval $[0,T]$ is defined as follows:

$$\varphi_i(t) = \begin{cases} 1 & \frac{iT}{N} \le t \le \frac{(i+1)T}{N}, \quad \text{for} \quad i = 0, \dots, N-1 \\ 0 & \text{elsewhere} \end{cases} \tag{A.1}$$

with a positive integer value for N. Also, $\varphi_i(t)$ is the ith Block-Pulse Functions. The most important properties of Block-Pulse Functions are disjointness, orthogonality and completeness.

So, a vector function $x(t)$ of n dimensional components which are square integrable in $[0,T]$ can be represented approximately by a finite block pulse series [14]:

$$x(t) \cong \sum_{i=0}^{N-1} x_i \varphi_i(t) = X_{1,N} S_N(t) \tag{A.2}$$

with

$$X_{1,N} = \left[\begin{array}{ccc} x_0 & \cdots & x_{N-1} \end{array}\right] \quad \text{and} \quad S_N(t) = \left[\begin{array}{ccc} \varphi_0(t) & \cdots & \varphi_{N-1}(t) \end{array}\right]^T.$$

The x_i are the block pulse coefficients of $x(t)$, as obtained from the orthogonality of the block pulse functions:

$$x_i = \frac{N}{T}\int_{\left(\frac{iT}{N}\right)}^{\left(\frac{(i+1)T}{N}\right)} x(t)\,\mathrm{d}t. \tag{A.3}$$

Operational matrix of integration

The integration matrix of the BPFs is given by [14]:

$$\int_0^t S_N(t)\,\mathrm{d}t = P_N S_N(t) \tag{A.4}$$

where

$$P_N = \frac{T}{2N}\begin{bmatrix} 1 & 2 & \dots & 2 \\ 0 & 1 & \dots & 2 \\ \vdots & \ddots & \ddots & 2 \\ 0 & \dots & 0 & 1 \end{bmatrix}.$$

Operational matrix of product

We notate e_i^p p dimensional unit vector which has 1 in the i^{th} element and zero elsewhere. The elementary matrix is defined by [4]:

$$E_{i,j}^{p\times q} = e_i^p \otimes e_j^{q^T} \tag{A.5}$$

where $\otimes$ is the symbol of the Kronecker product, then based on the disjointness property of BPFs, we have [27]:

$$S_N(t) \otimes S_N(t) = \begin{bmatrix} E_{1,1}^{N\times N} \\ \vdots \\ E_{N,N}^{N\times N} \end{bmatrix} S_N(t) = M_N S_N(t). \tag{A.6}$$

Appendix B

An important vector valued function of matrix denoted $vec()$ was defined in [4], as follows:

$$H = \left[\begin{array}{cccc} h_1 & h_2 & \dots & h_q \end{array}\right] \tag{B.1}$$

where for all $i \in \{1\dots q\}$, $h_i \in \mathbb{R}^p$ are the columns of H

$$vec(H) = \left[\begin{array}{cccc} {h_1}^T & {h_2}^T & \dots & {h_q}^T \end{array}\right]^T \in \mathbb{R}^{pq}. \tag{B.2}$$

The vectorization operator property is given as follows:

$$vec(F_1F_2F_3) = \left(F_3^T \otimes F_1\right) vec(F_2) \tag{B.3}$$

where F_1, F_2 and F_3 are arbitrary matrices with appropriate dimensions.

Appendix C

Let the matrices $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{p\times q}$, we have:

$$vec(A \otimes B) = \Pi_{m,n}(B) vec(A) \tag{C.1}$$

where

$$\Pi_{m,n}(B) = \begin{bmatrix} vec\left(E_{11}^{m\times n} \otimes B\right) \vdots \dots \vdots vec(E_{m1}^{m\times n} \otimes B) \vdots \\ vec(E_{22}^{m\times n} \otimes B) \vdots \dots \vdots vec(E_{m2}^{m\times n} \otimes B) \vdots \dots \vdots \\ vec(E_{1n}^{m\times n} \otimes B) \vdots \dots \vdots vec(E_{mn}^{m\times n} \otimes B) \end{bmatrix}. \tag{C.2}$$

Appendix D

Let $\alpha(t), \beta(t)$ and $v(t)$ be real-valued continuous functions defined on interval $[0, \infty)$. If $v(t)$ is differentiable in the interval $[0, \infty)$ and satisfies the differential inequality:

$$\frac{\mathrm{d}v(t)}{\mathrm{d}t} \prec \alpha(t)v(t) + \beta(t) \tag{D.1}$$

then $v(t)$ is bounded by

$$v(t) \prec e^{\int_0^t \alpha(\tau)\,\mathrm{d}\tau} \left[v(0) + \int_0^t \beta(\tau) e^{-\int_0^\tau \alpha(\mu) d\mu} \,\mathrm{d}\tau \right]. \tag{D.2}$$

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Bassem Iben Warrad, Corresponding author. Laboratory of Advanced Systems. Polytechnic High School of Tunisia. University of Carthage. BP 743, 2078 La Marsa. Tunisia.
e-mail: bassem.Ibenwarrad@ept.rnu.tn

Mohamed Karim Bouafoura, Laboratory of Advanced Systems. Polytechnic High School of Tunisia. University of Carthage. BP 743, 2078 La Marsa. Tunisia.
e-mail: mohamed.bouafoura@ept.rnu.tn

Naceur Benhadj Braiek, Laboratory of Advanced Systems. Polytechnic High School of Tunisia. University of Carthage. BP 743, 2078 La Marsa. Tunisia.
e-mail: naceur.benhadj@ept.rnu.tn