KYBERNETIKA — VOLUME 55 (2019), NUMBER 4, PAGES 714–726

MULTIAGENT OPINION DYNAMICS INFLUENCED BY INDIVIDUAL SUSCEPTIBILITY AND ANCHORING EFFECT

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This paper studies a new model of social opinion dynamics in multiagent system by counting in two important factors, individual susceptibility and anchoring effect. Different from many existing models only focusing on one factor, this model can exhibit not only agreement phenomena, but also disagreement phenomena such as clustering and fluctuation, during opinion evolution. Then we provide several conditions to show how individual susceptibility and anchoring effect work on steady-state behaviors in some specific situations, with strict mathematical analysis. Finally, we investigate the model for general situations via simulations.

Keywords: opinion dynamics, individual susceptibility, anchoring effect, steady-state behavior

Classification: 91C99, 91D30, 91E99

1. INTRODUCTION

Social opinion evolution takes place ubiquitously in our daily life [3, 11, 24]. In fact, social influence happens because of various reasons such as persuasion and conformity [8]. Numerous theories appear to describe the processes and reveal the underlying mechanisms [6, 8, 18]. With regarding human opinions as scalar or vector quantities, different mathematical models are proposed in the literature [3, 6, 9, 11, 13]. A great number of these models are agent-based and established in multiagent systems [6, 9, 11]. In addition to opinion agreement, there are disagreement phenomena such as clustering and fluctuation, which widely exist in everyday experience [2, 7, 10, 15, 25].

Individual susceptibility of exogenous information, which is independent of social networks, is a significant factor of opinion formation [9, 10]. A famous opinion dynamics concerning this factor is Friedkin–Johnsen model [9]. In this model, individuals modify their opinions based on the exogenous information and aggregate opinion from others. Individual susceptibility describes the influence degree of the exogenous information in opinion formation.

Anchoring effect demonstrates that a person has a cognitive bias towards exogenous information in social interactions [21]. It indicates that the exogenous information not only provides continuous direct influence, as in the Friedkin–Johnsen model, but also

DOI: 10.14736/kyb-2019-4-0714

serves as a constant reference point when individuals update their opinions. Specifically speaking, individuals may check, before updating, whether the aggregate opinions are acceptable based on their initial opinions. We can use thresholds to characterize these individual tolerances of the difference between aggregate opinions and reference points.

Because of diverse phenomena in reality, the analysis of opinion evolution is a popular and essential issue of opinion dynamics [4, 6, 10, 11]. The DeGroot model leads to an agreement across the entire society, when the network structure has sufficient connectivity [6]. We can find clustering phenomenon in the Friedkin–Johnsen model because of individual susceptibilities of exogenous information [10]. When the social network changes randomly, opinion fluctuations appear [1]. Thus, we can conclude that steady-state behaviors are determined by social network topologies, initial values and model parameters [6, 10, 11].

To our best knowledge, there are few results which combine individual susceptibility and anchoring effect. In this paper, we introduce an opinion dynamic model considering these two mechanisms together. We use individual susceptibility parameters and acceptable thresholds to describe individual susceptibility and anchoring effect. This model can exhibit not only consensus, but also clustering and fluctuation phenomena. We thus give some examples to describe the impacts of exogenous information, susceptibility parameters, and acceptable thresholds on steady-state behaviors. Then we establish several conditions on these three factors to classify steady-state behaviors. One main result is that the emergence of agreement relates to large acceptable threshold, and the disagreement is due to small one. We also prove a necessary condition of clustering based on both susceptibility parameters and acceptable thresholds. Finally, we give some simulations to further illustrate the combined effects of these two mechanisms. These results show that our effort to consider both individual susceptibility and anchoring effect is effective and deepen our understanding of opinion evolution processes.

2. MODEL DEFINITION

In this section, we introduce the multiagent opinion model influenced by individual susceptibility and anchoring effect of exogenous information, which is a generalization of model in [22].

Consider a social network with n individuals, which are called nodes or agents in the sequel, indexed by the elements in $\mathcal{V} = \{1, \dots, n\}$. The structure of the social network is represented by an undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where each edge $\{i, j\} \in \mathcal{E}$ is an unordered pair of two different nodes in $\mathcal{V}$. The graph $\mathcal{G}$ is assumed to be connected without loss of generality. Each $i \in \mathcal{V}$ holds an opinion $x_i(t) \in \mathbb{R}$ at slotted time $t = 0, 1, 2, ...$, and let $x(t)$ be the opinion vector. Node i interacts with neighbors in the set $\mathcal{N}_i := \big\{j \in \mathcal{V} : \{i, j\} \in \mathcal{E} \text{ or } j = i\big\}$. It is noteworthy that, for node i, the initial opinion $x_i(0)$ can be also regarded as exogenous information at the same time. Without loss of generality, we assume that $\max_{i\in\mathcal{V}} x_i(0) - \min_{i\in\mathcal{V}} x_i(0) > 0$, which means that the initial opinions are not the same value.

The interpersonal influence strength between two neighboring nodes i and j is represented by $w_{ij} > 0$. If $j \notin \mathcal{N}_i$, $w_{ij} = 0$. Suppose that $\sum_{j\in\mathcal{V}} w_{ij} = 1$, and let

$$s_i(t) := \sum_{j\in\mathcal{N}_i} w_{ij}x_j(t)$$

be the weighted aggregate opinion of neighbors for i at time t.

After receiving the aggregate opinion from neighbors, agents determine to what extent they accept it according to the anchoring effect. The reference point of every individual is his or her own exogenous information. To be more specific, individual i examines whether the aggregate opinion $s_i(t)$ lies in an acceptable interval which centers at the exogenous information $x_i(0)$ and has radius $c_i \geq 0$, i. e., $\mathcal{I}_0(i) := [x_i(0) - c_i, x_i(0) + c_i]$. We refer to c_i as the acceptable threshold of agent i. If the difference between the aggregate opinion and $x_i(0)$ is smaller than the acceptable threshold, that is, $s_i(t) \in \mathcal{I}_0(i)$, then individual i adopts $s_i(t)$ as its new opinion. Otherwise, i holds an opinion which is a linear combination of the aggregate opinion and the exogenous information, according to individual susceptibility mechanism. Therefore, we can describe this process mathematically as follows.

$$x_i(t+1) = \begin{cases} s_i(t), & \text{if } s_i(t) \in \mathcal{I}_0(i), \\ (1-h_i)s_i(t) + h_i x_i(0), & \text{if } s_i(t) \notin \mathcal{I}_0(i), \end{cases} \tag{1}$$

where $h_i \in [0,1]$ measures i's susceptibility to others, called susceptibility parameter of i. In reality, h_i reflects i's tendency to defer to others' opinions.

Clearly, if all individuals ignore the exogenous information, that is, $h_i = 0$ for all $i \in \mathcal{V}$, our model (1) turns into the DeGroot model in [6]. If acceptable thresholds based on anchoring effect are removed, that is $c_i = 0$ for all $i \in \mathcal{V}$, our model (1) becomes the Friedkin–Johnson model in [9] with $\Lambda = diag(1-h_1, ..., 1-h_n)$.

3. MODEL ANALYSIS

In this section, we focus on how individual susceptibility and anchoring effect influence the opinion dynamics and its steady-state behaviors, via c_i, h_i and $x_i(0)$ for all $i \in \mathcal{V}$. It is worth noting that conclusions are obtained not relying on the network information, the estimation of which takes great efforts [19].

3.1. Behaviors of the proposed model

First of all, Figure 1 shows three important kinds of phenomena found in our model, i. e., consensus, clustering, and fluctuation.

When acceptable thresholds $\{c_i\}_{i\in\mathcal{V}}$ are large, the group reaches a consensus, which accords with social psychological findings [11]. To be more specific, large c_i represents that individual i tends to be more acceptable of others' views. This may be due to difficulties of judgments or attempts to satisfy others [17].

The clustering phenomenon emerges when acceptable thresholds $\{c_i\}_{i\in\mathcal{V}}$ are small. It is a common phenomenon in opinion formation and studied in a lot of classical opinion models in depth [10, 11]. Reasons for this are possibly due to less unanimity of opinions, distrust of each other, or high relevance to oneself [17].

Fluctuation is another common phenomenon in social reality [1, 12, 25]. This happens in our model when acceptable thresholds c_i take moderate values. The opinion fluctuation may be attributed to that people cannot determine whether to follow others or to stick to their own biases under mild social pressure. Empirical studies for large population behaviors such as voting discover this phenomenon [14].

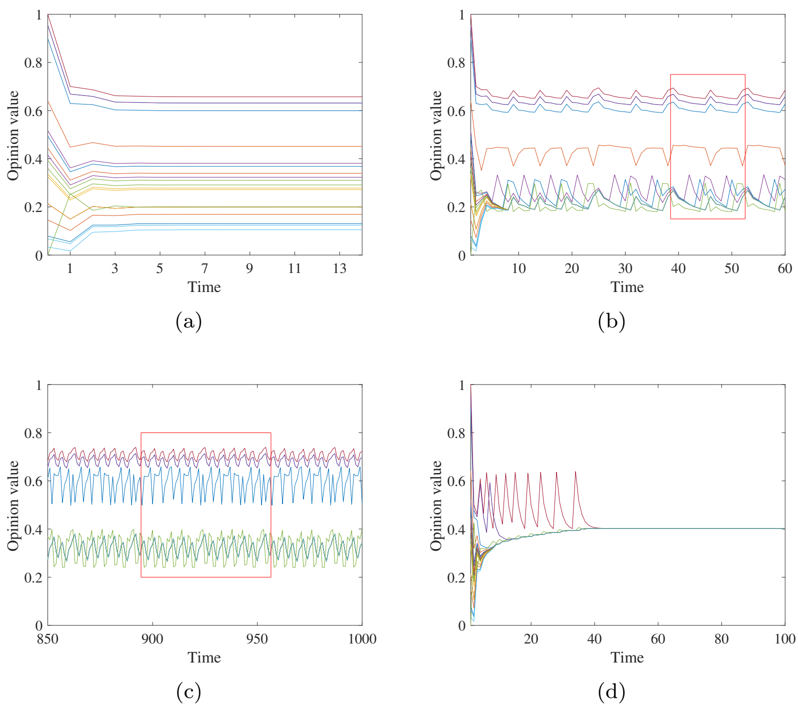
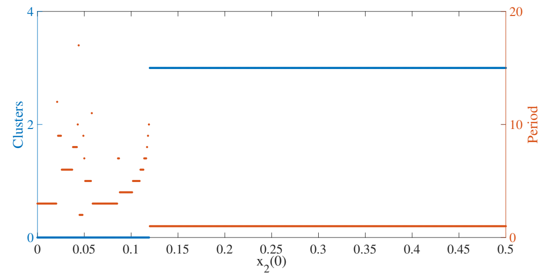


Fig. 1. Basic behaviors of system (1). There are three kinds of steady-state behaviors, namely, clustering in (a), fluctuations in (b) and (c) where red frames demonstrate one period of the trajectories, and consensus in (d).

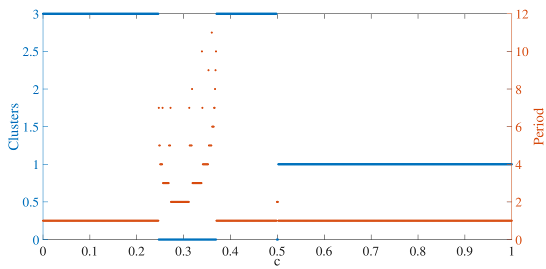
As we mentioned before, the whole system depends on the social network topology $\mathcal{G}$, exogenous information $\{x_i(0)\}_{i\in\mathcal{V}}$, acceptable thresholds $\{c_i\}_{i\in\mathcal{V}}$, and susceptibility parameters $\{h_i\}_{i\in\mathcal{V}}$. To emphasize individual susceptibility and anchoring effect in this model, we study steady-state behaviors without concerning the influence of different types of $\mathcal{G}$. We thus present a specific example to obtain intuitive understandings of the relation between steady-state behaviors and $x_i(0)$, c_i, as well as h_i, $i \in \mathcal{V}$.

In this example, the topology $\mathcal{G}$ is a complete graph with 3 nodes and $w_{ij} = 1/3$ for all $i, j \in \mathcal{V}$. Furthermore, suppose that for all $i \in \mathcal{V}$, $c_i = c \geq 0$ and $h_i = h \geq 0$. We show how these three factors influence the system. Without loss of generality, the largest initial opinion is taken as $x_1(0) = 1$, and the smallest initial opinion $x_3(0) = 0$. As in

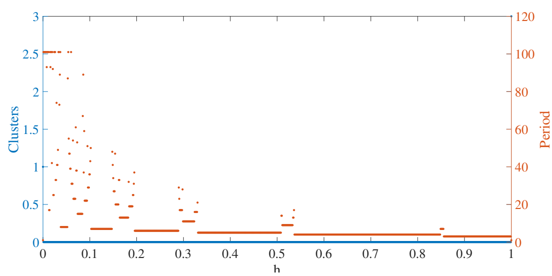
Figure 2(a), for fixed c and h, the model finally converges to a period orbit when $x_2(0)$ is smaller than 0.12, but the pattern of the period is complicated. For a large $x_2(0)$, the clustering phenomenon emerges and remains. In Figure 2(b), the initial values $x_i(0)$, h are fixed and c varies from 0 to 1. The model is able to reach an equilibrium of clustering for small c, but as c grows larger, fluctuation appears. The behavior of this system returns to clustering when c is close to 0.5, and finally becomes consensus when $c > 0.5$. If the initial values $x_i(0)$ and c are fixed, the period seems to decrease as h increases from 0 to 1, which is shown in Figure 2(c).



(a) The effect of initial value $x_2(0)$.



(b) The effect of acceptable threshold c.



(c) The effect of susceptibility parameter h.

Fig. 2. Bifurcation patterns when the initial value, acceptable threshold and susceptibility parameter change. Cluster number 0 represents fluctuation in each subfigure.

Theoretically, we can prove following results for system (1).

Lemma 3.1. $\min_{i\in\mathcal{V}} x_i(0) \le x_i(t) \le \max_{i\in\mathcal{V}} x_i(0)$ for all $i \in \mathcal{V}$ and $t \in \mathbb{N}$.

P r o o f. We use mathematical induction to prove this result. First, when $t = 0$, we have that $\min_{i\in\mathcal{V}} x_i(0) \le x_i(0) \le \max_{i\in\mathcal{V}} x_i(0)$ for all $i \in \mathcal{V}$. We suppose that $\min_{i\in\mathcal{V}} x_i(0) \le x_i(k) \le \max_{i\in\mathcal{V}} x_i(0)$ for all $i \in \mathcal{V}$, and consider the situation when $t = k + 1$. In view of $\sum_{j\in\mathcal{V}} w_{ij} = 1$ with $w_{ij} \ge 0$,

$$s_i(k+1) = \sum_{j\in\mathcal{N}_i} w_{ij} x_j(k) \in \left[\min_{i\in\mathcal{N}_i} x_i(k), \max_{i\in\mathcal{N}_i} x_i(k)\right].$$

From $\min_{i\in\mathcal{V}} x_i(0) \le x_i(k) \le \max_{i\in\mathcal{V}} x_i(0)$ for all $i \in \mathcal{V}$,

$$s_i(k+1) \in \left[\min_{i\in\mathcal{V}} x_i(0), \max_{i\in\mathcal{V}} x_i(0)\right].$$

According to (1),

$$x_i(k+1) \in \left[\min_{i\in\mathcal{V}} x_i(0), \max_{i\in\mathcal{V}} x_i(0)\right].$$

The conclusion follows from mathematical induction. □

This property can also be found in some pre-existing models, for example, the DeGroot model [6]. But the same result does not hold in other models, e. g. [4, 5]. More precisely, when all individuals hold the similar polarized opinions, all of them converge to corresponding extreme opinions finally [4].

Then, we provide a sufficient condition of reaching a consensus. It is also another situation, where our model can turn into the DeGroot model different from $h_i = 0$ for all $i \in \mathcal{V}$.

Theorem 3.2. If $\min_{i\in\mathcal{V}} c_i \ge \max_{i\in\mathcal{V}} x_i(0) - \min_{i\in\mathcal{V}} x_i(0)$, then $\lim_{t\to\infty} x_i(t) = x_\infty$ for all $i \in \mathcal{V}$ with a constant $x_\infty \in [\min_{i\in\mathcal{V}} x_i(0), \max_{i\in\mathcal{V}} x_i(0)]$.

P r o o f. From Lemma 3.1 we know that $s_i(t) \in [\min_{i\in\mathcal{V}} x_i(0), \max_{i\in\mathcal{V}} x_i(0)]$ for all $t \in \mathbb{N}$. According to $\max_{i\in\mathcal{V}} x_i(0) - \min_{i\in\mathcal{V}} x_i(0) \le \min_{i\in\mathcal{V}} c_i$, there holds that $s_i(t) \in [x_j(0) - c_j, x_j(0) + c_j]$ for all $i, j \in \mathcal{V}$ and $t \in \mathbb{N}$. Therefore, system (1) becomes

$$x_i(t+1) = s_i(t)$$

for all $i \in \mathcal{V}$.

Therefore, system (1) turns into a DeGroot model. In view of results in [6], $\lim_{t\to\infty} x_i(t) = x_\infty$ for all $i \in \mathcal{V}$, where $x_\infty \in [\min_{i\in\mathcal{V}} x_i(0), \max_{i\in\mathcal{V}} x_i(0)]$. □

Clearly, Theorem 3.2 provides a theoretical support to Figure 2(b) when c is large enough. On the other hand, we can prove an corresponding result illustrated in Figure 2(b) at the same time. In other words, all individuals never reach a consensus opinion when some of c_i are sufficient small.

Theorem 3.3. If $\cap_{i\in\mathcal{V}}[x_i(0)-c_i, x_i(0)+c_i] = \emptyset$, then there exists no constant $x_\infty \in \mathbb{R}$, s.t. $\lim_{t\to\infty} x_i(t) = x_\infty$ for all $i \in \mathcal{V}$.

Proof. According to Lemma 3.1, we suppose that there exists a number $x_\infty \in [0, 1]$ such that $\lim_{t\to\infty} x_i(t) = x_\infty$ for all $i \in \mathcal{V}$. Then $\lim_{t\to\infty} s_i(t) = x_\infty$ for all $i \in \mathcal{V}$. From (1), there holds that $x_\infty \in \mathcal{I}_0(i)$ for all $i \in \mathcal{V}$. In other words, $\cap_{i\in\mathcal{V}}[x_i(0)-c_i, x_i(0)+c_i] \neq \emptyset$, which contradicts the assumption. Therefore, there exists no consensus point x_∞ if $\cap_{i\in\mathcal{V}}[x_i(0) - c_i, x_i(0) + c_i] = \emptyset$. □

In Theorems 3.2 and 3.3, we provide a sufficient condition and a necessary one of reaching a consensus. Next, we introduce a necessary condition for clustering phenomenon. It provides a criterion for when the public opinions cluster into different groups, which is shown in Figure 1 (a).

To ease the presentation, we impose the following assumptions in the rest of our paper. First, $w_{ij} = \frac{1}{|\mathcal{N}_i|}$ for all $i \in \mathcal{V}$ and $j \in \mathcal{N}_i$. Furthermore, for all $i \in \mathcal{V}$, $c_i = c \geq 0$ and $h_i = h \geq 0$. Then we consider the system as follows,

$$x_i(t+1) = \begin{cases} s_i(t), & \text{if } s_i(t) \in \mathcal{I}_0(i), \\ (1-h)s_i(t) + hx_i(0), & \text{if } s_i(t) \notin \mathcal{I}_0(i), \end{cases} \tag{2}$$

where $s_i(t) = \sum_{j\in\mathcal{N}_i} \frac{1}{|\mathcal{N}_i|} x_j(t)$ and $\mathcal{I}_0(i) = [x_i(0) - c, x_i(0) + c]$.

Theorem 3.4. If $\max_{i\in\mathcal{V}} x_i(0) - \min_{i\in\mathcal{V}} x_i(0) \leq (2-h)c$, that is, $|\cap_{i\in\mathcal{V}}[x_i(0)-c, x_i(0)+c]| \geq hc$, then system (2) cannot cluster into different groups.

Proof. Let $\Omega := \cap_{i\in\mathcal{V}}[x_i(0)-c, x_i(0)+c] \neq \emptyset$ and $y \in \Omega$. Then $x_i(0) \in [y-c, y+c]$ for all $i \in \mathcal{V}$. Suppose that the final values of nodes in system (2) are $0 \leq x_1 < ... < x_k \leq 1$ where $k \leq n$.

Suppose that there is $m \in \mathcal{V}$ such that $\lim_{t\to\infty} x_m(t) = x_1$ and there is at least one node $p \in \mathcal{N}(m)$ such that $\lim_{t\to\infty} x_q(t) \neq x_1$. Define $s_m := \lim_{t\to\infty} s_m(t)$. Then $s_m > x_1$ in view of the definitions of x_1 and node m, and therefore $s_m \notin \mathcal{I}_0(m)$. That is $s_m < x_m(0) - c$ or $s_m > x_m(0) + c$.

If $s_m < x_m(0) - c$ and $x_m(0) - c > 0$, then

$$x_1 < s_m < x_m(0) - c < x_m(0).$$

Furthermore,

$$\begin{aligned} x_1 = (1-h)s_m + hx_m(0) &> (1-h)x_1 + hx_m(0) \\ &\Rightarrow x_1 > x_m(0). \end{aligned}$$

which leads to a contradiction. Thus $s_m \geq x_m(0) - c$.

Next we consider the case where $s_m > x_m(0) + c$ and $x_m(0) + c < 1$. In this case,

$$\begin{aligned} x_1 = x_m(\infty) &= (1-h)s_m + hx_m(0) \\ &> (1-h)(x_m(0)+c) + hx_m(0) \\ &= x_m(0) + (1-h)c \\ &\geq (y-c) + (1-h)c \\ &= y - hc \end{aligned}$$

where the first inequality holds for $s_m > x_m(0)+c$ and the second one holds for $x_m(0) \geq y - c$. Now we know that $x_1 > y - hc$.

Analogously, we can obtain that $x_k < y + hc$. Therefore, we know that

$$x_m(0) + c < s_m < y + hc,$$

and

$$\sup \Omega = \min_{i \in \mathcal{V}} \{x_i(0) + c\} < y + hc.$$

Let $y = \inf \Omega$. Then

$$|\Omega| = \sup \Omega - \inf \Omega < hc.$$

We have proved that $|\Omega| < hc$ is a necessary condition of the case where opinions in system (2) divide into different values. Therefore, the conclusion follows. □

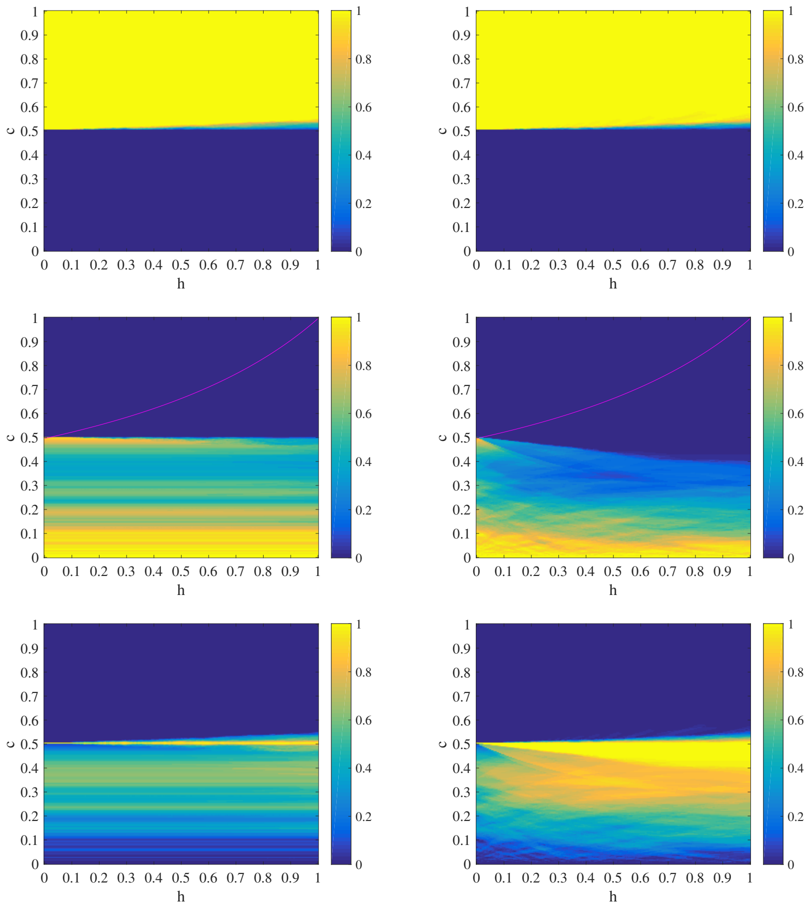
Remark 3.5. Notice that the only condition on the network topology in Lemma 3.1, Theorems 3.2, 3.3 and 3.4 is connectivity. Therefore, when the network topology is a series of switching graphs each of which is connected, all the results in this section also hold.

It is worth pointing out that the bound of the clustering condition in Theorem 3.4 does not depend singly on susceptibility parameter h or acceptable threshold c, but their product hc. This shows the combined impact of both individual susceptibility and anchoring effect in opinion evolution. Therefore, we are going to investigate c, h and their combined effects in depth.

3.2. The effect of c and h, and their combined effects

To examine the effect of susceptibility parameter h and acceptable threshold c, we carry out simulations over both two fundamental networks and two classical random graphs. Without loss of generality, we set $\hat{x} := \max_{i \in \mathcal{V}} x_i(0) - \min_{i \in \mathcal{V}} x_i(0) = 1$ for simplicity, because it can be verified that model (2) with parameters c, h and $x(0)$, behaves the same as that with parameters $c/\hat{x}$, h and $x(0)/\hat{x}$.

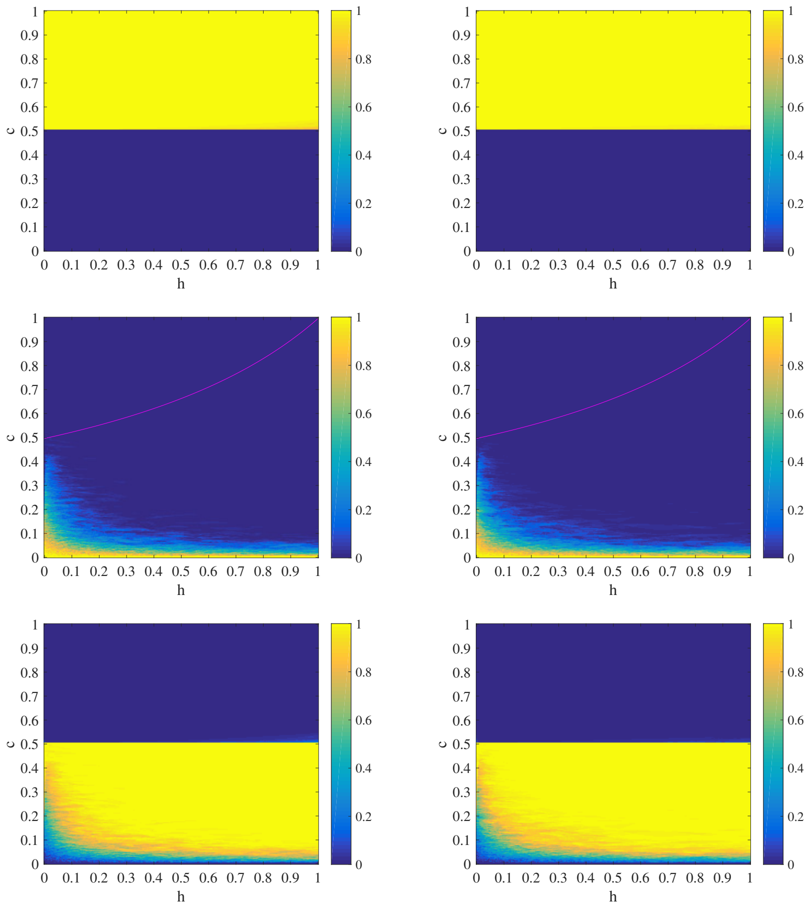
We take two fundamental graphs (namely, the complete graph and cycle graph) to demonstrate the results, and select 100 initial vectors randomly from uniform distribution to run the simulation. Figure 3 shows that the proportions of different steady-state



(a) Complete Graph. (b) Cycle Graph.

Fig. 3. The proportions of consensus (first row), clustering (second row) and fluctuation (third row) for complete graph and cycle graph.

behaviors with different c and h. The pink line in the first row of Figure 3 is the theoretical bound of clustering in Theorem 3.4, so we can comprehend this result intuitively. A natural conjecture base on both of three rows in Figure 3 is, the constraint $(2-h)c \geq 1$ is a sufficient condition of consensus. We prove a special case, in which $c > 0.5$ leads to consensus in [22]. On the other hand, consensus does not appear when $c < 0.5$ over neither complete graph nor cycle graph. This discovery matches results in previous sections. All six subfigures in Figure 3 illustrate that clustering and fluctuation phenomena are mixing together at the same time. Furthermore, comparing (a) and (b) in Figure 3, we may find that higher connectivity weakens the fluctuation phenomena.

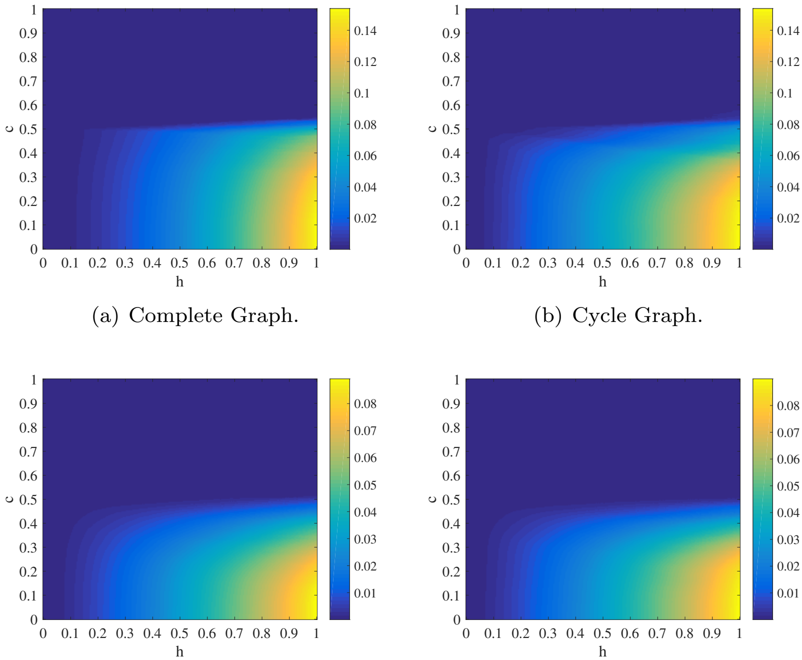


(a) Barabasi–Albert Network. (b) Erdos–Renyi Graph.

Fig. 4. The proportions of consensus (first row), clustering (second row) and fluctuation (third row) for two random graphs.

We also randomly select 50 initial vectors to conduct simulations over two random graphs with more nodes, i. e., Barabasi–Albert graphs and Erdos–Renyi graphs with 100 nodes and average degree 6 in Figure 4. All the observations and conjectures from complete graph and cycle graph still hold for these two graphs. Furthermore, fluctuation phenomena happen more often than clustering, which may due to the complexities of larger graphs.

In addition to above findings, we notice that when the steady states of the model are fluctuation and clustering, the dispersion of individual opinions has close connections with the susceptibility parameter h and acceptable threshold c. Figure 4 and 5 shows



(a) Complete Graph. (b) Cycle Graph.

(c) Barabasi–Albert Network. (d) Erdos–Renyi Graph.

Fig. 5. The variance of steady-state opinion vectors.

the pattern of the variance of steady-state opinions. In these simulations, if the model ends in fluctuating, we take the average of the variance of opinion vectors at the last 20 time steps as a characterization of the dispersion. As h varies from 0 to 1, the variance increases, which indicates the growth of opinion dispersion among a crowd. Furthermore, the smaller c is, the larger the variance is. Intuitively, these phenomena illustrate that the dispersion of steady-state opinions as well as steady-state category depends on individual tendency to change their ideas (represented by h) and intensity of the anchoring effect (represented by c).

4. CONCLUSIONS

In this paper, we studied a new model of multiagent opinion dynamics by counting in two important factors, individual susceptibility and anchoring effect of exogenous information. We showed that this model can exhibit not only consensus, but also clustering and fluctuation phenomena by simulations. Then we proved several sufficient or necessary conditions for steady-state behaviors of the opinion dynamics based on the exogenous information, susceptibility parameters and acceptable thresholds. We also provided several numerical examples to validate our study and investigate the model for general situations. Future work includes systematic classifications of steady-state behaviors in opinion evolution based on all the factors; they are the social network topology, exogenous information, individual susceptibility parameters and acceptable thresholds.

(Received April 21, 2019)

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