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MEAN ALMOST PERIODICITY AND MOMENT EXPONENTIAL STABILITY OF DISCRETE-TIME STOCHASTIC SHUNTING INHIBITORY CELLULAR NEURAL NETWORKS WITH TIME DELAYS

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By using the semi-discrete method of differential equations, a new version of discrete analogue of stochastic shunting inhibitory cellular neural networks (SICNNs) is formulated, which gives a more accurate characterization for continuous-time stochastic SICNNs than that by Euler scheme. Firstly, the existence of the $2p$th mean almost periodic sequence solution of the discrete-time stochastic SICNNs is investigated with the help of Minkowski inequality, Hölder inequality and Krasnoselskii's fixed point theorem. Secondly, the moment global exponential stability of the discrete-time stochastic SICNNs is also studied by using some analytical skills and the proof of contradiction. Finally, two examples are given to demonstrate that our results are feasible. By numerical simulations, we discuss the effect of stochastic perturbation on the almost periodicity and global exponential stability of the discrete-time stochastic SICNNs.

Keywords: semi-discrete method, stochastic, Krasnoselskii's fixed point theorem, almost periodicity, global exponential stability

Classification: 39A50, 39A24, 39A30, 92B20

1. INTRODUCTION

In [5], Bouzerdoum and Pinter propounded a new version of cellular neural networks [33, 35, 42, 43, 44], named by shunting inhibitory cellular neural networks (SICNNs), which have been widely applied in psychophysics, parallel computing, perception, robotics, adaptive pattern recognition, associative memory, image processing pattern recognition and combinatorial optimization. All of these applications heavily depend on the (almost) periodicity and global exponential stability. Specifically, there are lots of articles focus on the issue of the existence and global exponential stability of the equilibrium point, periodic and almost periodic solutions of SICNNs with time delays in the literature [6, 9, 19, 36, 38]. For instance, Yılmaz et al. [28] considered the following SICNNs:

$$\frac{\mathrm{d}x_{ij}(t)}{\mathrm{d}t} = -a_{ij}(t)x_{ij}(t) - F_{ij}(t,x)x_{ij}(t) + I_{ij}(t), \tag{1.1}$$

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where

$$F_{ij}(t,x) = \sum_{b^{hl}\in N_r(i,j)} b_{ij}^{hl}(t)f_{ij}(x_{hl}(t)) + \sum_{c^{hl}\in N_q(i,j)} c_{ij}^{hl}(t)g_{ij}(x_{hl}(t-\mu_{hl}(t))), \qquad (1.2)$$

where c_{ij} denotes the cell at the (i,j) position of the lattice. Denote by $N_r(i,j)$ the r-neighborhood of c_{ij} such that

$$N_r(i,j) = \{c_{hl} : \max\{|h-i|, |l-j|\} \le r, 1 \le h \le m, 1 \le l \le n\}.$$

$N_q(i,j)$ is similarly specified, x_{ij} is the activity of the cell c_{ij}, I_{ij} is the external input to c_{ij}, the constant a_{ij} represents the passive decay rate of the cell activity, b_{ij}^{hl} and c_{ij}^{hl} are the connection or coupling strength of postsynaptic activity of the cell transmitted to the cell c_{ij}, the activity functions $f_{ij}(\cdot)$ and $g_{ij}(\cdot)$ are continuous functions representing the output or firing rate of the cell c_{ij}, and the continuous function μ_{hl} corresponds to the transmission delay along the axon of the (h,l)th cell from the (i,j)th cell, $i = 1,2,\dots,m$, $j = 1,2,\dots,n$.

Uncertain models described by stochastic differential equations have caused great concerns since they have wide applications in practice such as engineering, physics, chemistry and biology [3, 4, 11, 18, 27, 46]. In the actual situations, uncertainties have a consequence on the performance of the neural networks. In neural networks, the connection weights of the neurons depend on certain resistance and capacitance values that include modeling errors or uncertainties during the parameter identification process. The uncertainties come mainly from the deviations and perturbations in parameters. In particular, when modeling neural networks, the parameters uncertainties should be taken into consideration. Therefore, we consider the following stochastic SICNNs [32, 37]:

$$\begin{aligned} \mathrm{d}x_{ij}(t) &= \Big[- a_{ij}(t)x_{ij}(t) - F_{ij}(t,x)x_{ij}(t) + I_{ij}(t)\Big]\mathrm{d}t \\ &\quad + \sum_{d^{hl}\in N_e(i,j)} d_{ij}^{hl}(t)\sigma_{ij}(x_{ij}(t))\mathrm{d}w_{ij}(t), \end{aligned} \qquad (1.3)$$

where d_{ij}^{hl} and $N_e(i,j)$ are similarly specified as that in system (1.1), F_{ij} is defined as that in (1.2), w_{ij} is the standard Brownian motion defined on a complete probability space, $i = 1,2,\dots,m$, $j = 1,2,\dots,n$.

Periodicity [47] often appears in implicit ways in various natural phenomena. This is the case when one studies the effects of fluctuating environments. Though one can deliberately periodically fluctuate environmental parameters in controlled laboratory experiments, fluctuations in nature are hardly periodic. Almost periodicity is more likely to accurately describe natural fluctuations [39, 40, 48, 41, 45]. The concept of almost periodicity is important in probability especially for investigations on stochastic processes. The interest in such a notion lies in its significance and applications arising in engineering, statistics, etc, see [1, 7, 16, 30, 31].

The discrete-time neural networks become more important than the continuous-time counterparts when implementing the neural networks in a digital way. In order to investigate the dynamical characteristics with respect to digital signal transmission, it is essential to formulate the discrete analog of neural networks. In recent years, many researches have been obtained for the dynamic analysis of discrete-time determinant or stochastic neural networks formulated by Euler scheme [2, 8, 10, 12, 20, 25, 26, 34].

Mohamad and Gopalsamy [22, 23] proposed a novel method (i. e., semi-discretization technique) in formulating a discrete-time analogue of the continuous-time neural networks, which faithfully preserved the characteristics of their continuous-time counterparts. In [22], the authors employed computer simulations to show that semi-discrete models give a more accurate characterization for the corresponding continuous-time models than that by Euler scheme. With the help of the semi-discretization technique [22], many scholars obtained the semi-discrete analogue of the continuous-time neural networks and some meaningful results were gained for the dynamic behaviours of the semi-discrete neural networks, such as periodic solutions, almost periodic solutions and global exponential stability, see [13, 14, 15, 21, 24, 49, 50].

However, the disquisitive models in literatures [13, 15, 14, 21, 24, 49, 50] are deterministic. Stimulated by this point, we should consider random factors in the studied models, such as system (1.3). By using the semi-discretization technique [22], Krasnoselskii's fixed point theorem and stochastic theory, the main aim of this paper is to establish some decision theorems for the existence of $2p$th mean almost periodic sequence solutions and pth moment global exponential stability for the semi-discrete analogue of uncertain system (1.3). The work of this paper is a continuation of that in [13, 14, 15] and the results in this paper complement the corresponding results in [13, 14, 15].

In order to get the discrete analogue of system (1.3) by the semi-discrete method [22], the following stochastic SICNNs with piecewise constant arguments corresponding to system (1.3) has been taken into account:

$$\begin{aligned} \mathrm{d}x_{ij}(t) = \Big[& - a_{ij}([t])x_{ij}(t) - F_{ij}([t],x)x_{ij}([t]) + I_{ij}([t]) \\ & + \sum_{d^{hl}\in N_e(i,j)} d_{ij}^{hl}([t])\sigma_{ij}(x_{ij}([t]))\Delta w_{ij}([t])\Big]\mathrm{d}t, \end{aligned}$$

where $[t]$ denotes the integer part of t, $i = 1, 2, \dots, m$, $j = 1, 2, \dots, n$. Here the discrete analogue of the stochastic parts of system (1.3) is obtained by Euler scheme, i. e., $\mathrm{d}w_{ij}(t) \approx \Delta w_{ij}([t])\mathrm{d}t = [w_{ij}([t]+1) - w_{ij}([t])]\mathrm{d}t$, $i = 1, 2, \dots, m$, $j = 1, 2, \dots, n$. For each $t \in \mathbb{R}$, there exists an integer $k \in \mathbb{Z}$ such that $k \le t < k+1$. Then the above equation becomes

$$\begin{aligned} \mathrm{d}x_{ij}(t) = \Big[& - a_{ij}(k)x_{ij}(t) - F_{ij}(k,x)x_{ij}(k) + I_{ij}(k) \\ & + \sum_{d^{hl}\in N_e(i,j)} d_{ij}^{hl}(k)\sigma_{ij}(x_{ij}(k))\Delta w_{ij}(k)\Big]\mathrm{d}t, \end{aligned}$$

where $i = 1, 2, \dots, m$, $j = 1, 2, \dots, n$. Integrating the above equation from k to t and letting $t \to k+1$, we achieve the discrete analogue of system (1.3) as follows:

$$\begin{aligned} x_{ij}(k+1) = {} & e^{-a_{ij}(k)}x_{ij}(k) \\ & - \frac{1 - e^{-a_{ij}(k)}}{a_{ij}(k)}\Big[F_{ij}(k,x)x_{ij}(k) - G_{ij}(k,x)\Delta w_{ij}(k) - I_{ij}(k)\Big], \end{aligned} \tag{1.4}$$

where $G_{ij}(k,x) = \sum_{d^{hl}\in N_e(i,j)} d_{ij}^{hl}(k)\sigma_{ij}(x_{ij}(k))$, $\Delta w_{ij}(k) = w_{ij}(k+1) - w_{ij}(k)$ and F_{ij} is defined as that in (1.2), $k \in \mathbb{Z}$, $i = 1, 2, \dots, m$, $j = 1, 2, \dots, n$.

Stimulated by the above discussions, the main purpose of this paper is to establish a set of sufficient conditions for the existence and global exponential stability of mean almost periodic sequence solutions of system (1.4). The paper is organized as follows. In Section 2, we give some basic definitions and necessary lemmas which will be used in later sections. In Section 3, we employ Krasnoselskii's fixed point theorem to obtain sufficient conditions for the existence of at least one $2p$th mean almost periodic sequence solution of system (1.4). In Section 4, we consider the global exponential stability of system (1.4). Two examples and simulations are also given to illustrate our main results.

Throughout this paper, we use the following notations. Let $\mathbb{R}$ and $\mathbb{Z}$ denote the sets of real numbers and integers, respectively. $\mathbb{R}^n$ denotes the n-dimensional real vector space. Let $(\Omega, \mathcal{F}, P)$ be a complete probability space. Denote by $BC(\mathbb{Z}, L^p(\Omega, \mathbb{R}^{m\times n}))$ the vector space of all bounded continuous functions from $\mathbb{Z}$ to $L^p(\Omega, \mathbb{R}^{m\times n})$, where $L^p(\Omega, \mathbb{R}^{m\times n})$ denotes the collection of all pth integrable $\mathbb{R}^{m\times n}$-valued random variables. Then $BC(\mathbb{Z}, L^p(\Omega, \mathbb{R}^{m\times n}))$ is a Banach space with the norm $\|X\|_p = \sup_{k\in\mathbb{Z}} |X|_p$, $|X|_p = \max_{(i,j)}(E|x_{ij}(k)|^p)^{\frac{1}{p}}$, $\forall X = \{x_{ij}\} := \{x_{11}, x_{12}, \dots, x_{mn}\} \in BC(\mathbb{Z}, L^p(\Omega, \mathbb{R}^{m\times n}))$, where $p > 1$ and $E(\cdot)$ stands for the expectation operator with respect to the given probability measure P. Set $\bar{f} = \sup k \in \mathbb{Z}|f(k)|$ and $\underline{f} = \inf k \in \mathbb{Z}|f(k)|$ for bounded real function f defined on $\mathbb{Z}$. $[a, b]_{\mathbb{Z}} = [a, b] \cap \mathbb{Z}$, $\forall a, b \in \mathbb{R}$.

2. PRELIMINARIES

Before we derive our main results, we shall introduce several basic definitions and important lemmas.

Definition 2.1. (Bezandry and Diagana [4]) A stochastic process $X \in BC(\mathbb{Z}; L^{2p}(\Omega; \mathbb{R}^{m\times n}))$ is said to be $2p$th mean almost periodic sequence if for each $\epsilon > 0$, there exists an integer $l(\epsilon) > 0$ such that each interval of length $l(\epsilon)$ contains an integer τ for which

$$E|X(k+\tau) - X(k)|_{2p} < \epsilon, \quad \forall k \in \mathbb{Z}.$$

A stochastic process X, which is 2-nd mean almost periodic sequence will be called square-mean almost periodic sequence. Like for classical almost periodic functions, the number τ will be called an ϵ-translation of X.

Lemma 2.2. (Kuang [17]) (Minkowski inequality) Assume that $p \geq 1$, $E|\xi|^p < \infty$, $E|\eta|^p < \infty$, then

$$(E|\xi + \eta|^p)^{1/p} \leq (E|\xi|^p)^{1/p} + (E|\eta|^p)^{1/p}.$$

Lemma 2.3. (Kuang [17]) (Hölder inequality) Assume that $p > 1$, then

$$\sum_k |a_k b_k| \leq \left[\sum_k |a_k|\right]^{1-1/p} \left[\sum_k |a_k||b_k|^p\right]^{1/p}.$$

If $p = 1$, then $\sum_k |a_k b_k| \leq (\sum_k |a_k|)(\sup_k |b_k|)$.

Lemma 2.4. $X = \{x_{ij}\}$ is a solution of system (1.4) if and only if

$$x_{ij}(k) = \prod_{s=k_0}^{k-1} e^{-a_{ij}(s)} x_{ij}(k_0) - \sum_{v=k_0}^{k-1} \prod_{s=v+1}^{k-1} \frac{e^{-a_{ij}(s)}[1-e^{-a_{ij}(v)}]}{a_{ij}(v)} \Big[F_{ij}(v,x)x_{ij}(v) - G_{ij}(v,x)\Delta w_{ij}(v) - I_{ij}(v)\Big], \tag{2.1}$$

where $k_0 \in \mathbb{Z}$, $k \in (k_0, +\infty)_{\mathbb{Z}}$, $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$.

Proof. By $\Delta[u(k)v(k)] = [\Delta u(k)]v(k) + u(k+1)[\Delta v(k)]$ and system (1.4), it gets

$$\Delta\Big[\prod_{s=0}^{k-1} e^{a_{ij}(s)} x_{ij}(k)\Big] = -\prod_{s=0}^{k} \frac{e^{a_{ij}(s)}[1-e^{-a_{ij}(k)}]}{a_{ij}(k)} \Big[F_{ij}(k,x)x_{ij}(k) - G_{ij}(k,x)\Delta w_{ij}(k) - I_{ij}(k)\Big],$$

where $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$, $k \in \mathbb{Z}$. So

$$\sum_{v=k_0}^{k-1} \Delta\Big[\prod_{s=0}^{v-1} e^{a_{ij}(s)} x_{ij}(v)\Big] = -\sum_{v=k_0}^{k-1} \prod_{s=0}^{v} \frac{e^{a_{ij}(s)}[1-e^{-a_{ij}(v)}]}{a_{ij}(v)} \Big[F_{ij}(v,x)x_{ij}(v) - G_{ij}(v,x)\Delta w_{ij}(v) - I_{ij}(v)\Big]$$

is equivalent to

$$\prod_{s=0}^{k-1} e^{a_{ij}(s)} x_{ij}(k) = \prod_{s=0}^{k_0-1} e^{a_{ij}(s)} x_{ij}(k_0) - \sum_{v=k_0}^{k-1} \prod_{s=0}^{v} \frac{e^{a_{ij}(s)}[1-e^{-a_{ij}(v)}]}{a_{ij}(v)} \Big[F_{ij}(v,x)x_{ij}(v) - G_{ij}(v,x)\Delta w_{ij}(v) - I_{ij}(v)\Big],$$

where $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$, $k \in \mathbb{Z}$. By the above equations, we can easily derive (2.1). This completes the proof. □

Lemma 2.5. (Hu at al. [11]) Suppose that $g \in L^2([a,b], \mathbb{R})$, then

$$E\Big[\sup_{t\in[a,b]} \Big|\int_a^t g(s)\,\mathrm{d}\omega(s)\Big|^p\Big] \le C_p E\Big[\int_a^b |g(t)|^2\,\mathrm{d}t\Big]^{\frac{p}{2}},$$

where

$$C_p = \begin{cases} (32/p)^{p/2}, & 0 < p < 2, \\ 4, & p = 2, \\ \Big[\frac{p^{p+1}}{2(p-1)^{(p-1)}}\Big]^{\frac{p}{2}}, & p > 2. \end{cases}$$

Lemma 2.6. Assume that $\{x(k) : k \in \mathbb{Z}\}$ is real-valued stochastic process and $w(k)$ is the standard Brownian motion, then

$$E\big|x(k)\Delta w(k)\big|^p \leq C_p E\big|x(k)\big|^p, \quad \forall k \in \mathbb{Z},$$

where C_p is defined as that in Lemma 2.5, $p > 0$.

Proof. By Lemma 2.5, it follows that

$$E\big|x(k)\Delta w(k)\big|^p = E\bigg|\int_k^{k+1} x(k)\,\mathrm{d}w(s)\bigg|^p \leq C_p E\bigg|\int_k^{k+1} x^2(k)\,\mathrm{d}s\bigg|^{\frac{p}{2}} \leq C_p E\big|x(k)\big|^p,\ k \in \mathbb{Z}.$$

This completes the proof. □

Lemma 2.7. (Smart [29]) Assume that Λ is a closed convex nonempty subset of a Banach space $\mathbb{X}$. Suppose further that $\mathcal{B}$ and $\mathcal{C}$ map Λ into $\mathbb{X}$ such that

(1) $x, y \in \Lambda$ implies that $\mathcal{B}x + \mathcal{C}y \in \Lambda$;

(2) $\mathcal{B}$ is continuous and $\mathcal{B}\Lambda$ is contained in a compact set;

(3) $\mathcal{C}$ is a contraction mapping.

Then there exists a $z \in \Lambda$ with $z = \mathcal{B}z + \mathcal{C}z$.

Throughout this paper, suppose that the following conditions are satisfied:

(H_1) a_{ij} is a bounded sequence defined on $\mathbb{Z}$ with $\underline{a}_{ij} > 0$, $i = 1, 2, \dots, m$, $j = 1, 2, \dots, n$.

(H_2) There exists several constants f_{ij}^*, g_{ij}^*, σ_{ij}^*, L_{ij}^f, L_{ij}^g and L_{ij}^σ such that $f_{ij}(k) \leq f_{ij}^*$, $g_{ij}(k) \leq g_{ij}^*$, $\sigma_{ij}(k) \leq \sigma_{ij}^*$ and

$$|f_{ij}(u) - f_{ij}(v)| \leq L_{ij}^f|u - v|, \quad |g_{ij}(u) - g_{ij}(v)| \leq L_{ij}^g|u - v|,$$

$$|\sigma_{ij}(u) - \sigma_{ij}(v)| \leq L_{ij}^\sigma|u - v|,$$

for all $u, v \in \mathbb{R}$, where $i = 1, 2, \dots, m$, $j = 1, 2, \dots, n$.

3. 2PTH MEAN ALMOST PERIODIC SEQUENCE SOLUTION

Define

$$\bar{a} := \max_{(i,j)} \bar{a}_{ij}, \quad \underline{a} := \min_{(i,j)} \underline{a}_{ij}, \quad \beta_{2p} := \frac{\alpha_{2p}}{1 - r_{2p}},$$

$$r_{2p} := \frac{(1 - e^{-\bar{a}})}{\underline{a}(1 - e^{-\underline{a}})} \max_{(i,j)} \Big\{D^* + K^* C_{2p}^{1/2p}\Big\},$$

$$\alpha_{2p} := \frac{(1 - e^{-\bar{a}})}{\underline{a}(1 - e^{-\underline{a}})} \max_{(i,j)} \Big[\bar{I}_{ij} + \sum_{d^{hl} \in N_e(i,j)} \bar{d}_{ij}^{hl} \sigma_{ij}^* C_{2p}^{1/2p}\Big], \quad 2p \geq 1,$$

$$D^* := \max_{(i,j)} \left\{ \sum_{b^{hl} \in N_r(i,j)} \bar{b}_{ij}^{hl} f_{ij}^* + \sum_{c^{hl} \in N_q(i,j)} \bar{c}_{ij}^{hl} g_{ij}^* \right\},$$

$$K^* := \max_{(i,j)} \left\{ \sum_{d^{hl} \in N_e(i,j)} \bar{d}_{ij}^{hl} L_{ij}^{\sigma}, \sum_{d^{hl} \in N_e(i,j)} \bar{d}_{ij}^{hl} \sigma_{ij}^* \right\}.$$

Theorem 3.1. Assume that all of the coefficients in system (1.4) are almost periodic sequences, (H_1)-(H_2) and the following condition are satisfied:

(H_3) $r_{2p} < 1$, $2p \geq 1$.

Then there exists a $2p$-mean almost periodic sequence solution X of system (1.4) with $\|X\|_{2p} \leq \beta_{2p}$.

Proof. Let $\Lambda \subseteq BC(\mathbb{Z}; L^{2p}(\Omega; \mathbb{R}^{m\times n}))$ be the collection of all $2p$-mean almost periodic sequences $X = \{x_{ij}\}$ satisfying $\|X\|_{2p} \leq \beta_{2p}$.

Firstly, $X = \{x_{ij}\}$ is described by

$$x_{ij}(k) = -\sum_{v=-\infty}^{k-1} \prod_{s=v+1}^{k-1} \frac{e^{-a_{ij}(s)}[1 - e^{-a_{ij}(v)}]}{a_{ij}(v)} \Big[F_{ij}(v,x)x_{ij}(v) - G_{ij}(v,x)\Delta w_{ij}(v) - I_{ij}(v)\Big], \tag{3.1}$$

where $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$, $k \in \mathbb{Z}$. Obviously, (3.1) is well defined and satisfies (2.1). So we define $\Phi X(k) = \mathcal{B}X(k) + \mathcal{C}X(k)$, where

$$\Phi X(k) = ((\Phi X)_{11}(k), (\Phi X)_{12}(k), \ldots, (\Phi X)_{mn}(k))^T,$$

$$(\Phi X)_{ij}(k) = (\mathcal{B}X)_{ij}(k) + (\mathcal{C}X)_{ij}(k), \tag{3.2}$$

$$(\mathcal{B}X)_{ij}(k) = -\sum_{v=-\infty}^{k-1} \prod_{s=v+1}^{k-1} \frac{e^{-a_{ij}(s)}[1 - e^{-a_{ij}(v)}]}{a_{ij}(v)} \Big[F_{ij}(v,x)x_{ij}(v) - I_{ij}(v)\Big], \tag{3.3}$$

$$(\mathcal{C}X)_{ij}(k) = \sum_{v=\infty}^{k-1} \prod_{s=v+1}^{k-1} \frac{e^{-a_{ij}(s)}[1 - e^{-a_{ij}(v)}]}{a_{ij}(v)} G_{ij}(v,x)\Delta w_{ij}(v), \tag{3.4}$$

where $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$, $k \in \mathbb{Z}$.

Let $X^0 = \{x_{ij}^0\}$ be defined as

$$x_{ij}^0(k) = \sum_{v=-\infty}^{k-1} \prod_{s=v+1}^{k-1} \frac{e^{-a_{ij}(s)}[1 - e^{-a_{ij}(v)}]}{a_{ij}(v)} \Big[\sum_{d^{hl} \in N_e(i,j)} d_{ij}^{hl}(v)\sigma_{ij}(0)\Delta w_{ij}(v) + I_{ij}(v)\Big],$$

where $i = 1, 2, \ldots, m,\ j = 1, 2, \ldots, n,\ k \in \mathbb{Z}$. By Minkoswki inequality in Lemma 2.2 and Hölder inequality in Lemma 2.3, we obtain

$$
\begin{aligned}
&\|X^0\|_{2p} \\
&\leq \max_{(i,j)} \sup_{k\in\mathbb{Z}} \left\{ \left[E\left| \sum_{v=-\infty}^{k-1} \prod_{s=v+1}^{k-1} \frac{e^{-a_{ij}(s)}[1-e^{-a_{ij}(v)}]}{a_{ij}(v)} I_{ij}(v) \right|^{2p} \right]^{1/2p} \right. \\
&\quad \left. + \left[E\left| \sum_{v=-\infty}^{k-1} \prod_{s=v+1}^{k-1} \frac{e^{-a_{ij}(s)}[1-e^{-a_{ij}(v)}]}{a_{ij}(v)} \sum_{d^{hl}\in N_e(i,j)} d_{ij}^{hl}(v)\sigma_{ij}(0)\Delta w_{ij}(v) \right|^{2p} \right]^{1/2p} \right\} \\
&\leq \max_{(i,j)} \sup_{k\in\mathbb{Z}} \left\{ \frac{(1-e^{-\bar{a}})\bar{I}_{ij}}{\underline{a}(1-e^{-\underline{a}})} + \sum_{d^{hl}\in N_e(i,j)} \bar{d}_{ij}^{hl}\sigma_{ij}^* \left[\sum_{v=-\infty}^{k-1} \prod_{s=v+1}^{k-1} \frac{e^{-a_{ij}(s)}[1-e^{-a_{ij}(v)}]}{a_{ij}(v)} \right]^{1-1/2p} \right. \\
&\quad \left. \times \left[\sum_{v=-\infty}^{k-1} \prod_{s=v+1}^{k-1} \frac{e^{-a_{ij}(s)}[1-e^{-a_{ij}(v)}]}{a_{ij}(v)} E|\Delta w_{ij}(v)|^{2p} \right]^{1/2p} \right\} \\
&\leq \frac{(1-e^{-\bar{a}})}{\underline{a}(1-e^{-\underline{a}})} \max_{(i,j)} \left[\bar{I}_{ij} + \sum_{d^{hl}\in N_e(i,j)} \bar{d}_{ij}^{hl}\sigma_{ij}^* C_{2p}^{1/2p} \right] = \alpha_{2p}. \qquad (3.5)
\end{aligned}
$$

Using Lemmas 2.2 and 2.3, it follows (3.2), (3.3) and (3.4) that

$$
\begin{aligned}
&\|\Phi X - X^0\|_{2p} \\
&\leq \max_{(i,j)} \sup_{k\in\mathbb{Z}} D^* \left\{ E\left[\sum_{v=-\infty}^{k-1} \prod_{s=v+1}^{k-1} \frac{e^{-a_{ij}(s)}[1-e^{-a_{ij}(v)}]}{a_{ij}(v)} |x_{ij}(v)| \right]^{2p} \right\}^{1/2p} \\
&\quad + \max_{(i,j)} \sup_{k\in\mathbb{Z}} K^* \left\{ E\left[\sum_{v=-\infty}^{k-1} \prod_{s=v+1}^{k-1} \frac{e^{-a_{ij}(s)}[1-e^{-a_{ij}(v)}]}{a_{ij}(v)} |x_{ij}(v)\Delta w_{ij}(v)| \right]^{2p} \right\}^{1/2p} \\
&\leq \max_{(i,j)} \sup_{k\in\mathbb{Z}} D^* \left\{ \left[\sum_{v=-\infty}^{k-1} \prod_{s=v+1}^{k-1} \frac{e^{-a_{ij}(s)}[1-e^{-a_{ij}(v)}]}{a_{ij}(v)} \right]^{2p-1} \right. \\
&\quad \left. \times \sum_{v=-\infty}^{k-1} \prod_{s=v+1}^{k-1} \frac{e^{-a_{ij}(s)}[1-e^{-a_{ij}(v)}]}{a_{ij}(v)} E\left|x_{ij}(v)\right|^{2p} \right\}^{1/2p} \\
&\quad + \max_{(i,j)} \sup_{k\in\mathbb{Z}} K^* \left\{ \left[\sum_{v=-\infty}^{k-1} \prod_{s=v+1}^{k-1} \frac{e^{-a_{ij}(s)}[1-e^{-a_{ij}(v)}]}{a_{ij}(v)} \right]^{2p-1} \right. \\
&\quad \left. \times \sum_{v=-\infty}^{k-1} \prod_{s=v+1}^{k-1} \frac{e^{-a_{ij}(s)}[1-e^{-a_{ij}(v)}]}{a_{ij}(v)} E\left|x_{ij}(v)\Delta w_{ij}(v)\right|^{2p} \right\}^{1/2p}.
\end{aligned}
$$

Applying Lemma 2.6 to the above inequality, it derives

$$
\|\Phi X - X^0\|_{2p} \leq \frac{(1-e^{-\bar{a}})}{\underline{a}(1-e^{-\underline{a}})} \max_{(i,j)} \left\{ D^* + K^* C_{2p}^{1/2p} \right\} \|X\|_{2p} = r_{2p}\|X\|_{2p} \leq \frac{r_{2p}\alpha_{2p}}{1-r_{2p}}. \quad (3.6)
$$

Hence, $\forall X = \{x_{ij}\} \in \Lambda$, it leads from (3.5) and (3.6) to

$$\|\Phi X\|_{2p} \le \|X^0\|_{2p} + \|\Phi X - X^0\|_{2p} \le \alpha_{2p} + \frac{r_{2p}\alpha_{2p}}{1-r_{2p}} = \frac{\alpha_{2p}}{1-r_{2p}} := \beta_{2p}. \tag{3.7}$$

From (3.7), $\mathcal{B}\Lambda$ is uniformly bounded. Together with the continuity of $\mathcal{B}$, for any bounded sequence $\{\varphi_n\}$ in Λ, we know that there exists a subsequence $\{\varphi_{n_k}\}$ in Λ such that $\{\mathcal{B}(\varphi_{n_k})\}$ is convergent in $\mathcal{B}(\Lambda)$. Therefore, $\mathcal{B}$ is compact on Λ. Then condition (2) of Lemma 2.7 is satisfied.

The next step is proving condition (1) of Lemma 2.7. Now, we consist in proving the $2p$th mean almost periodicity of $\mathcal{B}X(\cdot)$ and $\mathcal{C}X(\cdot)$. Since $X(\cdot)$ is a $2p$th mean almost periodic sequence and all the coefficients in system (1.4) are almost periodic sequences, for all $\epsilon > 0$ there exists $l_\epsilon > 0$ such that every interval of length $l_\epsilon > 0$ contains a τ with the property that

$$E|x_{ij}(k+\tau) - x_{ij}(k)|^{2p} < \epsilon, \quad |a_{ij}(k+\tau) - a_{ij}(k)| < \epsilon,$$

$$|b_{ij}^{hl}(k+\tau) - b_{ij}^{hl}(k)| < \epsilon, \quad |c_{ij}^{hl}(k+\tau) - c_{ij}^{hl}(k)| < \epsilon, \quad |I_{ij}(k+\tau) - I_{ij}(k)| < \epsilon,$$

where $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$, $k \in \mathbb{Z}$. In view of (3.3), it follows that

$$\begin{aligned}
&T_{ij}(k)\\
&= \big[E|(\mathcal{B}X)_{ij}(k+\tau) - (\mathcal{B}X)_{ij}(k)|^{2p}\big]^{1/2p}\\
&= \bigg\{E\bigg[\sum_{v=-\infty}^{k-1}\prod_{s=v+1}^{k-1}\bigg|\frac{e^{-a_{ij}(s+\tau)}[1-e^{-a_{ij}(v+\tau)}]}{a_{ij}(v+\tau)}\Big(F_{ij}(v+\tau, x_{ij}(v+\tau)) - I_{ij}(v+\tau)\Big)\\
&\quad - \frac{e^{-a_{ij}(s)}[1-e^{-a_{ij}(v)}]}{a_{ij}(v)}\Big(F_{ij}(v, x_{ij}(v)) - I_{ij}(v)\Big)\bigg|\bigg]^{2p}\bigg\}^{\frac{1}{2p}}\\
&\le \bigg[\frac{(1-e^{-\bar{a}})}{\underline{a}} + \frac{(e^{-\underline{a}}\bar{a} + e^{-\underline{a}} + 1)}{\underline{a}^2}\bigg]\\
&\quad \times \bigg\{E\bigg[\sum_{v=-\infty}^{k-1}\prod_{s=v+1}^{k-1} e^{-\underline{a}}(D^*|x_{ij}(v)| + \bar{I}_{ij})|a_{ij}(s+\tau) - a_{ij}(s)|\bigg]^{2p}\bigg\}^{1/2p}\\
&\quad + \frac{[1-e^{-\bar{a}}]}{\underline{a}}\bigg\{E\bigg[\sum_{v=-\infty}^{k-1}\prod_{s=v+1}^{k-1} e^{-\underline{a}}\big|I_{ij}(v+\tau) - I_{ij}(v)\big|\bigg]^{2p}\bigg\}^{\frac{1}{2p}}\\
&\quad + \frac{[1-e^{-\bar{a}}]}{\underline{a}}\bigg\{E\bigg[\sum_{v=-\infty}^{k-1}\prod_{s=v+1}^{k-1} e^{-\underline{a}}\big|F_{ij}(v+\tau, x_{ij}(v+\tau)) - F_{ij}(v, x_{ij}(v))\big|\bigg]^{2p}\bigg\}^{\frac{1}{2p}}\\
&= \bigg[\frac{(1-e^{-\bar{a}})}{\underline{a}} + \frac{(e^{-\underline{a}}\bar{a} + e^{-\underline{a}} + 1)}{\underline{a}^2}\bigg]T_{1ij}(k) + \frac{[1-e^{-\bar{a}}]}{\underline{a}}T_{2ij}(k) + \frac{[1-e^{-\bar{a}}]}{\underline{a}}T_{3ij}(k),
\end{aligned}$$

where

$$T_{1ij}(k) = \bigg\{E\bigg[\sum_{v=-\infty}^{k-1}\prod_{s=v+1}^{k-1} e^{-\underline{a}}(D^*|x_{ij}(v)| + \bar{I}_{ij})|a_{ij}(s+\tau) - a_{ij}(s)|\bigg]^{2p}\bigg\}^{1/2p},$$

$$T_{2ij}(k)=\bigg\{E\Big[\sum_{v=-\infty}^{k-1}\prod_{s=v+1}^{k-1}e^{-\underline{a}}|I_{ij}(v+\tau)-I_{ij}(v)|\Big]^{2p}\bigg\}^{1/2p},$$

$$\begin{aligned}
&T_{3ij}(k)\\
=&\bigg\{E\Big[\sum_{v=-\infty}^{k-1}\prod_{s=v+1}^{k-1}e^{-\underline{a}}\big|F_{ij}(v+\tau,x_{ij}(v+\tau))-F_{ij}(v,x_{ij}(v))\big|\Big]^{2p}\bigg\}^{1/2p}\\
\le&\bigg\{E\Big[\sum_{v=-\infty}^{k-1}\prod_{s=v+1}^{k-1}e^{-\underline{a}}\sum_{b^{hl}\in N_r(i,j)}\bar{b}_{ij}^{hl}f_{ij}^{*}\big|x_{ij}(v+\tau)-x_{ij}(v)\big|\Big]^{2p}\bigg\}^{1/2p}\\
&+\bigg\{E\Big[\sum_{v=-\infty}^{k-1}\prod_{s=v+1}^{k-1}e^{-\underline{a}}\sum_{b^{hl}\in N_r(i,j)}f_{ij}^{*}|x_{ij}(v)|\big|b_{ij}^{hl}(v+\tau)-b_{ij}^{hl}(v)\big|\Big]^{2p}\bigg\}^{1/2p}\\
&+\bigg\{E\Big[\sum_{v=-\infty}^{k-1}\prod_{s=v+1}^{k-1}e^{-\underline{a}}\sum_{b^{hl}\in N_r(i,j)}\bar{b}_{ij}^{hl}\sqrt{2f_{ij}^{*}L_{ij}^{f}}|x_{ij}(v)|\big|x_{hl}(v+\tau)-x_{hl}(v)\big|^{1/2}\Big]^{2p}\bigg\}^{1/2p}\\
&+\bigg\{E\Big[\sum_{v=-\infty}^{k-1}\prod_{s=v+1}^{k-1}e^{-\underline{a}}\sum_{c^{hl}\in N_r(i,j)}\bar{c}_{ij}^{hl}g_{ij}^{*}\big|x_{ij}(v+\tau)-x_{ij}(v)\big|\Big]^{2p}\bigg\}^{1/2p}\\
&+\bigg\{E\Big[\sum_{v=-\infty}^{k-1}\prod_{s=v+1}^{k-1}e^{-\underline{a}}\sum_{c^{hl}\in N_r(i,j)}g_{ij}^{*}|x_{ij}(v)|\big|b_{ij}^{hl}(v+\tau)-b_{ij}^{hl}(v)\big|\Big]^{2p}\bigg\}^{1/2p}\\
&+\bigg\{E\Big[\sum_{v=-\infty}^{k-1}\prod_{s=v+1}^{k-1}e^{-\underline{a}}\sum_{c^{hl}\in N_r(i,j)}\bar{c}_{ij}^{hl}\sqrt{2g_{ij}^{*}L_{ij}^{g}}|x_{ij}(v)|\\
&\times\big|x_{hl}(v-\mu_{hl}(v+\tau)+\tau)-x_{hl}(v-\mu_{hl}(v))\big|^{\frac{1}{2}}\Big]^{2p}\bigg\}^{\frac{1}{2p}},
\end{aligned}$$

where $i=1,2,\ldots,m,\ j=1,2,\ldots,n,\ k\in\mathbb{Z}$.

By Minkoswki inequality in Lemma 2.2 and similar to the arguments as that in (3.6), we obtain

$$\begin{aligned}
T_{1ij}(k)\le&\bigg\{E\Big[\sum_{v=-\infty}^{k-1}\prod_{s=v+1}^{k-1}e^{-\underline{a}}D^{*}|x_{ij}(v)|\Big]^{2p}\bigg\}^{1/2p}\epsilon+\bigg\{E\Big[\sum_{v=-\infty}^{k-1}\prod_{s=v+1}^{k-1}e^{-\underline{a}}\bar{I}_{ij}\Big]^{2p}\bigg\}^{1/2p}\epsilon\\
\le&\frac{(D^{*}\|X\|_p+\bar{I}_{ij})}{1-e^{-\underline{a}}}\epsilon\le\frac{(D^{*}\beta_{2p}+\bar{I}_{ij})}{1-e^{-\underline{a}}}\epsilon,
\end{aligned}\tag{3.8}$$

where $i=1,2,\ldots,m,\ j=1,2,\ldots,n,\ k\in\mathbb{Z}$. In (3.8), we use the Liapunov inequality $|X|_p\le|X|_{2p}\le\beta_{2p}$. Similarly,

$$T_{2ij}(k)\le\frac{1}{1-e^{-\underline{a}}}\epsilon,\tag{3.9}$$

$$
\begin{aligned}
&T_{3ij}(k)\\
\le\ & \frac{1}{1-e^{-\underline{a}}}\Big\{\sum_{b^{hl}\in N_r(i,j)} f_{ij}^*(\bar{b}_{ij}^{hl}+\|X\|_{2p})\epsilon+\sum_{c^{hl}\in N_r(i,j)} g_{ij}^*(\bar{c}_{ij}^{hl}+\|X\|_{2p})\epsilon\\
&+\sum_{b^{hl}\in N_r(i,j)}\bar{b}_{ij}^{hl}\sqrt{2f_{ij}^*L_{ij}^f}\Big[E\big(\big|x_{ij}(v)\big|^{2p}\big|x_{hl}(v+\tau)-x_{hl}(v)\big|^{p}\big)\Big]^{1/2p}\\
&+\sum_{c^{hl}\in N_r(i,j)}\bar{c}_{ij}^{hl}\sqrt{2g_{ij}^*L_{ij}^g}\Big[E\big(\big|x_{ij}(v)\big|^{2p}\big|x_{hl}(v-\sigma_{hl}+\tau)-x_{hl}(v-\sigma_{hl})\big|^{p}\big)\Big]^{1/2p}\Big\}\\
\le\ & \frac{1}{1-e^{-\underline{a}}}\Big\{\sum_{b^{hl}\in N_r(i,j)} f_{ij}^*(\bar{b}_{ij}^{hl}+\beta_{2p})\epsilon+\sum_{c^{hl}\in N_r(i,j)} g_{ij}^*(\bar{c}_{ij}^{hl}+\beta_{2p})\epsilon\\
&+\sum_{b^{hl}\in N_r(i,j)}\bar{b}_{ij}^{hl}\sqrt{2f_{ij}^*L_{ij}^f}\big[E\big|x_{ij}(v)\big|^{2p}\big]^{1/2p}\big[E\big|x_{hl}(v+\tau)-x_{hl}(v)\big|^{2p}\big]^{1/2p}\\
&+\sum_{c^{hl}\in N_r(i,j)}\bar{c}_{ij}^{hl}\sqrt{2g_{ij}^*L_{ij}^g}\\
&\big[E\big|x_{ij}(v)\big|^{2p}\big]^{\frac{1}{2p}}\big[E\big|x_{hl}(v-\mu_{hl}(v+\tau)+\tau)-x_{hl}(v-\mu_{hl}(v))\big|^{2p}\big]^{\frac{1}{2p}}\Big\}\\
\le\ & \frac{1}{1-e^{-\underline{a}}}\Big\{\sum_{b^{hl}\in N_r(i,j)} f_{ij}^*(\bar{b}_{ij}^{hl}+\beta_{2p})+\sum_{c^{hl}\in N_r(i,j)} g_{ij}^*(\bar{c}_{ij}^{hl}+\beta_{2p})\\
&+\sum_{b^{hl}\in N_r(i,j)}\bar{b}_{ij}^{hl}\sqrt{2f_{ij}^*L_{ij}^f}\beta_{2p}+\sum_{c^{hl}\in N_r(i,j)}\bar{c}_{ij}^{hl}\sqrt{2g_{ij}^*L_{ij}^g}\beta_{2p}\Big\}\epsilon,
\end{aligned}
\tag{3.10}
$$

where $i=1,2,\dots,m$, $j=1,2,\dots,n$, $k\in\mathbb{Z}$. In the second inequality from the bottom of (3.10), we use the Hölder inequality $E(|\xi\eta|)\le[E(|\xi|^2)]^{\frac{1}{2}}[E(|\eta|^2)]^{\frac{1}{2}}$. On the other hand, from (3.4) and Lemma 2.6, we get

$$
\begin{aligned}
&W_{ij}(k)\\
=\ &\big[E|(\mathcal{C}X)_{ij}(k+\tau)-(\mathcal{C}X)_{ij}(k)|^{2p}\big]^{1/2p}\\
=\ &\Big\{E\Big[\sum_{v=-\infty}^{k-1}\prod_{s=v+1}^{k-1}\Big|\frac{e^{-a_{ij}(s+\tau)}[1-e^{-a_{ij}(v+\tau)}]}{a_{ij}(v+\tau)}\\
&\sum_{d^{hl}\in N_e(i,j)} d_{ij}^{hl}(v+\tau)\sigma_{ij}(x_{ij}(v+\tau))\Delta w_{ij}(v+\tau)\\
&-\frac{e^{-a_{ij}(s)}[1-e^{-a_{ij}(v)}]}{a_{ij}(v)}\sum_{d^{hl}\in N_e(i,j)} d_{ij}^{hl}(v)\sigma_{ij}(x_{ij}(v))\Delta w_{ij}(v)\Big|\Big]^{2p}\Big\}^{1/2p}\\
\le\ &\Big[\frac{(1-e^{-\bar{a}})}{\underline{a}}+\frac{(e^{-\underline{a}}\bar{a}+e^{-\underline{a}}+1)}{\underline{a}^2}\Big]\sum_{d^{hl}\in N_e(i,j)}\bar{d}_{ij}^{hl}
\end{aligned}
$$

$$
\begin{aligned}
&\times\Big\{E\Big[\sum_{v=-\infty}^{k-1}\prod_{s=v+1}^{k-1}e^{-\underline{a}}\sigma_{ij}^*|a_{ij}(s+\tau)-a_{ij}(s)||\Delta w_{ij}(v)|\Big]^{2p}\Big\}^{1/2p}\\
&+\frac{[1-e^{-\bar{a}}]}{\underline{a}}\sum_{d^{hl}\in N_e(i,j)}\bar{d}_{ij}^{hl}\Big\{E\Big[\sum_{v=-\infty}^{k-1}\prod_{s=v+1}^{k-1}e^{-\underline{a}}\sigma_{ij}^*|\Delta w_{ij}(v+\tau)-\Delta w_{ij}(v)|\Big]^{2p}\Big\}^{1/2p}\\
&+\frac{[1-e^{-\bar{a}}]}{\underline{a}}\Big\{E\Big[\sum_{v=-\infty}^{k-1}\prod_{s=v+1}^{k-1}e^{-\underline{a}}L_{ij}^{\sigma}\sum_{d^{hl}\in N_e(i,j)}\bar{d}_{ij}^{hl}\big|x_{ij}(v+\tau)-x_{ij}(v)\big|\Delta w_{ij}(v)\Big]^{2p}\Big\}^{1/2p}\\
\le&\sum_{C_{2p}^{1/2p}d^{hl}\in N_e(i,j)}\frac{\bar{d}_{ij}^{hl}(1-e^{-\bar{a}})}{\underline{a}(1-e^{-\underline{a}})}\Big\{\Big[1+\frac{(e^{-\underline{a}}\bar{a}+e^{-\underline{a}}+1)}{\underline{a}(1-e^{-\bar{a}})}\Big]\sigma_{ij}^*+L_{ij}^{\sigma}\Big\}\epsilon, \qquad (3.11)
\end{aligned}
$$

where $i=1,2,\dots,m$, $j=1,2,\dots,n$, $k\in\mathbb{Z}$. From (3.8), (3.9), (3.10) and (3.11), $\mathcal{B}X(\cdot)$ and $\mathcal{C}X(\cdot)$ are $2p$th mean almost periodic processes. Further, by (3.7), it is easy to obtain that $\mathcal{B}X+\mathcal{C}Y\in\Lambda$, $\forall X,Y\in\Lambda$. Then condition (1) of Lemma 2.7 holds.

Finally, $\forall X=\{x_{ij}\},Y=\{y_{ij}\}\in\Lambda$, from (3.4), it yields

$$
\begin{aligned}
\|\mathcal{C}X-\mathcal{C}Y\|_{2p}\le&\frac{[1-e^{-\bar{a}}]}{\underline{a}}\max_{(i,j)}\sup_{k\in\mathbb{Z}}\Big\{E\Big[\sum_{v=-\infty}^{k-1}\prod_{s=v+1}^{k-1}e^{-\underline{a}}\\
&\times\sum_{d^{hl}\in N_e(i,j)}d_{ij}^{hl}(v)\big(\sigma_{ij}(x_{ij}(v))-\sigma_{ij}(y_{ij}(v))\big)\Delta w_{ij}(v)\Big]^{2p}\Big\}^{\frac{1}{2p}}\\
\le&\frac{[1-e^{-\bar{a}}]}{\underline{a}}\max_{(i,j)}\sup_{k\in\mathbb{Z}}K^*\Big\{\Big[\sum_{v=-\infty}^{k-1}\prod_{s=v+1}^{k-1}e^{-\underline{a}}\Big]^{2p-1}\\
&\times\sum_{v=-\infty}^{k-1}\prod_{s=v+1}^{k-1}e^{-\underline{a}}E\big|[x_{ij}(v)-y_{ij}(v)]\Delta w_{ij}(v)\big|^{2p}\Big\}^{\frac{1}{2p}}\\
\le&\frac{K^*C_{2p}^{1/2p}(1-e^{-\bar{a}})}{\underline{a}(1-e^{-\underline{a}})}\|X-Y\|_{2p}\\
\le&r_{2p}\|X-Y\|_{2p}. \qquad (3.12)
\end{aligned}
$$

In view of (H_3), $\mathcal{C}$ is a contraction mapping. Hence condition (3) of Lemma 2.7 is satisfied. Therefore, all the conditions in Lemma 2.7 hold. By Lemma 2.7, system (1.4) has a $2p$th mean almost periodic sequence solution. This completes the proof. □

4. PTH MOMENT GLOBAL EXPONENTIAL STABILITY

Theorem 4.1. Assume that (H_1)-(H_2) hold and system (1.4) has a stochastic sequence solution $X^*=\{x_{ij}^*\}$ with initial value $\varphi^*=\{\varphi_{ij}^*\}$. Suppose further that

(H_4) there exist positive constants M_{ij} such that $|x_{ij}^*(k)|\le M_{ij}$ for $k\in[-\bar{\mu},+\infty)_{\mathbb{Z}}$ and

$$\frac{(1-e^{-\bar{a}})(F^*+C_p^{\frac{1}{p}}K^*+H^*)}{\underline{a}[1-e^{-\underline{a}}]}<1,$$

where $p \geq 1$ and

$$F^* = \max_{(i,j)} \Big\{ D^* + \sum_{b^{hl} \in N_r(i,j)} \bar{b}_{ij}^{hl} L_{ij}^f M_{ij} \Big\}, \quad H^* = \max_{(i,j)} \sum_{c^{hl} \in N_q(i,j)} \bar{c}_{ij}^{hl} L_{ij}^g M_{ij}.$$

Then the stochastic sequence solution X^* of system (1.4) is pth moment globally exponentially stable.

Proof. Suppose that $X = \{x_{ij}\}$ with initial value $\varphi = \{\varphi_{ij}\}$ is an arbitrary solution of system (1.4). Then it follows from Lemma 2.4 and (3.10) that

$$\begin{aligned}
&|x_{ij}(k) - x_{ij}^*(k)| \\
\leq\ & \prod_{s=0}^{k-1} e^{-a_{ij}(s)} |\varphi_{ij}(0) - \varphi_{ij}^*(0)| \\
&+ \frac{(1 - e^{-\bar{a}})}{\underline{a}} \sum_{v=0}^{k-1} \prod_{s=v+1}^{k-1} e^{-a_{ij}(s)} \big| F_{ij}(v, x_{ij}(v)) - F_{ij}(v, x_{ij}^*(v)) \big| \\
&+ \frac{(1 - e^{-\bar{a}})}{\underline{a}} \sum_{v=0}^{k-1} \prod_{s=v+1}^{k-1} e^{-a_{ij}(s)} \sum_{d^{hl} \in N_e(i,j)} d_{ij}^{hl}(v) \big| [\sigma_{ij}(x_{ij}(v)) - \sigma_{ij}(x_{ij}^*(v))] \Delta w_{ij}(v) \big| \\
\leq\ & e^{-\underline{a}k} |\varphi_{ij}(0) - \varphi_{ij}^*(0)| \\
&+ \frac{(1 - e^{-\bar{a}})}{\underline{a}} \sum_{v=0}^{k-1} e^{-\underline{a}(k-v-1)} \Big\{ \sum_{b^{hl} \in N_r(i,j)} \bar{b}_{ij}^{hl} \big[f_{ij}^* + g_{ij}^* \big] |x_{ij}(v) - x_{ij}^*(v)| \\
&+ \sum_{b^{hl} \in N_r(i,j)} \bar{b}_{ij}^{hl} L_{ij}^f M_{ij} |x_{hl}(v) - x_{hl}^*(v)| \\
&+ \sum_{c^{hl} \in N_q(i,j)} \bar{c}_{ij}^{hl} L_{ij}^g M_{ij} |x_{hl}(v - \mu_{hl}(v)) - x_{hl}^*(v - \mu_{hl}(v))| \Big\} \\
&+ \frac{(1 - e^{-\bar{a}})}{\underline{a}} \sum_{v=0}^{k-1} e^{-\underline{a}(k-v-1)} \sum_{d^{hl} \in N_e(i,j)} \bar{d}_{ij}^{hl} L_{ij}^\sigma |[x_{ij}(v) - x_{ij}^*(v)] \Delta w_{ij}(v)|,
\end{aligned} \tag{4.1}$$

where $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$, $k \in [-\bar{\mu}, +\infty)_{\mathbb{Z}}$, $\bar{\mu} = \max_{(i,j)} \bar{\mu}_{ij}$.

Let $\gamma_p = \max_{(i,j)} \sup_{s \in [-\bar{\mu}, 0]_{\mathbb{Z}}} (E|\varphi_{ij}(s) - \varphi_{ij}^*(s)|^p)^{\frac{1}{p}}$, $Z = \{z_{ij}\}$, $z_{ij}(k) = x_{ij}(k) - x_{ij}^*(k)$, $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$, $k \in \mathbb{Z}$. Similar to the argument as that in (3.6), it gets from (4.1) that

$$\begin{aligned}
&|Z(k)|_p \\
=\ & |X(k) - X^*(k)|_p \\
\leq\ & e^{-\underline{a}k} \gamma_p + \max_{(i,j)} \frac{F^*(1 - e^{-\bar{a}})}{\underline{a}} \Big\{ \Big[\sum_{s=0}^{k-1} e^{-\underline{a}(k-s-1)} \Big]^{p-1}
\end{aligned}$$

$$\sum_{s=0}^{k-1} e^{-\underline{a}(k-s-1)} E\big|x_{ij}(s)-x_{ij}^*(s)\big|^p\Big\}^{\frac{1}{p}} + \max_{(i,j)} \frac{H^*(1-e^{-\bar{a}})}{\underline{a}}\Big\{\Big[\sum_{s=0}^{k-1} e^{-\underline{a}(k-s-1)}\Big]^{p-1}$$

$$\times \sum_{s=0}^{k-1} e^{-\underline{a}(k-s-1)} E\big|x_{ij}(s-\mu_{ij}(s))-x_{ij}^*(s-\mu_{ij}(s))\big|^p\Big\}^{\frac{1}{p}}$$

$$+ \max_{(i,j)} \frac{K^*(1-e^{-\bar{a}})}{\underline{a}}\Big\{\Big[\sum_{s=0}^{k-1} e^{-\underline{a}(k-s-1)}\Big]^{p-1}$$

$$\sum_{s=0}^{k-1} e^{-\underline{a}(k-s-1)} E\big|[x_{ij}(s)-x_{ij}^*(s)]\Delta w_{ij}(s)\big|^p\Big\}^{\frac{1}{p}}$$

$$\le e^{-\underline{a}k}\gamma_p + \frac{F^*(1-e^{-\bar{a}})}{\underline{a}}\Big\{\Big[\sum_{s=0}^{k-1} e^{-\underline{a}(k-s-1)}\Big]^{p-1}\sum_{s=0}^{k-1} e^{-\underline{a}(k-s-1)}|Z(s)|_p^p\Big\}^{\frac{1}{p}}$$

$$+\frac{H^*(1-e^{-\bar{a}})}{\underline{a}}\Big\{\Big[\sum_{s=0}^{k-1} e^{-\underline{a}(k-s-1)}\Big]^{p-1}\sum_{s=0}^{k-1} e^{-\underline{a}(k-s-1)}|Z(s-\mu_{ij}(s))|_p^p\Big\}^{\frac{1}{p}}$$

$$+\frac{C_p^{\frac{1}{p}}K^*(1-e^{-\bar{a}})}{\underline{a}}\Big\{\Big[\sum_{s=0}^{k-1} e^{-\underline{a}(k-s-1)}\Big]^{p-1}\sum_{s=0}^{k-1} e^{-\underline{a}(k-s-1)}|Z(s)|_p^p\Big\}^{\frac{1}{p}}, \tag{4.2}$$

where $k \in [-\bar{\mu}, +\infty)_{\mathbb{Z}}$.

Be aware of (H_4), there exists a constant $\lambda > 0$ small enough such that

$$\frac{e^{\lambda}(1-e^{-\bar{a}})(F^* + C_p^{\frac{1}{p}}K^* + e^{\bar{\mu}\lambda}H^*)}{\underline{a}[1-e^{-(\underline{a}-p\lambda)}]} \overset{\text{def}}{=} \rho \le 1.$$

Next, we claim that there exists a constant $M_0 > 1$ such that

$$|Z(k)|_p \le M_0\gamma_p e^{-\lambda k}, \quad \forall k \in [-\bar{\mu}, +\infty)_{\mathbb{Z}}. \tag{4.3}$$

If (4.3) is invalid, then there must exist $k_0 \in (0, +\infty)_{\mathbb{Z}}$ such that

$$|Z(k_0)|_p > M_0\gamma_p e^{-\lambda k_0} \tag{4.4}$$

and

$$|Z(k)|_p \le M_0\gamma_p e^{-\lambda k}, \quad \forall k \in [-\bar{\mu}, k_0)_{\mathbb{Z}}. \tag{4.5}$$

In view of (4.2), it follows from (4.5) that

$$|Z(k_0)|_p$$

$$\le e^{-\underline{a}k_0}\gamma_p$$

$$+\frac{(1-e^{-\bar{a}})(F^*+C_p^{\frac{1}{p}}K^*)}{\underline{a}}M_0\gamma_p\Big\{\Big[\sum_{s=0}^{k_0-1} e^{-\underline{a}(k_0-s-1)}\Big]^{p-1}\sum_{s=0}^{k_0-1} e^{-\underline{a}(k_0-s-1)}e^{-p\lambda s}\Big\}^{\frac{1}{p}}$$

$$+\frac{(1-e^{-\bar{a}})H^*}{\underline{a}}M_0\gamma_p\Big\{\Big[\sum_{s=0}^{k_0-1} e^{-\underline{a}(k_0-s-1)}\Big]^{p-1}\sum_{s=0}^{k_0-1} e^{-\underline{a}(k_0-s-1)}e^{-p\lambda(s-\bar{\mu})}\Big\}^{\frac{1}{p}}$$

$$
\begin{aligned}
&\le e^{-\underline{a}k_0}\gamma_p \\
&\quad+\frac{(1-e^{-\bar{a}})(F^*+C_p^{\frac{1}{p}}K^*)}{\underline{a}}M_0\gamma_p e^{-\lambda k_0}e^{\lambda}\left[\frac{1-e^{-\underline{a}k_0}}{1-e^{-\underline{a}}}\right]^{1-\frac{1}{p}}\left[\sum_{s=0}^{k_0-1}e^{-(\underline{a}-p\lambda)(k_0-s-1)}\right]^{\frac{1}{p}} \\
&\quad+\frac{(1-e^{-\bar{a}})H^*}{\underline{a}}M_0\gamma_p e^{-\lambda k_0}e^{(\bar{\mu}+1)\lambda}\left[\frac{1-e^{-\underline{a}k_0}}{1-e^{-\underline{a}}}\right]^{1-\frac{1}{p}}\left[\sum_{s=0}^{k_0-1}e^{-(\underline{a}-p\lambda)(k_0-s-1)}\right]^{\frac{1}{p}} \\
&\le e^{-\underline{a}k_0}\gamma_p \\
&\quad+\frac{(1-e^{-\bar{a}})(F^*+C_p^{\frac{1}{p}}K^*+e^{\bar{\mu}\lambda}H^*)}{\underline{a}}M_0\gamma_p e^{-\lambda k_0}e^{\lambda}\left[\frac{1-e^{-\underline{a}k_0}}{1-e^{-\underline{a}}}\right]^{1-\frac{1}{p}}\left[\frac{1-e^{-(\underline{a}-p\lambda)k_0}}{1-e^{-(\underline{a}-p\lambda)}}\right]^{\frac{1}{p}} \\
&\le M_0\gamma_p e^{-\lambda k_0}\left\{\frac{1}{M_0}e^{-(\underline{a}-\lambda)k_0}+\frac{e^{\lambda}(1-e^{-\bar{a}})(F^*+C_p^{\frac{1}{p}}K^*+e^{\bar{\mu}\lambda}H^*)}{\underline{a}[1-e^{-(\underline{a}-p\lambda)}]}[1-e^{-(\underline{a}-\lambda)k_0}]\right\} \\
&\le M_0\gamma_p e^{-\lambda k_0}\left\{e^{-(\underline{a}-\lambda)k_0}+\rho[1-e^{-(\underline{a}-\lambda)k_0}]\right\} \\
&\le M_0\gamma_p e^{-\lambda k_0}. \qquad (4.6)
\end{aligned}
$$

In the fourth inequality from the bottom of (4.6), we use the fact $[1-e^{-\underline{a}k_0}]^{1-\frac{1}{p}}[1-e^{-(\underline{a}-p\lambda)k_0}]^{\frac{1}{p}}\le 1-e^{-(\underline{a}-\lambda)k_0}$ and $[1-e^{-\underline{a}}]^{\frac{1}{p}}\ge[1-e^{-(\underline{a}-p\lambda)}]^{\frac{1}{p}}$. (4.6) contradicts (4.4). Hence, (4.3) is satisfied. Therefore, the stochastic sequence solution X^* of system (1.4) is pth moment globally exponentially stable. This completes the proof. □

According to Theorem 3.1, system (1.4) has a $2p$th mean almost periodic sequence solution $X^\diamond=\{x_{ij}^\diamond\}$. We obtain

Theorem 4.2. Assume that (H_1)-(H_3) hold. Suppose further that

(H_5) there exist positive constants $M_{ij}^\diamond$ such that $|x_{ij}^\diamond(k)|\le M_{ij}^\diamond$ for $k\in[-\bar{\mu},+\infty)_{\mathbb{Z}}$ and

$$\frac{(1-e^{-\bar{a}})(\tilde{F}^*+C_{2p}^{\frac{1}{2p}}K^*+\tilde{H}^*)}{\underline{a}[1-e^{-\underline{a}}]}<1,\quad 2p\ge 1,$$

where

$$\tilde{F}^*=\max_{(i,j)}\left\{D^*+\sum_{b^{hl}\in N_r(i,j)}\bar{b}_{ij}^{hl}L_{ij}^{f}M_{ij}^\diamond\right\},\quad \tilde{H}^*=\max_{(i,j)}\sum_{c^{hl}\in N_q(i,j)}\bar{c}_{ij}^{hl}L_{ij}^{g}M_{ij}^\diamond.$$

Then the $2p$th mean almost periodic sequence solution $X^\diamond$ of system (1.4) is $2p$th moment globally exponentially stable.

P r o o f. The result can be easily obtained by Theorem 4.1, so we omit it. This completes the proof. □

Remark 4.3. In literature [14], Huang et al. studied a simple semi-discrete cellular neural networks and obtained some sufficient conditions for the existence of a unique almost periodic sequence solution which is globally attractive. In [13], they considered the semi-discrete models for a class of general neural networks and studied the dynamics of 2^N almost periodic sequence solutions. But neither of them considered the uncertain factors. Therefore, the work in this paper complements the corresponding results in [13, 14].

Remark 4.4. Assume that $X(t) = (x_{11}(t), \dots, x_{1n}(t), \dots, x_{m1}(t), \dots, x_{mn}(t))$ is a solution of system (1.1), the length of $X(t)$ is usually measured by

$$\|X\|_\infty = \sup_{t\in\mathbb{R}} \max_{1\le i\le m, 1\le j\le n} |x_{ij}(t)|.$$

However, if $X(t)$ is a solution of stochastic system, its length should not be measured by $\|X\|_\infty$ because $X(t)$ is a random variable. In this paper, we use norm $\|X\|_{2p} = \max_{1\le i\le m, 1\le j\le n} \sup_{k\in\mathbb{Z}} (E|x_{ij}(k)|^{2p})^{\frac{1}{2p}}$ $(2p > 1)$ for random variable $X(k)$. Owing to the expectation E and order $2p$ in $\|X\|_{2p}$, the computing processes of this paper are more complicated than that in literatures [13, 15, 14, 21, 24, 49, 50]. It is worth mentioning that Minkoswki inequality in Lemma 2.2 and Hölder inequality in Lemma 2.3 are crucial to the computing processes. The facts above are obvious from the computations of (3.5), (3.10), (4.2) and (4.6) in Theorems 3.1 and 4.1. Further, the stochastic term $G_{ij}(k,x)\Delta w_{ij}$ $(i = 1, 2, \dots, n; j = 1, 2, \dots, m)$ in system (1.4) also increases the complexity of computing. This point is also clear from the computations of (3.5), (3.10), (4.2) and (4.6) in Theorems 3.1 and 4.1.

5. EXAMPLES AND SIMULATIONS

Example 5.1. Considering the following discrete stochastic SICNNs:

$$\begin{cases} x_{11}(k+1) = 0.4x_{11}(k) - 0.6\Big[0.01\cos(\sqrt{2}k)\cos(x_{11}(k))x_{11}(k) \\ \qquad +0.02\sin(\sqrt{3}k)\cos(x_{21}(k-1))x_{11}(k) + 0.1\Delta w_{11}(k)\Big], \\ x_{21}(k+1) = 0.4x_{21}(k) - 0.6\Big[0.01\sin(\sqrt{3}k)\sin(x_{21}(k))x_{21}(k) \\ \qquad +0.02\cos(\sqrt{5}k)\sin(x_{11}(k-1))x_{21}(k) + 0.1\Delta w_{21}(k)\Big], \end{cases} \tag{5.1}$$

where $k \in \mathbb{Z}$.

Corresponding to system (1.4), we have $\underline{a} = \bar{a} = 1$, $f_{ij}^* = g_{ij}^* = \sigma_{ij}^* = 1$, $L_{ij}^f = L_{ij}^g = L_{ij}^\sigma = 1$, $\bar{b}_{ij}^{hl} = 0.01$, $\bar{c}_{ij}^{hl} = 0.02$, $\bar{d}_{ij}^{hl} = 0.1$, $i = 1, 2$, $j = 1$. Taking $p = 1$, by simple calculation,

$$C_2^{1/2} = 4, \quad D^* \approx 0.06, \quad K^* \approx 0.2, \quad r_2 \approx 0.86 < 1.$$

According to Theorem 3.1, system (5.1) admits a square-mean almost periodic sequence solution, see Figure 1.

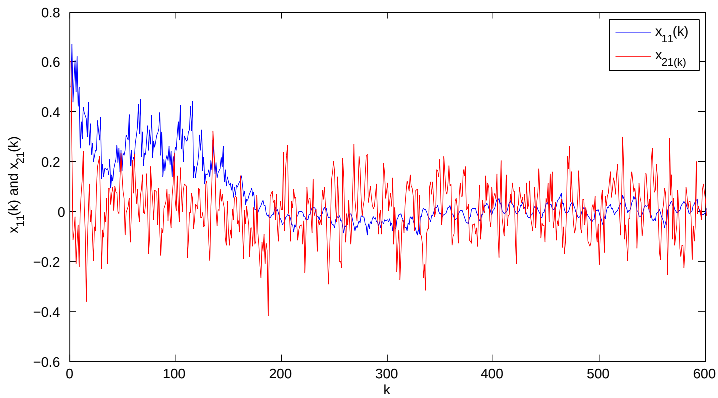

Fig. 1. Almost periodicity of state variables $(x_{11}, x_{21})^T$ in stochastic model (5.1).

Considering the corresponding determinant model of system (5.1) as follows:

$$\begin{cases} x_{11}(k+1) = 0.4x_{11}(k) - 0.6\Big[0.01\cos(\sqrt{2}k)\cos(x_{11}(k))x_{11}(k) \\ \qquad\qquad +0.02\sin(\sqrt{3}k)\cos(x_{21}(k-1))x_{11}(k) + 0.1\Big], \\ x_{21}(k+1) = 0.4x_{21}(k) - 0.6\Big[0.01\sin(\sqrt{3}k)\sin(x_{21}(k))x_{21}(k) \\ \qquad\qquad +0.02\cos(\sqrt{5}k)\sin(x_{11}(k-1))x_{21}(k) + 0.1\Big], \end{cases} \tag{5.2}$$

where $k \in \mathbb{Z}$. In Figures 2−3, we give results of contrast between stochastic model (5.1) and determinant model (5.2). Figures 2−3 indicate that the effect of stochastic perturbation on state variable x_{21} is more obvious than that on state variable x_{11}.

Example 5.2. Considering the following discrete stochastic SICNNs:

$$\begin{cases} x_{11}(k+1) = 0.4x_{11}(k) - 0.006\sin(x_{11}(k))x_{11}(k) \\ \qquad\qquad +0.6 + 0.006\sin 1 + 0.1(x_{21}(k) - 2)\Delta w_{11}(k), \\ x_{21}(k+1) = 0.4x_{21}(k) - 0.006\cos(x_{21}(k-1))x_{21}(k) \\ \qquad\qquad +1.2 + 0.012\cos 2 + 0.1(x_{11}(k) - 1)\Delta w_{21}(k), \end{cases} \tag{5.3}$$

where $k \in \mathbb{Z}$.

It is evident that system (5.3) has a equilibrium point $(1, 2)^T$. Taking $p = 2$, by simple calculation,

$$C_2^{1/2} = 2, \quad F^* \approx 0.03, \quad K^* \approx 0.1, \quad H^* \approx 0.024.$$

Hence,

$$\frac{(1 - e^{-\bar{a}})(F^* + C_2^{\frac{1}{2}} K^* + H^*)}{\underline{a}[1 - e^{-\underline{a}}]} \approx 0.254 < 1.$$

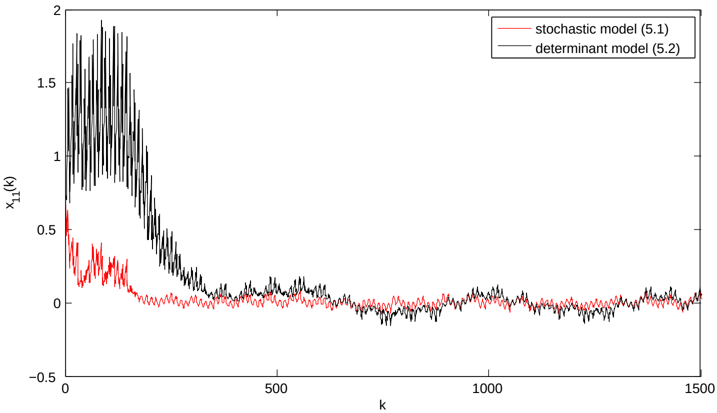

Fig. 2. State variable x_{11}'s comparison between stochastic model (5.1) and determinant model (5.2).

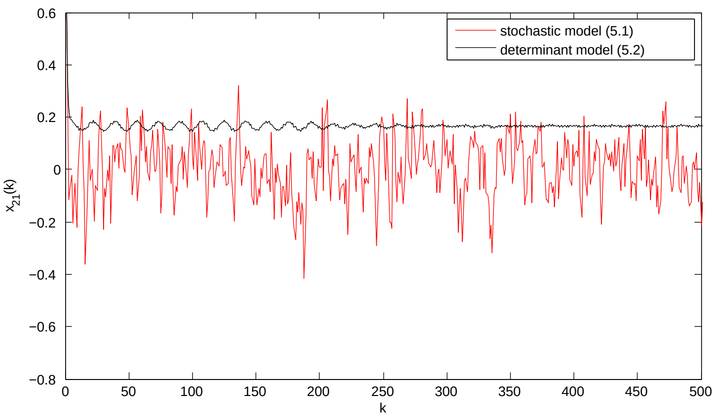

Fig. 3. State variable x_{21}'s comparison between stochastic model (5.1) and determinant model (5.2).

According to Theorem 4.1, the equilibrium point $(1,2)^T$ of system (5.3) is globally exponentially stable, see Figures 4−5.

Considering the corresponding determinant model of system (5.3) as follows:

$$\begin{cases} x_{11}(k+1) = 0.4x_{11}(k) \\ \qquad -0.006\sin(x_{11}(k))x_{11}(k) + 0.6 + 0.006\sin 1 + 0.1(x_{21}(k) - 2), \\ x_{21}(k+1) = 0.4x_{21}(k) \\ \qquad -0.006\cos(x_{21}(k-1))x_{21}(k) + 1.2 + 0.012\cos 2 + 0.1(x_{11}(k) - 1), \end{cases} \tag{5.4}$$

where $k \in \mathbb{Z}$. In Figures 6−7, we give results of globally exponentially stable contrast between stochastic model (5.3) and determinant model (5.4). Figures 6−7 reveal that the convergence speed of stochastic model (5.3) is faster than determinant model (5.4).

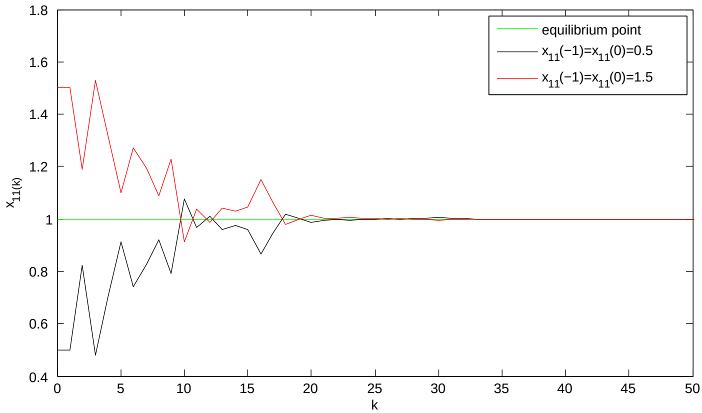

Fig. 4. Global exponential stability of equilibrium point $(1, 2)^T$ of stochastic model (5.3).

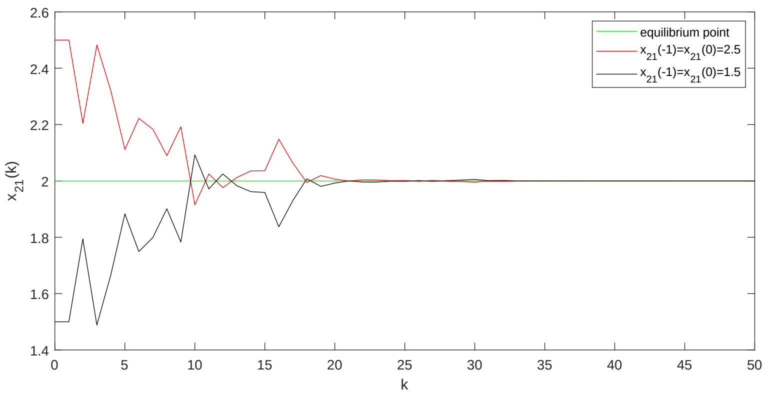

Fig. 5. Global exponential stability of equilibrium point $(1, 2)^T$ of stochastic model (5.3).

6. DISCUSSION

In recent years, the semi-discrete method of differential equations has been applied into the investigations of determinant neural networks. But few people employ this method to study stochastic neural networks. In this paper, we formulate a new kind of discrete analogue of stochastic shunting inhibitory cellular neural networks (SICNNs) by using semi-discrete method, which gives a more accurate characterization for continuous-time stochastic SICNNs than that by Euler scheme. Besides, the stability of discrete-time stochastic neural networks has been studied by many scholars in recent years and yet there are few people to consider the issue of almost periodicity and global exponential stability of discrete-time stochastic neural networks. Therefore, in this paper, we investigate the existence of mean almost periodic sequence solution and moment global

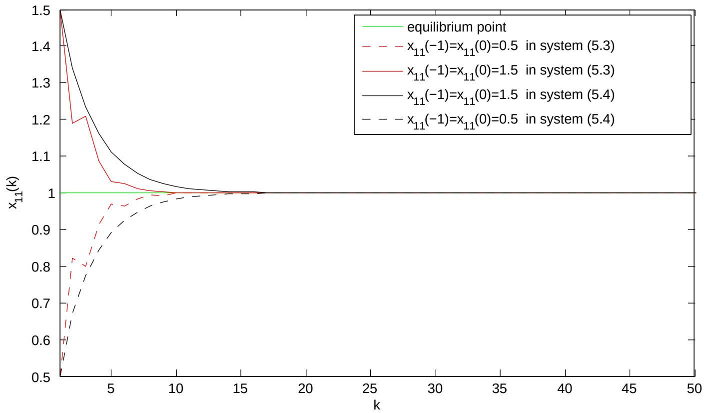

Fig. 6. State variable x_{11}'s convergence speed comparison between stochastic model (5.3) and determinant model (5.4).

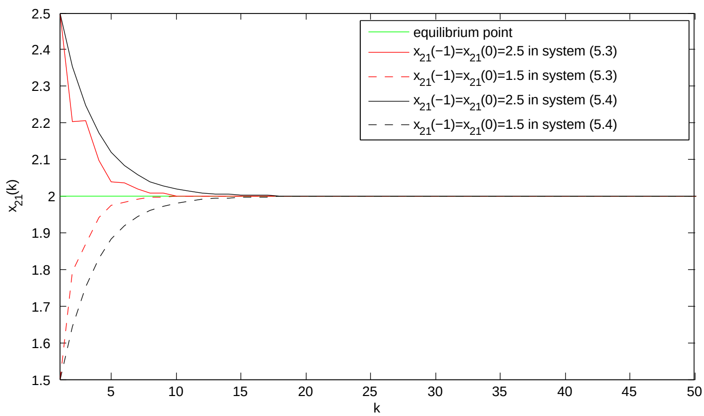

Fig. 7. State variable x_{21}'s convergence speed comparison between stochastic model (5.3) and determinant model (5.4).

exponential stability of a discrete-time stochastic SICNNs with the help of Minkowski inequality, Hölder inequality, Krasnoselskii's fixed point theorem and the proof of contradiction. The main results obtained in this paper are completely new and the methods used in this paper provide a possible technique to study the almost periodic sequence solution and global exponential stability of the semi-discrete models with stochastic perturbations.

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REFERENCES

[1] L. Arnold and C. Tudor: Stationary and almost periodic solutions of almost periodic affine stochastic differential equations. Stochastics and Stochast. Reports *64* (1998), 177–193. DOI:10.1080/17442509808834163

[2] A. Arunkumar, R. Sakthivel, K. Mathiyalagan, and J. H. Park: Robust stochastic stability of discrete-time fuzzy Markovian jump neural networks. ISA Transactions *53* (2014), 1006–1014. DOI:10.1016/j.isatra.2014.05.002

[3] I. Bashkirtseva: Controlling the stochastic sensitivity in thermochemical systems under incomplete information. Kybernetika *54* (2018), 96–109. DOI:10.14736/kyb-2018-1-0096

[4] P. H. Bezandry and T. Diagana: Almost Periodic Stochastic Processes. Springer, New York 2011. DOI:10.1007/978-1-4419-9476-9

[5] A. Bouzerdoum and R. B. Pinter: Shunting inhibitory cellular neural networks: derivation and stability analysis. IEEE Trans. Circuits Systems 1 – Fundament. Theory Appl. *40* (1993), 215–221. DOI:10.1109/81.222804

[6] A. Chen, J. Cao, and L. Huang: Almost periodic solution of shunting inhibitory CNNs with delays. Phys. Lett. A *298* (2002), 161–170. DOI:10.1016/s0375-9601(02)00469-3

[7] A. Y. Dorogovtsev and O. A. Ortega: On the existence of periodic solutions of a stochastic equation in a Hilbert space. Visnik Kiiv. Univ. Ser. Mat. Mekh. *30* (1988), 21–30.

[8] Bo Du, Y. R. Liu, and I. A. Abbas: Existence and asymptotic behavior results of periodic solution for discrete-time neutral-type neural networks. J. Frank. Inst. *353* (2016), 448–461. DOI:10.1016/j.jfranklin.2015.11.013

[9] Q. Y. Fan and J. Y. Shao: Positive almost periodic solutions for shunting inhibitory cellular neural networks with time-varying and continuously distributed delays. Commun. Nonlinear Sci. Numer. Simulat. *15* (2010), 1655–1663. DOI:10.1016/j.cnsns.2009.06.026

[10] M. F. Hu, J. D. Cao, and A. H. Hua: Mean square exponential stability for discrete-time stochastic switched static neural networks with randomly occurring nonlinearities and stochastic delay. Neurocomputing *129* (2014), 476–481. DOI:10.1016/j.neucom.2013.09.011

[11] S. G. Hu, C. M. Huang, and F. K. Wu: Stochastic Differential Equations. Science Press, Beijing 2008. DOI:10.1142/9789812774798_0002

[12] S. Hu and J. Wang: Global robust stability of a class of discrete-time interval neural networks. IEEE Trans. Circuits Syst. I Regul. Pap. *53* (2006), 129–138. DOI:10.1109/tcsi.2005.854288

[13] Z. K. Huang, S. Mohamad, and F. Gao: Multi-almost periodicity in semi-discretizations of a general class of neural networks. Math. Computers Simul. *101* (2014), 43–60. DOI:10.1016/j.matcom.2013.05.017

[14] Z. K. Huang, X. H. Wang, and F. Gao: The existence and global attractivity of almost periodic sequence solution of discrete-time neural networks. Physics Lett. A *350* (2006), 182–191. DOI:10.1016/j.physleta.2005.10.022

[15] Z. K. Huang, X. H. Wang, and Y. H. Xia: Exponential attractor of κ-almost periodic sequence solution of discrete-time bidirectional neural networks. Simul. Modell. Practice Theory *18* (2010), 317–337. DOI:10.1016/j.simpat.2009.11.007

[16] T. Kawata: Almost Periodic Weakly Stationary Processes, Statistics and Probability: Essays in Honor of C. R. Rao. North-Holland, Amsterdam 1982, pp. 383–396.

[17] J. C. Kuang: Applied Inequalities. Shandong Science and Technology Press, Shandong 2012.

[18] Q. X. Lan, H. W. Niu, Y. M. Liu, and H. F. Xu: Global output-feedback finite-time stabilization for a class of stochastic nonlinear cascaded systems. Kybernetika *53* (2017), 780–802. DOI:10.14736/kyb-2017-5-0780

[19] B. Liu and L. Huang: Existence and stability of almost periodic solutions for shunting inhibitory cellular neural networks with continuously distributed delays. Phys. Lett. A *349* (2006), 177–186. DOI:10.1016/j.physleta.2005.09.023

[20] D. Liu, L.J. Wang, Y. N. Pan, and H. Y. Ma: Mean square exponential stability for discrete-time stochastic fuzzy neural networks with mixed time-varying delay. Neurocomputing *171* (2016), 1622–1628. DOI:10.1016/j.neucom.2015.06.045

[21] S. Mohamad: Global exponential stability in continuous-time and discrete-time delayed bidirectional neural networks. Physica D *159* (2001), 233–251. DOI:10.1016/s0167-2789(01)00344-x

[22] S. Mohamad and K. Gopalsamy: Dynamics of a class of discrete-time neural networks and their continuous-time counterparts. Math. Computers Simul. *53* (2000), 1–39. DOI:10.1016/s0378-4754(00)00168-3

[23] S. Mohamad and K. Gopalsamy: Exponential stability of continuous-time and discrete-time cellular neural networks with delays. Appl. Math. Comput. *135* (2003), 17–38. DOI:10.1016/s0096-3003(01)00299-5

[24] S. Mohamad and A. G. Naim: Discrete-time analogues of integro-differential equations modelling bidirectional neural networks. J. Comput. Appl. Math. *138* (2002), 1–20. DOI:10.1016/s0377-0427(01)00366-1

[25] G. Nagamani, S. Ramasamy, and P. Balasubramaniam: Robust dissipativity and passivity analysis for discrete-time stochastic neural networks with time-varying delay. Complexity *21* (2014), 47–58. DOI:10.1002/cplx.21614

[26] Y. Ou, H. Liu, Y. Si, and Z. Feng: Stability analysis of discrete-time stochastic neural networks with time-varying delays. Neurocomputing *73* (2010), 740–748. DOI:10.1016/j.neucom.2009.10.017

[27] S. Raj, R. Ramachandran, S. Rajendiran, and J. D. Cao: Passivity analysis of uncertain stochastic neural network with leakage and distributed delays under impulsive perturbations. Kybernetika *54* (2018), 3–29. DOI:10.14736/kyb-2018-1-0003

[28] M. Şaylı and E. Yılmaz: Periodic solution for state-dependent impulsive shunting inhibitory CNNs with time-varying delays. Neural Networks *68* (2015), 1–11. DOI:10.1016/j.neunet.2015.04.004

[29] D. R. Smart: Fixed Point Theorems. Cambridge University Press, Cambridge 1980.

[30] R. J. Swift: Almost periodic harmonizable processes. Georgian Math. J. *3* (1996), 275–292. DOI:10.1007/bf02280009

[31] C. Tudor: Almost periodic solutions of affine stochastic evolutions equations. Stochastics and Stochast. Reports *38* (1992), 251–266. DOI:10.1080/17442509208833758

[32] P. Wang, B. Li, and Y. K. Li: Square-mean almost periodic solutions for impulsive stochastic shunting inhibitory cellular neural networks with delays. Neurocomputing *167* (2015), 76–82. DOI:10.1016/j.neucom.2015.04.089

[33] J. Wang, X. M. Zhang, and Q. L. Han: Event-Triggered generalized dissipativity filtering for neural networks with time-varying delays. IEEE Trans. Neural Networks Learning Systems *27* (2016), 77–88. DOI:10.1109/tnnls.2015.2411734

[34] W. Xiong and J. D. Cao: Global exponential stability of discrete-time Cohen–Grossberg neural networks. Neurocomputing *64* (2005), 433–446. DOI:10.1016/j.neucom.2004.08.004

[35] L. L. Xiong, J. Cheng, J. D. Cao, and Z. X. Liu: Novel inequality with application to improve the stability criterion for dynamical systems with two additive time-varying delays. Applied Math. Comput. *321* (2018), 672–688. DOI:10.1016/j.amc.2017.11.020

[36] C. J. Xu and P. L. Li: On anti-periodic solutions for neutral shunting inhibitory cellular neural networks with time-varying delays and D operator. Neurocomputing *275* (2018), 377–382. DOI:10.1016/j.neucom.2017.08.030

[37] L. Yang and Y. K. Li: Periodic solutions for stochastic shunting inhibitory cellular neural networks with distributed delays. Advances Difference Equations *2014* (2014), 1–37. DOI:10.1186/1687-1847-2014-37

[38] L. G. Yao: Global exponential convergence of neutral type shunting inhibitory cellular neural networks with d operator. Neural Process. Lett. *45* (2017), 401–409. DOI:10.1007/s11063-016-9529-7

[39] T. W. Zhang: Multiplicity of positive almost periodic solutions in a delayed Hassell–Varleytype predator-prey model with harvesting on prey. Math. Meth. Appl. Sci. *37* (2014), 686–697. DOI:10.1002/mma.2826

[40] T. W. Zhang: Almost periodic oscillations in a generalized Mackey-Glass model of respiratory dynamics with several delays. Int. J. Biomath. *7* (2014), 1450029 (22 pages). DOI:10.1002/mma.2826

[41] T. W. Zhang and X. R. Gan: Almost periodic solutions for a discrete fishing model with feedback control and time delays. Commun. Nonlinear Sci. Numer. Simul. *19* (2014), 150–163. DOI:10.1016/j.cnsns.2013.06.019

[42] X. M. Zhang and Q. L. Han: Global asymptotic stability for a class of generalized neural networks with interval time-varying delays. IEEE Trans. Neural Networks *22* (2011), 1180–1192. DOI:10.1109/tnn.2011.2147331

[43] X. M. Zhang and Q. L. Han: An overview of recent developments in Lyapunov–Krasovskii functionals and stability criteria for recurrent neural networks with time-varying delays. Neurocomputing *313* (2018), 392–401. DOI:10.1016/j.neucom.2018.06.038

[44] X. M. Zhang, Q. L. Han, and Z. G. Zeng: Hierarchical type stability criteria for delayed neural networks via canonical Bessel–Legendre inequalities. IEEE Trans. Cybernet. *48* (2018), 1660–1671. DOI:10.1109/tcyb.2017.2776283

[45] T. W. Zhang and Y. Z. Liao: Existence and global attractivity of positive almost periodic solutions for a kind of fishing model with pure-delay. Kybernetika *53* (2017), 612–629. DOI:10.14736/kyb-2017-4-0612

[46] H. Y. Zhang, Z. P. Qiu, and L. L. Xiong: Stochastic stability criterion of neutral-type neural networks with additive time-varying delay and uncertain semi-Markov jump. Neurocomputing *333* (2019), 395–406. DOI:10.1016/j.neucom.2018.12.028

[47] T. W. Zhang, L. L. Xiong: Periodic motion for impulsive fractional functional differential equations with piecewise Caputo derivative. Applied Mathematics Letters *101* (2020), 106072. DOI:10.1016/j.aml.2019.106072

[48] T. W. Zhang, L. Yang, and L. J. Xu: Stage-structured control on a class of predator-prey system in almost periodic environment. International Journal of Control *2019* (2019), in print. DOI:10.1080/00207179.2018.1513165

[49] Z. Q. Zhang and D. M. Zhou: Existence and global exponential stability of a periodic solution for a discrete-time interval general BAM neural networks. J. Frank. Inst. *347* (2010), 763–780. DOI:10.1016/j.jfranklin.2010.02.007

[50] H. Zhao, L. Sun, and G. Wang: Periodic oscillation of discrete-time bidirectional associative memory neural networks. Neurocomputing *70* (2007), 2924–2930. DOI:10.1016/j.neucom.2006.11.010

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