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ON APPLICATION OF ROTHE'S FIXED POINT THEOREM TO STUDY THE CONTROLLABILITY OF FRACTIONAL SEMILINEAR SYSTEMS WITH DELAYS

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The paper presents finite-dimensional dynamical control systems described by semilinear fractional-order state equations with multiple delays in the control and nonlinear function f. The relative controllability of the presented semilinear system is discussed. Rothe's fixed point theorem is applied to study the controllability of the fractional-order semilinear system. A control that steers the semilinear system from an initial complete state to a final state at time $t > 0$ is presented. A numerical example is provided to illustrate the obtained theoretical results and a practical example is given to show a possible application of the study.

Keywords: fractional systems, semilinear control systems, Rothe's fixed point theorem, delays in control, pseudo-transition matrix, the Caputo derivative

Classification: 93B05, 93C05, 93C10, 34G20

1. INTRODUCTION

The controllability of dynamical systems is one of the most important problems in the control theory. In general, controllability means that it is possible to steer a control system from an initial state into a final state with the aid of admissible controls. Many different controllability definitions have been formulated in literature, which depend on both a class of control systems and a set of admissible controls. A review of recently analyzed controllability problems for a wide class of dynamical systems is presented in [19] and especially for fractional-order systems in [2].

In recent years, the papers and monographs concerning the controllability of dynamical control systems have focused on systems defined by fractional-order differential equations. In many practical applications, fractional-order models have proven to describe the behavior of real-life processes more accurately. Control systems modeled by fractional differential equations occur, among others, in mechanical, biological and chemical problems. Comprehensive discussions of fractional differential equations and their practical applications can be found, for example, in [15, 25, 26, 29, 30, 31, 34].

The controllability of discrete-time fractional systems is studied in [1, 17, 18, 20, 21], positive fractional discrete-time systems are discussed in [40], positive fractional linear systems, both discrete- and continuous-time, are presented in [13] and [14]. Con-

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trollability of continuous time linear fractional systems is studied, among others, in [3, 6, 9, 32, 41].

In many processes, future states depend on both the present state and past states of a system. This means that models describing the processes involve delays in state or in control. If we have delays in the input function, we deal with control systems with delayed controls. In view of the apparent large number of mathematical models which describe dynamical systems with delays in control, solving controllability problems for such systems is of particular importance. Controllability problems for linear continuous-time fractional systems with delayed control are analyzed in [4, 7, 22, 35, 36, 37, 42]. Semilinear and nonlinear fractional-order systems with delays are discussed, among others, in [38] where the Frechet derivative method is applied, in [5, 8] and [27] the Schauder fixed point theorem is used, in [33] the Schaefer fixed point theorem is considered.

The aim of the paper is to study the controllability of continuous-time fractional-order semilinear control systems with multiple delays in control. A new relative controllability criterion for semilinear fractional systems with delays is formulated and proved based on Rothe's fixed point theorem. The Rothe fixed point method has been previously used in [23] for integer-order semilinear systems.

The paper is organized in the following manner. Section 2 recalls some preliminary definitions, formulas and notations. In Section 3, the mathematical model of the considered semilinear fractional systems with point delays in control is presented. The formula for a solution of the discussed system is presented and definitions of the relative controllability from given initial conditions, local relative controllability and (global) relative controllability of the system are formulated. Section 4 contains the main result of the paper – a criterion of relative controllability for the semilinear fractional systems with delays. The proof of the theorem is provided in detail. Rothe's fixed point theorem is applied. Some auxiliary theorems are also included. In Section 5 the theoretical results are illustrated by a numerical example. Finally, some concluding remarks and future work are presented in Section 6.

2. PRELIMINARIES AND NOTATION

Before we present a system description, let us recall some notions concerning fractional-order differential equations. Fractional-order differentiation is a generalization of the integer-order one. There are several definitions of fractional-order derivatives, among others: the Grünwald-Letnikov, the Riemann-Liouville, the Caputo fractional derivatives ([30]). Since in the Caputo approach the derivative of a constant is equal to zero which implies that initial conditions for fractional differential equations take on a similar form as for integer-order differential equations, in this paper we use the Caputo fractional derivatives.

The Caputo fractional differential operator of a fractional order was introduced by an Italian mathematician Michele Caputo in 1967. The Caputo fractional derivative of order α ($n < \alpha < n+1$, $n \in \mathbb{N} \cup \{0\}$) for a differentiable function $f\colon \mathbb{R}^+ \to \mathbb{R}$ is defined as the following integral

$$^{C}D^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha+1)} \int_0^t \frac{f^{(n+1)}(\tau)}{(t-\tau)^{\alpha-n}} \, \mathrm{d}\tau,$$

where Γ is a gamma function.

The Caputo fractional derivative is a linear operator, i. e.

$$^{C}D^{\alpha}\left(af(t)+g(t)\right)=a^{C}D^{\alpha}f(t)+^{C}D^{\alpha}g(t)$$

for any constant a, provided that both $^{C}D^{\alpha}f(t)$ and $^{C}D^{\alpha}g(t)$ exist. It is also obvious that for $\alpha\to n$, Caputo's derivative approaches the nth order conventional derivative of f, that is $\lim_{\alpha\to n}{}^{C}D^{\alpha}f(t)=f^{(n)}(t)$. However, since the fractional derivative is defined based on the definite integral, it is a non-local operator. It has a "memory property", which means that the present state depends not only on the time, but also on previous states.

Based on the definition of the Mittag–Leffler function ([13, 30])

$$E_{\alpha,\beta}(z)=\sum_{k=0}^{\infty}\frac{z^{k}}{\Gamma(\alpha k+\beta)},\ z\in\mathbb{C},\ \alpha>0,\ \beta>0,$$

for an arbitrary nth order square matrix A we can give the formula for a *pseudo-transition matrix* $\Phi_0(t)$ of the uncontrolled linear fractional system $^{C}D^{\alpha}(t)=Ax(t)$ ([13, 26])

$$\Phi_0(t)=E_{\alpha,1}(At^{\alpha})=\sum_{k=0}^{\infty}\frac{A^{k}t^{\alpha k}}{\Gamma(k\alpha+1)}.$$

Moreover, we introduce the following denotation

$$\Phi(t)=t^{\alpha-1}E_{\alpha,\alpha}(At^{\alpha})=t^{\alpha-1}\sum_{k=0}^{\infty}\frac{A^{k}t^{\alpha k}}{\Gamma((k+1)\alpha)}.$$

For $\alpha=1$ we obtain the classical transition matrix of ordinary differential equations

$$\Phi_0(t)=\sum_{k=0}^{\infty}\frac{A^{k}t^{k}}{\Gamma(k+1)}=\sum_{k=0}^{\infty}\frac{(At)^{k}}{k!}=e^{At}.$$

Therefore the pseudo-transition matrix $\Phi_0(t)$ is also called the matrix α-exponential function and is denoted by $\Phi_0(t)=e_{\alpha}^{At}$ ([13, 15]).

To compute the functions $\Phi_0(t)$ and $\Phi(t)$, several methods are applied, such as the inverse Laplace transform method, the Jordan matrix decomposition method and the Cayley–Hamilton method. All the methods are presented in [26]. In the provided example, the method based on the Cayley–Hamilton theorem is applied. The theorem states that a matrix A satisfies its own characteristic equation. That is, if

$$\det[s^{\alpha}I-A]=(s^{\alpha})^{n}+a_{n-1}(s^{\alpha})^{n-1}+\cdots+a_1s^{\alpha}+a_0,$$

then

$$A^{n}+a_{n-1}(A)^{n-1}+\cdots+a_1A^{\alpha}+a_0I=0.$$

The following notation is also used throughout the paper: $L^2([0,\infty),\ \mathbb{R}^m)$ is the Hilbert space of square integrable functions with values in $\mathbb{R}^m$, $L^2_{loc}([0,\infty),\ \mathbb{R}^m)$ is the linear space of locally square integrable functions with values in $\mathbb{R}^m$. Moreover, $||\cdot||_{L^2}$ means the norm in the space $L^2_{loc}([0,\infty),\ \mathbb{R}^m)$.

3. SYSTEM DESCRIPTION

In the paper the semilinear control systems with multiple delays in control described by the following fractional-order differential state equation are studied.

$$ {}^{C}D^{\alpha}x(t) = A\,x(t) + \sum_{i=0}^{M} B_i\,u(t-h_i) + f(x(t), u(t), u(t-h_1), \dots, u(t-h_M)) \qquad (1) $$

for $t \geq 0$ and $0 < \alpha < 1$, where

- $x(t) \in \mathbb{R}^n$ is a pseudo-state vector,
- $u \in L^2_{\text{loc}}([0,\infty), \mathbb{R}^m)$ is a control,
- A is a $(n \times n)$-matrix with real elements,
- B_i are $(n \times m)$-matrices with real elements for $i = 0, 1, \dots, M$,
- $h_i \colon [0,T] \to \mathbb{R}$, $i = 1, 2 \dots, M$ are constant point delays in control that satisfy the following inequalities
$$ 0 = h_0 < h_1 < \dots < h_i < \dots < h_{M-1} < h_M < +\infty, $$
- f is the nonlinear mapping $f \colon \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m \times \cdots \times \mathbb{R}^m \to \mathbb{R}^n$, continuously differentiable near the origin in the space $\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m \times \cdots \times \mathbb{R}^m$ such that $f(0,0,0,\dots,0) = 0$, there are real constants a, b, c and ξ such that $\frac{1}{2} \leq \xi < 1$, and f satisfies the condition
$$ ||f(x(t), u(t), u(t-h_1), \dots, u(t-h_M))||_{\mathbb{R}^n} \leq a||x(t)||_{\mathbb{R}^n} + b||u(t)||^{\xi}_{\mathbb{R}^m} + c. \qquad (2) $$
where $||u(t)||_{\mathbb{R}^m} = \sum_{i=0}^{M} ||u(t-h_i)||_{\mathbb{R}^m}$.

Let $z(0) = (x(0), u_0)$ be given initial conditions called the initial complete state. For time-delay systems, only the complete state $z(t) = (x(t), u_t(s))$, where $u_t(s) = u(s)$ for $s \in [t - h_M(t), t)$, completely describes the behavior of the system at time t.

Theorem 3.1. (Sikora and Klamka [38]) For the given initial conditions $z(0) = (x(0), u_0) \in \mathbb{R}^n \times L^2([-h_M, 0], \mathbb{R}^m)$ and a control $u \in L^2_{\text{loc}}([0,\infty), \mathbb{R}^m)$, there exists a unique solution $x(t) = x(t, z(0), u)$ of the semilinear fractional-order system (1), for each $t \geq 0$, taking the following form

$$ x(t) = \Phi_0(t)x(0) + \int_0^t \Phi(t-\tau) \sum_{i=0}^{M} B_i\,u(\tau - h_i)\,\mathrm{d}\tau \qquad (3) $$

$$ + \int_0^t \Phi(t-\tau) f(x(\tau), u(\tau), u(\tau-h_1), \dots, u(\tau-h_M))\,\mathrm{d}\tau. $$

As for integer-order dynamical systems, we can define a set of reachable states called also the attainable set for the fractional system (1).

Definition 3.2. The attainable set from the initial complete state $z(0) = (x(0), u_0)$ on $[0, t_1]$ for the time-delay fractional system (1) is the set

$$K(t_1) = \Big\{ x \in \mathbb{R}^n : x = x(t_1) = \Phi_0(t_1)x(0)$$

$$+ \int_0^{t_1} \Phi(t_1 - \tau) \sum_{i=0}^{M} B_i\, u(\tau - h_i)\, \mathrm{d}\tau \tag{4}$$

$$+ \int_0^{t} \Phi(t-\tau) f(x(\tau), u(\tau), u(\tau - h_1), \dots, u(\tau - h_M))\, \mathrm{d}\tau \colon u(t) \in U \text{ for } t \in [0, t_1] \Big\}.$$

For systems with delays in control, two types of controllability of dynamical systems are generally considered: relative controllability and absolute controllability [16]. In the case of relative controllability on $[t_0, t_1]$, the aim is to find a control u such that the state $x(t_1)$ can be reached using the control. In the case of absolute controllability, the aim is to reach a function. It means that the final segment of a trajectory (over the interval $[t_1 - v_M(t_1), t_1]$) should be a given function.

Since fixed point theorems give as an answer to the question whether a solution exists, which means that a system can be steered to a final state $x(t_1)$ (the state $x(t_1)$ can be reached), the relative controllability is studied in the paper.

Definitions of relative controllability from a given initial complete state, local relative and (global) relative controllability for the semilinear system (1) on $[0, t_1]$ are presented below. The definitions are a consequence of corresponding definitions for integer-order systems presented in [16].

Definition 3.3. The semilinear fractional-order system (1) is called relatively controllable on $[0, t_1]$ from the initial complete state $z(0) = (x(0), u_0)$ if for each vector $\widetilde{x} \in \mathbb{R}^n$, there exists a control $\widetilde{u} \in L^2([0, T], \mathbb{R}^m)$ such that

$$x(t_1) = x(t_1, z(0), \widetilde{u}) = \widetilde{x}.$$

Especially, the system (1) is relatively null controllable on $[0, t_1]$ from the initial complete state $z(0) = (x(0), u_0)$ if for each vector $\widetilde{x} \in \mathbb{R}^n$, there exists a control $\widetilde{u} \in L^2([0, T], \mathbb{R}^m)$ such that the solution $x = x(t)$ satisfies

$$x(t_1) = x(t_1, z(0), \widetilde{u}) = 0.$$

Definition 3.4. The semilinear fractional system (1) is called locally relatively controllable on $[0, t_1]$ if the attainable set $K(t_1)$ contains a certain neighborhood of zero in the space $\mathbb{R}^n$.

Definition 3.5. The semilinear fractional system (1) is called (globally) relatively controllable on $[0, t_1]$ if it is relatively controllable on $[0, t_1]$ for every initial complete state $z(0) = (x(0), u_0)$.

Remark 3.6. Definition 3.5 implies that the system (1) is (globally) relatively controllable on $[0, t_1]$ if $K(t_1) = \mathbb{R}^n$.

Lemma 3.7. (Iturriaga and Leiva [12]) Let W and Z be Hilbert spaces, $S \in L(W, Z)$ and $S^* \in L(Z, W)$ be the adjoined operator of S, and $\dim(Z) < +\infty$, then the following statements are equivalent

(i) $\mathrm{Rang}(S) = Z$,

(ii) $\ker(S^*) = \{0\}$,

(iii) $\exists_{\gamma>0} \langle SS^*x, x\rangle > \gamma||x||^2, x \neq 0$,

(iv) $\exists\, (SS^*)^{-1} \in L(Z)$,

where the symbol $\exists$ stands for the existential quantifier.

Lemma 3.8. (Leiva [23]) Let (X, Σ, μ) be a measure space with $\mu(X) < +\infty$ and $1 \leq q < p < +\infty$. Then $L^p(\mu) \subset L^q(\mu)$ and

$$\forall_{f\in L^p(\mu)} \quad ||f||_{L^q} \leq \mu(X)^{\frac{p-q}{pq}}||f||_{L^p}.$$

4. MAIN RESULTS – CONTROLLABILITY CRITERION

In this section we discuss relative controllability of the system (1). We prove, under the assumed conditions, that if the corresponding linear system

$$^{C}D^{\alpha}x(t) = A\,x(t) + \sum_{i=0}^{M} B_i\,u(t-h_i) \tag{5}$$

is relatively controllable on $[0, t_1]$, then the semilinear system (1) is also controllable on $[0, t_1]$. Moreover we give a control $\widetilde{u}$ that steers the semilinear system (1) from an initial complete state $z(0) = (x(0), u_0)$ into a final state $x_1 = x(t_1, z(0), \widetilde{u})$.

Rothe's fixed point theorem is used to prove the sufficient condition. Thus let us recall the theorem (see [11, 39]).

Theorem 4.1. (Rothe's fixed point theorem) Let E be a Banach space and $B \subset E$ be a closed convex subset such that zero of E is contained in the interior of B. Let $g\colon B \to E$ be a continuous mapping with $g(B)$ relatively compact (closure is compact) in E and $g(\partial B) \subset B$, where ∂B denotes the boundary of B. Then there is a point $x^* \in B$ such that $g(x^*) = x^*$.

Since the Mittag–Leffler function is of exponential order, the following inequality holds for $t > 0$

$$\exists_{M_1>0}\ \exists_{\varrho\geq 0}\ ||\Phi(t)|| \leq M_1 e^{\varrho t}.$$

Without loss of generality, let us assume zero initial conditions $z(0) = (0, 0)$ and $c = 0$. Taking into account the assumption (2), we may formulate the theorem below.

Lemma 4.2. The solution (3) satisfies the following estimation

$$||x(t)|| \leq \left(\int_0^{t_1} ||B||M_1 e^{\varrho(t_1-\tau)}||u(\tau)||\,\mathrm{d}\tau + \int_0^{t_1} bM_1 e^{\varrho(t_1-\tau)}||u(\tau)||^{\xi}\,\mathrm{d}\tau\right) e^{aM_1t_1}, \tag{6}$$

where $||B|| = \sum_{i=0}^{M} ||B_i|| = \sum_{i=0}^{M} \max |b_{ikj}|, 1 \leq k \leq n, 1 \leq j \leq m$, and $||u(t)|| = ||u(t)||_{\mathbb{R}^m} = \sum_{i=0}^{M} ||u(t-h_i)||_{\mathbb{R}^m}$.

Next, let us transform the solution (2) of the fractional system (1). By substitution and definite integral properties, the solution of (1) can be rewritten in the following form

$$x(t_1, z(0), u) = \Phi_0(t_1)q(z(0)) + \int_0^{t_1} \Phi(t_1-\tau)B_{t_1}u(\tau)\,\mathrm{d}\tau \tag{7}$$

$$+ \int_0^{t_1} \Phi(t_1-\tau)f(x(\tau), u(\tau), u(\tau-h_1), \ldots, u(\tau-h_M))\,\mathrm{d}\tau,$$

where

$$q(z(0)) = x(0) + \big(\Phi_0(t_1)\big)^{-1}\Bigg[\sum_{i=0}^{k}\int_{-h_i}^{0} \Phi(t_1-\tau-h_i)B_i u_0(\tau)\,\mathrm{d}\tau$$

$$+ \sum_{i=k+1}^{M}\int_{-h_i}^{t_1-h_i} \Phi(t_1-\tau-h_i)B_i u_0(\tau)\,\mathrm{d}\tau\Bigg],$$

$$B_{t_1}(t) = \big(\Phi(t_1-\tau)\big)^{-1}\sum_{j=0}^{i}\Phi(t_1-\tau-h_j)B_j,$$

for $t \in [t_1-h_{i+1}, t_1-h_i)$, $i = 0, 1, ..., k-1$. Since $q(z(0))$ depends only on initial conditions, if we take zero initial conditions, $q(z(0)) = 0$. To simplify the notation, we introduce the following definition.

Definition 4.3. For the semilinear fractional system (1) the functions $G, G_f\colon L^2([0,t_1],\mathbb{R}^m) \to \mathbb{R}^n$ defined by the formulas

$$G(u) = \int_0^{t_1} \Phi(t_1-\tau)B_{t_1}(\tau)u(\tau)\,\mathrm{d}\tau \tag{8}$$

$$G_f(u) = \int_0^{t_1} \Phi(t_1-\tau)B_{t_1}(\tau)u(\tau)\,\mathrm{d}\tau \tag{9}$$

$$+ \int_0^{t_1} \Phi(t_1-\tau)f(\widetilde{x}(\tau), u(\tau), u(\tau-h_1), \ldots, u(\tau-h_M))\,\mathrm{d}\tau$$

are called controllability operators for $t_1 > 0$, where $\widetilde{x}$ is the unique solution of the fractional differential equation (1).

Hence, the controllability operator $G_f(u)$ can be written as the following sum

$$G_f(u) = G(u) + H(u),$$

where $H\colon L^2([0,t_1],\mathbb{R}^m) \to \mathbb{R}^n$ is the nonlinear operator given by

$$H(u) = \int_0^{t_1} \Phi(t_1-\tau)f(\widetilde{x}(\tau), u(\tau), u(\tau-h_1), \ldots, u(\tau-h_M))\,\mathrm{d}\tau.$$

Consider the adjoined operator $G^*\colon \mathbb{R}^n \to L^2([0,t_1],\mathbb{R}^m)$ of the operator G

$$G^*(x) = B_{t_1}^*(\tau)\Phi^*(t_1-\tau)x = B_{t_1}^*(\tau)E_{\alpha,\alpha}(A^*(t_1-\tau)^\alpha)x, \tau \in [0,t_1].$$

Lemma 4.4. The system (5) is relatively controllable on $[0, t_1]$ if and only if $\mathrm{Rang}(G) = \mathbb{R}^n$. Moreover, the system (1) is relatively controllable on $[0, t_1]$ if and only if $\mathrm{Rang}(G) = \mathbb{R}^n$ and $\mathrm{Rang}(G_f) = \mathbb{R}^n$.

P r o o f. It follows immediately from formulas (8) and (9), and Remark 3.6. □

Theorem 4.5. If the linear system (5) is relatively controllable on $[0, t_1]$ and the following inequality holds

$$\frac{1}{\gamma\sqrt{2}}||B||^2 M_1^3 a\sqrt{t_1} e^{aM_1 t_1}\left(\frac{e^{2\varrho t_1}-1}{\varrho}\right)^{\frac{3}{2}} < 1, \tag{10}$$

then the semilinear system (1) is also relatively controllable on $[0, t_1]$. Moreover, a control steering the system (1) from the initial complete state $z(0) = (x(0), u_0)$ to a final state $\widetilde{x} = x(t_1)$ at time $t_1 > 0$ is given by the formula

$$u(t) = B^*_{t_1}(t)E_{\alpha,\alpha}(A^*(t_1-t)^\alpha)(GG^*)^{-1}\left(\widetilde{x} - \Phi_0(t_1)q(z(0)) - H(u)\right), t \in [0, t_1]. \tag{11}$$

P r o o f. For each $x \in \mathbb{R}^n$ fixed we define an operator $\Gamma\colon L^2([0, t_1], \mathbb{R}^m) \to L^2([0, t_1], \mathbb{R}^m)$ by the formula

$$\Gamma(u) = G^*(GG^*)^{-1}(x - H(u)).$$

Γ is defined properly, because $(GG^*)^{-1}$ exists which follows from Lemmma 3.7(iv). Moreover, from (iii)

$$||(GG^*)^{-1}x|| \le \gamma^{-1}||x||.$$

We shall prove that the operator Γ has a fixed point u that depends on x.

Since f is continuous, the operator H is continuous. Next, due to inequality (2), H is a compact operator.

For $u \in L^2([0, t_1], \mathbb{R}^m)$, applying the Hölder inequality and condition (2), we have the following estimation

$$||H(u)|| \le \int_0^{t_1} M_1 e^{\varrho(t_1-\tau)} f(\widetilde{x}(\tau), u(\tau), u(\tau-h_1), \ldots, u(\tau-h_M))\,\mathrm{d}\tau$$

$$\le \left(\int_0^{t_1} M_1^2 e^{2\varrho(t_1-\tau)}\mathrm{d}\tau\right)^{\frac{1}{2}}\left(\int_0^{t_1} ||f(\widetilde{x}(\tau), u(\tau), u(\tau-h_1), \ldots, u(\tau-h_M))||^2\,\mathrm{d}\tau\right)^{\frac{1}{2}}$$

$$= N\left(\int_0^{t_1} ||f(\widetilde{x}(\tau), u(\tau), u(\tau-h_1), \ldots, u(\tau-h_M))||^2\,\mathrm{d}tau\right)^{\frac{1}{2}}$$

$$\le N\left(\int_0^{t_1}\left(a||x(\tau)|| + b||u(\tau)||^\xi\right)^2\,\mathrm{d}\tau\right)^{\frac{1}{2}}$$

$$\le N\left(\int_0^{t_1}\left(4a^2||x(\tau)||^2 + 4b^2||u(\tau)||^{2\xi}\right)\,\mathrm{d}\tau\right)^{\frac{1}{2}}$$

$$\leq 2Na\left(\int_0^{t_1}||x(\tau)||^2\,\mathrm{d}\tau\right)^{\frac{1}{2}}+2Nb\left(\int_0^{t_1}||u(\tau)||^{2\xi}\,\mathrm{d}\tau\right)^{\frac{1}{2}}$$

$$\leq 2Na\left(\int_0^{t_1}\left(\int_0^{t_1}||B||M_1e^{\varrho(t_1-s)}||u(s)||\,\mathrm{d}s+\int_0^{t_1}bM_1e^{\varrho(t_1-s)}||u(s)||^{\xi}\,\mathrm{d}s\right)^2e^{2aM_1t_1}\,\mathrm{d}\tau\right)^{\frac{1}{2}}$$

$$+2Nb\left[\left(\int_0^{t_1}||u(\tau)||^{2\xi}\,\mathrm{d}\tau\right)^{\frac{1}{2\xi}}\right]^{\xi}$$

$$\leq 2Na\sqrt{t_1}\left(\int_0^{t_1}||B||M_1e^{\varrho(t_1-\tau)}||u(\tau)||\,\mathrm{d}\tau+\int_0^{t_1}bM_1e^{\varrho(t_1-\tau)}||u(\tau)||^{\xi}\,\mathrm{d}\tau\right)e^{aM_1t_1}$$

$$+2Nb\left(||u||_{L^{2\xi}}\right)^{\xi},$$

where $L^{2\xi}=L^{2\xi}([0,t_1],\mathbb{R}^m)$ and $N=\left(\int_0^{t_1}M_1^2e^{2\varrho(t_1-\tau)}\mathrm{d}\tau\right)^{\frac{1}{2}}$.

Since $\frac{1}{2}\leq\xi<1$, we obtain $1\leq\xi<2$. Applying Lemma 3.7 we have

$$||H(u)||\leq 2N^2a\sqrt{t_1}||B||e^{aM_1t_1}||u||_{L^2}+2Nbt_1^{\frac{1-\xi}{2}}(Na\sqrt{t_1}e^{aM_1t_1}+1)\left(||u||_{L^{2\xi}}\right)^{\xi},$$

which implies

$$\lim_{||u||_{L^2}\to+\infty}\frac{||H(u)||_{L^2}}{||u||_{L^2}}\leq\frac{M_1^2}{\varrho}a\sqrt{t_1}||B||e^{aM_1t_1}(e^{2\varrho t_1}-1).$$

Therefore,

$$\lim_{||u||_{L^2}\to\infty}\frac{||\Gamma(u)||_{L^2}}{||u||_{L^2}}\leq||G^*(GG^*)^{-1}||\frac{M_1^2}{\varrho}a\sqrt{t_1}||B||e^{aM_1t_1}(e^{2\varrho t_1}-1)$$

and, finally,

$$\lim_{||u||_{L^2}\to\infty}\frac{||\Gamma(u)||_{L^2}}{||u||_{L^2}}\leq\frac{1}{\gamma\sqrt{2}}||B||^2M_1^3a\sqrt{t_1}e^{aM_1t_1}\left(\frac{e^{2\varrho t_1}-1}{\varrho}\right)^{\frac{3}{2}}.$$

Let us denote $r=\frac{1}{\gamma\sqrt{2}}||B||^2M_1^3a\sqrt{t_1}e^{aM_1t_1}\left(\frac{e^{2\varrho t_1}-1}{\varrho}\right)^{\frac{3}{2}}$. By the assumption, $r<1$. From the above it follows that for a fixed ε, $r<\varepsilon<1$, there exists $r_0>0$ (big enough) such that

$$||\Gamma(u)||_{L^2}\leq\varepsilon||u||_{L^2}=\varepsilon r_0.$$

Let $B(0,r_0)$ denote the ball of center at zero and radius $r_0>0$, then $\Gamma(\partial B(0,r_0))\subset B(0,r_0)$. The operator Γ is compact and maps the sphere $\partial B(0,r_0)$ into the interior of the ball $B(0,r_0)$, the Rothe fixed point theorem can be applied. Hence, it follows that there exists a fixed point $u\in B(0,r_0)\subset L^2([0,t_1],\mathbb{R}^m)$ such that

$$u=G^*(GG^*)^{-1}(x-H(u)).$$

Since $G(u) = x - H(u)$, putting $x = \widetilde{x} - \Phi_0(t_1)q(z(0))$, we have

$$\widetilde{x} = \Phi_0(t_1)q(z(0)) + \int_0^{t_1} \Phi(t_1 - \tau)B_{t_1}u(\tau)\,\mathrm{d}\tau \tag{12}$$

$$+ \int_0^{t_1} \Phi(t_1 - \tau)f(x(\tau), u(\tau), u(\tau - h_1), \ldots, u(\tau - h_M))\,\mathrm{d}\tau.$$

Thus, $\widetilde{x}$ is the solution of the system (1), and it is easy to verify that $\widetilde{x} = x(t_1)$. Therefore the system is relatively controllable on $[0, t_1]$.

Finally, from the above, we obtain the control steering the system (1) from the initial complete state $z(0) = (x(0), u_0)$ to a final state $\widetilde{x} = x(t_1)$ at time $t_1 > 0$, given by the following formula

$$u(t) = B^*_{t_1}(t)E_{\alpha,\alpha}(A^*(t_1 - t)^\alpha)(GG^*)^{-1}\left(\widetilde{x} - \Phi_0(t_1)q(z(0)) - H(u)\right), t \in [0, t_1].$$

□

Remark 4.6. Criteria for relative controllability of linear fractional-order systems of the form (4) have been formulated and proved in [35] and [37].

Theorem 4.7. Let $u(t) \in U$, where $U \subset \mathbb{R}^m$ is a convex and compact set containing 0 in its interior. If the linear system (5) is relatively controllable on $[0, t_1]$, the inequality (10) holds and $|arg(\lambda_i)| > \alpha\frac{\pi}{2}$, $1 \leq i \leq n$, where λ_i are the eigenvalues of matrix A, then the semilinear system (1) is relatively null controllable on $[0, t_1]$.

Proof. Assume that $U \subset \mathbb{R}^m$ is a convex and compact set containing 0 in its interior and $u(t) \in U$. If the linear system (5) is relatively controllable on $[0, t_1]$ and the inequality (10) holds, then the semilinear fractional system (1) is relatively controllable on $[0, t_1]$ by Theorem 4.5. Moreover, if the eigenvalues of matrix A satisfy the condition $|arg(\lambda_i)| > \alpha\frac{\pi}{2}$, $1 \leq i \leq n$, then the fractional system (1) is asymptotically stable (see: [10, 28]). Owing the asymptotical stability assumption, $x = 0$ is the solution of the system (1) for the admissible control $u(t) = 0$. Using the null control $u(t) = 0$, the solution $x(t, z(t_0), 0)$ of (1) satisfies the conditions

$$\lim_{t \to +\infty} x(t, z(t_0), 0) = 0 \quad \text{and} \quad x(t_1, z(t_0), 0) \in N(0),$$

for some, finite $t_1 \in (0, +\infty)$, where $N(0)$ is a sufficiently small neighborhood of $0 \in \mathbb{R}^n$. Then, since $U \subset \mathbb{R}^m$ is a convex and compact set, the instantaneous state $x(t_1, z(t_0), 0)$ can be steered to $0 \in \mathbb{R}^n$ in finite time. Therefore the fractional system (1) is relatively null controllable on $[0, t_1]$. □

5. EXAMPLES

In this section a numerical example is presented to illustrate the obtained theoretical results. Moreover, an example of practical problem modeled by a fractional semilinear differential equation considered in the paper is given.

Example 5.1. Consider the following semilinear fractional dynamical system with delays in control

$$^{C}D^{\frac{1}{2}}x(t) = x(t) + u(t) + u(t-1) + u(t-2) + \sin x(t) + \cos u(t-2) - 1 \qquad (13)$$

for $t \in [0,2]$, with zero initial conditions $z(0) = (0,0)$.

Thus we have $A = 1$, $\forall_{i=0,\ldots,2}\, B_i = 1$, $(M = 2)$, and two delays in control $h_1 = 1, h_2 = 2$. The nonlinear function f is defined by

$$f(x(t), u(t-2)) = \sin x(t) + \cos u(t-2) - 1.$$

Of course $k = 2$ since $t_1 - h_2 = 0$.

The pseudo-transition matrix $\Phi_0(t)$ of the system (13) has the following form

$$\Phi_0(t) = \sum_{k=0}^{\infty} \frac{t^{\frac{k}{2}}}{\Gamma(\frac{k}{2}+1)}$$

and

$$\Phi(t) = t^{-\frac{1}{2}} \sum_{k=0}^{\infty} \frac{t^{\frac{k}{2}}}{\Gamma(\frac{1}{2}(k+1))}.$$

Using the Cayley-Hamilton method we calculate

$$\Phi(t) = t^{-\frac{1}{2}} E_{\frac{1}{2},\frac{1}{2}}(At^{\frac{1}{2}}) = t^{-\frac{1}{2}} \sum_{k=0}^{1} \frac{t^{\frac{k}{2}}}{\Gamma(\frac{1}{2}(k+1))} = t^{-\frac{1}{2}} \left(\frac{t^0}{\sqrt{\pi}} + \frac{t^{\frac{1}{2}}}{1} \right) = \frac{1}{\sqrt{t\pi}} + 1$$

and

$$(\Phi(t))^{-1} = \frac{\sqrt{t\pi}}{1+\sqrt{t\pi}}.$$

The corresponding linear fractional-order system is relatively controllable on $[0,2]$, since (see [37], Lemmma 4.1)

$$\text{rank} \sum_{i=0}^{2} \int_{-h_i}^{t_1-h_i} (t_1-\tau-h_i)^{-\frac{1}{2}} E_{\frac{1}{2},\frac{1}{2}}(A(t_1-\tau-h_i)^{\frac{1}{2}}) B_i B_i^T E_{\frac{1}{2},\frac{1}{2}}(A^T(t_1-\tau-h_i)^{\frac{1}{2}}) \mathrm{d}\tau = n = 1.$$

We also see that f satisfies the condition (2)

$$|f(x(t), u(t-2))| = |\sin x(t) + \cos u(t-2) - 1|$$

$$\leq |\sin x(t)| + |\cos u(t-2)| - 1 \leq |x(t)| + |u(t)|^{\frac{1}{2}}.$$

We have: $a = 1, b = 1, \xi = \frac{1}{2}$ and $||B|| = 3$, so

$$\frac{1}{\gamma\sqrt{2}} ||B||^2 M_1^3 a\sqrt{t_1} e^{aM_1t_1} \left(\frac{e^{2\varrho t_1} - 1}{\varrho} \right)^{\frac{3}{2}} = \frac{9}{\gamma\sqrt{2}} M_1^3 \sqrt{2} e^{2M_1} \left(\frac{e^{4\varrho} - 1}{\varrho} \right)^{\frac{3}{2}}$$

$$= \frac{9}{\gamma} M_1^3 e^{2M_1} \left(\frac{e^{4\varrho} - 1}{\varrho} \right)^{\frac{3}{2}} = \frac{72}{\gamma} M_1^3 e^{2M_1} \left(\frac{e^{4\varrho} - 1}{4\varrho} \right)^{\frac{3}{2}}.$$

For $\varrho \to 0$, $\left(\frac{e^{4\varrho}-1}{4\varrho}\right)^{\frac{3}{2}} \to 1$, so it is possible to choose γ and M_1 such that $|\Phi(2)| = |\frac{1}{\sqrt{2\pi}} + 1| \le M_1$ and $72M_1^3 e^{2M_1} < \gamma$. Then the condition (10) holds, and applying Theorem 4.5 we conclude that the semilinear fractional-order system (13) is relatively controllable on $[0, 2]$.

Moreover, since

$$B_2(t) = \big(\Phi(2-t)\big)^{-1} \sum_{j=0}^{i} \Phi(2-t-h_j)B_j,$$

for $t \in [2-h_{i+1}, 2-h_i)$, $i = 0, 1$, we have

$$B_2(t) = \begin{cases} \big(\Phi(2-t)\big)^{-1}\big(\Phi(2-t)B_0 + \Phi(2-t-1)B_1\big), & t \in [0,1), \\ \big(\Phi(2-t)\big)^{-1}\Phi(2-t)B_0, & t \in [1,2). \end{cases}$$

It follows that

$$B_2(t) = \begin{cases} 1 + \frac{1+\sqrt{(1-t)\pi}}{1+\sqrt{(2-t)\pi}} \cdot \sqrt{\frac{2-t}{1-t}}, & t \in [0,1), \\ 1, & t \in [1,2), \end{cases}$$

and we may determine

$$u(t) = B_2^*(t) E_{\frac{1}{2},\frac{1}{2}}(A^*(2-t)^\alpha)(GG^*)^{-1}\left(\widetilde{x} - H(u)\right), t \in [0,2],$$

steering the system (13) from the initial complete state $z(0) = (0,0)$ to a final state $\widetilde{x} = x(2)$, given by the formula (12).

Example 5.2. An example of fractional-order semilinear system with delays in control considered in the paper can be a system of chemical solution control composed of a parallel connection of two fully filled mixers and two reactors. Simplifying the partial differential equation that describes the reactors [24], we obtain the following fractional version of semilinear state equations

$$\begin{cases} {}^C D^\alpha c_1(t) = -\frac{Q^*}{V} c_1(t) + \frac{Q^*}{V} c_{we1}(t-h_1) + f_1(c_1(t), c_{we1}(t-h_1)) \\ {}^C D^\alpha c_2(t) = -\frac{Q^*}{V} c_1(t) + \frac{Q^*}{V} c_{we1}(t-h_2) + f_2(c_1(t), c_2(t), c_{we1}(t-h_1), c_{we2}(t-h_2)) \end{cases}$$

where $0 < \alpha \le 1$, $c_1(t), c_2(t)$ are strengths of solutions in the mixers (states), $c_{we1}(t)$, $c_{we2}(t)$ are input concentrations of a product (controls), Q^* is the constant intensity of flow, V is the volume of each mixer (for simplicity, the same volume of both mixers is assumed), h_1, h_2 ($h_1 < h_2$) are constant delays in control, and f_1, f_2 are functions determining changes in the concentration of the substance during the mixing in the corresponding mixers.

6. CONCLUDING REMARKS

The relative controllability of semilinear fractional control systems with delays in control has been discussed in the paper. The nonlinear function f was assumed to be smooth enough and to satisfy the condition (2). Definitions of relative controllability from a given initial complete state, local relative controllability and relative controllability on the interval $[0, t_1]$ (called also global relative controllability) for the systems have been formulated. The formula for solution of the fractional-order differential equation (1) has been presented. Moreover, it has been rewritten into a form allowing to indicate an admissible control u.

The main result of the paper is the new controllability criterion of relative controllability of time-delay fractional semilinear systems described by the equation (1), which has been established and proved. The criterion (Theorem 4.5) is based on the Rothe fixed point theorem. Example 5.1 has been presented to illustrate how to verify the relative controllability of the discussed systems with the aid of the established criterion (Example 5.1). Moreover, a practical example has been included to show a possible application of the study (Example 5.2).

We deal with semilinear systems with delays in control also in modeling the process of steel rolling, where thickness can only be measured at some distances from rolls (this leads to measurement delays), or in metal cutting modeling, where delays depend on full rotation time. In future, we plan to propose a fractional model and find an optimal control for the steel rolling process, since fractional calculus provides more accurate models of the systems under consideration.

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