KYBERNETIKA — VOLUME *55* (2019), NUMBER 4, PAGES 641–667

CONSTRUCTION METHODS FOR IMPLICATIONS ON BOUNDED LATTICES

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In this paper, the ordinal sum construction methods of implications on bounded lattices are studied. Necessary and sufficient conditions of an ordinal sum for obtaining an implication are presented. New ordinal sum construction methods on bounded lattices which generate implications are discussed. Some basic properties of ordinal sum implications are studied.

Keywords: ordinal sum, implication, bounded lattice

Classification: 03E72, 03B52

1. INTRODUCTION

In fuzzy logic, fuzzy implications having a significant role in many applications, such as fuzzy control, approximate reasoning, and decision support systems [2, 8, 18, 19, 24] are one of the most important operations whose truth values belong to the unit interval $[0, 1]$. This is the reason to built and investigate new families of implications. In the literature, there are some methods to generate new families of aggregation operators. In this sense, the ordinal sum construction method is one of the most commonly used generating methods for such operators. For more results on ordinal sum of triangular norms, triangular conorms, uninorms, fuzzy implications on the unit interval $[0, 1]$, see [2, 5, 6, 11, 14, 16, 21, 22, 23, 26, 27].

Introducing and researching of logical operators on more general structures than the unit interval $[0, 1]$ have become topics of interest to many researchers [3, 17]. In this context, the ordinal sum of logical operators on bounded lattices, like t-norms, t-conorms, uninorms, copulas, has been studied by researchers [10, 20, 25]. In [26], the ordinal sum of implications on $[0, 1]$ has been introduced similar to the concept to the ordinal sum of t-norms and the properties of ordinal sum implications have been investigated. As it can be seen in [26], the introduced function for the ordinal sum of implications on $[0, 1]$ has been shown to be an implication under a special condition. This means that the construction method need not generate a fuzzy implication on $[0, 1]$ without the additional conditions. In [1, 5, 6], new ordinal sum construction methods for fuzzy implication which generate fuzzy implications have been proposed.

DOI: 10.14736/kyb-2019-4-0641

Even though, the ordinal sum construction successes to preserve the logical structures on the unit interval, they may fail on bounded lattices as seen in [20, 25].

In the present paper, based on [5, 6], the ordinal sum construction methods of implications on bounded lattices are studied. The paper is organized as follows. In Section 2, we review some basic concepts and notations which will be used in the paper. In Section 3, we introduce the ordinal sum of implications on bounded lattices and investigate the introduced function yields again an implication on the lattice under which conditions. We present necessary and sufficient conditions for an ordinal sum being an implication on a bounded lattice. In Section 4, we propose new ordinal sum construction methods for implications on bounded lattices which generate implications. We investigate some properties of ordinal sum implications.

2. NOTATIONS, DEFINITIONS AND A REVIEW OF PREVIOUS RESULTS

In this section, we recall some basic notions and results.

Throughout the paper we will use the notation $(L, \leq, 0, 1)$ for a bounded lattice. For any $a, b \in L$ if a and b are not comparable, we will denote this case by $a \| b$. For any $a, b \in L$ with $a \leq b$, a subinterval $[a, b]$ of L is defined as

$$[a, b] = \{x \in L |\ a \leq x \leq b\}.$$

Similarly, $(a, b] = \{x \in L |\ a < x \leq b\}$, $[a, b) = \{x \in L |\ a \leq x < b\}$ and $(a, b) = \{x \in L |\ a < x < b\}$.

Definition 2.1. (Baczyński and Jayaram [2]) A function $I : L^2 \to L$ on a bounded lattice $(L, \leq, 0, 1)$ is called an implication if it satisfies the following conditions:

(I1) I is a decreasing operation on the first variable, that is, for every $a, b \in L$ with $a \leq b$, $I(b, y) \leq I(a, y)$ for all $y \in L$.

(I2) I is an increasing operation on the second variable, that is, for every $a, b \in L$ with $a \leq b$, $I(x, a) \leq I(x, b)$ for all $x \in L$.

(I3) $I(0, 0) = 1$.

(I4) $I(1, 1) = 1$.

(I5) $I(1, 0) = 0$.

Example 2.2. (Baczyński and Jayaram [2]) The following are well-known implications on the unit interval $[0, 1]$.

$$I_{LK}(x, y) = \min(1, 1 - x + y), \qquad I_{RC}(x, y) = 1 - x + xy,$$

$$I_{KD}(x, y) = \max(1 - x, y), \qquad I_{GD}(x, y) = \begin{cases} 1 & x \leq y, \\ y & x > y, \end{cases}$$

$$I_{GG}(x, y) = \begin{cases} 1 & x \leq y, \\ \frac{y}{x} & x > y, \end{cases} \qquad I_{RS}(x, y) = \begin{cases} 1 & x \leq y, \\ 0 & x > y, \end{cases}$$

$$I_{YG}(x,y) = \begin{cases} 1 & x = 0 \quad \text{and} \quad y = 0, \\ y^x & x > 0 \quad \text{or} \quad y > 0, \end{cases} \qquad I_{WB}(x,y) = \begin{cases} 1 & x < 1, \\ y & x = 1, \end{cases}$$

$$I_{FD}(x,y) = \begin{cases} 1 & x \leq y, \\ \max(1-x,y) & x > y. \end{cases}$$

The least and the greatest implications are respectively given by:

$$I_0(x,y) = \begin{cases} 1 & x = 0 \quad \text{or} \quad y = 1, \\ 0 & x > 0 \quad \text{and} \quad y < 1, \end{cases} \qquad I_1(x,y) = \begin{cases} 1 & x < 1 \quad \text{or} \quad y > 0, \\ 0 & x = 1 \quad \text{and} \quad y = 0. \end{cases}$$

Definition 2.3. (Baczyński and Jayaram [2], Kesicioğlu and Mesiar [13], Ma and Wu [17]) Let $(L, \leq, 0, 1)$ be a bounded lattice. A decreasing function $N : L \to L$ is called a negation if $N(0) = 1$ and $N(1) = 0$. A negation N on L is called strong if it is an involution, i. e., $N(N(x)) = x$, for all $x \in L$.

The weakest and strongest negations on L are given by respectively

$$N_{D_1}(x) = \begin{cases} 0 & x \neq 0, \\ 1 & x = 0, \end{cases} \qquad N_{D_2}(x) = \begin{cases} 1 & x \neq 1, \\ 0 & x = 1. \end{cases}$$

The natural negation of an implication I on a bounded lattice is the function $N_I : L \to L$ defined by $N_I(x) = I(x, 0)$, for all $x \in L$.

Definition 2.4. (Baczyński and Jayaram [2]) An implication I on L is said to satisfy

(i) the left neutrality property, if

$$I(1,y) = y, \qquad y \in L; \tag{NP}$$

(ii) the identity principle, if

$$I(x,x) = 1, \qquad x \in L; \tag{IP}$$

(iii) the order principle, if

$$I(x,y) = 1 \iff x \leq y, \qquad x, y \in L; \tag{OP}$$

(iv) exchange principle, if

$$I(x, I(y,z)) = I(y, I(x,z)), \qquad x, y, z \in L; \tag{EP}$$

(v) consequent boundary, if

$$I(x,y) \geq y, \qquad x, y \in L. \tag{CB}$$

Definition 2.5. (Su et al. [26]) Let $\{I_k\}_{k \in A}$ be a family of implications on $[0,1]$ and $\{[a_k, b_k]\}_{k \in A}$ be a family of pairwise disjoint close subintervals of $[0,1]$ with $0 < a_k < b_k$ for all $k \in A$, where A is a finite or infinite index set. Define the mapping $I : [0,1]^2 \to [0,1]$ given by

$$I(x,y) = \begin{cases} a_k + (b_k - a_k) I_k\left(\frac{x - a_k}{b_k - a_k}, \frac{y - a_k}{b_k - a_k}\right) & x, y \in [a_k, b_k], \\ I_{GD}(x,y) & \text{otherwise}, \end{cases} \tag{1}$$

is called an ordinal sum of fuzzy implications $\{I_k\}_{k \in A}$.

The next theorem characterizes the ordinal sum I given by (1) as a fuzzy implication

Theorem 2.6. (Su et al. [26]) Let $\{I_k\}_{k\in A}$ be a family of implications on $[0,1]$. Then I given by (1) in Definition 2.5 is a fuzzy implication if and only if I_k satisfies (CB) whenever $k \in A$ and $b_k < 1$.

As we can see easily, every fuzzy implication can not be used to generate a fuzzy implication in the sense of (1). Thus, Drygaś and Król [5] introduced the ordinal sum of any fuzzy implications having no additional conditions which generate again a fuzzy implication.

Definition 2.7. (Drygaś and Król [5]) Let $\{I_k\}_{k\in A}$ be a family of implications on $[0,1]$ and $\{[a_k,b_k]\}_{k\in A}$ be a family of pairwise disjoint close subintervals of $[0,1]$ with $0 < a_k < b_k$ for all $k \in A$, where A is a finite or countably infinite index set. Let us consider an operation $I : [0,1]^2 \to [0,1]$ given by the following formula

$$I(x,y) = \begin{cases} a_k + (b_k - a_k) I_k\left(\frac{x-a_k}{b_k-a_k}, \frac{y-a_k}{b_k-a_k}\right) & x, y \in [a_k, b_k], \\ I_{RS}(x,y) & \text{otherwise.} \end{cases} \tag{2}$$

Then, the operation I given by (2) is a fuzzy implication.

3. ORDINAL SUM OF IMPLICATIONS ON BOUNDED LATTICE

In this section, we introduce the ordinal sum of implications on bounded lattices based on [5]. We give some counterexample to show that the introduced function need not be an implication on a bounded lattice. We present necessary and sufficient conditions for an ordinal sum being an implication on a bounded lattice.

Definition 3.1. Let $(L, \leq, 0, 1)$ be a bounded lattice and A be an index set. Let $([a_i,b_i])_{i\in A}$ be a family of pairwise disjoint subinterval of L with $0 < a_i < b_i$ for all $i \in A$ and $(I^{[a_i,b_i]})_{i\in A}$ a family of implications on the corresponding intervals $([a_i,b_i])_{i\in A}$. Then, the ordinal sum $I_1 = (\langle a_i, b_i, I^{[a_i,b_i]}\rangle)_{i\in A} : L^2 \to L$ is given by

$$I_1(x,y) = \begin{cases} I^{[a_i,b_i]}(x,y) & x, y \in [a_i, b_i], \\ (I_{RS})_1(x,y) & \text{otherwise,} \end{cases} \tag{3}$$

where $(I_{RS})_1(x,y) = \begin{cases} 0 & x > y, \\ 1 & \text{otherwise.} \end{cases}$

Definition 3.2. Let $(L, \leq, 0, 1)$ be a bounded lattice and A be an index set. Let $([a_i,b_i])_{i\in A}$ be a family of pairwise disjoint subinterval of L with $0 < a_i < b_i$ for all $i \in A$ and $(I^{[a_i,b_i]})_{i\in A}$ a family of implications on the corresponding intervals $([a_i,b_i])_{i\in A}$. Then, the ordinal sum $I_2 = (\langle a_i, b_i, I^{[a_i,b_i]}\rangle)_{i\in A} : L^2 \to L$ is given by

$$I_2(x,y) = \begin{cases} I^{[a_i,b_i]}(x,y) & x, y \in [a_i, b_i], \\ (I_{RS})_2(x,y) & \text{otherwise,} \end{cases} \tag{4}$$

where $(I_{RS})_2(x,y) = \begin{cases} 1 & x \leq y, \\ 0 & \text{otherwise.} \end{cases}$

Remark 3.3. Let $(L, \leq, 0, 1)$ be a bounded lattice.

(i) Note that the functions $(I_{RS})_1$ and $(I_{RS})_2$ given in Definition 3.1 and Definition 3.2 are obviously implications on a bounded lattice L. Now, let us verify that $(I_{RS})_2$ is an implication.

(I1) Let $x_1 \leq x_2$ for $x_1, x_2 \in L$. For any $y \in L$, if $x_2 \leq y$, then $(I_{RS})_2(x_1, y) = 1 = (I_{RS})_2(x_2, y)$. If $x_2 > y$ or $x_2 || y$, then $(I_{RS})_2(x_2, y) = 0 \leq (I_{RS})_2(x_1, y)$. Then, $(I_{RS})_2$ is a decreasing function in the first variable.

(I2) Let $y_1 \leq y_2$ for $y_1, y_2 \in L$. For any $x \in L$, if $x \leq y_1$, since $y_1 \leq y_2$, we have that $(I_{RS})_2(x, y_1) = 1 = (I_{RS})_2(x, y_2)$. If $x > y_1$ or $x || y_1$, then it is clear that $(I_{RS})_2(x, y_1) = 0 \leq (I_{RS})_2(x, y_2)$. Then, $(I_{RS})_2$ is an increasing function in the second place.

Also, it is obvious that $(I_{RS})_2(0, 0) = (I_{RS})_2(1, 1) = 1$ and $(I_{RS})_2(1, 0) = 0$. Thus, $(I_{RS})_2$ is an implication on L.

Similarly, it can be easily seen that $(I_{RS})_1$ is an implication on a bounded lattice L.

(ii) Even though the functions given by (3) and (4) are implications on the unit interval $[0, 1]$, they need not be implications on any bounded lattices. We shall give the following examples.

Example 3.4. Let $(L = \{0, a, b, c, d, e, 1\}, \leq, 0, 1)$ be a bounded lattice whose lattice diagram is as in Figure 1:

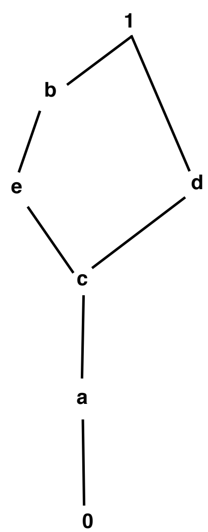

Fig. 1. $(L, \leq)$.

Define the function $I^{'} : [a, b]^2 \to [a, b]$ as in Table 1:

I'	a	c	e	b
a	b	b	b	b
c	b	b	b	b
e	b	b	b	b
b	a	c	e	b

Tab. 1. The function I'.

It can be easily seen that I' is an implication on $[a,b]$. But, the function $I : L^2 \to L$ defined by

$$I(x,y) = \begin{cases} I'(x,y) & x,y \in [a,b], \\ (I_{RS})_2(x,y) & \text{otherwise,} \end{cases}$$

is not an implication on L. Indeed, as seen in Table 1, $I(e,c) = I'(e,c) = b$ and $I(e,d) = (I_{RS})_2(e,d) = 0$. Even though $c \le d$, $I(e,c) = b \not\le 0 = I(e,d)$. That is, I is not an increasing function in the second variable.

Example 3.5. Let $(L = \{0,a,b,c,d,1\}, \le, 0, 1)$ be a bounded lattice depicted in Figure 2:

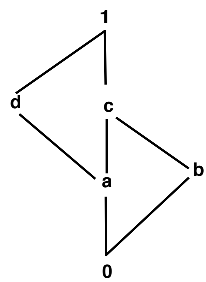

Fig. 2. $(L, \le)$.

Define the function $I' : [a,1]^2 \to [a,1]$ as in Table 2:

I'	a	d	c	1
a	1	1	1	1
d	a	1	a	1
c	a	a	1	1
1	a	a	a	1

Tab. 2. The function I'.

Obviously, it can be seen that $I^{'}$ is an implication on $[a, 1]$. Let us consider the function $I : L^2 \to L$ given by

$$I(x, y) = \begin{cases} I^{'}(x, y) & x, y \in [a, 1], \\ (I_{RS})_1(x, y) & \text{otherwise.} \end{cases}$$

As seen in Figure 2, $b \leq c$. Since $I(d, b) = (I_{RS})_1(d, b) = 1$ and $I(d, c) = I^{'}(d, c) = a$, I is not an increasing function in the second variable. Then, I is not an implication on L.

Theorem 3.6. Let $(L, \leq, 0, 1)$ be a bounded lattice. The function I given by (4) is an implication on L iff the following conditions hold:

(i) For all $x \in L$, x is comparable to a_i when it is comparable to b_i.

(ii) For all $x \in L$, x is comparable to b_i when it is comparable to a_i.

Proof.

$:\Rightarrow$ Let I be an implication on L. Suppose that there exists an element $x \in L$ such that it is comparable to a_i for some $i \in A$, but not to b_i. If $x < a_i$, it would be $x < b_i$ since $a_i < b_i$, a contradiction. Then, it must be $x \geq a_i$. Since I is an implication, it is increasing in the second variable. That is, for every $y \in L$, $I(y, a_i) \leq I(y, x)$. Especially, if we take $y = b_i$, it must be

$$I(b_i, a_i) \leq I(b_i, x).$$

Since $I(b_i, x) = (I_{RS})_2(b_i, x) = 0$, we have that $I(b_i, a_i) = 0$. Since $a_i \leq I^{[a_i,b_i]}(b_i, a_i) = I(b_i, a_i) = 0$, it must be $a_i = 0$, which contradicts that for all $i \in A$, $a_i > 0$. Thus, each element compared to a_i must be also comparable to b_i.

Now, assume that there exists an element $x \in L$ compared to b_i but not to a_i. If $x > b_i$, it would be $x > b_i > a_i$, which is a contradiction. Then, it must be $x \leq b_i$. Since I is decreasing in the first place, $I(b_i, a_i) \leq I(x, a_i)$. Thus, we have that

$$a_i \leq I^{[a_i,b_i]}(b_i, a_i) \leq I(x, a_i) = (I_{RS})_2(x, a_i) = 0,$$

which is $a_i = 0$. This is a contradiction to $a_i > 0$ for all i. Thus, each element compared to b_i must be also comparable to a_i.

$\Leftarrow$: Suppose that the conditions (i) and (ii) hold. Let us show that the function I given by (4) is an implication. It is clear that $I(0, 0) = (I_{RS})_2(0, 0) = 1$ and $I(1, 0) = (I_{RS})_2(1, 0) = 0$.

Let $b_i = 1$ for some $i \in A$. Then,

$$I(1, 1) = I^{[a_i,b_i]}(1, 1) = 1.$$

If $b_i \neq 1$ for all $i \in A$, then

$$I(1, 1) = (I_{RS})_2(1, 1) = 1.$$

Now, let us show that I is decreasing in the first place under the conditions (i) and (ii).
(I1) Let $x_1 \leq x_2$ for $x_1, x_2 \in L$.

1. If $y \notin [a_i, b_i]$ for all $i \in A$, then

$$I(x_1, y) = (I_{RS})_2(x_1, y) \geq (I_{RS})_2(x_2, y) = I(x_2, y).$$

2. Suppose that $y \in [a_i, b_i]$ for some $i \in A$.

2.1. Let $x_1 \in [a_i, b_i]$. Then, $a_i \leq x_1 \leq x_2$. Since x_2 is comparable to a_i, by (ii) it is also comparable to b_i. Thus, either $x_2 \leq b_i$ or $x_2 > b_i$. If $x_2 \leq b_i$, then it is clear that

$$I(x_1, y) = I^{[a_i, b_i]}(x_1, y) \geq I^{[a_i, b_i]}(x_2, y) = I(x_2, y).$$

Let $x_2 > b_i$. Since $x_2 > b_i \geq y$, we have that

$$I(x_2, y) = (I_{RS})_2(x_2, y) = 0 \leq I^{[a_i, b_i]}(x_1, y) = I(x_1, y).$$

2.2. Let $x_1 \notin [a_i, b_i]$. Suppose that $x_2 \in [a_i, b_i]$. Since $x_1 \leq x_2 \leq b_i$, x_1 is comparable to b_i. By (i), x_1 is comparable to a_i. Since $x_1 \notin [a_i, b_i]$, it must be $x_1 < a_i$. Thus, $x_1 < a_i < y$, whence it is obtained that

$$I(x_1, y) = (I_{RS})_2(x_1, y) = 1 \geq I^{[a_i, b_i]}(x_2, y) = I(x_2, y).$$

Let $x_2 \notin [a_i, b_i]$. Then, it is clear that

$$I(x_1, y) = (I_{RS})_2(x_1, y) \geq (I_{RS})_2(x_2, y) = I(x_2, y).$$

Thus, I is decreasing in the first variable.
(I2) Let $y_1 \leq y_2$ for $y_1, y_2 \in L$.

1. If $x \notin [a_i, b_i]$ for all $i \in A$, then

$$I(x, y_1) = (I_{RS})_2(x, y_1) \leq (I_{RS})_2(x, y_2) = I(x, y_2).$$

2. Suppose that $x \in [a_i, b_i]$ for some i.

2.1. Let $y_1 \in [a_i, b_i]$. Since $a_i \leq y_1 \leq y_2$, y_2 is comparable to a_i. By (ii), y_2 is comparable to b_i. Then, either $y_2 \leq b_i$ or $y_2 > b_i$. Suppose that $y_2 \leq b_i$. Then, we have that

$$I(x, y_1) = I^{[a_i, b_i]}(x, y_1) \leq I^{[a_i, b_i]}(x, y_2) = I(x, y_2).$$

Let $y_2 > b_i$. Since $y_2 > b_i \geq x$, it is obtained that

$$I(x, y_2) = (I_{RS})_2(x, y_2) = 1 \geq I^{[a_i, b_i]}(x, y_1) = I(x, y_1).$$

2.2. Let $y_1 \notin [a_i, b_i]$. Suppose that $y_2 \in [a_i, b_i]$. Since $y_1 \leq y_2 \leq b_i$ and y_1 is comparable to b_i, by (i) y_1 is comparable to a_i. Also, since $y_1 \notin [a_i, b_i]$, it must be $y_1 < a_i$. From $y_1 < a_i \leq x$, we have that

$$I(x, y_1) = (I_{RS})_2(x, y_1) = 0 \leq I^{[a_i, b_i]}(x, y_2) = I(x, y_2).$$

Let $y_2 \notin [a_i, b_i]$. Then, we have that

$$I(x, y_1) = (I_{RS})_2(x, y_1) \leq (I_{RS})_2(x, y_2) = I(x, y_2).$$

Thus, I is increasing in the second place. Hence, I is an implication on L. $\square$

Similar assertions are also true for the function I given by (3). Let us look at the next theorem.

Theorem 3.7. Let $(L, \leq, 0, 1)$ be a bounded lattice. If the following conditions

(i) For all $x \in L$, x is comparable to a_i when it is comparable to b_i.

(ii) For all $x \in L$, x is comparable to b_i when it is comparable to a_i.

hold, the function I given by (3) is an implication on L. Conversely, if I given by (3) is an implication with for all $i \in A$, $b_i \neq 1$, the conditions (i) and (ii) are satisfied.

Proof. If I satisfies the conditions (i) and (ii), then it can be shown that I is an implication in a similar way to the proof of Theorem 3.6.

Let I given by (3) be an implication with $b_i \neq 1$ for all $i \in A$. Suppose that there exists an element $x \in L$ such that it is comparable to a_i for some $i \in A$, but not to b_i. If $x < a_i$, it would be $x < b_i$ since $a_i < b_i$, a contradiction. Then, it must be $x \geq a_i$. Since I is decreasing in the first variable, for all $y \in L$, $I(x, y) \leq I(a_i, y)$. Especially, if we take $y = b_i$, we have that $1 = (I_{RS})_1(x, b_i) = I(x, b_i) \leq I(a_i, b_i) = I^{[a_i,b_i]}(a_i, b_i) = b_i$, whence $b_i = 1$, contradiction.

Assume that there exists an element $x \in L$ such that it is comparable to b_i for some $i \in A$, but not to a_i. Then, it must be $x \leq b_i$. Since I is increasing in the second place, $I(y, x) \leq I(y, b_i)$. Especially, if we take $y = a_i$, we would have $b_i = 1$, contradiction.

□

In Theorem 3.7, if we move the condition $b_i \neq 1$ for all i, we can find an example for implications in shape of (3) need not satisfy (i) and (ii).

Example 3.8. Consider the lattice $L = \{0, a, b, 1\}$ with $0 < a < 1$, $0 < b < 1$ and $a \| b$. Take the function defined by

$$I(x, y) = \begin{cases} I_1(x, y) & x, y \in [a, 1], \\ (I_{RS})_1(x, y) & \text{otherwise,} \end{cases}$$

where I_1 is given by Table 3:

I_1	a	1
a	1	1
1	a	1

Tab. 3. The function I_1.

Obviously, I is an implication but the condition (i) in Theorem 3.7 does not satisfy.

Proposition 3.9. Let $(L, \leq, 0, 1)$ be a bounded lattice, $a \in L \setminus \{0, 1\}$ and I' be an implication on the corresponding interval $[a, 1]$. If a is a co-atom, then the function $I : L^2 \to L$ defined by

$$I(x, y) = \begin{cases} I'(x, y) & x, y \in [a, 1], \\ (I_{RS})_1(x, y) & \text{otherwise} \end{cases}$$

is an implication on L.

If a is not a co-atom, the function I given in Proposition 3.9 need not be an implication. Let us look at the following example.

Example 3.10. Let L be a bounded lattice as in Figure 3:

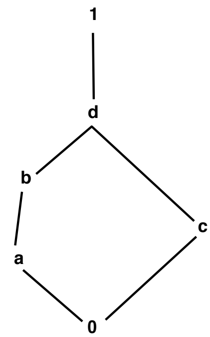

Fig. 3. $(L, \leq)$.

Consider the function $I_1 : [a, 1]^2 \to [a, 1]$ defined in Table 4

I_1	a	b	d	1
a	1	1	1	1
b	d	d	d	1
d	b	d	d	1
1	a	b	d	1

Tab. 4. The function I_1.

It is clear that I_1 is an implication on $[a, 1]$. Although $c \leq d$,

$$I(b, c) = (I_{RS})_1(b, c) = 1 \nleq d = I_1(b, d) = I(b, d),$$

whence I is not an implication on L.

4. SOME CONSTRUCTION METHODS FOR IMPLICATIONS ON BOUNDED LATTICES

By the methods given in the previous section, we see that an implication is generated by implications $I^{[a_i,b_i]}$ on the corresponding subintervals $[a_i, b_i]$ such that each element in L compared to a_i is also comparable to b_i, and vice versa.

In this section, we give some construction methods for implications whose summands are implications $I^{[a_i,b_i]}$ on the corresponding subintervals $[a_i, b_i]$ such that each comparable element to a_i need not to be comparable to b_i, and vice versa. The importance of

Proposition 4.3 and Theorem 4.5 comes from their applicability to any bounded lattice since any bounded lattice has the top element which is comparable to each element.

In this paper, denote by $\mathcal{L}$ the set of all comparable elements to each element of L. Then,

$$\mathcal{L} = \{x \in L| \quad x \leq y \quad \text{or} \quad y \leq x \quad \text{for every} \quad y \in L\}.$$

Proposition 4.1. Let $(L, \leq, 0, 1)$ be a bounded lattice, A be an index set. Let $([a_i, b_i])_{i\in A}$ be a family of pairwise disjoint subintervals of L with $0 < a_i < b_i$, for all $i \in A$ and $(I^{[a_i,b_i]})_{i\in A}$ a family of implications on the corresponding intervals $([a_i, b_i])_{i\in A}$. If $a_i \in \mathcal{L}$, then

$$I(x,y) = \begin{cases} I^{[a_i,b_i]}(x,y) & x, y \in [a_i, b_i], \\ (I_{RS})_1(x,y) & x \in [a_i, b_i] \quad \text{and} \quad y \notin [a_i, b_i], \\ (I_{RS})_2(x,y) & \text{otherwise}, \end{cases}$$

is an implication on L.

Proof.

- Since $a_i \neq 0$ for all $i \in A$, $I(0,0) = (I_{RS})_2(0,0) = 1$.
- Let $b_i = 1$ for some $i \in A$. Then,

$$I(1,0) = (I_{RS})_1(1,0) = 0.$$

If $b_i \neq 1$ for all $i \in A$, then we have that

$$I(1,0) = (I_{RS})_2(1,0) = 0.$$

- Suppose that $b_i = 1$ for some $i \in A$. Then,

$$I(1,1) = I^{[a_i,1]}(1,1) = 1.$$

Let $b_i \neq 1$ for all $i \in A$. Then,

$$I(1,1) = (I_{RS})_2(1,1) = 1.$$

Now, let us verify that I satisfies the conditions (I1) and (I2).

(I2) Let $y_1 \leq y_2$ for $y_1, y_2 \in L$.

1. Suppose that $x \notin [a_i, b_i]$ for all $i \in A$. Then,

$$I(x, y_1) = (I_{RS})_2(x, y_1) \leq (I_{RS})_2(x, y_2) = I(x, y_2).$$

2. Let $x \in [a_i, b_i]$ for some $i \in A$.

2.1. Let $y_2 \in [a_i, b_i]$.

2.1.1. If $y_1 \in [a_i, b_i]$, then

$$I(x, y_1) = I^{[a_i,b_i]}(x, y_1) \leq I^{[a_i,b_i]}(x, y_2) = I(x, y_2).$$

2.1.2. Let $y_1 \notin [a_i, b_i]$. Since a_i is comparable to each element of L, either $y_1 \geq a_i$ or $y_1 < a_i$. If $y_1 \geq a_i$, it would be $a_i \leq y_1 \leq y_2 \leq b_i$, which is a contradiction since $y_1 \notin [a_i, b_i]$. Then, it must be $y_1 < a_i$. It follows $y_1 < x$ from $a_i \leq x$. Thus, we obtain that

$$I(x, y_1) = (I_{RS})_1(x, y_1) = 0 \leq I^{[a_i,b_i]}(x, y_2) = I(x, y_2).$$

2.2. Let $y_2 \notin [a_i, b_i]$.

2.2.1. If $y_1 \notin [a_i, b_i]$, then

$$I(x, y_1) = (I_{RS})_1(x, y_1) \leq (I_{RS})_1(x, y_2) = I(x, y_2).$$

2.2.2. Let $y_1 \in [a_i, b_i]$. If $x > y_2$, it would be $y_2 \in [a_i, b_i]$ since $a_i \leq y_1 \leq y_2 < x \leq b_i$, a contradiction. Then, we have that

$$I(x, y_2) = (I_{RS})_1(x, y_2) = 1 \geq I^{[a_i,b_i]}(x, y_1) = I(x, y_1).$$

Thus, I is increasing in the second variable.

(I1) Let $x_1 \leq x_2$ for $x_1, x_2 \in L$.

1. Suppose that $y \in [a_i, b_i]$ for some $i \in A$.

1.1. Let $x_2 \in [a_i, b_i]$.

1.1.1. If $x_1 \in [a_i, b_i]$, then

$$I(x_1, y) = I^{[a_i,b_i]}(x_1, y) \geq I^{[a_i,b_i]}(x_2, y) = I(x_2, y).$$

1.1.2. Let $x_1 \notin [a_i, b_i]$. Since a_i is comparable to each element of L, $a_i \leq x_1$ or $x_1 < a_i$. If $a_i \leq x_1$, it would be $x_1 \in [a_i, b_i]$ from $x_1 \leq x_2 \leq b_i$, a contradiction. Then, $x_1 < a_i$. Since $x_1 < a_i \leq y$, we have that

$$I(x_1, y) = (I_{RS})_2(x_1, y) = 1 \geq I^{[a_i,b_i]}(x_2, y) = I(x_2, y).$$

1.2. Let $x_2 \notin [a_i, b_i]$.

1.2.1. If $x_1 \notin [a_i, b_i]$, then

$$I(x_1, y) = (I_{RS})_2(x_1, y) \geq (I_{RS})_2(x_2, y) = I(x_2, y).$$

1.2.2. Let $x_1 \in [a_i, b_i]$. If $x_2 \leq y$, it would be $x_2 \in [a_i, b_i]$ from $a_i \leq x_1 \leq x_2 \leq y \leq b_i$, a contradiction. Then, it is not possible the case $x_2 \leq y$. Thus,

$$I(x_2, y) = (I_{RS})_2(x_2, y) = 0 \leq I^{[a_i,b_i]}(x_1, y) = I(x_1, y)$$

is obtained.

2. Let $y \notin [a_i, b_i]$ for all $i \in A$.

2.1. Let $x_2 \in [a_j, b_j]$ for some $j \in A$.

2.1.1. If $x_1 \in [a_j, b_j]$, then it is clear that

$$I(x_1, y) = (I_{RS})_1(x_1, y) \geq (I_{RS})_1(x_2, y) = I(x_2, y).$$

2.1.2. Let $x_1 \notin [a_j, b_j]$. Since a_j is comparable to each element of L, either $x_1 \geq a_j$ or $x_1 < a_j$. If $x_1 \geq a_j$, it would be $x_1 \in [a_j, b_j]$ from $a_j \leq x_1 \leq x_2 \leq b_j$, a contradiction. Then, it must be $x_1 < a_j$. On the other hand, by the comparability of

a_j, either $a_j \leq y$ or $y < a_j$. If $a_j \leq y$, then $x_1 < y$ since $x_1 < a_j \leq y$. In this case, it is clear that

$$I(x_1, y) = (I_{RS})_2(x_1, y) = 1 \geq I(x_2, y).$$

Let $y < a_j$. Since $y < a_j \leq x_2$, we have that

$$I(x_2, y) = (I_{RS})_1(x_2, y) = 0 \leq I(x_1, y).$$

2.2 Let $x_2 \notin [a_j, b_j]$ for all $j \in A$.

2.2.1. Suppose that $x_1 \in [a_i, b_i]$ for some $i \in A$. Since $a_i \in \mathcal{L}$, either $a_i \leq y$ or $a_i > y$. Let $a_i \leq y$. If $y < x_1$, it would be $y \in [a_i, b_i]$ since $y < x_1 < b_i$, a contradiction. That is, it is not possible the case $y < x_1$. Then,

$$I(x_1, y) = (I_{RS})_1(x_1, y) = 1 \geq I(x_2, y)$$

holds. Let $y < a_i$. Since $y < a_i \leq x_1 \leq x_2$, we have that

$$I(x_1, y) = (I_{RS})_1(x_1, y) = 0 = (I_{RS})_2(x_2, y) = I(x_2, y).$$

2.2.2. Let $x_1 \notin [a_i, b_i]$ for all $i \in A$. Then,

$$I(x_1, y) = (I_{RS})_2(x_1, y) \geq (I_{RS})_2(x_2, y) = I(x_2, y).$$

Then, I is a decreasing function in the first place. Thus, I is an implication on L.

□

The converse of Proposition 4.1 need not be true. Let us investigate the following example.

Example 4.2. Let $(L, \leq, 0, 1)$ be a bounded lattice whose lattice diagram is displayed in Figure 4:

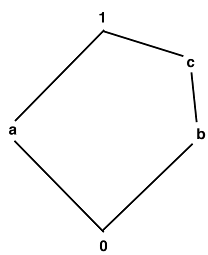

Fig. 4. $(L, \leq, 0, 1)$.

Consider the function $I_1 : [b, c]^2 \to [b, c]$ defined as Table 5:
Obviously, the function I_1 is an implication on $[b, c]$. Take the function $I : L^2 \to L$ defined as

$$I(x, y) = \begin{cases} I_1(x, y) & x, y \in [b, c], \\ (I_{RS})_1(x, y) & x \in [b, c] \quad \text{and} \quad y \notin [b, c], \\ (I_{RS})_2(x, y) & \text{otherwise}. \end{cases}$$

Even though, b is not comparable to each element of L, I is an implication on L.

I_1	b	c
b	c	c
c	b	c

Tab. 5. The function I_1.

Proposition 4.3. Let $(L, \leq, 0, 1)$ be a bounded lattice, A be an index set. Let $([a_i, b_i])_{i\in A}$ be a family of pairwise disjoint subintervals of L with $0 < a_i < b_i$, for all $i \in A$ and $(I^{[a_i,b_i]})_{i\in A}$ a family of implications on the corresponding intervals $([a_i, b_i])_{i\in A}$. If $b_i \in \mathcal{L}$, then

$$I(x,y) = \begin{cases} I^{[a_i,b_i]}(x,y) & x, y \in [a_i, b_i], \\ (I_{RS})_1(x,y) & x \notin [a_i, b_i] \quad \text{and} \quad y \in [a_i, b_i], \\ (I_{RS})_2(x,y) & \text{otherwise,} \end{cases}$$

is an implication on L.

Proof. The proof is similar to Proposition 4.1. □

Similarly, the converse of Proposition 4.3 may fail. Look at the following example.

Example 4.4. Take the same lattice $(L, \leq, 0, 1)$ and the same implication I_1 in Example 4.2. Even though c is not comparable to each element of L, it is clear that the function given as

$$I(x,y) = \begin{cases} I_1(x,y) & x, y \in [b, c], \\ (I_{RS})_1(x,y) & x \notin [b, c] \quad \text{and} \quad y \in [b, c], \\ (I_{RS})_2(x,y) & \text{otherwise,} \end{cases}$$

is an implication on L.

As a generalization of the methods given in Proposition 4.1 and Proposition 4.3, we give the following construction method. Note that, this method can be applied to any bounded lattices since they have the top elements 1 which is comparable to each elements of bounded lattices.

Theorem 4.5. Let $(L, \leq, 0, 1)$ be a bounded lattice, A be an index set. Let $([a_i, b_i])_{i\in A}$ be a family of pairwise disjoint subintervals of L with $0 < a_i < b_i$ such that $a_i \in \mathcal{L}$ or $b_i \in \mathcal{L}$. Let $(I^{[a_i,b_i]})_{i\in A}$ be a family of implications on the corresponding intervals $([a_i, b_i])_{i\in A}$. Then,

$$I(x,y) = \begin{cases} I^{[a_i,b_i]}(x,y) & x, y \in [a_i, b_i], \\ (I_{RS})_1(x,y) & (a_i \in \mathcal{L}, x \in [a_i, b_i] \quad \text{and} \quad y \notin [a_i, b_i]) \\ & \text{or} \quad (b_i \in \mathcal{L}, x \notin [a_i, b_i] \quad \text{and} \quad y \in [a_i, b_i]), \\ (I_{RS})_2(x,y) & \text{otherwise,} \end{cases} \tag{5}$$

is an implication on L.

P r o o f. Let us verify the axioms of implications.

(I2) Let $y_1 \leq y_2$ for $y_1, y_2 \in L$.

1. Suppose that $x \in [a_i, b_i]$ for some $i \in A$.

1.1. Let $y_2 \in [a_i, b_i]$.

1.1.1. If $y_1 \in [a_i, b_i]$, then it is clear that

$$I(x, y_1) \leq I(x, y_2).$$

1.1.2. Let $y_1 \notin [a_i, b_i]$. If $a_i \in \mathcal{L}$ or $b_i \in \mathcal{L}$, then it is clear that $I(x, y_1) \leq I(x, y_2)$ by (I2) with the case 2.1.2 in Proposition 4.1 and Proposition 4.3, respectively.

1.2. Let $y_2 \notin [a_i, b_i]$.

1.2.1. Suppose that $y_1 \in [a_i, b_i]$. If $a_i \in \mathcal{L}$ or $b_i \in \mathcal{L}$, then it is clear that $I(x, y_1) \leq I(x, y_2)$ by (I2) with the case 2.2.2 in Proposition 4.1 and Proposition 4.3, respectively.

1.2.2. Let $y_1 \notin [a_i, b_i]$. If $a_i \in \mathcal{L}$ or $b_i \in \mathcal{L}$, then it is clear that $I(x, y_1) \leq I(x, y_2)$ by (I2) with the case 2.2.1 in Proposition 4.1 and Proposition 4.3, respectively.

2. Let $x \notin [a_i, b_i]$ for all $i \in A$.

2.1. Let $y_2 \in [a_j, b_j]$ for some $j \in A$.

2.1.1. Suppose that $y_1 \in [a_j, b_j]$. If $a_j \in \mathcal{L}$ or $b_j \in \mathcal{L}$, then it is clear that $I(x, y_1) \leq I(x, y_2)$ by (I2) with the case 2 in Proposition 4.1 and Proposition 4.3, respectively.

2.1.2. Let $y_1 \in [a_k, b_k]$, for $k \in A$ with $k \neq j$.

• Let $a_j \in \mathcal{L}$. Then, $a_j < a_k$ or $a_k < a_j$. If $a_j < a_k$, since $a_j < a_k \leq y_1 \leq y_2 \leq b_j$, we have that $a_k \in [a_j, b_j] \cap [a_k, b_k] \neq \emptyset$, contradiction. Thus, it must be $a_k < a_j$, whence $a_k < a_j < b_j$.

•• Let $b_k \in \mathcal{L}$. Then, $b_k > a_j$ or $a_j > b_k$. If $b_k > a_j$, since $a_k < a_j < b_k$, $a_j \in [a_k, b_k] \cap [a_j, b_j] \neq \emptyset$, a contradiction. Then, it must be $b_k < a_j$, whence we have that $a_k \leq y_1 \leq b_k < a_j \leq y_2 \leq b_j$. Since $a_j, b_k \in \mathcal{L}$ and $b_k < a_j$, there are three possible cases for any $x \in L$: $a_j < x$ or $b_k < x < a_j$ or $x < b_k < a_j$.

Let $a_j < x$. Since $x > a_j > y_1$, we have that

$$I(x, y_1) = (I_{RS})_1(x, y_1) = 0 \leq (I_{RS})_2(x, y_2) = I(x, y_2).$$

Let $b_k < x < a_j$. Since $y_1 \leq b_k < x < a_j \leq y_2$, it is clear that

$$I(x, y_1) = (I_{RS})_1(x, y_1) = 0 \leq I(x, y_2).$$

Let $x < b_k$. Since $x < b_k < a_j < y_2$, we have that

$$I(x, y_1) \leq 1 = (I_{RS})_2(x, y_2) = I(x, y_2).$$

•• Let $a_k \in \mathcal{L}$. In this case, it is obvious the condition by (I2) with the case 2 in Proposition 4.1.

• Let $b_j \in \mathcal{L}$.

•• If $b_k \in \mathcal{L}$, then it is clear that $I(x, y_1) \leq I(x, y_2)$ by Proposition 4.3.

•• Let $a_k \in \mathcal{L}$. If $x > y_2$, then $I(x, y_1) = (I_{RS})_2(x, y_1) = 0 = (I_{RS})_1(x, y_2) = I(x, y_2)$. Otherwise, $I(x, y_2) = (I_{RS})_1(x, y_2) = 1 \geq (I_{RS})_2(x, y_1) = I(x, y_1)$ is satisfied.

2.1.3. Let $y_1 \notin [a_i, b_i]$ for all $i \in A$. If $a_j \in \mathcal{L}$, we obtain that $I(x, y_2) = (I_{RS})_2(x, y_2) \geq (I_{RS})_2(x, y_1) = I(x, y_1)$ by (I2) with the case 2 in Proposition 4.1. Let $b_j \in \mathcal{L}$. By Proposition 4.3, the condition is clear.

2.2. Let $y_2 \notin [a_j, b_j]$ for all $j \in A$.

2.2.1. Suppose that $y_1 \in [a_i, b_i]$ for some i. If $a_i \in \mathcal{L}$ or $b_i \in \mathcal{L}$, the condition holds by (I2) with the case 2 in Proposition 4.1 and Proposition 4.3, respectively.

2.2.2. Let $y_1 \notin [a_i, b_i]$ for all $i \in A$. Then, it is clear that

$$I(x, y_1) = (I_{RS})_2(x, y_1) \leq (I_{RS})_2(x, y_2) = I(x, y_2).$$

(I1) Let $x_1 \leq x_2$ for $x_1, x_2 \in L$.

1. Suppose that $y \in [a_i, b_i]$ for some $i \in A$.

1.1. Let $x_2 \in [a_i, b_i]$.

1.1.1. If $x_1 \in [a_i, b_i]$, then

$$I(x_2, y) = I^{[a_i,b_i]}(x_2, y) \leq I^{[a_i,b_i]}(x_1, y) = I(x_1, y).$$

1.1.2. Let $x_1 \notin [a_i, b_i]$. If $a_i \in \mathcal{L}$ or $b_i \in \mathcal{L}$, the condition holds by (I1) with the case 1.1.2 in Proposition 4.1 and Proposition 4.3, respectively.

1.2. Let $x_2 \notin [a_i, b_i]$.

1.2.1. Let $x_1 \in [a_i, b_i]$. If $a_i \in \mathcal{L}$ or $b_i \in \mathcal{L}$, the the condition holds by (I1) with the case 1.2.2 in Proposition 4.1 and Proposition 4.3, respectively.

1.2.2. Let $x_1 \notin [a_i, b_i]$. If $a_i \in \mathcal{L}$ or $b_i \in \mathcal{L}$, the condition holds by (I1) with the case 1.2.1 in Proposition 4.1 and Proposition 4.3, respectively.

2. Let $y \notin [a_i, b_i]$ for all $i \in A$.

2.1. Let $x_2 \in [a_j, b_j]$ for some $j \in A$.

2.1.1. Assume that $x_1 \in [a_j, b_j]$. If $a_j \in \mathcal{L}$ or $b_j \in \mathcal{L}$, the condition holds by (I1) with the case 2.1.1 in Proposition 4.1 and Proposition 4.3, respectively.

2.1.2. Suppose that $x_1 \in [a_k, b_k]$ for $k \in A$ with $k \neq j$.

• Let $a_j \in \mathcal{L}$.

•• If $a_k \in \mathcal{L}$, then it is clear that

$$I(x_2, y) = (I_{RS})_1(x_2, y) \leq (I_{RS})_1(x_1, y) = I(x_1, y).$$

•• Let $b_k \in \mathcal{L}$. Since $a_j \in \mathcal{L}$, either $a_j < a_k$ or $a_j > a_k$. If $a_j < a_k$, then we would have $a_k \in [a_j, b_j] \cap [a_k, b_k]$ since $a_j < a_k \leq x_1 \leq x_2 \leq b_j$, a contradiction. Then, it must be $a_k < a_j$, whence $a_k < a_j < b_j$. Since $b_k \in \mathcal{L}$, either $b_k > a_j$ or $b_k < a_j$. If $b_k > a_j$, it would be $a_j \in [a_k, b_k] \cap [a_j, b_j]$, contradiction. Then, it must be $b_k < a_j$, whence $a_k < b_k < a_j < b_j$. Thus, there exists the following relations between the elements x_1 and x_2:

$$a_k \leq x_1 \leq b_k < a_j \leq x_2 \leq b_j.$$

Since $a_j, b_k \in \mathcal{L}$ and $b_k < a_j$, there are three possible cases for any $y \in L$: $a_j < y$ or $b_k < y < a_j$ or $y < b_k$.
Let $y > a_j$. Since $y > a_j > x_1$, it is clear that

$$I(x_1, y) = (I_{RS})_2(x_1, y) = 1 \geq I(x_2, y).$$

If $b_k < y < a_j$, since $x_1 \leq b_k < y < a_j$, we have that

$$I(x_1, y) = (I_{RS})_2(x_1, y) = 1 \geq I(x_2, y).$$

Let $y < b_k$. By $y < b_k < a_j \leq x_2$, it is obtained that

$$I(x_2, y) = (I_{RS})_1(x_2, y) = 0 \leq I(x_1, y).$$

• Let $b_j \in \mathcal{L}$.

•• If $b_k \in \mathcal{L}$, then it is clear that

$$I(x_1, y) = (I_{RS})_2(x_1, y) \geq (I_{RS})_2(x_2, y) = I(x_2, y).$$

•• Let $a_k \in \mathcal{L}$. If $x_2 \leq y$, since $x_1 \leq x_2 \leq y$, we have that

$$I(x_1, y) = (I_{RS})_1(x_1, y) = 1 = (I_{RS})_2(x_2, y).$$

If $x_2 > y$ or $x_2 \| y$, then

$$I(x_2, y) = (I_{RS})_2(x_2, y) = 0 \leq I(x_1, y).$$

2.1.3. Suppose that $x_1 \notin [a_i, b_i]$ for all $i \in A$. If $a_j \in \mathcal{L}$ or $b_j \in \mathcal{L}$, the condition holds by (I1) in Proposition 4.1 and Proposition 4.3, respectively.

2.2. Let $x_2 \notin [a_j, b_j]$ for all $j \in A$.

2.2.1. Let $x_1 \in [a_i, b_i]$ for some $i \in A$. If $a_i \in \mathcal{L}$ or $b_i \in \mathcal{L}$, it is obvious the condition holds by (I1) in Proposition 4.1 and Proposition 4.3, respectively.

2.2.2. Let $x_1 \notin [a_i, b_i]$ for all $i \in A$. Then, it is clear that

$$I(x_1, y) = (I_{RS})_2(x_1, y) \geq (I_{RS})_2(x_2, y) = I(x_2, y).$$

Thus, I is decreasing in the first place.

(I3) Since $0 < a_i < b_i$ for all $i \in A$, $0 \notin [a_i, b_i]$. Then, it is clear that

$$I(0, 0) = (I_{RS})_2(0, 0) = 1.$$

(I4) If $b_i = 1$ for some $i \in A$, then it is clear that $I(1, 1) = I^{[a_i,1]}(1, 1) = 1$. If $b_i \neq 1$ for all i, $I(1, 1) = (I_{RS})_2(1, 1) = 1$ holds.

(I5) Since $0 \notin [a_i, b_i]$ for all i, $(1, 0) \notin [a_i, b_i]^2$.

• Suppose that $b_i = 1$ for some i. If $a_i \in \mathcal{L}$, then $I(1, 0) = (I_{RS})_1(1, 0) = 0$ and if $b_i \in \mathcal{L}$, it is clear that $I(1, 0) = (I_{RS})_2(1, 0) = 0$.

• Let $b_i \neq 1$ for all $i \in A$. Then,

$$I(1, 0) = (I_{RS})_2(1, 0) = 0.$$

Thus, I is an implication L. □

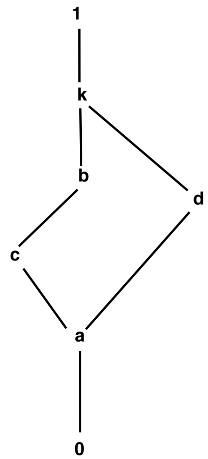

Fig. 5. $(L, \leq)$.

Example 4.6. Consider the lattice $(L, \leq, 0, 1)$ whose lattice diagram is displayed in Figure 5. Let $I_1 : [a, b]^2 \to [a, b]$ and $I_2 : [d, k]^2 \to [d, k]$ be two implications. Then, by Theorem 4.5,

$$I(x,y) = \begin{cases} I_1(x,y) & (x,y) \in [a,b]^2, \\ I_2(x,y) & (x,y) \in [d,k]^2, \\ (I_{RS})_1(x,y) & (x \in [a,b] \quad \text{and} \quad y \notin [a,b]) \\ & \text{or} \quad (x \notin [d,k] \quad \text{and} \quad y \in [d,k]), \\ (I_{RS})_2(x,y) & \text{otherwise}, \end{cases}$$

is an implication on L.

Remark 4.7. Note that, by the construction methods given in Proposition 4.3 and Theorem 4.5, we can generate an implication on any bounded lattice since any bounded lattice has the top element which is comparable to each element. Let us look at the following illustrating example.

Example 4.8. Let $(L, \leq, 0, 1)$ be a bounded lattice. For any implication $I^{'}$ on $[u, 1]$ with $u > 0$, the following function defined by

$$I(x,y) = \begin{cases} I^{'}(x,y) & x, y \in [u,1], \\ (I_{RS})_1(x,y) & (u \in \mathcal{L}, x \in [u,1] \quad \text{and} \quad y \notin [u,1]) \\ & \text{or} \quad (x \notin [u,1] \quad \text{and} \quad y \in [u,1]), \\ (I_{RS})_2(x,y) & \text{otherwise}, \end{cases}$$

is an implication on L.

Proposition 4.9. Let $(I^{[a_i,b_i]})_{i\in A}$ be a family of implications on $([a_i, b_i])_{i\in A}$ which is a family of pairwise disjoint subintervals of a bounded lattice L with $0 < a_i < b_i$ and I be the implication given by (5) in Theorem 4.5.

(i) $N_I = N_{D_1}$.

(ii) If $b_i < 1$ for all $i \in A$, then I does not satisfy (NP).

(iii) If there exists $i \in A$ such that $b_i = 1$, then I satisfies (NP) if and only if $I^{[a,1]}$ with $a > 0$ satisfies (NP) and it must be $x = 0$ when $x \notin [a, 1]$.

Proof.

(i) The proof is straighforward.

(ii) Let $b_i < 1$ for all $i \in A$. For any $0 \neq y \in L$, since

$$I(1,y) = \begin{cases} (I_{RS})_1(1,y) & b_i \in \mathcal{L}, y \in [a_i, b_i] \\ (I_{RS})_2(1,y) & \text{otherwise,} \end{cases} = 0 \neq y,$$

I does not satisfy (NP).

(iii) Let $b_i = 1$ for some $i \in A$. Assume that I satisfies (NP). Then, $I(1, y) = y$, for all $y \in L$. Especially for any $y \in [a, 1]$ with $a > 0$,

$$y = I(1, y) = I^{[a,1]}(1, y)$$

holds. Thus, $I^{[a,1]}$ satisfies (NP).

Suppose that there exists an element $x \in L$ such that $x \notin [a, 1]$ and $x \neq 0$. Since I satisfies (NP), we have that

$$x = I(1,x) = \begin{cases} (I_{RS})_1(1,x) & a_i \in \mathcal{L} \\ (I_{RS})_2(1,x) & \text{otherwise,} \end{cases} = 0,$$

contradiction. Thus, if I satisfies (NP), then it must be $x = 0$ for all $x \notin [a, 1]$.

Conversely, for any element $y \in L$, let us show that $I(1, y) = y$. If $y \in [a, 1]$, since $I^{[a,1]}$ satisfies (NP), we have that

$$I(1, y) = I^{[a,1]}(1, y) = y.$$

Let $y \notin [a, 1]$. Then, it must be $y = 0$. Thus, it is clear that

$$I(1, y) = I(1, 0) = 0 = y$$

holds. This completes the proof. □

Proposition 4.10. Let $(I^{[a_i,b_i]})_{i\in A}$ be a family of implications on $([a_i, b_i])_{i\in A}$ which is a family of pairwise disjoint subintervals of a bounded lattice L with $0 < a_i < b_i$ and I be the implication given by (5) in Theorem 4.5. *I satisfies (IP) if and only if the family $([a_i, b_i])_{i\in A}$ has only one member with $b_i = 1$ and the corresponding implication $I^{[a,1]}$ satisfies (IP).*

Proof. Let I satisfy (IP). Suppose that the family $([a_i, b_i])_{i\in A}$ has at least two members. Then, there exists $j \in A$ such that $i \neq j$ such that

$$[a_i, b_i] \cap [a_j, b_j] = \emptyset.$$

If there exists $i \in A$ such that $b_i \neq 1$, since I satisfies (IP), for any $x \in [a_i, b_i]$, we have that

$$1 = I(x,x) = I^{[a_i,b_i]}(x,x) \leq b_i < 1,$$

contradiction. Thus, for all $i \in A$, it must be $b_i = 1$. We have that

$$1 \in [a_i, 1] \cap [a_j, 1] = [a_i, b_i] \cap [a_i, b_i] = \emptyset,$$

which is a contradiction. Then, the family $([a_i, b_i])_{i\in A}$ has only one interval $[a, 1]$ with $a > 0$. Thus, the implication I is in the form of

$$I(x,y) = \begin{cases} I^{[a,1]}(x,y) & x, y \in [a,1], \\ (I_{RS})_1(x,y) & (a \in \mathcal{L}, x \in [a,1] \quad \text{and} \quad y \notin [a,1]) \\ & \text{or} \quad (x \notin [a,1] \quad \text{and} \quad y \in [a,1]), \\ (I_{RS})_2(x,y) & \text{otherwise.} \end{cases}$$

Since I satisfies (IP), for any $x \in [a, 1]$, we have that

$$1 = I(x,x) = I^{[a,1]}(x,x),$$

showing that $I^{[a,1]}$ satisfies (IP).

Conversely, suppose that the family $([a_i, b_i])_{i\in A}$ has only one member with $b_i = 1$ and the corresponding implication $I^{[a,1]}$, $a > 0$ satisfies (IP). If $x \in [a, 1]$ for $a > 0$, then $I(x,x) = I^{[a,1]}(x,x) = 1$ since $I^{[a,1]}$, $a > 0$ satisfies (IP). Let $x \notin [a, 1]$. Then, $I(x,x) = (I_{RS})_2(x,x) = 1$. Thus, I satisfies (IP). □

Proposition 4.11. Let $(I^{[a_i,b_i]})_{i\in A}$ be a family of implications on $([a_i, b_i])_{i\in A}$ which is a family of pairwise disjoint subintervals of a bounded lattice L with $0 < a_i < b_i$ and I be the implication given by (5) in Theorem 4.5. I satisfies (OP) if and only if

$$I(x,y) = \begin{cases} I^{[a,1]}(x,y) & x, y \in [a,1], \\ (I_{RS})_1(x,y) & (a \in \mathcal{L}, x \in [a,1] \quad \text{and} \quad y \notin [a,1]) \\ & \text{or} \quad (x \notin [a,1] \quad \text{and} \quad y \in [a,1]), \\ (I_{RS})_2(x,y) & \text{otherwise.} \end{cases}$$

and the implication $I^{[a,1]}$ on the corresponding interval $[a, 1]$ with $a > 0$ satisfies (OP).

Proof. Let I satisfy (OP). If there exists $i \in A$ such that $b_i \neq 1$, since I satisfies (OP), for the elements $a_i < b_i$, we have that

$$1 = I(a_i, b_i) = I^{[a_i,b_i]}(a_i, b_i) = b_i < 1,$$

contradiction. Thus, for all $i \in A$, it must be $b_i = 1$. If there exists a subinterval different from $[a, 1]$ with $a > 0$, we would have $\emptyset = [a_i, b_i] \cap [a_i, b_i] = [a_i, 1] \cap [a_j, 1] \ni 1$, contradiction. Thus, the implication I must be in the form of

$$I(x,y) = \begin{cases} I^{[a,1]}(x,y) & x, y \in [a,1], \\ (I_{RS})_1(x,y) & (a \in \mathcal{L}, x \in [a,1] \quad \text{and} \quad y \notin [a,1]) \\ & \text{or} \quad (x \notin [a,1] \quad \text{and} \quad y \in [a,1]), \\ (I_{RS})_2(x,y) & \text{otherwise.} \end{cases}$$

Since I satisfies (OP), for any $x, y \in [a,1]$, we have that

$$x \le y \Leftrightarrow 1 = I(x,y) = I^{[a,1]}(x,y).$$

Thus, $I^{[a,1]}$ satisfies (OP).

Conversely, let the implication I be in the form of

$$I(x,y) = \begin{cases} I^{[a,1]}(x,y) & x, y \in [a,1], \\ (I_{RS})_1(x,y) & (a \in \mathcal{L}, x \in [a,1] \quad \text{and} \quad y \notin [a,1]) \\ & \text{or} \quad (x \notin [a,1] \quad \text{and} \quad y \in [a,1]), \\ (I_{RS})_2(x,y) & \text{otherwise,} \end{cases}$$

and let the implication $I^{[a,1]}$ on the corresponding interval $[a,1]$ with $a > 0$ satisfy (OP). For any $x, y \in [a,1]$,

$$x \le y \Leftrightarrow 1 = I^{[a,1]}(x,y) = I(x,y).$$

Let $x \notin [a,1]$ or $y \notin [a,1]$. Then,

$$x \le y \Leftrightarrow 1 = \begin{cases} (I_{RS})_1(x,y) & (a \in \mathcal{L}, x \in [a,1] \quad \text{and} \quad y \notin [a,1]) \\ & \text{or} \quad (x \notin [a,1] \quad \text{and} \quad y \in [a,1]), \\ (I_{RS})_2(x,y) & \text{otherwise,} \end{cases} = I(x,y).$$

Thus, I satisfies (OP).

□

In general, the implications on bounded lattices generated by the method given in (5) need not to satisfy the exchange principle (EP). Let us look at the following illustrating example.

Example 4.12. Consider the lattice L in Figure 5 and take the implication I in Example 4.6. Since $I(1, I(c,d)) = I(1, (I_{RS})_1(c,d)) = I(1,1) = 1$ and $I(c, I(1,d)) = I(c, (I_{RS})_1(1,d)) = I(c,0) = (I_{RS})_1(c,0) = 0$, the implication I doesn't satisfy (EP).

Lemma 4.13. Let $(L, \le, 0, 1)$ be a bounded lattice and $L \setminus \{0\} = [a,1]$ for $a > 0$. Then, the implication defined by

$$I(x,y) = \begin{cases} I^{[a,1]}(x,y) & x, y \in [a,1], \\ (I_{RS})_1(x,y) & (a \in \mathcal{L}, x \in [a,1] \quad \text{and} \quad y \notin [a,1]) \\ & \text{or} \quad (x \notin [a,1] \quad \text{and} \quad y \in [a,1]), \\ (I_{RS})_2(x,y) & \text{otherwise,} \end{cases} \tag{6}$$

satisfies (EP) iff $I^{[a,1]}$ satisfies (EP).

Proof. Let I defined by (6) satisfy (EP). Then, for any elements $x, y, z \in L$

$$I(x, I(y,z)) = I(y, I(x,z)).$$

Especially, for $x, y, z \in [a,1]$, since

$$\begin{aligned} I^{[a,1]}(x, I^{[a,1]}(y,z)) &= I(x, I(y,z)) = I(y, I(x,z)) \\ &= I^{[a,1]}(y, I^{[a,1]}(x,z)), \end{aligned}$$

$I^{[a,1]}$ satisfies (EP).

Conversely, let $I^{[a,1]}$ satisfy (EP).

1. Suppose that $x \in [a,1]$ for $a > 0$.

1.1. Let $y \in [a,1]$. If $z \in [a,1]$, the proof is clear. Let $z \notin [a,1]$. Since $L \setminus \{0\} = [a,1]$, $z = 0$. Then,

$$\begin{aligned} I(x, I(y,z)) &= I(x, I(y,0)) = I(x,0) \\ &= 0 = I(y,0) = I(y, I(x,0)) \\ &= I(y, I(x,z)). \end{aligned}$$

1.2. Let $y \notin [a,1]$. Then, $y = 0$. Thus,

$$\begin{aligned} I(x, I(y,z)) &= I(x, I(0,z)) = I(x,1) \\ &= 1 = I(0, I(x,z)) \\ &= I(y, I(x,z)). \end{aligned}$$

2. Let $x \notin [a,1]$. Then, it must be $x = 0$. Thus,

$$\begin{aligned} I(x, I(y,z)) &= I(0, I(y,z)) = 1 \\ &= I(y,1) = I(y, I(0,z)) \\ &= I(y, I(x,z)). \end{aligned}$$

Hence, I satisfies (EP).

□

Theorem 4.14. Let $(L, \leq, 0, 1)$ be a bounded lattice and $L \setminus \{0\} = \oplus_{i\in A}[a_i, b_i]$ with $a_i < b_i$. The implication I given by (5) in Theorem 4.5 satisfies (EP) iff $1 \leq |A| \leq 2$ and for all $i \in A$, $I^{[a_i,b_i]}$ satisfies (EP).

Proof. $\Leftarrow$: If $x = 0$ (similarly, $y = 0$ or $z = 0$), it is clear that I satisfies (EP). Let $x \neq 0$, $y \neq 0$ and $z \neq 0$. Since $1 \in L \setminus \{0\} = \oplus_{i\in A}[a_i, b_i]$, there exists $i \in A$ such that $b_i = 1$. If $|A| = 1$, then $L \setminus \{0\} = [a,1]$. Thus, by Lemma 4.13, we have that I satisfies (EP). Let $|A| = 2$. Then, $L \setminus \{0\} = \oplus_{i=1}^{2}[a_i, b_i] = [a_1, b_1] \oplus [a_2, 1]$.

If $a_1 \in \mathcal{L}$, then either $a_1 \geq a_2$ or $a_1 < a_2$. If $a_1 \geq a_2$, it would be a contradiction, since $a_1 \in [a_2, 1]$. Then, it must be

$$a_1 < a_2. \tag{7}$$

If $b_1 \in \mathcal{L}$, either $b_1 \geq a_2$ or $b_1 < a_2$. Let $b_1 \geq a_2$. Then, we have that $1 \geq b_1 \geq a_2$, which is a contradiction since $[a_2, 1] \cap [a_1, b_1] = \emptyset$. Thus, it must be $b_1 < a_2$. In this case, we have that

$$a_1 < b_1 < a_2 < 1. \tag{8}$$

1. Let $x \in [a_1, b_1]$.
   1.1. Let $y \in [a_1, b_1]$.
      1.1.1. If $z \in [a_1, b_1]$, then I satisfies (EP) since $I^{[a_1,b_1]}$ satisfies (EP).
      1.1.2. Let $z \notin [a_1, b_1]$. Then, it must be $z \in [a_2, 1]$. Let $a_1 \in \mathcal{L}$. If $y > z$, we would have $y \in [a_2, 1]$ since $a_2 \leq z < y < 1$, which is a contradiction. Similarly, if $x > z$, we would have $x \in [a_2, 1]$ since $a_2 \leq z < x < 1$, which is a contradiction. Thus,

$$I(x, I(y, z)) = 1 = I(y, I(x, z)).$$

Let $b_1 \in \mathcal{L}$. Then, it is clear that $a_1 < b_1 < a_2 < 1$ by (8). Thus, $a_1 \leq y \leq b_1 < a_2 \leq z \leq 1$. In this case,

$$\begin{aligned} I(x, I(y, z)) &= I(x, (I_{RS})_2(y, z)) \\ &= 1 = I(y, (I_{RS})_2(x, z)) \\ &= I(y, I(x, z)). \end{aligned}$$

   1.2. Let $y \notin [a_1, b_1]$. Then, $y \in [a_2, 1]$.
      1.2.1. Let $z \in [a_1, b_1]$. If $a_1 \in \mathcal{L}$, then $a_1 < a_2$ by (7). If $y \leq z$, we would have $z \in [a_2, 1]$ since $a_2 \leq y \leq z \leq 1$, whence we have a contradiction. Then, the case $y \leq z$ is not possible. If $y \leq I^{[a_1,b_1]}(x, z)$, we would have $a_2 \in [a_1, b_1]$ since $a_1 < a_2 \leq y \leq I^{[a_1,b_1]}(x, z) \leq b_1$, contradiction. Thus, $y \leq I^{[a_1,b_1]}(x, z)$ is not possible. Thus,

$$\begin{aligned} I(x, I(y, z)) &= I(x, (I_{RS})_2(y, z)) = 0 \\ &= (I_{RS})_2(y, I^{[a_1,b_1]}(x, z)) \\ &= I(y, I^{[a_1,b_1]}(x, z)) \\ &= I(y, I(x, z)). \end{aligned}$$

Let $b_1 \in \mathcal{L}$. Then, $a_1 < b_1 < a_2 < 1$ by (8). Since $a_1 \leq z \leq b_1 < a_2 \leq y \leq 1$, $z < y$. Also, since $a_1 \leq I^{[a_1,b_1]}(x, z) \leq b_1 < a_2 \leq y \leq 1$, $I^{[a_1,b_1]}(x, z) < y$. Thus,

$$\begin{aligned} I(x, I(y, z)) &= I(x, (I_{RS})_1(y, z)) = 0 \\ &= (I_{RS})_1(y, I^{[a_1,b_1]}(x, z)) \\ &= I(y, I^{[a_1,b_1]}(x, z)) \\ &= I(y, I(x, z)). \end{aligned}$$

1.2.2. Let $z \notin [a_1, b_1]$. Then, it must be $z \in [a_2, 1]$. Let $a_1 \in \mathcal{L}$. If $x > I^{[a_2,1]}(y,z)$, we would have $1 > x > I^{[a_2,1]}(y,z) \geq a_2$, contradiction. If $x > z$, we would have $a_2 \leq z < x \leq 1$, contradiction. Thus,

$$\begin{aligned} I(x, I(y,z)) &= I(x, I^{[a_2,1]}(y,z)) \\ &= (I_{RS})_1(x, I^{[a_2,1]}(y,z)) \\ &= 1 = I(y, (I_{RS})_1(x,z)) = I(y, I(x,z)). \end{aligned}$$

Let $b_1 \in \mathcal{L}$. Then, $a_1 < b_1 < a_2 < 1$ by (8). Since $a_1 \leq x \leq b_1 < a_2 \leq I^{[a_2,1]}(y,z) \leq 1$, we have that

$$\begin{aligned} I(x, I(y,z)) &= I(x, I^{[a_2,1]}(y,z)) \\ &= (I_{RS})_2(x, I^{[a_2,1]}(y,z)) \\ &= 1 = I(y,1) = I(y, (I_{RS})_2(x,z)) = I(y, I(x,z)). \end{aligned}$$

2. Let $x \notin [a_1, b_1]$. Then, $x \in [a_2, 1]$.

2.1. $y \in [a_1, b_1]$.

2.1.1. $z \in [a_1, b_1]$. The proof is clear by the case 1.2.1.

2.1.2. $z \notin [a_1, b_1]$. Then, it must be $z \in [a_2, 1]$. The proof is clear by the case 1.2.2.

2.2. Let $y \notin [a_1, b_1]$. Then, $y \in [a_2, 1]$.

2.2.1 Suppose that $z \in [a_1, b_1]$. Let $a_1 \in \mathcal{L}$. If $y \leq z$, we would have $z \in [a_2, 1]$ since $a_2 \leq y \leq z \leq 1$, contradiction. If $x \leq z$, we would have $z \in [a_2, 1]$, since $a_2 \leq x \leq z < 1$, contradiction. Thus,

$$\begin{aligned} I(x, I(y,z)) &= I(x, (I_{RS})_2(y,z)) \\ &= 0 = I(y, (I_{RS})_2(x,z)) \\ &= I(y, I(x,z)). \end{aligned}$$

Let $b_1 \in \mathcal{L}$. Then, $a_1 < b_1 < a_2 < 1$ by (8). Since $a_1 \leq z \leq b_1 < a_2 \leq x(y) < 1$, we have that

$$\begin{aligned} I(x, I(y,z)) &= I(x, (I_{RS})_1(y,z)) \\ &= 0 = I(y,0) = I(y, (I_{RS})_1(x,z)) \\ &= I(y, I(x,z)). \end{aligned}$$

2.2.2. Let $z \notin [a_1, b_1]$. Then, it must be $z \in [a_2, 1]$. Since $I^{[a_2,1]}$ satisfies (EP), it is clear that

$$\begin{aligned} I(x, I(y,z)) &= I^{[a_2,1]}(x, I^{[a_2,1]}(y,z)) \\ &= I^{[a_2,1]}(y, I^{[a_2,1]}(x,z)) \\ &= I(y, I(x,z)). \end{aligned}$$

$\Rightarrow$: Suppose that I satisfies (EP) and $|A| \geq 3$. Since $1 \in L \setminus \{0\} = \oplus_{i\in A}[a_i, b_i]$, it is clear that there exists $k \in A$ such that $1 \in [a_k, b_k]$. Then, $1 \leq b_k$, whence it must be $b_k = 1$. Thus, $L \setminus \{0\} = \oplus_{k\neq i}[a_i, b_i] \oplus [a_k, 1]$. Since $|A| \geq 3$, there exist at least two elements $i, j \in A$ such that $i, j \neq k$ and $i \neq j$.

1. Let $a_i \in \mathcal{L}$.

1.1. $a_j \in \mathcal{L}$. Then, either $a_i < a_j$ or $a_j < a_i$. Let $a_i < a_j$. If we take $x = 1$, $z = a_j$ and $y = a_i$, we have that

$$I(y, I(x, z)) = 0 \neq 1 = I(x, I(y, z)),$$

contradiction. Let $a_j < a_i$. For $x = 1$, $y = a_j$ and $z = a_i$, we have that

$$I(y, I(x, z)) = 0 \neq 1 = I(x, I(y, z)),$$

contradiction again.

1.2. Let $b_j \in \mathcal{L}$. Then, either $a_i < b_j$ or $b_j < a_i$. Let $a_i < b_j$. If we take $x = 1$, $z = b_j$ and $y = a_i$, we have that

$$I(y, I(x, z)) = 0 \neq 1 = I(x, I(y, z)),$$

contradiction. Let $b_j < a_i$. For $x = 1$, $z = a_i$ and $y = b_j$, we have that

$$I(y, I(x, z)) = 0 \neq 1 = I(x, I(y, z)),$$

contradiction.

2. $b_i \in \mathcal{L}$.

2.1. Let $a_j \in \mathcal{L}$. Then, either $a_j < b_i$ or $b_i < a_j$. Let $a_j < b_i$. For $x = 1$, $z = b_i$ and $y = a_j$, it is clear that

$$I(y, I(x, z)) = 0 \neq 1 = I(x, I(y, z)),$$

contradiction. Let $b_i < a_j$. For $x = 1$, $z = a_j$ and $y = b_i$, we have that

$$I(y, I(x, z)) = 0 \neq 1 = I(x, I(y, z)),$$

contradiction.

2.2. Let $b_j \in \mathcal{L}$. Then, either $b_i < b_j$ or $b_j < b_i$. Let $b_i < b_j$. Then, if we consider $x = 1$, $z = b_j$ and $y = b_i$, we have that

$$I(y, I(x, z)) = 0 \neq 1 = I(x, I(y, z)),$$

contradiction. Let $b_j < b_i$. Then, for $x = 1$, $z = b_i$ and $y = b_j$, we have that

$$I(y, I(x, z)) = 0 \neq 1 = I(x, I(y, z)),$$

contradiction. Thus, if $|A| \geq 3$, I does not satisfy (EP). Hence, it must be $|A| \leq 2$. On the other hand, since $1 \in L \setminus \{0\} = \oplus_{i\in A}[a_i, b_i]$, there exists $k \in A$ such that $b_k = 1$. Thus, $|A| \geq 1$. If I satisfies (EP), it is clear that for all $i \in A$, $I^{[a_i,b_i]}$ satisfies (EP). $\square$

5. CONCLUDING REMARKS

Yong Su et al.[26] introduced the ordinal sum of fuzzy implications on the unit interval $[0,1]$ in a similar way to the concept to the ordinal sum of t-norms and they presented the necessary and sufficient condition for the ordinal sum being a fuzzy implication on $[0,1]$. This means that the ordinal sum of every fuzzy implications need not be a fuzzy implication without some special conditions. In this sense, in [5, 6], Drygaś and Król introduced some construction methods generating again a fuzzy implication by means of the ordinal sum of fuzzy implications having no additional conditions. In this paper, we introduced the ordinal sum of implications on bounded lattices based on [5, 25]. We showed that it need not an implication on a bounded lattice and presented some necessary and sufficient conditions for the ordinal sum of implications on bounded lattices being again an implication. Also, we gave some construction methods for implications on bounded lattices built from the implications defined on the subintervals of bounded lattices. We investigated their basic properties.

ACKNOWLEDGEMENT

The author is indebted to anonymous referees for their comments that have helped improving the paper. The author was supported by grants from Recep Tayyip Erdogan University, project number FBA-2018-919.

(Received February 14, 2019)

REFERENCES

[1] M. Baczyński, P. Drygaś, A. Król, and R. Mesiar: New types of ordinal sum of fuzzy implications. In: Fuzzy systems (FUZZ-IEEE), 2017 IEEE International Conference, 2017. DOI:10.1109/fuzz-ieee.2017.8015700

[2] M. Baczyński and B. Jayaram: Fuzzy Implications. Studies in Fuzziness and Soft Computing *231*, Springer, Berlin, Heidelberg, 2008.

[3] G. Birkhoff: Lattice Theory. Third edition. Providence, 1967. DOI:10.1090/coll/025

[4] G. D. Çaylı: On a new class of t-norms and t-conorms on bounded lattices. Fuzzy Sets and Systems *132* (2018), 129–143. DOI:10.1016/j.fss.2017.07.015

[5] P. Drygaś and A. Król: Various kinds of fuzzy implications. In: Novel Developments in Uncertainty Represent, and Processing (K. T. Atanassov et al., eds.), Advences in Intelligent Systems and Computing *401*, Springer Internat. Publ. AG, 2016, pp. 37–49. DOI:10.1007/978-3-319-26211-6_4

[6] P. Drygaś and A. Król: Generating fuzzy implications by ordinal sums. Tatra Mt. Math. Publ. *66* (2016), 39–50. DOI:10.1515/tmmp-2016-0018

[7] D. Dubois and H. Prade: A review of fuzzy set aggregation connectives. Inform. Sci. *36* (1985), 85–121. DOI:10.1016/0020-0255(85)90027-1

[8] D. Dubois and H. Prade: Fuzzy sets in approximate reasoning, Part 1: inference with possibility distributions. Fuzzy Sets and Systems *40* (1991), 143–202. DOI:10.1016/0165-0114(91)90050-z

[9] Ü. Ertuğrul, M. N. Kesicioğlu, and F. Karaçal: Ordering based on uninorms. Inform. Sci. *330* (2016), 315–327. DOI:10.1016/j.ins.2015.10.019

[10] Ü. Ertuğrul, F. Karaçal, and R. Mesiar: Modified ordinal sums of triangular norms and triangular conorms on bounded lattices. Int. J. Intell. Systems *30* (2015), 807–817. DOI:10.1002/int.21713

[11] J. Fodor and I. J. Rudas: Migrative t-norms with respect to continuous ordinal sums. Inform. Sci. *181* (2011), 4860–4866. DOI:10.1016/j.ins.2011.05.014

[12] M. Grabisch, J.-L. Marichal, R. Mesiar, and E. Pap: Aggregation Functions. Cambridge University Press, 2009. DOI:10.1109/sisy.2008.4664901

[13] M. N. Kesicioğlu and R. Mesiar: Ordering based on implications. Inform. Sci. *276* (2014), 377–386. DOI:10.1016/j.ins.2013.12.047

[14] E. P. Klement and R. Mesiar (eds.): Logical, Algebraic, Analytic and Probabilistic Aspects of Triangular Norms. Elsevier, Amsterdam 2005.

[15] E. P. Klement, R. Mesiar, and E. Pap: Triangular Norms. Kluwer Academic Publishers, Dordrecht 2000.

[16] E. P. Klement, R. Mesiar, and E. Pap: Triangular norms as ordinal sums of semigroups in the sense of A. H. Clifford. Semigroup Forum *65* (2002), 71–82. DOI:10.1007/s002330010127

[17] Z. Ma and W. M. Wu: Logical operators on complete lattices. Inform. Sci. *55* (1991), 77–97. DOI:10.1016/0020-0255(91)90007-h

[18] M. Mas, M. Monserrat, and J. Torrens: The law of importation for discrete implications. Inform. Sci. *179* (2009), 4208–4218. DOI:10.1016/j.ins.2009.08.028

[19] M. Mas, M. Monserrat, J. Torrens, and E. Trillas: A survey on fuzzy implication functions. IEEE Trans. Fuzzy Syst. *15* (2007), 1107–1121. DOI:10.1109/tfuzz.2007.896304

[20] J. Medina: Characterizing when an ordinal sum of t-norms is a t-norm on bounded lattices. Fuzzy Sets and Systems *202* (2012), 75–88. DOI:10.1016/j.fss.2012.03.002

[21] R. Mesiar and A. Mesiarová: Residual implications and left-continuous t-norms which are ordinal sum of semigroups. Fuzzy Sets and Systems *143* (2004), 47–57. DOI:10.1016/j.fss.2003.06.008

[22] A. Mesiarová-Zemánková: Ordinal sum construction for uninorms and generalized uninorms. Int. J. Approx. Reason. *76* (2016), 1–17. DOI:10.1016/j.ijar.2016.04.007

[23] A. Mesiarová-Zemánková: Ordinal sums of representable uninorms. Fuzzy Sets and Systems *308* (2017), 42–53. DOI:10.1016/j.fss.2016.07.006

[24] J. V. Riera and J. Torrens: Residual implications on the set of discrete fuzzy numbers. Inform. Sci. *247* (2013), 131–143. DOI:10.1016/j.ins.2013.06.008

[25] S. Saminger: On ordinal sum of triangular norms on bounded lattices. Fuzzy Sets and Systems *157* (2006), 1403–1416. DOI:10.1016/j.fss.2005.12.021

[26] Y. Su, A. Xie, and H. Liu: On ordinal sum implications. Inform. Sci. *293* (2015), 251–262. DOI:10.1016/j.ins.2014.09.021

[27] A. Xie, H. Liu, F. Zhang, and C. Li: On the distributivity of fuzzy implications over continuous Archimedean t-conorms and continuous t-conorms given as ordinal sums. Fuzzy Sets and Systems *205* (2012), 76–100. DOI:10.1016/j.fss.2012.01.009

[28] R. R. Yager: Aggregation operators and fuzzy systems modelling. Fuzzy Sets and Systems *67* (1994), 129–145. DOI:10.1016/0165-0114(94)90082-5

[29] R. R. Yager and A. Rybalov: Uninorm aggregation operators. Fuzzy Sets and Systems *80* (1996), 111–120. DOI:10.1016/0165-0114(95)00133-6

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