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A NOTE ON RESOLVING THE INCONSISTENCY OF ONE-SIDED MAX-PLUS LINEAR EQUATIONS

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When a system of one-sided max-plus linear equations is inconsistent, its right-hand side vector may be slightly modified to reach a consistent one. It is handled in this note by minimizing the sum of absolute deviations in the right-hand side vector. It turns out that this problem may be reformulated as a mixed integer linear programming problem. Although solving such a problem requires much computational effort, it may propose a solution that just modifies few elements of the right-hand side vector, which is a desired property in some practical situations.

Keywords: max-plus algebra, max-plus linear systems, mixed integer programming

Classification: 15A80, 90C11

1. INTRODUCTION

Max-plus algebra is the dioid $(\mathbb{R}_{\max}, \oplus, \otimes)$ endowed with the maximization operation as addition and the usual sum as multiplication, that is, $\mathbb{R}_{\max} = \mathbb{R} \cup \{-\infty\}$ and

$$a \oplus b = \max\{a, b\}, \quad a \otimes b = a + b, \quad \forall a, b \in \mathbb{R}_{\max}.$$

Analogously, min-plus algebra is the dioid $(\mathbb{R}_{\min}, \oplus', \otimes')$ with $\mathbb{R}_{\min} = \mathbb{R} \cup \{+\infty\}$ and

$$a \oplus' b = \min\{a, b\}, \quad a \otimes' b = a + b, \quad \forall a, b \in \mathbb{R}_{\min}.$$

The pair of operations $(\oplus, \otimes)$, as well as that of $(\oplus', \otimes')$, can be extended to vectors and matrices in the same way as in linear algebra, preserving the analogous commutative, associative, and distributive properties for the matrices of compatible sizes.

Max-plus algebra, as well as some analogous algebraic structures, has been intensively investigated and widely applied since it provides an attractive approach to formulating many real-world problems of scheduling, production, transportation, decision making, etc. For more details on the theory and application of max-plus algebra, the reader may refer to Baccelli et al. [3], Heidergott et al. [12], Gondran and Minoux [11], Butkovič [4], and references therein.

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A basic problem in max-plus algebra is to solve a system of one-sided max-plus linear equations of the form

$$\bigoplus_{j=1}^{n} a_{ij} \otimes x_j = \max_{j\in N}\{a_{ij} + x_j\} = b_i, \quad \forall i \in M,$$

or equivalently, in its matrix form

$$A \otimes \mathbf{x} = \mathbf{b},$$

where $M = \{1, 2, \ldots, m\}$ and $N = \{1, 2, \ldots, n\}$ are two index sets, the coefficient matrix $A = (a_{ij})_{m\times n} \in \mathbb{R}^{m\times n}_{\max}$ and the right-hand side vector $\mathbf{b} = (b_i)_{m\times 1} \in \mathbb{R}^{m}_{\max}$ are given, while $\mathbf{x} = (x_j)_{n\times 1} \in \mathbb{R}^{n}_{\max}$ is a vector to be determined. Without loss of generality, A is assumed to be doubly $\mathbb{R}$-astic, i. e., A has at least one finite element in each row and in each column, and $\mathbf{b}$ consists of only finite elements. Furthermore, noticing that $\max_{j\in N}\{a_{ij} + x_j\} = b_i$ is equivalent to $\max_{j\in N}\{a_{ij} - b_i + x_j\} = 0$ for each $i \in M$, any system of one-sided max-plus linear equations may be normalized such that all the right-hand side constants are zero.

A system $A \otimes \mathbf{x} = \mathbf{b}$ is said to be consistent if its solution set, denoted by $S(A, \mathbf{b})$, is nonempty and inconsistent otherwise. The principal solution $\widehat{\mathbf{x}} = (\widehat{x}_1, \widehat{x}_2, \ldots, \widehat{x}_n)^T$ of a given system $A \otimes \mathbf{x} = \mathbf{b}$ is constructed as

$$\widehat{x}_j = \min_{i\in M}\{-a_{ij} + b_i\}, \quad \forall j \in N,$$

that is,

$$\widehat{\mathbf{x}} = A^\sharp \otimes' \mathbf{b},$$

where $A^\sharp = -A^T$ is the conjugate matrix of A in the context of max-plus algebra, obtained by negation and transposition. It is well known that $A\otimes\widehat{\mathbf{x}} \leq \mathbf{b}$ and the system $A \otimes \mathbf{x} = \mathbf{b}$ is consistent if and only if $A \otimes \widehat{\mathbf{x}} = \mathbf{b}$. However, the consistency of max-plus linear equations is somewhat sensitive to the perturbation or noise in the data. It happens quite often that the system obtained in a practical situation may not possess an exact solution. In such a case, the principal solution $\widehat{\mathbf{x}}$ provides only a best under approximation $A \otimes \widehat{\mathbf{x}}$ to the right-hand side vector $\mathbf{b}$, which is not necessarily desirable in the process of modeling.

For resolving the inconsistency, one of the possible approaches is to modify the vector $\mathbf{b}$ slightly to reach a consistent system. This issue is concerned in the context of max-plus algebra as to minimize $\|A \otimes \mathbf{x} - \mathbf{b}\|_\infty$ where $\|\cdot\|_\infty$ denotes the conventional L_∞ norm for vectors, that is, to find a best L_∞ approximation to the vector $\mathbf{b}$ such that the corresponding system is consistent. Note that throughout this text, the max-plus matrix multiplication takes precedence over the conventional vector addition and subtraction whenever they appear simultaneously. It turns out that this optimization problem of minimizing the maximum absolute deviation in $\mathbf{b}$ can be readily solved. An optimal solution may be constructed in the closed form as $\widehat{\mathbf{x}} + \Delta$ where Δ is the vector of length n with each element being $\frac{1}{2}\|A\otimes\widehat{\mathbf{x}} - \mathbf{b}\|_\infty$. See, e. g., Cechlárová and Diko [8], Cechlárová and Cuninghame-Green [7], Krivulin [13], and Zimmermann [16] for more details. Besides, the inconsistency can be resolved efficiently as well with respect to the L_∞

norm for one-sided linear equations defined over some other dioid structures, see, e. g., Cuninghame-Green and Cechlárová [10], Tharwat and Zimmermann [15], Cechlárová [6], Li and Fang [14], and Cimler et al. [9].

Although the L_∞ norm based optimization criterion has a sound interpretation in max-plus algebra, it may lead to a consistent system $A \otimes \mathbf{x} = A \otimes (\widehat{\mathbf{x}} + \Delta)$ with most, if not all, elements of $\mathbf{b}$ modified as illustrated in Section 3, since all the elements in Δ are equal and hence $A \otimes (\widehat{\mathbf{x}} + \Delta) = A \otimes \widehat{\mathbf{x}} + \Delta$. However, in some situations, it is desired that the number of modified elements in the right-hand side vector keeps as few as possible. This property may be partially captured by minimizing $\|A \otimes \mathbf{x} - \mathbf{b}\|_1$ where $\|\cdot\|_1$ is the conventional L_1 norm for vectors, i.e, by minimizing the sum of absolute deviations in $\mathbf{b}$. This problem, to the best of our knowledge, has not been explicitly tackled in the related literature of max-plus algebra. However, the concerned optimization problem can be reformulated as

$$\begin{aligned} \min \quad & \|\mathbf{y} - \mathbf{b}\|_1 \\ \text{s.t.} \quad & A \otimes \mathbf{x} = \mathbf{y}, \end{aligned}$$

and considered as a nonlinear programming problem constrained by a system of two-sided max-plus linear equations with separated variables. Such a problem has been investigated in its more general form, see, e. g., Butkovič and Aminu [5], Aminu and Butkovič [2], and Allamigeon et al. [1]. In this note, a mixed integer linear programming formulation is proposed for the problem of minimizing $\|A \otimes \mathbf{x} - \mathbf{b}\|_1$ taking advantage of the particular properties of the L_1 norm for vectors.

This note proceeds as follows. An equivalent formulation is provided in Section 2 for minimizing $\|A \otimes \mathbf{x} - \mathbf{b}\|_1$ within the framework of mixed integer linear programming. Some numerical examples are presented in Section 3 to demonstrate this procedure and to compare the results with those given by minimizing $\|A \otimes \mathbf{x} - \mathbf{b}\|_\infty$. Some concluding remarks are addressed in Section 4.

2. DECOMPOSITION AND REFORMULATION

Given an inconsistent system of one-sided max-plus linear equations $A \otimes \mathbf{x} = \mathbf{b}$ with the principal solution $\widehat{\mathbf{x}}$, it holds that $A \otimes \widehat{\mathbf{x}} \leq \mathbf{b}$ and $A \otimes \mathbf{x} \leq A \otimes \widehat{\mathbf{x}}$ whenever $\mathbf{x} \leq \widehat{\mathbf{x}}$. Hence, for any $\mathbf{x} \leq \widehat{\mathbf{x}}$,

$$\|A \otimes \widehat{\mathbf{x}} - \mathbf{b}\|_1 \leq \|A \otimes \mathbf{x} - \mathbf{b}\|_1.$$

Moreover, since $A \otimes \widehat{\mathbf{x}} \leq \mathbf{b}$, it holds for any $\mathbf{x} \in \mathbb{R}^n_{\max}$ that

$$\|(A \otimes \widehat{\mathbf{x}}) \oplus (A \otimes \mathbf{x}) - \mathbf{b}\|_1 \leq \|A \otimes \mathbf{x} - \mathbf{b}\|_1.$$

This implies that

$$\|A \otimes (\widehat{\mathbf{x}} \oplus \mathbf{x}) - \mathbf{b}\|_1 \leq \|A \otimes \mathbf{x} - \mathbf{b}\|_1,$$

by the max-plus distributive property

$$A \otimes (\widehat{\mathbf{x}} \oplus \mathbf{x}) = (A \otimes \widehat{\mathbf{x}}) \oplus (A \otimes \mathbf{x}).$$

Consequently, due to $\widehat{\mathbf{x}} \oplus \mathbf{x} \geq \widehat{\mathbf{x}}$, one may impose the restriction $\mathbf{x} \geq \widehat{\mathbf{x}}$ to figure out the best L_1 approximation to the vector $\mathbf{b}$ in order to resolve the inconsistency.

Let $\boldsymbol{\delta} = (\delta_1, \delta_2, \dots, \delta_n)^T$ and $\mathbf{t} = (t_1, t_2, \dots, t_m)^T$ be two nonnegative vectors in the conventional sense. Then, $\|\mathbf{t}\|_1 = \mathbf{e}^T\mathbf{t}$ where $\mathbf{e} = (1, 1, \dots, 1)^T$ of the compatible size. The problem to minimize $\|A \otimes \mathbf{x} - \mathbf{b}\|_1$ subject to $\mathbf{x} \geq \widehat{\mathbf{x}}$ can be formulated equivalently as

$$\begin{aligned} \min \quad & \mathbf{e}^T\mathbf{t} \\ \text{s.t.} \quad & \\ & -\mathbf{t} + \mathbf{b} \leq A \otimes (\widehat{\mathbf{x}} + \boldsymbol{\delta}) \leq \mathbf{t} + \mathbf{b}, \\ & \mathbf{t} \geq 0, \ \boldsymbol{\delta} \geq 0. \end{aligned}$$

This formulation involves the nonlinear constraint $-\mathbf{t} + \mathbf{b} \leq A \otimes (\widehat{\mathbf{x}} + \boldsymbol{\delta}) \leq \mathbf{t} + \mathbf{b}$, which can be further decomposed by the techniques routinely used in mixed integer linear programming.

Note that $A \otimes (\widehat{\mathbf{x}} + \boldsymbol{\delta}) \leq \mathbf{t} + \mathbf{b}$ stands for

$$\max_{j \in N}\{a_{ij} + \widehat{x}_j + \delta_j\} \leq t_i + b_i, \quad \forall i \in M,$$

which can be equivalently represented by a system of linear inequalities

$$a_{ij} + \widehat{x}_j + \delta_j \leq t_i + b_i, \quad \forall i \in M, \ j \in N.$$

On the other hand, $A \otimes (\widehat{\mathbf{x}} + \boldsymbol{\delta}) \geq -\mathbf{t} + \mathbf{b}$ stands for

$$\max_{j \in N}\{a_{ij} + \widehat{x}_j + \delta_j\} \geq -t_i + b_i, \quad \forall i \in M,$$

which means that for each $i \in M$ there exists at least one index $j_i \in N$ such that

$$a_{ij_i} + \widehat{x}_{j_i} + \delta_{j_i} \geq -t_i + b_i.$$

By introducing a group of mn auxiliary binary variables to record such associations, $A \otimes (\widehat{\mathbf{x}} + \boldsymbol{\delta}) \geq -\mathbf{t} + \mathbf{b}$ can be reformulated as

$$a_{ij} + \widehat{x}_j + \delta_j + K(1 - z_{ij}) \geq -t_i + b_i, \quad \forall i \in M, \ j \in N,$$

where K is a suitably large positive constant and

$$\begin{aligned} & \textstyle\sum_{j \in N} z_{ij} = 1, \quad \forall i \in M, \\ & z_{ij} \in \{0, 1\}, \quad \forall i \in M, \ j \in N. \end{aligned}$$

Consequently, the problem of minimizing $\|A \otimes \mathbf{x} - \mathbf{b}\|_1$ may be reformulated into a mixed integer linear programming problem as

$$\begin{aligned} \min \quad & t_1 + t_2 + \ldots + t_m \\ \text{s.t.} \quad & \\ & -t_i + \delta_j \leq -a_{ij} - \widehat{x}_j + b_i, \quad \forall i \in M, \ j \in N, \\ & t_i + \delta_j + K(1 - z_{ij}) \geq -a_{ij} - \widehat{x}_j + b_i, \quad \forall i \in M, \ j \in N, \\ & \textstyle\sum_{j \in N} z_{ij} = 1, \quad \forall i \in M, \\ & t_i \geq 0, \quad \forall i \in M, \\ & \delta_j \geq 0, \quad \forall j \in N, \\ & z_{ij} \in \{0, 1\}, \quad \forall i \in M, \ j \in N. \end{aligned}$$

It then may be solved to optimality by some widely available solvers for mixed integer linear programming, often based on branch-and-bound methods. Other than the restrictions on the total $mn + m + n$ decision variables, this formulation consists of $2mn$ inequality constraints and m equality constraints but possesses somewhat a sparse structure. Some additional information may also be used for preprocessing in order to simplify this formulation, for instance, the rule that $a_{ij} = -\infty$ implies $z_{ij} = 0$ may be applied whenever A contains such elements. Nevertheless, a more compact formulation or an efficient direct solving algorithm, if possible, is still desirable.

3. NUMERICAL EXAMPLES

Consider the following normalized system $A \otimes \mathbf{x} = \mathbf{b}$ of one-sided max-plus linear equations

$$\begin{pmatrix} 3 & 1 & 5 \\ 4 & 4 & 6 \\ 7 & 7 & 3 \end{pmatrix} \otimes \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

of which the equations involved are

$$\begin{aligned} \max\{3 + x_1, 1 + x_2, 5 + x_3\} &= 0, \\ \max\{4 + x_1, 4 + x_2, 6 + x_3\} &= 0, \\ \max\{7 + x_1, 7 + x_2, 3 + x_3\} &= 0. \end{aligned}$$

The associated principal solution is $\widehat{\mathbf{x}} = (-7, -7, -6)^T$ and $A \otimes \widehat{\mathbf{x}}$ is

$$\begin{pmatrix} 3 & 1 & 5 \\ 4 & 4 & 6 \\ 7 & 7 & 3 \end{pmatrix} \otimes \begin{pmatrix} -7 \\ -7 \\ -6 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} \leq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Hence, the given system is inconsistent and the right-hand side vector has to be modified to achieve the consistency. If the best L_∞ approximation is desired, it follows that $\|A \otimes \widehat{\mathbf{x}} - \mathbf{b}\|_\infty = 1$ and $\Delta = (0.5, 0.5, 0.5)^T$ so that $A \otimes (\widehat{\mathbf{x}} + \Delta)$ is

$$\begin{pmatrix} 3 & 1 & 5 \\ 4 & 4 & 6 \\ 7 & 7 & 3 \end{pmatrix} \otimes \begin{pmatrix} -6.5 \\ -6.5 \\ -5.5 \end{pmatrix} = \begin{pmatrix} -0.5 \\ 0.5 \\ 0.5 \end{pmatrix}$$

and $\|A \otimes (\widehat{\mathbf{x}} + \Delta) - \mathbf{b}\|_\infty = 0.5$. This instance has been illustrated by Zimmermann [16] for resolving the inconsistency of one-sided max-plus linear equations under the criterion of L_∞ norm of vectors.

However, since $\|A \otimes (\widehat{\mathbf{x}} + \Delta) - \mathbf{b}\|_1 = 1.5$ while $\|A \otimes \widehat{\mathbf{x}} - \mathbf{b}\|_1 = 1$ for this instance, the best L_∞ approximation $A \otimes (\widehat{\mathbf{x}} + \Delta)$ obtained is not optimal with respect to the L_1 norm, which is usually the case as expected. Actually, it can be verified later that for this instance $A \otimes \widehat{\mathbf{x}}$ happens to be a best L_1 approximation to the right-hand side vector.

According to Section 2, the corresponding mixed integer linear programming problem to find the best L_1 approximation can be constructed as

$$\begin{aligned} \min \quad & t_1 + t_2 + t_3 \\ \text{s.t.} \quad & \\ & -t_1 + \delta_1 \le 4, \quad -t_1 + \delta_2 \le 6, \quad -t_1 + \delta_3 \le 1, \\ & -t_2 + \delta_1 \le 3, \quad -t_2 + \delta_2 \le 3, \quad -t_2 + \delta_3 \le 0, \\ & -t_3 + \delta_1 \le 0, \quad -t_3 + \delta_2 \le 0, \quad -t_3 + \delta_3 \le 3, \\ & t_1 + \delta_1 + K(1 - z_{11}) \ge 4, \quad t_1 + \delta_2 + K(1 - z_{12}) \ge 6, \quad t_1 + \delta_3 + K(1 - z_{13}) \ge 1, \\ & t_2 + \delta_1 + K(1 - z_{21}) \ge 3, \quad t_2 + \delta_2 + K(1 - z_{22}) \ge 3, \quad t_2 + \delta_3 + K(1 - z_{23}) \ge 0, \\ & t_3 + \delta_1 + K(1 - z_{31}) \ge 0, \quad t_3 + \delta_2 + K(1 - z_{32}) \ge 0, \quad t_3 + \delta_3 + K(1 - z_{33}) \ge 3, \\ & z_{11} + z_{12} + z_{13} = 1, \quad z_{21} + z_{22} + z_{23} = 1, \quad z_{31} + z_{32} + z_{33} = 1, \\ & t_1, t_2, t_3, \delta_1, \delta_2, \delta_3 \ge 0, \quad z_{11}, z_{12}, \ldots, z_{33} \in \{0, 1\}. \end{aligned}$$

By calling a mixed integer linear programming solver, it can be shown that the optimality is achieved with $(t_1^*, t_2^*, t_3^*)^T = (1, 0, 0)^T$ and $(\delta_1^*, \delta_2^*, \delta_3^*)^T = (0, 0, 0)^T$, indicating that $A \otimes \widehat{\mathbf{x}} = (-1, 0, 0)^T$ is a best L_1 approximation to achieve the consistency. Furthermore, the best L_1 approximation is usually not unique as expected. For this instance, let $\boldsymbol{\delta} = (0, 0, \delta_3)^T$ with $0 \le \delta_3 \le 1$. It follows that

$$\begin{pmatrix} 3 & 1 & 5 \\ 4 & 4 & 6 \\ 7 & 7 & 3 \end{pmatrix} \otimes \begin{pmatrix} -7 \\ -7 \\ -6 + \delta_3 \end{pmatrix} = \begin{pmatrix} -1 + \delta_3 \\ \delta_3 \\ 0 \end{pmatrix},$$

and hence $\|A \otimes (\widehat{\mathbf{x}} + \boldsymbol{\delta}) - \mathbf{b}\|_1 = \|A \otimes \widehat{\mathbf{x}} - \mathbf{b}\|_1 = 1$. Consequently, when $\delta_3 = 0.5$, the resulted vector $A \otimes (\widehat{\mathbf{x}} + \boldsymbol{\delta}) = (-0.5, 0.5, 0)^T$ becomes a both best L_1 and best L_∞ approximation to the right-hand side vector.

It is intuitive that $A \otimes \widehat{\mathbf{x}}$ may not necessarily be the best L_1 approximation. For instance, consider $A \otimes \mathbf{x} = \mathbf{b}$ with

$$\begin{pmatrix} 3 & 1 & 3 \\ 4 & 4 & 6 \\ 7 & 7 & 7 \end{pmatrix} \otimes \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

of which $\widehat{\mathbf{x}} = (-7, -7, -7)^T$ and $A \otimes \widehat{\mathbf{x}} = (-4, -1, 0)^T$. By solving the corresponding mixed integer linear programming problem, it can be shown that $\boldsymbol{\delta}^* = (3, 0, 0)^T$ offers a best L_1 approximation $A \otimes (\widehat{\mathbf{x}} + \boldsymbol{\delta}^*) = (-1, 0, 3)^T$ such that

$$\|A \otimes (\widehat{\mathbf{x}} + \boldsymbol{\delta}^*) - \mathbf{b}\|_1 = 4 < 5 = \|A \otimes \widehat{\mathbf{x}} - \mathbf{b}\|_1.$$

Besides, $\|A \otimes \widehat{\mathbf{x}} - \mathbf{b}\|_\infty = 4$ and hence $\Delta = (2, 2, 2)^T$ such that $\|A \otimes (\widehat{\mathbf{x}} + \Delta) - \mathbf{b}\|_\infty = 2$. It is clear that $A \otimes (\widehat{\mathbf{x}} + \boldsymbol{\delta}^*)$ is not a best L_∞ approximation since $\|A \otimes (\widehat{\mathbf{x}} + \boldsymbol{\delta}^*) - \mathbf{b}\|_\infty = 3$.

However, with $\widehat{\boldsymbol{\delta}} = (2,0,1)^T$ obtained by imposing the upper bound constraint $\mathbf{t} \leq \Delta$, the vector $A \otimes (\widehat{\mathbf{x}} + \widehat{\boldsymbol{\delta}})$ becomes

$$\begin{pmatrix} 3 & 1 & 3 \\ 4 & 4 & 6 \\ 7 & 7 & 7 \end{pmatrix} \otimes \begin{pmatrix} -5 \\ -7 \\ -6 \end{pmatrix} = \begin{pmatrix} -2 \\ 0 \\ 2 \end{pmatrix},$$

which is still a best L_1 approximation and simultaneously a best L_∞ approximation.

With these two numerical examples, it is somewhat tempting to conjecture that there exists a simultaneously best L_1 and L_∞ approximation to the right-hand side vector whenever a system of one-sided max-plus linear equations is inconsistent. However, this is not the case. As a counterexample, consider $A \otimes \mathbf{x} = \mathbf{b}$ with

$$\begin{pmatrix} 5 & 1 & 0 \\ 4 & 4 & 5 \\ 7 & 7 & 7 \end{pmatrix} \otimes \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

of which $\widehat{\mathbf{x}} = (-7,-7,-7)^T$ and $A \otimes \widehat{\mathbf{x}} = (-2,-2,0)^T$. So, with $\Delta = (1,1,1)^T$, a best L_∞ approximation is given by $A \otimes (\widehat{\mathbf{x}} + \Delta)$ as

$$\begin{pmatrix} 5 & 1 & 0 \\ 4 & 4 & 5 \\ 7 & 7 & 7 \end{pmatrix} \otimes \begin{pmatrix} -6 \\ -6 \\ -6 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}$$

and $\|A \otimes (\widehat{\mathbf{x}} + \Delta) - \mathbf{b}\|_\infty = 1$. By solving the corresponding mixed integer linear programming problem, a best L_1 approximation may be constructed by $A \otimes (\widehat{\mathbf{x}} + \boldsymbol{\delta}^*)$ with $\boldsymbol{\delta}^* = (2,0,2)^T$ such that

$$\begin{pmatrix} 5 & 1 & 0 \\ 4 & 4 & 5 \\ 7 & 7 & 7 \end{pmatrix} \otimes \begin{pmatrix} -5 \\ -7 \\ -5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}$$

and $\|A \otimes (\widehat{\mathbf{x}} + \boldsymbol{\delta}^*) - \mathbf{b}\|_1 = 2$. Note that only one component in the right-hand side vector is modified to reach the consistency largely due to the applied L_1 norm criterion. However, with the upper bound constraint $\mathbf{t} \leq \Delta$, the resulting best L_1 approximation $A \otimes (\widehat{\mathbf{x}} + \widehat{\boldsymbol{\delta}})$ becomes

$$\begin{pmatrix} 5 & 1 & 0 \\ 4 & 4 & 5 \\ 7 & 7 & 7 \end{pmatrix} \otimes \begin{pmatrix} -6 \\ -7 \\ -6 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$$

with $\|A \otimes (\widehat{\mathbf{x}} + \widehat{\boldsymbol{\delta}}) - \mathbf{b}\|_1 = 3$ where $\widehat{\boldsymbol{\delta}} = (1,0,1)^T$. This means that for this instance there exists no vector that is simultaneously a best L_1 and L_∞ approximation to the right-hand side vector for resolving the inconsistency. This example also illustrates that the inconsistency of one-sided max-plus linear equations may be resolved by modifying possibly few components in the right-hand side vector.

Note that all the mixed integer linear programming problems associated with the numerical examples in this section are solved with the free linear programming solver `lp_solve` 5.5 using `lpSolve` package in `R` 3.0.1.

4. CONCLUSIONS

The problem of resolving inconsistency of one-sided max-plus linear equations is tackled in this note by changing the right-hand side vector as slightly as possible such that the resulting equations are consistent. By minimizing the sum of absolute deviations, this problem is reformulated equivalently as a mixed integer linear programming problem, which may be solved to optimality by some well developed techniques. Compared with the known computationally tractable method that minimizes the maximum absolute deviation, this approach requires much more computational effort but may propose a solution that just modifies few elements of the right-hand side vector. Besides, these two methods may be naturally combined as illustrated in the presented numerical examples. Nevertheless, the approach developed in this note provides an alternative perspective on resolving the inconsistency of one-sided max-plus linear equations. It may be extended as well to handle the programming problems constrained by two-sided max-plus linear equations as considered in Butkovič and Aminu [5] and Aminu and Butkovič [2].

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