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SOME NOTES ON U-PARTIAL ORDER

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In this paper, an equivalence on the class of uninorms on a bounded lattice is discussed. Some relationships between the equivalence classes of uninorms and the equivalence classes of their underlying t-norms and t-conorms are presented. Also, a characterization for the sets admitting some incomparability w.r.t. the U-partial order is given.

Keywords: uninorm, bounded lattice, partial order, equivalence, T-norm

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1. INTRODUCTION

The aggregation functions characterized by the non-decreasing monotonicity and fulfilling boundary conditions have been a valuable field of study for researchers in recent years. Their popularity come from their applicability to many areas [4, 22, 23].

Triangular norms (t-norms), triangular conorms (t-conorms), uninorms and nullnorms, which are extensively studied in [7, 8, 10, 18], are some of aggregation functions. Especially, the importance of uninorms is thanks to their special algebraic structures generalizing both t-norms and t-conorms. As a natural, it is not surprising that many problems for triangular norms (or t-conorm) are invesigated for uninorms [5, 9, 19].

In recent years, inducing an order from logical operators has been an interesting problem for many researchers. In this sense, the triangular order denoted by $\preceq_T$ has been introduced by Kesicioğlu and Karaçal in [11]. As an extension of t- partial order, the U- partial order $\preceq_U$ induced by uninorms, has been given in [6]. In the papers [12, 13, 17], some properties of $\preceq_U$ have been studied.

In this paper, we investigate an equivalence on the class of uninorms on a bounded lattice based on the equality of the sets admitting some incomparability w.r.t. the U-partial order. The paper consists of 4 main parts. In the second part, we shortly recall some basic notions and results. In the third part, Some relationships between the equivalence classes of uninorms and the equivalence classes of their underlying t-norms and t-conorms are presented. We show that two idempotent uninorms are equivalent. We determine the equivalence classes of the greatest and smallest uninorms. We give some relationships between the sets admitting incomparability w.r.t. the U-partial order and its ϕ-conjugate. Finally, we characterize the set admitting incomparability w.r.t. the U-partial order under some special conditions.

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2. NOTATIONS, DEFINITIONS AND A REVIEW OF PREVIOUS RESULTS

Definition 2.1. (Birkhoff [3]) Let $(L, \leq, 0, 1)$ be a bounded lattice. If $x \leq y$ or $y \leq x$, the elements x and y are called comparable. Otherwise, x and y are called incomparable and the notation $x||y$ is used.

Definition 2.2. (Karaçal and Kesicioğlu [11]) An operation T (S) on a bounded lattice L is called a triangular norm (triangular conorm) if it is commutative, associative, increasing with respect to the both variables and has a neutral element 1 (0).

The following are the four basic t-norms T_M, T_P, T_L and T_D given by respectively:

$T_M(x,y) = \min(x,y)$,
$T_P(x,y) = x.y$,
$T_L(x,y) = \max(x+y-1, 0)$,

$$T_D(x,y) = \begin{cases} 0 & (x,y) \in [0,1)^2, \\ \min(x,y) & \text{otherwise.} \end{cases}$$

Definition 2.3. (Karaçal and Kesicioğlu [11], Kesicioğlu et al. [14]) A t-norm T (or a t-conorm S) on a bounded lattice L is divisible if the following condition holds:

For all $x, y \in L$ with $x \leq y$ there is $z \in L$ such that $x = T(y,z)$ (or $y = S(x,z)$).

Definition 2.4. (Karaçal and Mesiar [10]) Let $(L, \leq, 0, 1)$ be a bounded lattice. An operation $U : L^2 \to L$ is called a uninorm on L, if it is commutative, associative, increasing with respect to the both variables and has a neutral element $e \in L$.

In this study, the notation $\mathcal{U}(e)$ will be used for the set of all uninorms on L with a neutral element $e \in L$.

Corollary 2.5. (Karaçal and Mesiar [10]) Let $(L, \leq, 0, 1)$ be a bounded lattice and $e \in L \setminus \{0,1\}$. Then the following uninorms $U_{T_\wedge} : L^2 \to L$ and $U_{S_\vee} : L^2 \to L$, respectively, are the greatest and the smallest uninorm on L with neutral element e.

$$U_{T_\wedge}(x,y) = \begin{cases} x \wedge y & , \text{ if } (x,y) \in [0,e]^2 \\ x \vee y & , \text{ if } (x,y) \in [0,e] \times (e,1] \cup (e,1] \times [0,e] \\ y & , \text{ if } x \in [0,e]\,,\ y \| e \\ x & , \text{ if } y \in [0,e]\,,\ x \| e \\ 1 & , \text{ otherwise,} \end{cases}$$

$$U_{S_\vee}(x,y) = \begin{cases} x \vee y & , \text{ if } (x,y) \in [e,1]^2 \\ x \wedge y & , \text{ if } (x,y) \in [0,e) \times [e,1] \cup [e,1] \times [0,e) \\ y & , \text{ if } x \in [e,1]\,,\ y \| e \\ x & , \text{ if } y \in [e,1]\,,\ x \| e \\ 0 & , \text{ otherwise.} \end{cases}$$

Proposition 2.6. (Karaçal and Mesiar [10]) Let $(L, \leq, 0, 1)$ be a bounded lattice, and U a uninorm with a neutral element $e \in L$. Then,

(i) $T_U = U\mid_{[0,e]^2} : [0,e]^2 \to [0,e]$ is a t-norm on $[0,e]$.

(ii) $S_U = U\mid_{[e,1]^2} : [e,1]^2 \to [e,1]$ is a t-conorm on $[e,1]$.

S_U and T_U given in Proposition 2.6 are called the underlying t-conorm and t-norm of U, respectively.

Definition 2.7. (Grabisch et al. [8]) Let $(L, \leq, 0, 1)$ be a bounded lattice and $U \in \mathcal{U}(e)$. An element $x \in L$ is called an idempotent element of U if $U(x, x) = x$.

Moreover, a uninorm is called an idempotent uninorm whenever $U(x, x) = x$ for all $x \in L$.

Definition 2.8. (Baczyński and Jayaram [2], Kesicioğlu, R. Mesiar [15]) Let $(L, \leq, 0, 1)$ be a bounded lattice. A decreasing function $N : L \to L$ is called a negation if $N(0) = 1$ and $N(1) = 0$. A negation N on L is called strong if it is an involution, i.e., $N(N(x)) = x$, for all $x \in L$.

Definition 2.9. (Baczyński and Jayaram [2]) Let T be a t-norm on a bounded lattice L and N be a strong negation on L. The t-conorm S defined by

$$S(x, y) = N(T(N(x), N(y))),\ x, y \in L$$

is called the N-dual t-conorm to T on L.

Definition 2.10. (Ertuğrul et al. [6]) Let $(L, \leq, 0, 1)$ be a bounded lattice and $U \in \mathcal{U}(e)$. Define the following relation: For every $x, y \in L$

$$x \preceq_U y \Leftrightarrow \begin{cases} \text{if} \quad x, y \in [0, e] \quad \text{and there exists} \quad k \in [0, e] \\ \quad \text{such that} \quad U(k, y) = x \quad \text{or,} \\ \text{if} \quad x, y \in [e, 1] \quad \text{and there exists} \quad \ell \in [e, 1] \\ \quad \text{such that} \quad U(x, \ell) = y \quad \text{or,} \\ \text{if} \quad (x, y) \in L^* \quad \text{and} \quad x \leq y, \end{cases}$$

where $I_e = \{x \in L \mid x \| e\}$ and $L^* = [0, e] \times [e, 1] \cup [0, e] \times I_e \cup [e, 1] \times [0, e] \cup [e, 1] \times I_e \cup I_e \times [0, e] \cup I_e \times [e, 1] \cup I_e \times I_e$.

This relation is a partial order and called U-partial order.

Definition 2.11. (Klement et al. [16]) If T is a t-norm on the unit interval $[0, 1]$ and $\phi : [0, 1] \to [0, 1]$ an order-preserving bijection, then the operation $T_\phi : [0, 1]^2 \to [0, 1]$ given by

$$T_\phi(x, y) = \phi^{-1}(T(\phi(x), \phi(y)))$$

is also a t-norm. This t-norm is called ϕ-conjugate of T.

The ϕ-conjugate of a uninorm (t-conorm) on a bounded lattice is defined as similar to Definition 2.11.

A lattice is directly related to the order on it. Therefore, it is also interesting to examine the change when the order on the lattice changes. To better observe the relationship between the natural order on the lattice and the order determined by an operator given, let's define the set K_M for the operator M on the lattice.

Definition 2.12. (Aşıcı F. Karaçal [1], Kesicioğlu et al. [14], Kesicioğlu et al. [13], Lu et al. [17]) Let $(L, \leq, 0, 1)$ be a bounded lattice and M be a t-norm (t-conorm, uninorm). Define the set K_M as follow:

$K_M = \{x \in L \backslash \{0, 1\}|$ for some $y \in L \backslash \{0, 1\}$, $[x < y$ and $x \not\preceq_M y]$ or $[y < x$ and $y \not\preceq_M x]$ or $x \| y\}$.

Definition 2.13. (Aşıcı and Karaçal [1], Kesicioğlu et al. [13], Kesicioğlu et al. [14], Lu et al. [17]) Let $(L, \leq, 0, 1)$ be a given bounded lattice. Define the relations $\sim$ and $\sim_K$ on the class of all t-norms (t-conorms, uninorms with a neutral element e) on L:

$$M_1 \sim M_2 \text{ if and only if the } \preceq_{M_1} = \preceq_{M_2},$$

$$M_1 \sim_K M_2 \text{ if and only if the } K_{M_1} = K_{M_2}.$$

Lemma 2.14. (Aşıcı and Karaçal [1], Kesicioğlu et al. [13], Kesicioğlu et al. [14], Lu et al. [17]) The relations $\sim$ and $\sim_K$ given in Definition 2.13 are the equivalence relations.

In this paper, we will use the notation $\overline{M}$ for the equivalence class linked to M w.r.t. the $\sim_K$, i.e.

$$\overline{M} = \{M^{'} | M^{'} \sim_K M\}.$$

Definition 2.15. (Birkhoff [3]) Let $(L, \leq, 0, 1)$ be a bounded lattice. If there exists an element $y \in L$ for an element $x \in L$ such that $x \wedge y = 0$ and $x \vee y = 1$, then the element y is called a complement of x.

L is called a complemented lattice if all elements have complements.

L is called relatively complemented if all intervals are complemented.

Proposition 2.16. (Kesicioğlu [12]) Let T_1 and T_2 be two t-norms on a bounded lattice $(L, \leq, 0, 1)$. If $T_1 \sim T_2$, then $T_1 \sim_K T_2$.

Proposition 2.16 is also true for two t-conorms.

Proposition 2.17. (Kesicioğlu [12]) Let $(L, \leq, 0, 1)$ be a bounded lattice, S_1 and S_2 the N-dual t-conorms of two t-norms T_1 and T_2 on L, respectively. Then, $T_1 \sim_K T_2$ iff $S_1 \sim_K S_2$.

Proposition 2.18. (Kesicioğlu [12]) Let $(L, \leq, 0, 1)$ be a bounded lattice, $a \in L$ and N be a strong negation on L with $N(a) = a$. If $S : [0, a]^2 \to [0, a]$ is a t-conorm, then for any $x, y \in [a, 1]$

$$T(x, y) = N(S(N(x), N(y)))$$

is a t-norm on $[a, 1]$.

Similarly, if $T : [a, 1]^2 \to [a, 1]$ is a t-norm, then for any $x, y \in [0, a]$

$$S(x, y) = N(T(N(x), N(y)))$$

is a t-conorm on $[0, a]$. Then, T (S) is called the N-dual t-norm (t-conorm) of S (T).

Proposition 2.19. (Kesicioğlu [12]) Let T be a t-norm on a bounded lattice L and ϕ an order-preserving bijection on L. Then, $x \in K_T$ iff $\phi^{-1}(x) \in K_{T_\phi}$.

3. THE EQUIVALENCE CLASS

In this section, an equivalence relation for the class of all uninorms on a bounded lattice is investigated. In this sense, a relationship between the equivalence of two uninorms and the equivalence of their underlying t-norms and t-conorms are examined.

Proposition 3.1. (Kesicioğlu et al. [13]) Let $(L, \leq, 0, 1)$ be a complemented lattice and $U \in \mathcal{U}(e)$. Then, $K_U = L \setminus \{0, 1\}$.

Proposition 3.2. Let $(L, \leq, 0, 1)$ be a complemented lattice and $U_1, U_2 \in \mathcal{U}(e)$. Then, $U_1 \sim_K U_2$.

Proof. Let $(L, \leq, 0, 1)$ be a complemented lattice. $K_{U_1} = L \setminus \{0, 1\} = K_{U_2}$ is obtained from Proposition 3.1. Thus, $U_1 \sim_K U_2$. □

Theorem 3.3. (Kesicioğlu et al. [13]) Let $(L, \leq, 0, 1)$ be a bounded lattice and $U \in \mathcal{U}(e)$. Then,

$$K_U = K_{T_U} \cup K_{S_U} \cup I_e \cup M,$$

where $M = \{x \in L \quad | \quad x \| y \quad \text{for some} \quad y \in I_e\}$.

Corollary 3.4. (Kesicioğlu et al. [13]) Let $(L, \leq, 0, 1)$ be a bounded lattice and $U \in \mathcal{U}(e)$. If $I_e = \emptyset$, then $K_U = K_{T_U} \cup K_{S_U}$.

Remark 3.5. Let $(L, \leq, 0, 1)$ be a bounded lattice and $U \in \mathcal{U}(e)$. If $I_e \neq \emptyset$, then K_U is a set with at least two elements.

Proof. Let $I_e \neq \emptyset$. Therefore, there exists at least an element $x \in I_e$. We have that $e \in M$ from the definition of the set M. Thus, $\{x, e\} \subseteq I_e \cup M \subseteq K_U$. So, K_U has at least two elements. □

Theorem 3.6. (Kesicioğlu et al. [13]) Let $(L, \leq, 0, 1)$ be a bounded lattice and $U_1, U_2 \in \mathcal{U}(e)$. Then, $T_{U_1} \sim T_{U_2}$ and $S_{U_1} \sim S_{U_2}$ iff $U_1 \sim U_2$.

Proposition 3.7. Let $(L, \leq, 0, 1)$ be a bounded lattice and $U_1, U_2 \in \mathcal{U}(e)$. If $T_{U_1} \sim_K T_{U_2}$ and $S_{U_1} \sim_K S_{U_2}$, then $U_1 \sim_K U_2$.

Proof. Let $T_{U_1} \sim_K T_{U_2}$ and $S_{U_1} \sim_K S_{U_2}$. Thus, we have that $K_{T_{U_1}} = K_{T_{U_2}}$ and $K_{S_{U_1}} = K_{S_{U_2}}$. Then,

$$\begin{aligned} K_{U_1} &= K_{T_{U_1}} \cup K_{S_{U_1}} \cup I_e \cup M \\ &= K_{T_{U_2}} \cup K_{S_{U_2}} \cup I_e \cup M = K_{U_2}, \end{aligned}$$

whence $U_1 \sim_K U_2$. □

Corollary 3.8. Let $(L, \leq, 0, 1)$ be a bounded lattice and $U_1, U_2 \in \mathcal{U}(e)$. If $U_1 \sim U_2$, then $U_1 \sim_K U_2$.

P r o o f. Let $U_1 \sim U_2$. We obtained that $T_{U_1} \sim T_{U_2}$ and $S_{U_1} \sim S_{U_2}$ from $U_1 \sim U_2$. Thus, we have that $T_{U_1} \sim_K T_{U_2}$ and $S_{U_1} \sim_K S_{U_2}$ by Proposition 2.16. Thus, by Proposition 3.7, it is obtained that $U_1 \sim_K U_2$. □

Proposition 3.9. Let U_1 and U_2 be two idempotent uninorms on $[0,1]$. Then, $U_1 \sim_K U_2$.

P r o o f. Since U_1 and U_2 are two idempotent uninorms,

$$T_{U_1} = T_{U_2} = T_M \text{ and } S_{U_1} = S_{U_2} = S_M.$$

It is obtained that $\preceq_{U_1} = \preceq_{U_2} = \leq$, i.e., $U_1 \sim U_2$ since $T_{U_1}, T_{U_2}, S_{U_1}$ and S_{U_2} are continuous (see Corollary 8 in [6]). Then, by Corollary 3.8, we have that $U_1 \sim_K U_2$. □

Remark 3.10. The converse of Proposition 3.7 and Corollary 3.8 may not be true. Let us investigate the following example.

Example 3.11. Consider the lattice $(L = \{0, a, b, c, d, 1\}, \leq, 0, 1)$ whose lattice diagram given in Figure 1.

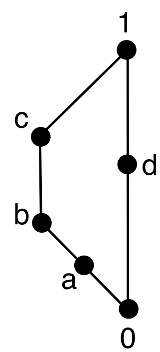

Fig. 1. $(L, \leq)$.

Take the drastic product T_D on $[0, c]$ and T_M on $[0, c]$, the functions U_1 and U_2 are uninorms on L with neutral element c by [10]. U_1 and U_2 can be seen in detail in Table 1 and Table 2:

U_1	0	a	b	c	d	1
0	0	0	0	0	d	1
a	0	0	0	a	d	1
b	0	0	0	b	d	1
c	0	a	b	c	d	1
d	d	d	d	d	1	1
1	1	1	1	1	1	1

Tab. 1. The uninorm U_1 on L.

U_2	0	a	b	c	d	1
0	0	0	0	0	d	1
a	0	a	a	a	d	1
b	0	a	b	b	d	1
c	0	a	b	c	d	1
d	d	d	d	d	1	1
1	1	1	1	1	1	1

Tab. 2. The uninorm U_2 on L.

The orders $\preceq_{U_1}$ and $\preceq_{U_2}$ obtained from U_1 and U_2, respectively, are depicted on Figure 2 and Figure 3.

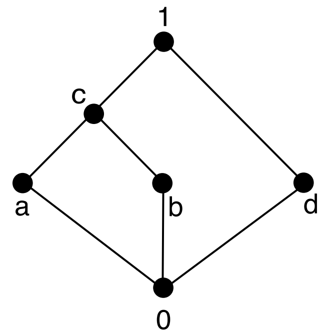

Fig. 2. $(L, \preceq_{U_1})$

and

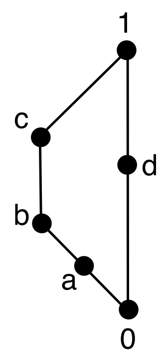

Fig. 3. $(L, \preceq_{U_2})$

Obviously, $K_{U_1} = \{a, b, c, d\} = K_{U_2}$ but $K_{T_{U_1}} = \{a, b\} \neq \emptyset = K_{T_{U_2}}$. That means, $U_1 \sim_K U_2$ but $K_{T_{U_1}} \sim_K K_{T_{U_2}}$ does not satisfy.

Proposition 3.12. Let $(L, \leq, 0, 1)$ be a bounded lattice, $U_1, U_2 \in \mathcal{U}(e)$ and $I_e = \emptyset$. Then, $T_{U_1} \sim_K T_{U_2}$ and $S_{U_1} \sim_K S_{U_2}$ iff $U_1 \sim_K U_2$.

Proof. $\Rightarrow$: It is clear by Proposition 3.7.

$\Leftarrow$: Suppose that $U_1 \sim_K U_2$. Then, we have that $K_{U_1} = K_{U_2}$. We know that for any uninorm U, $K_U = K_{T_U} \cup K_{S_U}$ by Corollary 3.4. Thus, it is obtained that

$$K_{T_{U_1}} \cup K_{S_{U_1}} = K_{U_1} = K_{U_2} = K_{T_{U_2}} \cup K_{S_{U_2}}.$$

Since $K_{T_{U_1}}, K_{T_{U_2}} \subseteq [0, e]$ and $K_{S_{U_1}}, K_{S_{U_2}} \subseteq [e, 1]$, we have that

$$\begin{aligned}
K_{T_{U_1}} &= K_{T_{U_1}} \cap [0,e] = \emptyset \cup (K_{T_{U_1}} \cap [0,e]) \\
&= (K_{T_{U_1}} \cap [0,e]) \cup (K_{S_{U_1}} \cap [0,e]) \\
&= (K_{T_{U_1}} \cup K_{S_{U_1}}) \cap [0,e] \\
&= (K_{T_{U_2}} \cup K_{S_{U_2}}) \cap [0,e] \\
&= (K_{T_{U_2}} \cap [0,e]) \cup (K_{S_{U_2}} \cap [0,e]) \\
&= (K_{T_{U_2}} \cap [0,e]) \cup \emptyset = K_{T_{U_2}} \cap [0,e] = K_{T_{U_2}}.
\end{aligned}$$

Then, it is obtained that $T_{U_1} \sim_K T_{U_2}$. Similarly, it can be shown that $S_{U_1} \sim_K S_{U_2}$. $\square$

Corollary 3.13. Let U_1 and U_2 be two uninorms on $[0,1]$. Then, $T_{U_1} \sim_K T_{U_2}$ and $S_{U_1} \sim_K S_{U_2}$ iff $U_1 \sim_K U_2$.

Corollary 3.14. Let $(L, \leq, 0, 1)$ be a bounded lattice and $U_1, U_2 \in \mathcal{U}(e)$. Suppose that S_{U_1} and S_{U_2} are the N-dual t-conorms of T_{U_1} and T_{U_2}, respectively. If $T_{U_1} \sim_K T_{U_2}$, then $U_1 \sim_K U_2$.

Remark 3.15. The converse of Corollary 3.14 may not be true. If we consider Example 3.11, $K_{U_1} = \{a,b,c,d\} = K_{U_2}$ but $K_{T_{U_1}} = \{a,b\} \neq \emptyset = K_{T_{U_2}}$, i.e., $U_1 \sim_K U_2$ but T_{U_1} and T_{U_2} are not equivalent under relation $\sim_K$.

Corollary 3.16. Let $(L, \leq, 0, 1)$ be a bounded lattice, $U \in \mathcal{U}(e)$ and N be a strong negation on L with $N(e) = e$. Let S_U be the N-dual t-conorm of T_U. If $I_e = \emptyset$, then

$$K_U = \{x | x \in K_{T_U} \quad \text{or there exists an element} \quad y \in K_{T_U} \quad \text{such that} \quad x = N(y)\}.$$

Example 3.17. The equivalence classes of the greatest uninorm $U_e^{(\vee)}$ and the smallest uninorm $U_e^{(\wedge)}$ with a neutral element $e \in (0,1)$ w.r.t. the relation $\sim_K$ are given by

$$\overline{U_e^{(\vee)}} = \{U | K_{T_U} = (0,e) \quad \text{and} \quad S_U \quad \text{is continuous}\}$$

and

$$\overline{U_e^{(\wedge)}} = \{U | K_{S_U} = (e,1) \quad \text{and} \quad T_U \quad \text{is continuous}\},$$

respectively.

Proof. It can be easily seen that $U_e^{(\vee)}|_{[0,e]^2} = T_D$ and $U_e^{(\vee)}|_{[e,1]^2} = S_M$. Then, $K_{T_{U_e^{(\vee)}}} = K_{T_D} = (0,e)$ and $K_{S_{U_e^{(\vee)}}} = K_{S_M} = \emptyset$.

Let $U \in \overline{U_e^{(\vee)}}$, i.e., $U \sim_K U_e^{(\vee)}$. Thus, it is obtained that $K_U = K_{U_e^{(\vee)}}$. Since $K_U = K_{T_U} \cup K_{S_U} = K_{T_{U_e^{(\vee)}}} \cup K_{S_{U_e^{(\vee)}}} = (0,e) \cup \emptyset = (0,e)$, it must be $K_{T_U} = (0,e)$ and $K_{S_U} = \emptyset$. It can be shown that S_U is continuous iff $K_{S_U} = \emptyset$ using Lemma 3 in [14]. Thus, $K_{T_U} = (0,e)$ and S_U is continuous.

Conversely, it is clear that $K_U = K_{U_e^{(\vee)}}$ for any uninorm $U \in \mathcal{U}(e)$ with $K_{T_U} = (0, e)$ and S_U is continuous.

Similarly, it can be easily shown that

$$\overline{U_e^{(\wedge)}} = \{U | K_{S_U} = (e, 1) \quad \text{and} \quad T_U \quad \text{is continuous}\}.$$

□

Proposition 3.18. Let $(L, \leq, 0, 1)$ be a bounded lattice, $U \in \mathcal{U}(e)$ and ϕ an order-preserving bijection with $\phi(e) = e$. If $T_U \sim_K T_{U_\phi}$ and $S_U \sim_K S_{U_\phi}$, then $U \sim_K U_\phi$.

Proposition 3.19. Let $(L, \leq, 0, 1)$ be a bounded lattice, $U \in \mathcal{U}(e)$ and ϕ an order-preserving bijection on L with $\phi(e) = e$. If T_U and S_U are divisible, then $U \sim_K U_\phi$.

Proof. If T_U and S_U are divisible, it is obtained that T_{U_ϕ} and S_{U_ϕ} are also divisible. Thus, we have that $\preceq_{T_{U_\phi}} = \leq = \preceq_{T_U}$ and $\preceq_{S_{U_\phi}} = \leq = \preceq_{S_U}$. So, it is clear that $T_{U_\phi} \sim T_U$ and $S_{U_\phi} \sim S_U$. We obtain that $U \sim U_\phi$ by Theorem 3.6. We have that $U \sim_K U_\phi$ by Corollary 3.8. □

Proposition 3.20. Let $(L, \leq, 0, 1)$ be a bounded lattice, $U \in \mathcal{U}(e)$ and ϕ be an order-preserving bijection on L satisfying $\phi(e) = e$. Then, $x \in K_U$ iff $\phi^{-1}(x) \in K_{U_\phi}$.

Proof. Let $U|_{[0,e]^2} = T_U$ and $U|_{[e,1]^2} = S_U$. It is clear that $U_\phi|_{[0,e]^2} = T_{U_\phi} = (T_U)_\phi$ and $U_\phi|_{[e,1]^2} = S_{U_\phi} = (S_U)_\phi$.

Let $x \in K_{T_U} \cup K_{S_U} \cup I_e \cup M$.

- If $x \in K_{T_U}$, $\phi^{-1}(x) \in K_{(T_U)_\phi} = K_{T_{U_\phi}} \subseteq K_{U_\phi}$ by Proposition 2.19. Thus, we have that $\phi^{-1}(x) \in K_{U_\phi}$.
- If $x \in K_{S_U}$, then we obtain that $\phi^{-1}(x) \in K_{U_\phi}$ since $\phi^{-1}(x) \in K_{(S_U)_\phi} = K_{S_{U_\phi}} \subseteq K_{U_\phi}$.
- Let $x \in I_e$. If $\phi^{-1}(x) \leq e$, it would be $\phi^{-1}(x) \leq e = \phi^{-1}(e)$, which contradicts that $x \in I_e$. Similarly, if $e \leq \phi^{-1}(x)$, a contradiction again. Thus, it must be $\phi^{-1}(x) \| e$, i.e., $\phi^{-1}(x) \in I_e \subseteq K_{U_\phi}$. We obtain that $\phi^{-1}(x) \in K_{U_\phi}$.
- Let $x \in M$. Then, there exists an element $y \in I_e$ such that $x \| y$. If $\phi^{-1}(y) \leq e$, it would be $\phi^{-1}(y) \leq e = \phi^{-1}(e)$, which contradicts that $y \in I_e$. We similarly obtain a contradiction if $e \leq \phi^{-1}(y)$. Thus, it must be $\phi^{-1}(y) \in I_e$. Now, let us show that $\phi^{-1}(x) \| \phi^{-1}(y)$. If we consider $\phi^{-1}(x) \leq \phi^{-1}(y)$, then we would have a contradiction since x and y are not comparable. Similarly, if $\phi^{-1}(y) \leq \phi^{-1}(x)$, we would have a contradiction again. Thus, it must be $\phi^{-1}(x) \| \phi^{-1}(y)$. Then, we have that $\phi^{-1}(x) \in M \subseteq K_{U_\phi}$. Thus, $\phi^{-1}(x) \in K_{U_\phi}$ for any $x \in K_U$.

Conversely, suppose that $\phi^{-1}(x) \in K_{U_\phi}$. Let us denote $\psi = \phi^{-1}$. Then, $\psi(e) = \phi^{-1}(e) = e$. We have that $\psi^{-1}(\phi^{-1}(x)) \in K_{(U_\phi)_\psi} = K_U$ for $\phi^{-1}(x) \in K_{U_\phi}$. Then, we obtain that

$$x = \phi(\phi^{-1}(x)) = \psi^{-1}(\phi^{-1}(x)) \in K_U.$$

□

Corollary 3.21. Let $(L, \leq, 0, 1)$ be a bounded lattice, $U \in \mathcal{U}(e)$ and ϕ an order-preserving bijection on L with $\phi(e) = e$. Then, $U \sim_K U_\phi$ iff $\phi^{-1}(x) \in K_U$ for any $x \in K_U$.

Theorem 3.22. Let $(L, \leq, 0, 1)$ be a bounded lattice and $U \in \mathcal{U}(e)$. If $K_{T_U} = \emptyset$ and $K_{S_U} = \emptyset$, then

$$K_U = \bigcup_{t \in I_e} (e \wedge t, e \vee t).$$

Proof. If $I_e = \emptyset$, then $M = \emptyset$. Clearly, $K_U = \emptyset$. Then, it is obvious that

$$K_U = \emptyset = \bigcup_{\emptyset} (e \wedge t, e \vee t).$$

Now, suppose that $I_e \neq \emptyset$. We have that $K_U = I_e \cup M$ since $K_U = K_{T_U} \cup K_{S_U} \cup I_e \cup M$ and $K_{T_U} = \emptyset = K_{S_U}$ by Theorem 3.3.

Let $x \in K_U$ be arbitrary.

- If $x \in I_e$, then it is obvious that $x \in \bigcup_{t \in I_e} (e \wedge t, e \vee t)$ since $x \in (e \wedge x, e \vee x)$.
- Let $x \notin I_e$. Then, it must be $x \in M$. There exists an element $y \in I_e$ such that $x \| y$.

  Suppose that $x \notin (e \wedge y, e \vee y)$.

  Let $x \| (e \wedge y)$. It must be $x \leq e$ or $e \leq x$ since $x \notin I_e$. If $x \leq e$, it would be $x \in K_{T_U}$ since $x \| (e \wedge y)$, but it contradicts that $K_{T_U} = \emptyset$. If $e \leq x$, we would get a contradiction with $x \| (e \wedge y)$ since $e \wedge y \leq e \leq x$.

  Let $x \| (e \vee y)$. It can be shown that this situation is not possible in a similar way to the previous situation.
  If $x \notin (e \wedge y, e \vee y)$, then it must be $x \leq e \wedge y$ or $x \geq e \vee y$. Let $x \leq e \wedge y$. Since $x \leq e \wedge y \leq y$, this contradicts that $x \| y$. If $e \vee y \leq x$, then we have a contradiction again since $y \leq e \vee y \leq x$. Thus, it must be $x \in (e \wedge y, e \vee y)$ for $y \in I_e$. Then, we have that

  $$x \in \bigcup_{t \in I_e} (e \wedge t, e \vee t).$$

  Thus,

  $$K_U \subseteq \bigcup_{t \in I_e} (e \wedge t, e \vee t).$$

  Conversely, suppose that $x \in \bigcup_{t \in I_e} (e \wedge t, e \vee t)$. We need to show that $x \in K_U$. If $x \in \bigcup_{t \in I_e} (e \wedge t, e \vee t)$, then there exists an element $t \in I_e$ such that $x \in (e \wedge t, e \vee t)$. If $x \in I_e$, then we obtain that $x \in K_U$ since $I_e \subseteq K_U$. Let $x \notin I_e$. Then, $x \leq e$ or $e \leq x$.

  Let $x \leq e$. If $x \leq t$, then we would have a contradiction since $x \leq (e \wedge t)$. If $t \leq x$, it is a contradiction since $t \in I_e$. Thus, it must be $x \| t$. Then, by the definition of K_U, we have that $x \in K_U$.

Let $e \leq x$. If $x \leq t$, then we would have $e \leq t$, which contradicts that $t \in I_e$. If $t \leq x$, it would be $e \vee t \leq x$. It is a contradiction. Then, it must be $x \| t$. Thus, it is clear that $x \in K_U$. Therefore,

$$\bigcup_{t \in I_e} (e \wedge t, e \vee t) \subseteq K_U.$$

□

Proposition 3.23. Let $(L, \leq)$ be a complete lattice and $U \in \mathcal{U}(e)$. If $K_{T_U} = \emptyset$ and $K_{S_U} = \emptyset$, then for any family $\{t_i | i \in I\} \subseteq I_e$,

$$K_U = (e \wedge (\bigwedge_{i \in I} t_i), e \vee (\bigvee_{i \in I} t_i)).$$

Proof. We know that $K_U = \bigcup_{t \in I_e} (e \wedge t, e \vee t)$ by Theorem 3.22.

Let $x \in \bigcup_{t \in I_e} (e \wedge t, e \vee t)$. Then, there exists an element $t \in I_e$ such that $x \in (e \wedge t, e \vee t)$. Since $e \wedge (\bigwedge_{i \in I} t_i) \leq e \wedge t < x < e \vee t \leq e \vee (\bigvee_{i \in I} t_i)$, we have that $x \in (e \wedge (\bigwedge_{i \in I} t_i), e \vee (\bigvee_{i \in I} t_i))$, whence

$$\bigcup_{t \in I_e} (e \wedge t, e \vee t) \subseteq (e \wedge (\bigwedge_{i \in I} t_i), e \vee (\bigvee_{i \in I} t_i)).$$

Conversely, let $x \in (e \wedge (\bigwedge_{i \in I} t_i), e \vee (\bigvee_{i \in I} t_i))$ be arbitrary. If $x = e$, then we have that $x \in \bigcup_{t \in I_e} (e \wedge t, e \vee t)$ since $e \wedge t < e < e \vee t$ for any $t \in I_e$. Let $x \neq e$. Then, there are three possible cases for x: $x \| e$ or $x < e$ or $e < x$.

- Let $x \| e$. Since $e \wedge x < x < e \vee x$ and $x \in I_e$, we obtain that $x \in \bigcup_{t \in I_e} (e \wedge t, e \vee t)$.
- Let $x < e$. Suppose that $x \notin (e \wedge t, e \vee t)$ for any $t \in I_e$. Since $x < e$ and $x \notin (e \wedge t, e \vee t)$, either $x \leq e \wedge t$ or $x \| (e \wedge t)$. Suppose that $x \leq e \wedge t$. For any $t \in I_e$, $x \leq e \wedge t \leq t$, whence $x \leq t$. $x \leq \bigwedge_i t_i$, $t_i \in I_e$ holds since $x \leq t$ for any $t \in I_e$. By $x = x \wedge e \leq e \wedge (\bigwedge_i t_i)$, a contradiction $x \in (e \wedge (\bigwedge_{i \in I} t_i), e \vee (\bigvee_{i \in I} t_i))$ is obtained. Let $x \| (e \wedge t)$ for any $t \in I_e$. Since $x, e \wedge t < e$ and $x \| (e \wedge t)$, it is clear that $x \in K_{T_U}$, it is a contradiction since $K_{T_U} = \emptyset$. Then, there must exist an element $t \in I_e$ such that $x \in (e \wedge t, e \vee t)$. Thus, $x \in \bigcup_{t \in I_e} (e \wedge t, e \vee t)$.
- Let $x > e$. Suppose that $x \notin (e \wedge t, e \vee t)$ for any $t \in I_e$. Since $x > e$ and $x \notin (e \wedge t, e \vee t)$, either $x \geq e \vee t$ or $x \| (e \vee t)$ for any $t \in I_e$. Let $x \geq e \vee t$. Then, for any $t \in I_e$, we obtain that $t \leq e \vee t \leq x$, whence $t \leq x$. Since $t \leq x$ for any $t \in I_e$, $\bigvee_i t_i \leq x$. By the monotonicity of the supremum, we have that

$$e \vee (\bigvee_i t_i) \leq e \vee x = x,$$

  which contradicts that $x \in (e \wedge (\bigwedge_{i \in I} t_i), e \vee (\bigvee_{i \in I} t_i))$. Let $x \| (e \vee t)$ for any $t \in I_e$. Since $x, e \vee t \geq e$ and $x \| (e \vee t)$, it is clear that $x \in K_{S_U}$. Since $K_{S_U} = \emptyset$, this is a contradiction. Then, there must exist an element $t \in I_e$ such that $x \in (e \wedge t, e \vee t)$, whence we have that $x \in \bigcup_{t \in I_e} (e \wedge t, e \vee t)$.

Thus, it is obtained that

$$K_U = \bigcup_{t \in I_e} (e \wedge t, e \vee t) = (e \wedge (\bigwedge_{i \in I} t_i), e \vee (\bigvee_{i \in I} t_i))$$

for any $\{t_i | i \in I\} \subseteq I_e$. □

4. CONCLUSIONS

After the definition of T-partial order on a bounded lattice, studies on the order obtained from uninorms have gained importance. In this paper, based on the U-partial order obtained from a uninorm, we study on the equivalence classes of uninorms on bounded lattice. In this respect, we present some relationships between the equivalence classes of uninorms and the equivalence classes of their underlying t-norms and t-conorms and we show that two idempotent uninorms are equivalent. Moreover, the equivalence classes of the greatest and smallest uninorms are determined and also some relationships between the sets admitting incomparability w.r.t. the U-partial order and its ϕ-conjugate are given. Finally, the set admitting incomparability w.r.t. the U-partial order under some special conditions is characterized.

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