KYBERNETIKA — VOLUME 55 (2019), NUMBER 3, PAGES 472–494

EXTENSIONS OF FUZZY CONNECTIVES ON ACDL

Hui Liu and Bin Zhao

The main goal of this paper is to construct fuzzy connectives on algebraic completely distributive lattice(ACDL) by means of extending fuzzy connectives on the set of completely join-prime elements or on the set of completely meet-prime elements, and discuss some properties of the new fuzzy connectives. Firstly, we present the methods to construct t-norms, t-conorms, fuzzy negations valued on ACDL and discuss whether De Morgan triple will be kept. Then we put forward two ways to extend fuzzy implications and also make a study on the behaviors of R-implication and reciprocal implication. Finally, we construct two classes of infinitely $\bigvee$-distributive uninorms and infinitely $\bigwedge$-distributive uninorms.

Keywords: extensions, algebraic completely distributive lattices, fuzzy connectives

Classification: 06D10, 03E72, 03B52

1. INTRODUCTION

Fuzzy logic has awakened the interest and curiosity of innumerable researchers in a variety of scientific areas due to its broad scope and the fact that it provides a good framework for constructing models which better approach reality. Emerging from the important work "Fuzzy Sets Theory" proposed by Zadeh in 1965, fuzzy logic typically considers for membership degree valued in the unit interval $[0, 1]$, but in modern fuzzy logic, lattices are used to range these degrees due to the close connections between fuzzy set theory and order theory. Since fuzzy connectives, such as conjunction, disjunction, negation and implication, play a significant role in the application of fuzzy logic, these connectives valued on lattices have been considered [2, 7, 9, 12, 17, 19, 20].

When there is a need to aggregate a large number of data in a given system with a single output, lattice-valued aggregation functions can be considered. Three classes of important lattice-valued aggregation functions are t-norms, t-conorms and uninorms. Uninorms were introduced as a generalization of both triangular norms and triangular conorms. Uninorm allows for a neutral element lying anywhere in the lattice rather than at the top element or the bottom element. Recently, Karaçal et al. studied the uninorms on bounded lattice and gave their characterizations in [10, 11]. Some other studies related to uninorms can also be found in [3–5, 8].

Various constructions of fuzzy connectives with certain definitions or by transformations from known functions generate important examples and classes of fuzzy connec-

DOI: 10.14736/kyb-2019-3-0472

tives. Particularly, fuzzy connectives are functions, then, if M is a subset of the bounded lattice L, it is reasonable to consider that under what conditions can fuzzy connectives defined on M be extended to L? The pioneer work in this framework was started by Saminger-Platz et al. who provided a method to extend t-norms from a complete sublattice M to a bounded lattice L in [18]. Then Palmeira and Bedregal [14, 15] gave extensions of t-norms (t-conorms, negations and fuzzy implications) via retractions. In 2014, Palmeira et al. [16] presented a way to extend fuzzy connectives by means of a special mapping named e-operator. Recently, based on the theory that every element of a finite distributive lattice has a unique irredundant $\vee$-decomposition and a unique irredundant $\wedge$-decomposition, Yılmaz and Kazancı [21] constructed t-norm(t-conorm) from a given behavior on the set of $\vee$-irreducible elements($\wedge$-irreducible elements) in the finite distributive lattice.

However, [21] only gave the constructions of t-norms and t-conorms on finite distributive lattices. What about the infinite cases? And what properties will the new t-norms have? Whether we can construct any other fuzzy connectives? The main idea of [21] is to construct t-norm on a finite distributive lattice from a given t-norm on the set of $\vee-$irreducible elements. In a finite distributive lattice L, the set of $\vee-$irreducible elements is join-dense in L, and the property of $\vee-$irreducible element can be cleverly applied to prove the associativity of the new t-norm. While for an infinite distributive lattice L, even if the set of $\vee-$irreducible elements is join-dense in L, the associativity can not be obtained. That is to say, $\vee-$irreducible element is no longer appropriate for the infinite cases. Fortunately, we discover that completely join-prime element is effective to prove the law of associativity, and the $\vee-$irreducible element is exactly the completely join-prime element in a finite distributive lattice. So we consider to use completely join-prime element to deal with the infinite cases. For an algebraic completely distributive lattice L, the set of completely join-prime elements is actually its join-dense subset, and it is exactly a finite distributive lattice when L is finite. Therefore, algebraic completely distributive lattice (ACDL) is the key for us to solve all the problems.

We begin in Section 2 with a specific formalization of the main concepts used throughout the paper such as algebraic completely distributive lattice (ACDL), t-norm, fuzzy implication, uninorm and so on. Inspired by the idea in [21], Section 3 is devoted to the study of completely join-prime element (completely meet-prime element) and the presentation of the approaches to extending t-norms, t-conorms and negations. Especially, the properties of De Morgan triple are discussed. Within the framework of extension, in Section 4 we extend fuzzy implications and make a study on the behaviors of two special classes of fuzzy implications, namely, R-implication and reciprocal implication. Finally, in Section 5, we pay our attention to the discussion about the extension methods for lattice-valued uninorms.

2. PRELIMINARIES

In this section, we will present and discuss some main concepts and results we are leading in this work which constitute the framework of our studies. For a further reading about such concepts, we recommend[1, 6, 8, 11, 13–15].

Given any ordered set P we can form a new ordered set P^{∂}(the dual of P) by defining $x \leq y$ to hold in P^{∂} if and only if $y \leq x$ holds in P. We consider a lattice to be an ordered structure $(L, \leq)$ in which for every two-element subset $\{x, y\}$ there exists the greatest lower bound or "meet" denoted by $x \wedge y$ and the least upper bound or "join" denoted by $x \vee y$. A lattice is complete if for every subset there exist the meet and the join. A lattice which possesses the smallest (the bottom) and the greatest (the top) element, 0 and 1, respectively is bounded. Given an arbitrary subset A of a lattice L and $x \in L$, we define $\downarrow A = \{y \in L \mid (\exists x \in A) y \leq x\}$, $\downarrow x = \downarrow \{x\}$.

Definition 2.1. (Davey and Priestley [6]) Let P be an ordered set and let $x, y \in P$. We say x is covered by y (or y covers x), and write $x \prec y$ or $y \succ x$, if $x < y$ and $x \leq z < y$ implies $z = x$.

Definition 2.2. (Davey and Priestley [6]) Let P be an ordered set and let $Q \subseteq P$. Then Q is called join-dense in P if for every element $a \in P$ there is a subset A of Q such that $a = \bigvee_P A$. The dual of join-dense is meet-dense.

Theorem 2.3. (Davey and Priestley [6]) Let L be a complete lattice and $Q \subseteq L$, then Q is join-dense in L if and only if $a = \bigvee_L (\downarrow a \cap Q)$ for all $a \in L$.

Definition 2.4. (Davey and Priestley [6]) An element x of a complete lattice L is called completely join-prime if for every subset S of L, $x \leq \bigvee S$ implies that $x \leq s$ for some $s \in S$; in particular, $x \neq 0$. Completely meet-prime is defined dually.

We denote the set of completely join-prime elements of L by $\mathcal{J}_p(L)$ and the set of completely meet-prime elements of L by $\mathcal{M}_p(L)$.

Definition 2.5. (Davey and Priestley [6]) Let L be a complete lattice and $k \in L$. k is said to be compact if, for every subset S of L, $k \leq \bigvee S \Longrightarrow k \leq \bigvee T$ for some finite subset T of S. The set of compact elements of L is denoted $K(L)$.

Definition 2.6. (Davey and Priestley [6]) A complete lattice L is said to be algebraic if, for each $a \in L$,

$$a = \bigvee \{k \in K(L) \mid k \leq a\}.$$

Definition 2.7. (Davey and Priestley [6]) A complete lattice L is said to be completely distributive if, for any doubly indexed subset $\{x_{ij}\}_{i \in I, j \in J}$ of L, we have

$$\bigwedge_{i \in I} (\bigvee_{j \in J} x_{ij}) = \bigvee_{\alpha : I \to J} (\bigwedge_{i \in I} x_{i\alpha(i)}),$$

where $\alpha : I \to J$ is a function.

It should be noted that a complete lattice is called algebraic completely distributive lattice if it is algebraic and completely distributive. Moreover, in order to ascertain further subject, we shall denote the algebraic completely distributive lattice by ACDL.

Definition 2.8. (Davey and Priestley [6]) A complete lattice L is said to satisfy the Join-Infinite Distributive Law (JID) if, for any subset $\{y_j\}_{j \in J}$ of L and any $x \in L$,

$$x \wedge \bigvee_{j \in J} y_j = \bigvee_{j \in J} x \wedge y_j.$$

The dual condition is the Meet-Infinite Distributive Law (MID).

Theorem 2.9. (Davey and Priestley [6]) Let L be a lattice. Then the following are equivalent:

(1) L is distributive and both L and L^{∂} are algebraic;
(2) L is complete, L satisfies (JID) and the completely join-prime elements are join-dense;
(3) L is complete, L satisfies (MID) and the completely meet-prime elements are meet-dense;
(4) L is an ACDL.

Definition 2.10. (Palmeira and Bedregal [14]) Let $(L, \leq, 0, 1)$ be a bounded poset.

(1) A triangular norm (i. e. t-norm) is a binary operation on L that is monotone, commutative, associative and with neutral element 1.
(2) A triangular conorm (i. e. t-conorm) is a binary operation on L that is monotone, commutative, associative and with neutral element 0.

The weakest t-norm and the strongest t-conorm, respectively, on a bounded poset L are

$$T_D(x, y) = \begin{cases} \min(x, y) & \text{if } y = 1 \text{ or } x = 1, \\ 0 & \text{otherwise,} \end{cases}$$

$$S_D(x, y) = \begin{cases} \max(x, y) & \text{if } y = 0 \text{ or } x = 0, \\ 1 & \text{otherwise.} \end{cases}$$

Definition 2.11. (Palmeira and Bedregal [14]) Let $(L, \leq, 0, 1)$ be a bounded poset. A mapping $N : L \to L$ is a fuzzy negation on L if the following properties are satisfied: for all $x, y \in L$,
(1) If $x \leq y$ then $N(y) \leq N(x)$,
(2) $N(0) = 1$ and $N(1) = 0$.
Moreover, a fuzzy negation N is strong if it also satisfies the involution property, i. e. $N(N(x)) = x$ for each $x \in L$.

The following are the weakest fuzzy negation and the strongest fuzzy negation, respectively, on a bounded poset L:

$$N_0(x) = \begin{cases} 1 & \text{if } x = 0, \\ 0 & \text{otherwise,} \end{cases}$$

$$N_1(x) = \begin{cases} 0 & \text{if } x = 1, \\ 1 & \text{otherwise.} \end{cases}$$

Definition 2.12. (Palmeira and Bedregal [14]) Let T be a t-norm, S be a t-conorm and N be a fuzzy negation, all of which are defined on bounded poset L. A triple $\langle T, S, N\rangle$ is a De Morgan triple if, for all $x, y \in L$ we have $N(T(x, y)) = S(N(x), N(y))$ and $N(S(x, y)) = T(N(x), N(y))$.

Definition 2.13. (Palmeira [15]) Let $(L, \leq, 0, 1)$ be a bounded poset. An implication I on L is a hybrid monotonous (with decreasing first and increasing second partial mapping) binary operation that satisfies the corner conditions: $I(0, 0) = I(1, 1) = 1$ and $I(1, 0) = 0$.

Moreover, we define the following properties of fuzzy implication I:

(LB) $I(0, y) = 1$ for all $y \in L$ (left boundary condition);

(RB) $I(x, 1) = 1$ for all $x \in L$ (right boundary condition);

(IP) $I(x, x) = 1$ for each $x \in L$ (identity principle);

(NP) $I(1, y) = y$ for each $y \in L$ (left neutrality principle);

(LOP) $\forall\ x, y \in L$, if $x \leq y$ then $I(x, y) = 1$ (left ordering property);

(CP) $\forall\ x, y \in L$, $I(x, y) = I(N(y), N(x))$ with N being a strong negation (contrapositivity property);

(L-CP) $\forall\ x, y \in L$, $I(N(x), y) = I(N(y), x)$ with N being a strong negation (left contrapositivity law).

Definition 2.14. (Baczyński and Jayaram [1]) Let S be a t-conorm and N be a fuzzy negation. We say that the pair (S, N) satisfies the law of excluded middle if

$$S(N(x), x) = 1, \qquad x \in L.$$

Definition 2.15. (Palmeira [15]) Let $(L, \leq, 0, 1)$ be a complete lattice. A function $I : L^2 \to L$ is called an R-implication if there exists a t-norm T such that for all $x, y \in L$ we have

$$I(x, y) = \bigvee \{t \in L \mid T(x, t) \leq y\}.$$

We denote this implication generated from a t-norm T by I_T.

Definition 2.16. (Palmeira [15]) Let $(L, \leq, 0, 1)$ be a bounded poset. If I is a fuzzy implication and N is a fuzzy negation on L, then the function $I_N : L^2 \to L$ defined by $I_N(x, y) = I(N(y), N(x))$ is a fuzzy implication on L and is called the N-reciprocal of I. When N is strong, then I_N is called the reciprocal implication of I.

Definition 2.17. (Baczyński and Jayaram [1]) Let $(L, \leq, 0, 1)$ be a bounded poset. If I is a fuzzy implication on L, then the function $N_I : L \to L$ defined for each $x \in L$ by $N_I(x) = I(x, 0)$ is a fuzzy negation on L and is called the natural negation of I.

Definition 2.18. (Deschrijver [8]) Let $(L, \leq, 0, 1)$ be a bounded poset. An associative, commutative and increasing operation $U : L^2 \to L$ is called a uninorm if it has a neutral element $e \in L$, i. e. $U(e, x) = x$, for all $x \in L$. Moreover, if $U(0, 1) = 0$ then U is called conjunctive uninorm; if $U(0, 1) = 1$ then U is called disjunctive uninorm.

3. EXTENSION OF T-NORMS, T-CONORMS AND NEGATIONS

There is a close relationship between the structure of a lattice and the t-norms which can be defined on the lattice. Inspired by the idea from part to whole, Yılmaz and Kazancı presented a method to construct new t-norms on finite distributive lattice from given t-norms on the set of $\vee$-irreducible elements in [21] without discussing any properties. While we find that this method is not only applicable to finite cases but also to some infinite situations. The following example provides an illustration:

Example 3.1. Consider an ordinal sum L of lattices $(L_k, \leq_k, 0_k, 1_k)$, $k \in K$, where the index set K is an infinite bounded chain with top element 1_K and bottom element 0_K such that $L_{k_1} \bigcap L_{k_2}$ is a singleton coinciding with $1_{min(k_1,k_2)}$ and with $0_{\max(k_1,k_2)}$, and for each $k \in K$, $L_k = \{0_k, a_k, b_k, 1_k\}$ is diamond lattice, i. e., L_k can be seen as a horizontal sum of two chain $\{0_k, a_k, 1_k\}$ and $\{0_k, b_k, 1_k\}$. Then $L = \bigcup_{k\in K} L_k$, and $x \leq y$ whenever $x \in L_{k_1}$, $y \in L_{k_2}$, and either $k_1 < k_2$ or $k_1 = k_2 = k$ and $x \leq y$. Clearly, L is an infinite distributive lattice and a simple inference shows that the set of $\vee$-irreducible elements is $J(L) = \bigcup_{k\in K}\{a_k, b_k\}$, which coincides with $\mathcal{J}_p(L)$ and is join-dense in L. Concerning the set $J^\star(L) = \mathcal{J}_p(L) \bigcup \{0, 1\}$, it suffices to define t-norms on it. Take any t-norm T on $J^\star(L)$, by a simple inference, we can obtain that for all $x, y \in L$, $\overline{T}(x, y) = \bigvee_{a\in(\downarrow x \bigcap J^\star(L))} \bigvee_{b\in(\downarrow y \bigcap J^\star(L))} T(a, b)$ is a t-norm on L. That is to say, the construction of Theorem 5.4 in [21] is applicable for this infinite lattice.

In fact, the lattice given in Example 3.1 is an ACDL, and the set of $\vee-$irreducible elements coincides with the set of completely join-prime elements. Not every ACDL has the property that $\mathcal{J}_p(L) = J(L)$. The following is a counterexample:

Example 3.2. Let $L = \{1\} \bigcup \{1 + \frac{1}{n}\}_{n\in N^+} \bigcup \{1 - \frac{1}{n}\}_{n\in N^+}$. One can verify that L is an ACDL, $J(L) = L \setminus \{0\}$, $\mathcal{J}_p(L) = \{1 + \frac{1}{n}\}_{n\in N^+} \bigcup \{1 - \frac{1}{n}\}_{n\in N^+\setminus\{1\}}$. Obviously, $J(L) \neq \mathcal{J}_p(L)$.

Now we consider to construct new t-norm from a given t-norm defined on the set of completely join-prime elements for every ACDL and discuss some related properties. As we all know, t-norms should be defined on the bounded posets. That is to say, for an ACDL L, if we want to define a t-norm on $\mathcal{J}_p(L)$, it is necessary for $\mathcal{J}_p(L)$ to have a top element and a bottom element. By the definition of completely join-prime element, we know that 0 is not a completely join-prime element. We have the following statements about $\mathcal{J}_p(L)$:

Proposition 3.3. Let L be an ACDL. If $x \in \mathcal{J}_p(L)$, then there exists a unique $p \in L$ such that $p \prec x$.

Proof. Let $x \in \mathcal{J}_p(L)$, then $x \neq 0$. So there exists $p \in L$ such that $p \prec x$. Assume $p_1 \neq p$ and $p_1 \prec x$, then it holds that $x = p \vee p_1$, which leads to an obvious contradiction with $x \in \mathcal{J}_p(L)$. Thus, there exists a unique $p \in L$ such that $p \prec x$. □

Proposition 3.4. Let L be an ACDL. If $\mathcal{J}_p(L)$ has a top element, then $1 \in \mathcal{J}_p(L)$.

Proof. Let x be the top element of $\mathcal{J}_p(L)$, then $a \leq x$ for all $a \in \mathcal{J}_p(L)$. It follows that $\bigvee \mathcal{J}_p(L) \leq x$. By Theorem 2.9 and Theorem 2.3, $1 = \bigvee(\downarrow 1 \bigcap \mathcal{J}_p(L)) = \bigvee \mathcal{J}_p(L)$. Thus, $x = 1$. □

By Proposition 3.4, if $1 \notin \mathcal{J}_p(L)$, then $\mathcal{J}_p(L)$ fails to have a top element. It can also be explained by the following example:

Example 3.5. Let L be the lattice in Example 3.1, then $\mathcal{J}_p(L) = \bigcup_{k\in K}\{a_k, b_k\}$. Clearly, $1 \notin \mathcal{J}_p(L)$ and $\mathcal{J}_p(L)$ has neither a top element nor a bottom element.

In order to make up this situation, we put elements 0 and 1 into the set $\mathcal{J}_p(L)$. Thus, we present the following definition.

Definition 3.6. (Yılmaz and Kazancı [21]) The set $\mathcal{J}_p(L)^\star = \mathcal{J}_p(L)\bigcup\{0,1\}$ is called the extended set of completely join-prime elements of L.

Similarly, we can discuss the construction of t-conorm by means of completely meet-prime elements, and all the dual statements hold for completely meet-prime elements.

Proposition 3.7. Let L be an ACDL. If $x \in \mathcal{M}_p(L)$, then there exists a unique $p \in L$ such that $x \prec p$.

P r o o f. It can be proved in a similar way as Proposition 3.3. $\square$

Proposition 3.8. Let L be an ACDL. If $\mathcal{M}_p(L)$ has a bottom element, then $0 \in \mathcal{M}_p(L)$.

P r o o f. Analogously to Proposition 3.4. $\square$

Definition 3.9. (Yılmaz and Kazancı [21]) The set $\mathcal{M}_p(L)^\star = \mathcal{M}_p(L)\bigcup\{0,1\}$ is called the extended set of completely meet-prime elements of L.

Remark 3.10. When L is an ACDL, $\mathcal{J}_p(L)^\star$ and $\mathcal{M}_p(L)^\star$ are bounded posets, not necessarily lattices.

Example 3.11. Let L be the lattice given in Example 3.1, then $\mathcal{J}_p(L)^\star = \mathcal{M}_p(L)^\star = \bigcup_{k\in K}\{a_k, b_k\}\bigcup\{0,1\}$. Take any $k_0 \in K \setminus \{0_K, 1_K\}$, because a_{k_0} and b_{k_0} have no least upper bound or greatest lower bound, $a_{k_0} \vee b_{k_0}$ and $a_{k_0} \wedge b_{k_0}$ don't exist. That is to say, $\mathcal{J}_p(L)^\star$ and $\mathcal{M}_p(L)^\star$ are not lattices.

Lemma 3.12. Let L be an ACDL. Then every element x of L has a representation as $x = \bigvee \eta(x)$, where $\eta(x) = \{j \in \mathcal{J}_p(L)^\star \mid j \le x\}$.

P r o o f. For all $x \ne 1$, it holds that $\eta(x) = \{j \in \mathcal{J}_p(L) \mid j \le x\} \cup \{0\}$. We know from Theorem 2.9 that $\mathcal{J}_p(L)$ is join-dense in L. So combined with Theorem 2.3, it shows that $x = \bigvee_L\{j \in \mathcal{J}_p(L) \mid j \le x\}$ for all $x \in L$. Thus, $\bigvee \eta(x) = \bigvee_L(\downarrow x \cap \mathcal{J}_p(L)) = x$. If $x = 1$, then $\bigvee \eta(1) = \bigvee(\{j \in \mathcal{J}_p(L) \mid j \le 1\} \cup \{0,1\}) = 1$. $\square$

Example 3.13. Let L be the lattice mentioned in Example 3.1. For any $x \in L$,
if $x = 1$, then $\eta(x) = \eta(1) = \mathcal{J}_p(L)^\star = \bigcup_{k\in K}\{a_k, b_k\}\bigcup\{0,1\}$;
if $x = a_k$, then $\eta(x) = \eta(a_k) = \bigcup_{i<k}\{a_i, b_i\}\bigcup\{0, a_k\}$;
if $x = b_k$, then $\eta(x) = \eta(b_k) = \bigcup_{i<k}\{a_i, b_i\}\bigcup\{0, b_k\}$;
if $x = 0_k$, then $\eta(x) = \eta(0_k) = \bigcup_{i<k}\{a_i, b_i\}\bigcup\{0\}$.
By a simple calculation, $x = \bigvee \eta(x)$ for all $x \in L$.

Theorem 3.14. Let L be an ACDL and T be a t-norm on $\mathcal{J}_p(L)^\star$. Then the binary operation $\overline{T}$ on L defined by

$$\overline{T}(x,y) = \bigvee_{j\in\eta(x)} \bigvee_{h\in\eta(y)} T(j,h)$$

is a t-norm on L.

P r o o f. Clearly, $\overline{T}$ is well-defined.

(1) For all $x \in L$, $\overline{T}(x,1) = \bigvee j \in \eta(x) \bigvee_{h\in\eta(1)} T(j,h) = \bigvee_{j\in\eta(x)} T(j,1) = \bigvee_{j\in\eta(x)} j = x$.

(2) By the commutativity of T, we can get that $\overline{T}$ is commutative.

(3) Let $x,y,z \in L$ and $y \le z$, then $\eta(y) \subseteq \eta(z)$. Thus, $\overline{T}(x,y) \le \overline{T}(x,z)$. That is to say, $\overline{T}$ is monotone.

(4) Take any $x,y,z \in L$.

If $\overline{T}(x,y) = 1$, then $x = y = 1$. Obviously, $\overline{T}(\overline{T}(x,y),z) = \overline{T}(x,\overline{T}(y,z))$.
If $\overline{T}(x,y) < 1$, we have

$$\begin{aligned}\overline{T}(\overline{T}(x,y),z) &= \bigvee_{p\in\eta(\overline{T}(x,y))} \bigvee_{k\in\eta(z)} T(p,k)\\ &= \bigvee_{p\in\eta(\bigvee_{j\in\eta(x)} \bigvee_{h\in\eta(y)} T(j,h))} \bigvee_{k\in\eta(z)} T(p,k).\end{aligned}$$

Let p be an element of $\mathcal{J}_p(L)^\star$ and $p \le \bigvee_{j\in\eta(x)} \bigvee_{h\in\eta(y)} T(j,h)$, then there exist $j \in \eta(x)$, $h \in \eta(y)$ such that $p \le T(j,h)$. By the associativity of T, it holds that $T(p,k) \le T(j,T(h,k))$ for all $k \in \eta(z)$. Since $T(h,k) \in \eta(\overline{T}(y,z))$, we have $T(p,k) \le \bigvee_{j\in\eta(x)} \bigvee_{q\in\eta(\overline{T}(y,z))} T(j,q) = \overline{T}(x,\overline{T}(y,z))$. Similarly, it can be proved that $\overline{T}(x,\overline{T}(y,z)) \le \overline{T}(\overline{T}(x,y),z)$. That is, $\overline{T}(\overline{T}(x,y),z) = \overline{T}(x,\overline{T}(y,z))$.

Thus, $\overline{T}$ is a t-norm on L. □

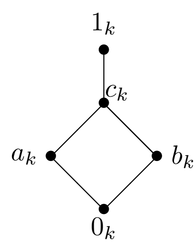

Fig. 1.

Example 3.15. Let L be the ordinal sum of lattices $(L_k, \le_k, 0_k, 1_k)$, $k \in K$, where the index set K is an infinite bounded chain with top element 1_K and bottom element 0_K such that $L_{k_1} \bigcap L_{k_2}$ is a singleton coinciding with $1_{\min(k_1,k_2)}$ and with $0_{\max(k_1,k_2)}$, and for each $k \in K$, $L_k = \{0_k, a_k, b_k, c_k, 1_k\}$ is the lattice shown in Figure 1. Then $L = \bigcup_{k\in K} L_k$, and $x \leq y$ whenever $x \in L_{k_1}$, $y \in L_{k_2}$, and either $k_1 < k_2$ or $k_1 = k_2 = k$ and $x \leq y$. Clearly, L is a complete lattice and satisfies (JID), and $\mathcal{J}_p(L) = \bigcup_{k\in K}\{0_k, a_k, b_k, 1_k\}\backslash\{0\}$ is join-dense in L. By Theorem 2.9, L is an ACDL. It follows from Definition 3.6 that $\mathcal{J}_p(L)^\star = \bigcup_{k\in K}\{0_k, a_k, b_k, 1_k\}$. Now, let $k_0 \in K \setminus \{0_K, 1_K\}$, we define a t-norm on $\mathcal{J}_p(L)^\star$ by

$$T(x,y) = \begin{cases} 0_{k_0}, & \text{if } (x,y) \in \{a_{k_0}, b_{k_0}\}^2, \\ x \wedge y, & \text{otherwise.} \end{cases}$$

Thus, by Theorem 3.14,

$$\overline{T}(x,y) = \begin{cases} 0_{k_0}, & \text{if } (x,y) \in \{a_{k_0}, b_{k_0}, c_{k_0}\}^2, \\ x \wedge y, & \text{otherwise,} \end{cases}$$

is a t-norm on L.

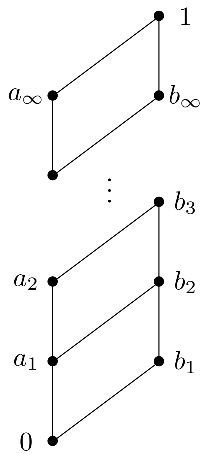

Fig. 2.

Example 3.16. Consider the lattice L as shown in Figure 2. It is easy to check that L is an ACDL. Let $L_1 = \bigcup_{n=1}^{\infty} a_n$ and T be a t-norm on L_1, then we can define a t-norm on $\mathcal{J}_p(L)^\star = \{0, b_1, 1\} \bigcup (\bigcup_{n=1}^{\infty} a_n)$ by

$$T^{'}(x,y) = \begin{cases} T(x,y) & \text{if } x, y \in L_1, \\ x \wedge y & \text{otherwise.} \end{cases}$$

Thus, by Theorem 3.14,

$$\overline{T^{'}}(x,y) = \begin{cases} T(x,y) & \text{if } x, y \in L_1, \\ T(a_i, a_{j-1}) & \text{if } x = a_i, y = b_j, i, j \in N_+, j \geq 2, \\ T(a_{i-1}, a_{j-1}) \vee b_1 & \text{if } x = b_i, y = b_j, i, j \in N_+, i, j \geq 2, \\ x \wedge y & \text{otherwise,} \end{cases}$$

is a t-norm on L.

Remark 3.17. The lattice L shown in Figure 3 is also an ACDL, and $\mathcal{J}_p(L)^\star = \{0,1\}\bigcup(\bigcup_{n=1}^{\infty} a_n)\bigcup(\bigcup_{n=1}^{\infty} b_n)$. For any t-norm on $\mathcal{J}_p(L)^\star$, by Theorem 3.14, we can construct t-norm on L.

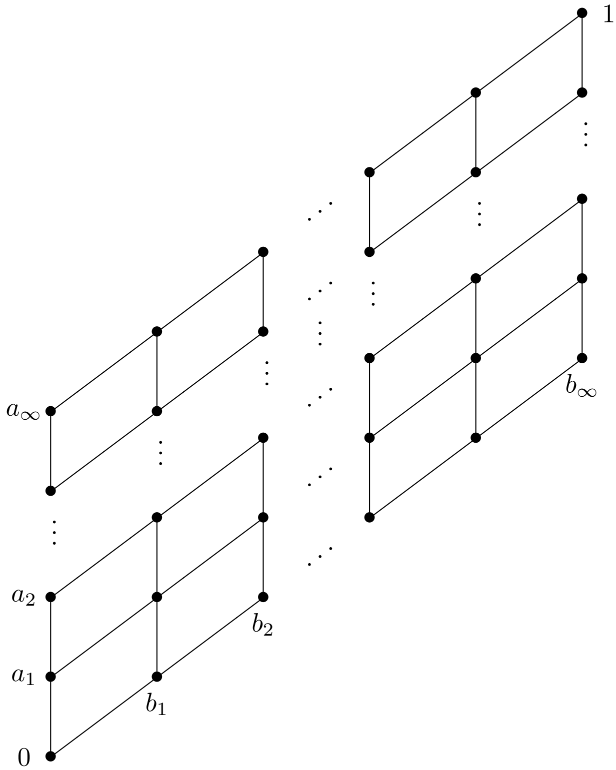

Fig. 3.

Dually, t-conorm can be constructed by means of completely meet-prime elements.

Lemma 3.18. Let L be an ACDL. Then every element x of L has a representation as $x = \bigwedge \xi(x)$, where $\xi(x) = \{h \in \mathcal{M}_p(L)^\star \mid h \geq x\}$.

Proof. It can be proved with Lemma 3.12 in an analogous way. □

Theorem 3.19. Let L be an ACDL and S be a t-conorm on $\mathcal{M}_p(L)^\star$. Then the binary operation $\overline{S}$ on L defined by

$$\overline{S}(x,y) = \bigwedge_{j\in\xi(x)} \bigwedge_{h\in\xi(y)} S(j,h)$$

is a t-conorm on L.

Proof. Analogously to Theorem 3.14. □

After describing how to extend t-norms and t-conorms, a natural question that arises is to discuss the De Morgan triple. To start with, we extend fuzzy negation.

Theorem 3.20. Let L be an ACDL and N be a fuzzy negation on $\mathcal{J}_p(L)^\star$. Then the unary operation $\overline{N}$ on L defined by

$$\overline{N}(x) = \bigwedge_{j \in \eta(x)} N(j)$$

is a fuzzy negation on L.

P r o o f. Obviously, $\overline{N}$ is well-defined.

(1) Let $x, y \in L$ and $x \leq y$, then $\eta(x) \subseteq \eta(y)$. Thus, $\{N(j)\}_{j \in \eta(x)} \subseteq \{N(j)\}_{j \in \eta(y)}$. Hence, $\overline{N}(x) \geq \overline{N}(y)$.

(2) $\overline{N}(0) = \bigwedge_{j \in \eta(0)} N(j) = N(0) = 1$.

(3) $\overline{N}(1) = \bigwedge_{j \in \eta(1)} N(j) = N(1) = 0$.

Thus, $\overline{N}$ is a fuzzy negation on L. □

Theorem 3.21. Let L be an ACDL and N be a fuzzy negation on $\mathcal{M}_p(L)^\star$. Then the unary operation $\widetilde{N}$ on L defined by

$$\widetilde{N}(x) = \bigvee_{h \in \xi(x)} N(h)$$

is a fuzzy negation on L.

P r o o f. The proof is similar to that of Theorem 3.20. □

Example 3.22. Let L be the lattice in Example 3.1, then $\mathcal{J}_p(L)^\star = \mathcal{M}_p(L)^\star$.

(1) If the fuzzy negation on $\mathcal{M}_p(L)^\star$ is the weakest fuzzy negation, then by Theorem 3.20 and Theorem 3.21, we can get that $\overline{N} = \widetilde{N}$ is the weakest fuzzy negation on L.

(2) Let $k_0 \in K \setminus \{0_K, 1_K\}$, we define a fuzzy negation on $\mathcal{M}_p(L)^\star$ by

$$N(u) = \begin{cases} 1 & \text{if } u < 0_{k_0}, \\ 0 & \text{otherwise,} \end{cases}$$

then $\overline{N}$ and $\widetilde{N}$ can be given as follows: for all $x \in L$,

$$\overline{N}(x) = \begin{cases} 1 & \text{if } x \leq 0_{k_0}, \\ 0 & \text{otherwise,} \end{cases}$$

$$\widetilde{N}(x) = \begin{cases} 1 & \text{if } x < 0_{k_0}, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, $\overline{N} \neq \widetilde{N}$.

From the above example, we know that $\overline{N}$ can be different from $\widetilde{N}$ even if $\mathcal{J}_p(L)^\star = \mathcal{M}_p(L)^\star$ and they have the same N on $\mathcal{J}_p(L)^\star$.

While discussing De Morgan triple (T, S, N), every operation should be defined on the same bounded poset. Determined by the ways we construct $\overline{T}$ and $\overline{S}$, it is necessary to consider whether $\mathcal{J}_p(L)^\star = \mathcal{M}_p(L)^\star$ holds for an ACDL L.

Example 3.23.

(1) Consider the lattice L in Example 3.15, then $\mathcal{J}_p(L)^\star = \bigcup_{k\in K}\{0_k, a_k, b_k, 1_k\}$. It can be checked that $\mathcal{M}_p(L)^\star = \bigcup_{k\in K}\{a_k, b_k, c_k\}\bigcup\{0,1\}$. Clearly, $\mathcal{J}_p(L)^\star \neq \mathcal{M}_p(L)^\star$.

(2) Let L be the lattice in Example 3.16, then $\mathcal{J}_p(L)^\star = \{0, b_1, 1\}\bigcup(\bigcup_{n=1}^{\infty} a_n)$. By Definition 3.9, $\mathcal{M}_p(L)^\star = \{0, a_\infty, 1\}\bigcup(\bigcup_{n=1}^{\infty} b_n)$. Thus, $\mathcal{J}_p(L)^\star \neq \mathcal{M}_p(L)^\star$.

We know from Example 3.23 that not every ACDL satisfies $\mathcal{J}_p(L)^\star = \mathcal{M}_p(L)^\star$. So it is important to discuss when such a condition in fact works.

Proposition 3.24. Let L be an ACDL. Then the following statements are equivalent:
(1) $\mathcal{J}_p(L)^\star = \mathcal{M}_p(L)^\star$;
(2) $\forall\ a \in L \setminus \{0,1\}$,
$$a = \bigvee\{x \in L \mid x < a\} \text{ if and only if } a = \bigwedge\{x \in L \mid x > a\}.$$

Proof. (1) $\Rightarrow$ (2) For all $a \in L\setminus\{0,1\}$, if $a = \bigvee\{x \in L \mid x < a\}$, then $a \notin \mathcal{J}_p(L)^\star$, that is $a \notin \mathcal{M}_p(L)^\star$. Since L is an ACDL, we have $a = \bigwedge(\uparrow a \bigcap \mathcal{M}_p(L)) = \bigwedge\{t \in \mathcal{M}_p(L) \mid t > a\} = \bigwedge\{t \in L \mid t > a\}$. Similarly, the left part can be proved.

(2) $\Rightarrow$ (1) Let $x \in \mathcal{J}_p(L)\setminus\{1\}$. We will show $x \in \mathcal{M}_p(L)\setminus\{0\}$. In fact, $x \neq 0$ and $x \neq 1$, and for every subset $\{x_i\}_{i\in I} \subseteq L$, if $x \geq \bigwedge_{i\in I} x_i$, then $x = x \vee (\bigwedge_{i\in I} x_i) = \bigwedge_{i\in I}(x \vee x_i)$. Suppose $x \ngeq x_i$ for all $i \in I$, then $x < x_i$ or $x \parallel x_i$. It follows that $x \vee x_i > x$ for all $i \in I$. Thus, $x = \bigwedge_{i\in I}(x \vee x_i) = \bigwedge\{t \in L \mid t > x\}$. By assumption, $x = \bigvee\{t \in L \mid t < x\}$, which is an obvious contradiction with $x \in \mathcal{J}_p(L)$. So there exists $i \in I$ such that $x \geq x_i$. Thus, $x \in \mathcal{M}_p(L) \setminus \{0\}$. Consequently, it holds that $\mathcal{J}_p(L)^\star \subseteq \mathcal{M}_p(L)^\star$. $\mathcal{M}_p(L)^\star \subseteq \mathcal{J}_p(L)^\star$ can be proved similarly. □

Subsequently, we will discuss whether De Morgan triple will be preserved when N is a strong fuzzy negation.

Lemma 3.25. Let L be an ACDL satisfying $\mathcal{J}_p(L)^\star = \mathcal{M}_p(L)^\star$, and N be a strong fuzzy negation on $\mathcal{J}_p(L)^\star$. Then the following statements hold:
(1) If $\overline{N}(x) = 0$, then $x = 1$;
(2) $\overline{N} = \widetilde{N}$.

Proof. (1) Let $\overline{N}(x) = 0$. Suppose $x \neq 1$, since N is a strong fuzzy negation on $\mathcal{M}_p(L)^\star$, it holds that $x \notin \mathcal{M}_p(L)^\star$. Thus, $h > x$ for all $h \in \xi(x)$. By the monotonicity of $\overline{N}$, $N(h) \leq \overline{N}(x) = 0$ for all $h \in \xi(x)$. As N is strong on $\mathcal{J}_p(L)^\star$, $h = 1$ for all $h \in \xi(x)$. It follows that $x = \bigwedge_{h\in\xi(x)} h = 1$, a contradiction. Thus, $x = 1$ holds.

(2) Suppose that N is a strong fuzzy negation on $\mathcal{J}_p(L)^\star$. Since $\mathcal{J}_p(L)^\star = \mathcal{M}_p(L)^\star$, we have $\overline{N}(x) = \widetilde{N}(x)$ for all $x \in \mathcal{J}_p(L)^\star$.

For each $x \in L \setminus \mathcal{J}_p(L)^\star$, then $j < h$ for all $j \in \eta(x)$, $h \in \xi(x)$. Since N is strong, then $N(j) > N(h)$. By Theorem 3.20 and 3.21, $\overline{N}(x) \geq \widetilde{N}(x)$.

Since $x \notin \mathcal{J}_p(L)^\star$ and N is strong, we can claim $\overline{N}(x) \notin \mathcal{J}_p(L)^\star$. If $\overline{N}(x) = 0$, then we can get $x = 1$ by (1), which contradicts with $x \in L \setminus \mathcal{J}_p(L)^\star$. If $\overline{N}(x) = 1$, then by the construction of $\overline{N}$, we have $x = 0$, a contradiction. If $\overline{N}(x) \in \mathcal{J}_p(L) \setminus \{1\}$, then $\overline{N}(x) \in \mathcal{M}_p(L) \setminus \{0\}$. Thus, there exits $a \in \eta(x)$ such that $\overline{N}(x) = N(a)$. Since $\overline{N}$ is a fuzzy negation, then $N(j) \geq \overline{N}(x)$ holds for all $j \in \eta(x)$. It means that $N(j) \geq N(a)$ for all $j \in \eta(x)$. As N is strong, then $j \leq a$ for all $j \in \eta(x)$. It follows that $x = a$, a contradiction. Consequently, $\overline{N}(x) \notin \mathcal{J}_p(L)^\star$.

As L is an ACDL, we have $\overline{N}(x) = \bigvee \eta(\overline{N}(x))$. For each $a \in \eta(\overline{N}(x))$, then $a \in \mathcal{J}_p(L)^\star$ and $a < \overline{N}(x)$. Since N is strong and $\overline{N}$ is a fuzzy negation, it holds that $N(a) > x$. That is, $N(a) \in \xi(x)$. Thus, $\{N(a) \mid a \in \eta(\overline{N}(x))\} \subseteq \xi(x)$. It follows that $\bigvee \eta(\overline{N}(x)) \leq \bigvee_{h\in\xi(x)} N(h)$, i.e., $\overline{N}(x) \leq \widetilde{N}(x)$. Thus, $\overline{N}(x) = \widetilde{N}(x)$ for all $x \in L \setminus \mathcal{J}_p(L)^\star$.

□

Theorem 3.26. Let L be an ACDL with $\mathcal{J}_p(L)^\star = \mathcal{M}_p(L)^\star$. If N is a strong fuzzy negation on $\mathcal{J}_p(L)^\star$, then $\overline{N}$ is a strong fuzzy negation on L.

Proof. For all $x \in L$, we shall prove $\overline{N}(\overline{N}(x)) = x$.

Case 1. If $\overline{N}(x) = 0$, then by Lemma 3.25(1), we have $x = 1$. Thus, $\overline{N}(\overline{N}(x)) = \overline{N}(0) = 1 = x$.

Case 2. If $\overline{N}(x) > 0$, by Lemma 3.25(2), we can get

$$\overline{N}(\overline{N}(x)) = \widetilde{N}(\overline{N}(x)) = \bigvee_{j\in\xi(\overline{N}(x))} N(j) = \bigvee_{j\in\xi(\bigwedge_{h\in\eta(x)} N(h))} N(j).$$

On the one hand, for each $j \in \xi(\bigwedge_{h\in\eta(x)} N(h))$, then $j \geq \bigwedge h \in \eta(x)N(h)$ and $j \in \mathcal{M}_p(L)^\star$. Thus, there exists $h \in \eta(x)$ such that $j \geq N(h)$. It follows that $N(j) \leq x$. Therefore, $\overline{N}(\overline{N}(x)) \leq x$. On the other hand, for all $h \in \eta(x)$, then $N(h) \in \xi(\overline{N}(x))$. So it holds that $h = N(N(h)) \leq \bigvee_{j\in\xi(\overline{N}(x))} N(j)$ for all $h \in \eta(x)$. Thus, $x = \bigvee_{h\in\eta(x)} h \leq \bigvee_{j\in\xi(\overline{N}(x))} N(j) = \overline{N}(\overline{N}(x))$.

Consequently, $\overline{N}$ is a strong fuzzy negation on L. □

Theorem 3.27. Let L be an ACDL and $\mathcal{J}_p(L)^\star = \mathcal{M}_p(L)^\star$. Suppose that (T, S, N) is De Morgan triple on $\mathcal{J}_p(L)^\star$ and N is strong, then $(\overline{T}, \overline{S}, \overline{N})$ is De Morgan triple on L.

Proof. For all $x, y \in L$, we claim $\overline{S}(\overline{N}(x), \overline{N}(y)) = \overline{N}(\overline{T}(x, y))$.

If $\overline{T}(x, y) = 1$, then $x = y = 1$. Since $\overline{N}$ is a fuzzy negation, we have

$$\overline{N}(\overline{T}(x, y)) = \overline{N}(1) = 0$$

and

$$\overline{S}(\overline{N}(x), \overline{N}(y)) = \overline{S}(\overline{N}(1), \overline{N}(1)) = \overline{S}(0, 0) = 0.$$

Thus, $\overline{S}(\overline{N}(x), \overline{N}(y)) = \overline{N}(\overline{T}(x, y))$.

If $\overline{T}(x, y) < 1$, by Lemma 3.25(2), we have

$$\begin{aligned}\overline{S}(\overline{N}(x), \overline{N}(y)) &= \overline{S}(\widetilde{N}(x), \widetilde{N}(y)) \\ &= \bigwedge_{j\in\xi(\widetilde{N}(x))} \bigwedge_{h\in\xi(\widetilde{N}(y))} S(j, h) \\ &= \bigwedge_{j\in\xi(\bigvee_{a\in\xi(x)} N(a))} \bigwedge_{h\in\xi(\bigvee_{b\in\xi(y)} N(b))} S(j, h)\end{aligned}$$

and

$$\overline{N}(\overline{T}(x, y)) = \bigwedge_{l\in\eta(\overline{T}(x,y))} N(l) = \bigwedge_{l\in\eta(\bigvee_{m\in\eta(x)} \bigvee_{n\in\eta(y)} T(m,n))} N(l).$$

Let $l \in \mathcal{J}_p(L)^\star$ and $l \leq \bigvee_{m\in\eta(x)} \bigvee_{n\in\eta(y)} T(m, n) < 1$. There exist $m \in \eta(x)$ and $n \in \eta(y)$ such that $l \leq T(m, n)$. Thus, $N(m) \in \xi(\widetilde{N}(x))$ and $N(n) \in \xi(\widetilde{N}(y))$. Since (T, S, N) is De Morgan triple on $\mathcal{J}_p(L)^\star$, we have $N(l) \geq N(T(m, n)) = S(N(m), N(n)) \geq \overline{S}(\overline{N}(x), \overline{N}(y))$. Therefore, $\overline{N}(\overline{T}(x, y)) \geq \overline{S}(\overline{N}(x), \overline{N}(y))$.

Conversely, take any $j \in \xi(\bigvee_{a\in\xi(x)} N(a))$ and $h \in \xi(\bigvee_{b\in\xi(y)} N(b))$, then $j \geq N(a)$ for all $a \in \xi(x)$ and $h \geq N(b)$ for all $b \in \xi(y)$. Since N is strong, $N(j) \leq x$ and $N(h) \leq y$. It follows from the monotonicity of T and the property of De Morgan triple that $S(j, h) = S(N(N(j)), N(N(h))) = N(T(N(j), N(h))) \geq \overline{N}(\overline{T}(x, y)))$. That is, $\overline{S}(\overline{N}(x), \overline{N}(y)) \geq \overline{N}(\overline{T}(x, y))$. Thus, $\overline{S}(\overline{N}(x), \overline{N}(y)) = \overline{N}(\overline{T}(x, y))$.

Take any $x, y \in L$, by Theorem 3.26(3), we have

$$\begin{aligned}\overline{T}(\overline{N}(x), \overline{N}(y)) &= (\overline{N} \circ \overline{N})(\overline{T}(\overline{N}(x), \overline{N}(y))) \\ &= \overline{N}(\overline{S}(\overline{N}(\overline{N}(x)), \overline{N}(\overline{N}(y)))) \\ &= \overline{N}(\overline{S}(x, y)).\end{aligned}$$

Consequently, $(\overline{T}, \overline{S}, \overline{N})$ is De Morgan triple on L. $\square$

Example 3.28. Let $L = \{0, a, b, c, d, e, 1\}$ be the lattice shown in Figure 4. Obviously, it is an ACDL, and $\mathcal{J}_p(L)^\star = \mathcal{M}_p(L)^\star = \{0, a, b, c, d, 1\}$. we can define operations T, S and N on $\mathcal{J}_p(L)^\star$ as follows:

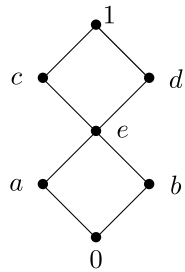

Fig. 4.

N	
0	1
a	c
b	d
c	a
d	b
1	0

T	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	0	0	0	0	a
b	0	0	b	b	b	b
c	0	0	b	c	b	c
d	0	0	b	b	d	d
1	0	a	b	c	d	1

S	0	a	b	c	d	1
0	0	a	b	c	d	1
a	a	a	d	1	d	1
b	b	d	b	1	d	1
c	c	1	1	1	1	1
d	d	d	d	1	d	1
1	1	1	1	1	1	1

Then it can be checked that T is a t-norm, S is a t-conorm, N is a strong fuzzy negation and the triple (T, S, N) is De Morgan triple. By Theorem 3.14, 3.19, 3.20 and 3.21, we can get that $\overline{T}$, $\overline{S}$ and $\overline{N}$ are as follows:

$\overline{T}$	0	a	b	c	d	e	1
0	0	0	0	0	0	0	0
a	0	0	0	0	0	0	a
b	0	0	b	b	b	b	b
c	0	0	b	c	b	b	c
d	0	0	b	b	d	b	d
e	0	0	b	b	b	b	e
1	0	a	b	c	d	e	1

$\overline{S}$	0	a	b	c	d	e	1
0	0	a	b	c	d	e	1
a	a	a	d	1	d	d	1
b	b	d	b	1	d	d	1
c	c	1	1	1	1	1	1
d	d	d	d	1	d	d	1
e	e	d	d	1	d	d	1
1	1	1	1	1	1	1	1

$\overline{N}$	0	a	b	c	d	e	1
	1	c	d	a	b	e	0

One can verify that triple $(\overline{T}, \overline{S}, \overline{N})$ is De Morgan triple.

4. EXTENSION OF FUZZY IMPLICATIONS

In this section, we present some ways to construct fuzzy implications and investigate some properties of the new implications.

Theorem 4.1. Let L be an ACDL and I be a fuzzy implication on $\mathcal{J}_p(L)^\star$. Then the binary operation $\overline{I}$ on L defined by

$$\overline{I}(x,y) = \bigwedge_{j\in\eta(x)} \bigvee_{h\in\eta(y)} I(j,h)$$

is a fuzzy implication on L.

P r o o f. Clearly, $\overline{I}$ is well-defined.

(1) Let $x, y, z \in L$ and $x \le y$, then $\{\bigvee_{k\in\eta(z)} I(j,k)\}_{j\in\eta(x)} \subseteq \{\bigvee_{k\in\eta(z)} I(j,k)\}_{j\in\eta(y)}$. Obviously, $\bigwedge_{j\in\eta(x)} \bigvee_{k\in\eta(z)} I(j,k) \ge \bigwedge_{j\in\eta(y)} \bigvee_{k\in\eta(z)} I(j,k)$. Whence, $\overline{I}(x,z) \ge \overline{I}(y,z)$.

(2) Let $x, y, z \in L$ and $y \le z$, we have $\eta(y) \subseteq \eta(z)$. Hence, $\forall\, j \in \eta(x)$, $\bigvee_{h\in\eta(y)} I(j,h) \le \bigvee_{k\in\eta(z)} I(j,k)$. Therefore, $\bigwedge_{j\in\eta(x)} \bigvee_{h\in\eta(y)} I(j,h) \le \bigwedge j \in \eta(x) \bigvee_{k\in\eta(z)} I(j,k)$, i. e., $\overline{I}(x,y)$ $\le \overline{I}(x,z)$.

(3) $\overline{I}(1,1) = \bigwedge_{j\in\eta(1)} \bigvee_{h\in\eta(1)} I(j,h) = \bigwedge_{j\in\eta(1)} I(j,1) = I(1,1) = 1.$

(4) $\overline{I}(0,0) = \bigwedge_{j\in\eta(0)} \bigvee_{h\in\eta(0)} I(j,h) = I(0,0) = 1$.
(5) $\overline{I}(1,0) = \bigwedge_{j\in\eta(1)} \bigvee_{h\in\eta(0)} I(j,h) = \bigwedge_{j\in\eta(1)} I(j,0) = I(1,0) = 0$.
Thus, $\overline{I}$ is a fuzzy implication on L. □

Proposition 4.2. Under the same condition as in Theorem 4.1, if I satisfies some of properties(LB),(RB), (IP), (NP), (LOP), then $\overline{I}$ is an implication on L which satisfies the same properties.

Proof. Suppose that I is an implication on $\mathcal{J}_p(L)^\star$. For all $x, y \in L$,

(LB) By hypothesis, $I(0,a) = 1$ for all $a \in \mathcal{J}_p(L)^\star$. Then $\overline{I}(0,y) = \bigwedge_{j\in\eta(0)} \bigvee_{h\in\eta(y)} I(j,h)$ $= \bigvee_{h\in\eta(y)} I(0,h) = \bigvee_{h\in\eta(y)} 1 = 1$.

(RB) Now, considering that $I(a,1) = 1$ for all $a \in \mathcal{J}_p(L)^\star$, then we have $\overline{I}(x,1) = \bigwedge_{j\in\eta(x)} \bigvee_{h\in\eta(1)} I(j,h) = \bigwedge_{j\in\eta(x)} I(j,1) = \bigwedge_{j\in\eta(x)} 1 = 1$.

(IP) Suppose $I(a,a) = 1$ for each $a \in \mathcal{J}_p(L)^\star$, then $\overline{I}(y,y) = \bigwedge_{j\in\eta(y)} \bigvee_{h\in\eta(y)} I(j,h) \geq \bigwedge_{j\in\eta(x)} I(j,j) = \bigwedge_{j\in\eta(x)} 1 = 1$.

(NP) According to the assumption, we have $I(1,a) = a$ for all $a \in \mathcal{J}_p(L)^\star$, then $\overline{I}(1,y) = \bigwedge_{j\in\eta(1)} \bigvee_{h\in\eta(y)} I(j,h) = \bigvee_{h\in\eta(y)} I(1,h) = \bigvee_{h\in\eta(y)} h = y$.

(LOP) Let $x, y \in L$ and $x \leq y$, then $\eta(x) \subseteq \eta(y)$. Thus, $\overline{I}(x,y) = \bigwedge_{j\in\eta(x)} \bigvee_{h\in\eta(y)} I(j,h)$ $\geq \bigwedge_{j\in\eta(x)} I(j,j) = 1$. □

Proposition 4.3. Let L be an ACDL. If I is a fuzzy implication on $\mathcal{J}_p(L)^\star$, then $N_{\overline{I}} = \overline{N_I}$.

Proof. For all $x \in L$, by Definition 2.17,

$$N_{\overline{I}}(x) = \overline{I}(x,0) = \bigwedge_{j\in\eta(x)} I(j,0) = \bigwedge_{j\in\eta(x)} N_I(j) = \overline{N_I}(x).$$

□

Theorem 4.4. Let L be an ACDL and I be a fuzzy implication on $\mathcal{M}_p(L)^\star$. Then the binary operation $\widetilde{I}$ on L defined by

$$\widetilde{I}(x,y) = \bigvee_{j\in\xi(x)} \bigwedge_{h\in\xi(y)} I(j,h)$$

is a fuzzy implication on L.

Proof. The proof follows similarly to the one given in Theorem 4.1. □

The following example shows that $\overline{I}$ can be different from $\widetilde{I}$ even if $\mathcal{J}_p(L)^\star = \mathcal{M}_p(L)^\star$ and they have the same I on $\mathcal{J}_p(L)^\star$.

Example 4.5. Let L be the lattice in Example 3.1, then $\mathcal{J}_p(L)^\star = \mathcal{M}_p(L)^\star$. If the fuzzy implication on $\mathcal{M}_p(L)^\star$ is defined by

$$I(u,v) = \begin{cases} 1 & \text{if } u \leq v, \\ v & \text{otherwise,} \end{cases}$$

then by Theorem 4.1 and 4.4, we can get that

$$\overline{I}(x,y) = \begin{cases} 1 & \text{if } x \leq y, \\ y & \text{otherwise,} \end{cases}$$

and

$$\widetilde{I}(x,y) = \begin{cases} 1 & \text{if } x \leq y, \\ b_k & \text{if } x = a_k \text{ and } y = 0_k,\ k \in K, \\ a_k & \text{if } x = b_k \text{ and } y = 0_k,\ k \in K, \\ y & \text{otherwise,} \end{cases}$$

are both fuzzy implications on L. Clearly, $\overline{I} \neq \widetilde{I}$.

Proposition 4.6. Let L be an ACDL. If the pair (S,N) satisfies the law of excluded middle on $\mathcal{M}_p(L)^\star$, then $(\overline{S},\widetilde{N})$ satisfies the law of excluded middle on L.

Proof. For all $x \in L$, we have

$$\overline{S}(\widetilde{N}(x),x) = \bigwedge_{m\in\xi(\widetilde{N}(x))} \bigwedge_{j\in\xi(x)} S(m,j) = \bigwedge_{m\in\xi(\bigvee_{j\in\xi(x)} N(j))} \bigwedge_{j\in\xi(x)} S(m,j).$$

Take any $m \in \xi(\bigvee_{j\in\xi(x)} N(j))$, then $m \geq \bigvee_{j\in\xi(x)} N(j)$. It can be written as $m \geq N(j)$ for all $j \in \xi(x)$. Thus, $S(m,j) \geq S(N(j),j) = 1$. Consequently, $\overline{S}(\widetilde{N}(x),x) = 1$ for all $x \in L$. □

Proposition 4.7. Let L be an ACDL with $\mathcal{J}_p(L)^\star = \mathcal{M}_p(L)^\star$, I be a fuzzy implication on $\mathcal{J}_p(L)^\star$ and N be a strong fuzzy negation on $\mathcal{J}_p(L)^\star$. If I satisfies CP(N), then $\widetilde{I}(\overline{N}(y),\overline{N}(x)) = \overline{I}(x,y)$ for all $x,y \in L$.

Proof. For all $x,y \in L$.

If $\overline{N}(y) = 0$, then $y = 1$. Thus, $\widetilde{I}(\overline{N}(y),\overline{N}(x)) = \overline{I}(x,y) = 1$.

If $\overline{N}(x) = 0$, then $x = 1$. We have

$$\widetilde{I}(\overline{N}(y),\overline{N}(x)) = \widetilde{I}(\overline{N}(y),0) = \bigvee_{m\in\xi(\overline{N}(y))} I(m,0)$$

and

$$\overline{I}(x,y) = \overline{I}(1,y) = \bigvee_{h\in\eta(y)} I(1,h).$$

On the one hand, suppose $m \in \mathcal{M}_p(L)^\star$ and $m \geq \overline{N}(y)$. Then there exists $h \in \eta(y)$ such that $m \geq N(h)$. It follows that $I(m,0) \leq I(N(h),0)$. As N is strong and I satisfies

CP(N), it holds that $I(m,0) \leq I(1,h)$. Thus, $\widetilde{I}(\overline{N}(y),0) \leq \overline{I}(1,y)$. On the other hand, for all $h \in \eta(y)$, we have $N(h) \in \xi(\overline{N}(y))$. It follows that $I(N(h),0) \leq \widetilde{I}(\overline{N}(y),0)$. Since N is strong and I satisfies CP(N), we have $I(1,h) \leq \widetilde{I}(\overline{N}(y),0)$ for all $h \in \eta(y)$. Thus, $\overline{I}(1,y) \leq \widetilde{I}(\overline{N}(y),0)$. Therefore, $\widetilde{I}(\overline{N}(y),0) = \overline{I}(\overline{N}(0),y)$.

If $\overline{N}(x) \neq 0$ and $\overline{N}(y) \neq 0$, we have

$$\begin{aligned}\widetilde{I}(\overline{N}(y),\overline{N}(x)) &= \bigvee_{m\in\xi(\overline{N}(y))}\ \bigwedge_{n\in\xi(\overline{N}(x))} I(m,n)\\ &= \bigvee_{m\in\xi(\bigwedge_{h\in\eta(y)} N(h))}\ \bigwedge_{n\in\xi(\bigwedge_{j\in\eta(x)} N(j))} I(m,n)\end{aligned}$$

and

$$\overline{I}(x,y) = \bigwedge_{j\in\eta(x)}\ \bigvee_{h\in\eta(y)} I(j,h).$$

For every $m \in \xi(\bigwedge_{h\in\eta(y)} N(h))$ and $n \in \xi(\bigwedge_{j\in\eta(x)} N(j))$, there exist $h \in \eta(y)$ and $j \in \eta(x)$ such that $m \geq N(h)$ and $n \geq N(j)$. It follows that $N(m) \in \eta(y)$ and $N(n) \in \eta(x)$. Since I satisfies CP(N), we have $I(m,n) = I(N(N(m)),N(N(n))) = I(N(n),N(m)) \in \{I(j,h)|j \in \eta(x), h \in \eta(y)\}$. Thus, $\{I(m,n)|m \in \xi(\overline{N}(y)), n \in \xi(\overline{N}(x))\} \subseteq \{I(j,h)|j \in \eta(x), h \in \eta(y)\}$. Conversely, for all $j \in \eta(x)$, $h \in \eta(y)$, it follows that $j,h \in \mathcal{J}_p(L)^\star$ and $N(j) \in \xi(\overline{N}(x))$, $N(h) \in \xi(\overline{N}(y))$. Thus, $I(j,h) = I(N(h),N(j)) \in \{I(m,n)|m \in \xi(\overline{N}(y)), n \in \xi(\overline{N}(x))\}$. That is to say, $\{I(j,h)|j \in \eta(x), h \in \eta(y)\} \subseteq \{I(m,n)|m \in \xi(\overline{N}(y)), n \in \xi(\overline{N}(x))\}$. Thus, by the one to one correspondence of $\eta(x)$ and $\xi(\overline{N}(x))$, $\eta(y)$ and $\xi(\overline{N}(y))$, we have $\widetilde{I}(\overline{N}(y),\overline{N}(x)) = \overline{I}(x,y)$. □

Proposition 4.8. Let L be an ACDL. If $\mathcal{J}_p(L)^\star$ is a complete lattice and T is a t-norm on $\mathcal{J}_p(L)^\star$, then $I_{\overline{T}} \leq \overline{I_T}$.

Proof. For any $x,y \in L$, by Definition 2.15 and Theorem 4.1, we have

$$I_{\overline{T}}(x,y) = \bigvee\{z \in L|\overline{T}(z,x) \leq y\}$$

and

$$\overline{I_T}(x,y) = \bigwedge_{j\in\eta(x)}\ \bigvee_{h\in\eta(y)} I_T(j,h).$$

We claim $I_{\overline{T}}(x,y) = \bigvee\{k \in \mathcal{J}_p(L)^\star|\overline{T}(k,x) \leq y\}$. Obviously, it holds that $\bigvee\{k \in \mathcal{J}_p(L)^\star|\overline{T}(k,x) \leq y\} \leq \bigvee\{z \in L|\overline{T}(z,x) \leq y\}$. Take any $u \in \{z \in L|\overline{T}(z,x) \leq y\}$, then $\overline{T}(u,x) \leq y$. Thus, $\eta(u) \subseteq \{k \in \mathcal{J}_p(L)^\star|\overline{T}(k,x) \leq y\}$. That is, $u \leq \bigvee\{k \in \mathcal{J}_p(L)^\star|\overline{T}(k,x) \leq y\}$. Hence, $\bigvee\{z \in L|\overline{T}(z,x) \leq y\} \leq \bigvee\{k \in \mathcal{J}_p(L)^\star|\overline{T}(k,x) \leq y\}$. Consequently, $\bigvee\{z \in L|\overline{T}(z,x) \leq y\} = \bigvee\{k \in \mathcal{J}_p(L)^\star|\overline{T}(k,x) \leq y\}$.

Take any $v \in \{k \in \mathcal{J}_p(L)^\star|\overline{T}(k,x) \leq y\}$, then $\overline{T}(v,x) \leq y$. That is $\bigvee_{j\in\eta(x)} T(v,j) \leq y$. Thus, for every $j \in \eta(x)$, there exists $h_j \in \eta(y)$ such that $T(v,j) \leq h_j$. It follows that $v \leq \bigvee_{h\in\eta(y)} I_T(j,h)$ for all $j \in \eta(x)$. Thus, $v \leq \bigwedge_{j\in\eta(x)} \bigvee_{h\in\eta(y)} I_T(j,h)$. It yields that $\bigvee\{k \in \mathcal{J}_p(L)^\star|\overline{T}(k,x) \leq y\} \leq \bigwedge_{j\in\eta(x)} \bigvee_{h\in\eta(y)} I_T(j,h)$. That is, $I_{\overline{T}} \leq \overline{I_T}$. □

Example 4.9. Consider the lattice L in Example 3.15, $\mathcal{J}_p(L)^\star = \bigcup_{k\in K}\{0_k, a_k, b_k, 1_k\}$ is a complete lattice. Let $k_0 \in K \setminus \{0_K, 1_K\}$, we have defined a t-norm T on $\mathcal{J}_p(L)^\star$. By calculation, we can get

$$I_T(x,y) = \begin{cases} 1, & x \le y,\\ 1_{k_0}, & x = a_{k_0}, y \in \{0_{k_0}, b_{k_0}\} \text{ or } x = b_{k_0}, y \in \{0_{k_0}, a_{k_0}\},\\ b_k, & x = a_k, y = 0_k, k \neq k_0,\\ a_k, & x = b_k, y = 0_k, k \neq k_0,\\ y, & \text{otherwise.} \end{cases}$$

Thus, by Theorem 4.1,

$$\overline{I_T}(x,y) = \begin{cases} 1, & x \le y,\\ 1_{k_0}, & x = a_{k_0}, y \in \{0_{k_0}, b_{k_0}\} \text{ or } x = b_{k_0}, y \in \{0_{k_0}, a_{k_0}\},\\ & \text{or } x = c_{k_0}, y \in \{0_{k_0}, a_{k_0}, b_{k_0}\},\\ b_k, & x = a_k, y = 0_k, k \neq k_0,\\ a_k, & x = b_k, y = 0_k, k \neq k_0,\\ y, & \text{otherwise.} \end{cases}$$

By Definition 2.15, we have

$$I_{\overline{T}}(x,y) = \begin{cases} 1, & x \le y,\\ c_{k_0}, & x = a_{k_0}, y \in \{0_{k_0}, b_{k_0}\} \text{ or } x = b_{k_0}, y \in \{0_{k_0}, a_{k_0}\},\\ & \text{or } x = c_{k_0}, y \in \{0_{k_0}, a_{k_0}, b_{k_0}\},\\ b_k, & x = a_k, y = 0_k, k \neq k_0,\\ a_k, & x = b_k, y = 0_k, k \neq k_0,\\ y, & \text{otherwise.} \end{cases}$$

Obviously, $I_{\overline{T}} < \overline{I_T}$.

Proposition 4.10. Let L be an ACDL and $\mathcal{J}_p(L)^\star = \mathcal{M}_p(L)^\star$. If I is a fuzzy implication on $\mathcal{J}_p(L)^\star$ and N is a strong fuzzy negation on $\mathcal{J}_p(L)^\star$, then $\widetilde{I_N}(x,y) = \overline{I}_{\overline{N}}(x,y)$ for all $x, y \in L$.

Proof. Let $x, y \in L$, we have

$$\widetilde{I_N}(x,y) = \bigvee_{j\in\xi(x)} \bigwedge_{h\in\xi(y)} I_N(j,h) = \bigvee_{j\in\xi(x)} \bigwedge_{h\in\xi(y)} I(N(h), N(j)),$$

and

$$\begin{aligned} \overline{I}_{\overline{N}}(x,y) = \overline{I}(\overline{N}(y), \overline{N}(x)) &= \bigwedge_{m\in\eta(\overline{N}(y))} \bigvee_{n\in\eta(\overline{N}(x))} I(m,n)\\ &= \bigwedge_{m\in\eta(\bigwedge_{u\in\eta(y)} N(u))} \bigvee_{n\in\eta(\bigwedge_{v\in\eta(x)} N(v))} I(m,n). \end{aligned}$$

Obviously, $m \in \eta(\bigwedge_{u\in\eta(y)} N(u))$ if and only if $N(m) \in \xi(y)$, $n \in \eta(\bigwedge_{v\in\eta(x)} N(v))$ if and only if $N(n) \in \xi(x)$. Since N is strong, we have $I(m,n) = I(N(N(m)), N(N(n)))$. Whence, $\widetilde{I_N}(x,y) = \overline{I}_{\overline{N}}(x,y)$ for all $x, y \in L$. □

5. EXTENSION OF UNINORMS

This section aims at constructing infinitely $\bigvee$-distributive uninorms and infinitely $\bigwedge$-distributive uninorms.

Theorem 5.1. Let L be an ACDL and U be a uninorm on $\mathcal{J}_p(L)^\star$ with neutral element e. If $1 \in \mathcal{J}_p(L)$, then the binary operation $\overline{U}$ on L defined by

$$\overline{U}(x,y) = \bigvee_{j\in\eta(x)} \bigvee_{h\in\eta(y)} U(j,h)$$

is a uninorm on L with neutral element e.

Proof. Clearly, $\overline{U}$ is well-defined. The commutativity and monotonicity of $\overline{U}$ can be proved on the analogy of Theorem 3.14.

(1) For all $x \in L$, $\overline{U}(x,e) = \bigvee_{j\in\eta(x)} \bigvee_{h\in\eta(e)} U(j,h) = \bigvee_{j\in\eta(x)} U(j,e) = \bigvee_{j\in\eta(x)} j = x$. Thus, $\overline{U}(x,e) = x$.

(2) Let $x, y, z \in L$, then

$$\begin{aligned}
&\overline{U}(\overline{U}(x,y),z)\\
&=\overline{U}(\bigvee_{j\in\eta(x)} \bigvee_{h\in\eta(y)} U(j,h), z)\\
&= \bigvee_{p\in\eta(\bigvee_{j\in\eta(x)} \bigvee_{h\in\eta(y)} U(j,h))} \bigvee_{k\in\eta(z)} U(p,k).
\end{aligned}$$

On the one hand, let $p \in \mathcal{J}_p(L)^\star$ and $p \le \bigvee_{j\in\eta(x)} \bigvee_{h\in\eta(y)} U(j,h)$, then $p \le U(j,h)$ for some $j \in \eta(x)$ and $h \in \eta(y)$. So it holds that $U(p,k) \le U(U(j,h),k)$ for all $k \in \eta(z)$. That is $U(p,k) \le \bigvee_{j\in\eta(x)} \bigvee_{h\in\eta(y)} \bigvee_{k\in\eta(z)} U(U(j,h),k)$ for all $k \in \eta(z)$. Whence, $\overline{U}(\overline{U}(x,y),z) \le \bigvee_{j\in\eta(x)} \bigvee_{h\in\eta(y)} \bigvee_{k\in\eta(z)} U(U(j,h),k)$. On the other hand, since $U(j,h) \le \overline{U}(x,y)$ for all $j \in \eta(x)$ and $h \in \eta(y)$, we can get $U(U(j,h),k) \le \overline{U}(\overline{U}(x,y),z)$ for all $k \in \eta(z)$. So it holds that $\bigvee_{j\in\eta(x)} \bigvee_{h\in\eta(y)} \bigvee_{k\in\eta(z)} U(U(j,h),k) \le \overline{U}(\overline{U}(x,y),z)$. Thus, $\overline{U}(\overline{U}(x,y),z) = \bigvee_{j\in\eta(x)} \bigvee_{h\in\eta(y)} \bigvee_{k\in\eta(z)} U(U(j,h),k)$. Similarly, we can prove $\overline{U}(x,\overline{U}(y,z)) = \bigvee_{j\in\eta(x)} \bigvee_{h\in\eta(y)} \bigvee_{k\in\eta(z)} U(j,U(h,k))$. Since U is associative, it holds that $\overline{U}(\overline{U}(x,y),z) = \overline{U}(x,\overline{U}(y,z))$. That is, $\overline{U}$ is associative.

Consequently, $\overline{U}$ is a uninorm on L. □

Example 5.2. Consider the ordinal sum $\widetilde{L}$ of lattices L_m, $m = 1, 2, 3$, where $L_1 = L$ is the lattice in Example 3.1, $L_2 = \{0_2, 1_2\}$ is a chain, $L_3 = \{0_3, a_3, b_3, 1_3\}$ is a diamond lattice which is the horizontal sum of chain $\{0_3, a_3, 1_3\}$ and chain $\{0_3, b_3, 1_3\}$, $L_m \bigcap L_n$

is a singleton coinciding with $1_{\min(m,n)}$ and with $0_{\max(m,n)}$. Then $\widetilde{L} = \bigcup_{m=1}^{3} L_m$, and $x \leq y$ whenever $x \in L_m$, $y \in L_n$, and either $m < n$ or $m = n$ and $x \leq y$. Adding a new top element to $\widetilde{L}$, we can obtain a new lattice $\overline{L} = \widetilde{L} \oplus 1$. It can be checked that $\overline{L}$ is an ACDL, and $\mathcal{J}_p(\overline{L}) = (\bigcup_{k\in K}\{a_k, b_k\}) \bigcup \{0_3, a_3, b_3\} \bigcup \{1\}$. If the uninorm on $\mathcal{J}_p(\overline{L})^\star$ is defined by: for all $v, w \in \mathcal{J}_p(\overline{L})^\star$,

$$U(v,w) = \begin{cases} 1 & \text{if } v > 0_3, w > 0_3, \\ v & \text{if } v < 0_3,\ 0_3 < w, \\ w & \text{if } 0_3 < v, w < 0_3, \\ 0 & \text{if } v < 0_3,\ w < 0_3, \\ w & \text{if } v = 0_3, \\ v & \text{if } w = 0_3, \end{cases}$$

then we can get that

$$\overline{U}(x,y) = \begin{cases} 1 & \text{if } x > 0_3, y > 0_3, \\ x & \text{if } x < 0_3,\ 0_3 \leq y, \\ y & \text{if } 0_3 \leq x, y < 0_3, \\ 0 & \text{if } x < 0_3,\ y < 0_3, \\ x \vee y & \text{otherwise}, \end{cases}$$

is a uninorm on $\overline{L}$.

Theorem 5.3. Let L be an ACDL and U be a uninorm on $\mathcal{M}_p(L)^\star$ with neutral element e. If $0 \in \mathcal{M}_p(L)$, then the binary operation $\widetilde{U}$ on L defined by

$$\widetilde{U}(x,y) = \bigwedge_{j\in\xi(x)} \bigwedge_{h\in\xi(y)} U(j,h)$$

is a uninorm on L with neutral element e.

Proof. The proof follows similarly to that of Theorem 5.1. □

Theorem 5.4. Let L be an ACDL and U be a conjunctive uninorm on $\mathcal{J}_p(L)^\star$ with neutral element e. If $1 \in \mathcal{J}_p(L)$, then $\overline{U}$ is infinitely $\bigvee$-distributive.

Proof. Suppose $a \in L$ and $\{b_i\}_{i\in I}$. Then we distinguish two cases:

(1) If $I = \emptyset$, then $\overline{U}(a, \bigvee_{i\in I} b_i) = \overline{U}(a, 0) = \bigvee_{m\in\eta(a)} U(m, 0) = 0$.

(2) If $I \neq \emptyset$, then

$$\overline{U}(a, \bigvee_{i\in I} b_i) = \bigvee_{m\in\eta(a)} \bigvee_{l\in\eta(\bigvee_{i\in I} b_i)} U(m,l).$$

On the one hand, for all $l \in \eta(\bigvee_{i\in I} b_i)$, it follows that $l \leq \bigvee_{i\in I} b_i$ and $l \in \mathcal{J}_p(L)^\star$, from which we get that there exists $i \in I$ such that $l \leq b_i$. That is $l \in \eta(b_i)$. Thus, $U(m,l) \leq \overline{U}(m, b_i)$ for all $m \in \eta(a)$. It follows that $\bigvee_{m\in\eta(a)} U(m,l) \leq \bigvee_{m\in\eta(a)} \overline{U}(m, b_i)$. Hence, $\bigvee_{l\in\eta(\bigvee_{i\in I} b_i)} \bigvee_{m\in\eta(a)} U(m,l) \leq \bigvee_{i\in I}(\bigvee_{m\in\eta(a)} \overline{U}(m, b_i)) = \bigvee_{i\in I} \overline{U}(a, b_i)$. That is, $\overline{U}(a, \bigvee_{i\in I} b_i) \leq \bigvee_{i\in I} \overline{U}(a, b_i)$. On the other hand, the monotonicity of $\overline{U}$ implies that $\bigvee_{i\in I} \overline{U}(a, b_i) \leq \overline{U}(a, \bigvee_{i\in I} b_i)$. Consequently, $\overline{U}(a, \bigvee_{i\in I} b_i) = \bigvee_{i\in I} \overline{U}(a, b_i)$. □

Corollary 5.5. Let L be an ACDL and U be a disjunctive uninorm on $\mathcal{M}_p(L)^\star$ with neutral element e. If $0 \in \mathcal{M}_p(L)$, then $\widetilde{U}$ is infinitely $\bigwedge$-distributive.

Proof. The proof is similar to that of Theorem 5.3. □

6. CONCLUSION

In this paper, some fuzzy connectives, such as t-norms, t-conorms, fuzzy negations, fuzzy implications, uninorms, have been constructed on ACDL by means of completely join-prime elements and completely meet-prime elements. We prove that De Morgan triple can be preserved when N is a strong negation, and some properties of fuzzy implication can be kept. Moreover, the behaviors of these extensions for two special classes of fuzzy implications, namely, R-implication and reciprocal implication, are discussed. Finally, we prove that the uninorms constructed by completely join-prime elements are infinitely $\bigvee$-distributive and the uninorms constructed by completely meet-prime elements are infinitely $\bigwedge$-distributive.

For future projects, we would like to investigate some further properties of the extended uninorms and propose a version of the extension, as is shown in this paper, for other aggregation functions, such as copula, overlap function, grouping function and 2-uninorm. Moreover, we are interested in investigating questions involving extension and a generalized notion of additive generators.

ACKNOWLEDGEMENT

The authors thank the editors and the anonymous reviewers for their valuable suggestions in improving this paper. This work was supported by a grant of National Natural Science Foundation of China (11531009).

(Received May 22, 2018)

REFERENCES

[1] M. Baczyński and B. Jayaram: Fuzzy Implications. Studies in Fuzziness and Soft Computing, Springer-Verlag, Berlin Heidelberg 2008.

[2] G. D. Çaylı: On a new class of t-norms and t-conorms on bounded lattices. Fuzzy Sets Syst. *332* (2018), 129–143. DOI:10.1016/j.fss.2017.07.015

[3] G. D. Çaylı and P. Drygaś: Some properties of idempotent uninorms on a special class of bounded lattices. Inform. Sci. *422* (2018), 352–363. DOI:10.1016/j.ins.2017.09.018

[4] G. D. Çaylı and F. Karaçal: Construction of uninorms on bounded lattices. Kybernetika *53* (2017), 394–417. DOI:10.14736/kyb-2017-3-0394

[5] G. D. Çaylı, F. Karaçal, and R. Mesiar: On a new class of uninorms on bounded lattices. Inform. Sci. *367–368* (2016), 221–231. DOI:10.1016/j.ins.2016.05.036

[6] B. A. Davey and H. A. Priestley: Introduction to lattices and Order. Cambridge University Press, Cambridge 1990.

[7] B. De Baets and R. Mesiar: Triangular norms on product lattices. Fuzzy Sets Syst. *104* (1999), 61–75. DOI:10.1016/s0165-0114(98)00259-0

[8] G. Deschrijver: Uninorms which are neither conjunctive nor disjunctive in interval-valued fuzzy set theory. Inform. Sci. *244* (2013), 48–59. DOI:10.1016/j.ins.2013.04.033

[9] S. Jenei and B. De Baets: On the direct decomposability of t-norms on product lattices. Fuzzy Sets Syst. *139* (2003), 699–707. DOI:10.1016/s0165-0114(03)00125-8

[10] F. Karaçal, Ü. Ertuğrul, and R. Mesiar: Characterization of uninorms on bounded lattices. Fuzzy Sets Syst. *308* (2017), 54–71. DOI:10.1016/j.fss.2016.05.014

[11] F. Karaçal and R. Mesiar: Uninorms on bounded lattices. Fuzzy Sets Syst. *261* (2015), 33–43. DOI:10.1016/j.fss.2014.05.001

[12] F. Karaçal and Y. Sağiroğlu: Infinitely $\bigvee$-distributive t-norms on complete lattices and pseudo-complements. Fuzzy Sets Syst. *160* (2009), 32–43. DOI:10.1016/j.fss.2008.03.022

[13] E. P. Klement, R. Mesiar, and E. Pap: Triangular Norms. Kluwer Academic Publishers, Dordrecht 2000.

[14] E. S. Palmeira and B. C. Bedregal: Extensions of fuzzy logic operators defined on bounded lattices via retraction. Comput. Math. Appl. *63* (2012), 1026–1038. DOI:10.1016/j.camwa.2011.12.007

[15] E. S. Palmeira and B. C. Bedregal: On the extension of lattice-valued implications via retractions. Fuzzy Sets Syst. *240* (2014), 66–85. DOI:10.1016/j.fss.2013.07.023

[16] E. S. Palmeira, B. C. Bedregal, R. Mesiar, and J. Fernandez: A new way to extend t-norms, t-conorms and negations. Fuzzy Sets Syst. *240* (2014), 1–21. DOI:10.1016/j.fss.2013.05.008

[17] S. Saminger-Platz: On ordinal sums of triangular norms on bounded lattices. Fuzzy Sets Syst. *157* (2006), 1403–1416. DOI:10.1016/j.fss.2005.12.021

[18] S. Saminger-Platz, E. P. Klement, and R. Mesiar: On extensions of triangular norms on bounded lattices. Indag. Math. *19* (2008), 1, 135–150. DOI:10.1016/s0019-3577(08)80019-5

[19] Z. D. Wang and J. X Fang: On the direct decomposability of pseudo-t-norms, t-norms and implication operators on product lattices. Fuzzy Sets Syst. *158* (2007), 2494–2503. DOI:10.1016/j.fss.2007.06.011

[20] Z. D. Wang and Y. D Yu: Pseudo-t-norms and implication operators on a complete Brouwerian lattice. Fuzzy Sets Syst. *132* (2002), 113–124. DOI:10.1016/s0165-0114(01)00210-x

[21] Ş. Yılmaz and O. Kazancı: Constructions of triangular norms on lattices by means of irreducible elements. Inform. Sci. *397–398* (2017), 110–117. DOI:10.1016/j.ins.2017.02.041

Hui Liu, Department of Mathematics and Information Science, Shaanxi Normal University, Xi'an, 710119. P. R. China.
e-mail: liuhui123@snnu.edu.cn

Bin Zhao, Corresponding author. Department of Mathematics and Information Science, Shaanxi Normal University, Xi'an, 710119. P. R. China.
e-mail: zhaobin@snnu.edu.cn