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FUNCTIONAL OBSERVERS DESIGN FOR NONLINEAR DISCRETE-TIME SYSTEMS WITH INTERVAL TIME-VARYING DELAYS

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This paper is concerned with the functional observer design for a class of Multi-Input Multi-Output discrete-time systems with mixed time-varying delays. Firstly, using the Lyapunov–Krasovskii functional approach, we design the parameters of the delay-dependent observer. We establish the sufficient conditions to guarantee the exponential stability of functional observer error system. In addition, for design purposes, delay-dependent sufficient conditions are proposed in terms of matrix inequalities to guarantee that the functional observer error system is exponentially stable. Secondly, we presented the sufficient conditions of the existence of internal-delay independent functional observer to ensure the estimated error system is asymptotically stable. Furthermore, some sufficient conditions are obtained to guarantee that the internal-delay independent functional observer error system is exponentially stable. Finally, simulation examples are provided to demonstrate the effectiveness of the proposed method.

Keywords: functional observer, discrete-time systems, exponential stability, interval time-varying delays, Lyapunov–Krasovskii functional

Classification: 93C55, 93C10, 93D09, 93D05

1. INTRODUCTION

The state estimation of dynamic systems that include time delays in their models has received considerable attention in the last few decades [6, 15]. This is justified by the fact that the time-delay is a common phenomenon encountered in various practical systems and the existence of the time-delay may significantly affect performances and causes instability in dynamic systems. In recent years, the problem of observer design for time-delay systems has been a subject of intensive research. Some results were presented in [2, 7, 14].

Functional observers are a class of observers that deal with estimating one or multiple functions of the states of a system [9, 13]. This type of observers has a wide range of applications in system monitoring, fault diagnosis, and observer based control of dynamic systems [8, 1]. Unlike Luenberger type full order observer design for delay systems ([11]), functional observer design for time-delay systems is a relatively new field of research. To date, a significant portion of the papers on the design of functional observers for

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time delay systems has been devoted to systems with state delays [3, 4, 12]. In [3], a method of design of the linear functional observer for time-delay systems was presented. In [4], an algebraic method for functional observer design was given. In [12], the design of reduced-order unknown input functional observers for a class of nonlinear Lipschitz systems was investigated. However, to the best of our knowledge, the observer design for discrete-time nonlinear systems with time-varying delays is relatively rare.

This paper investigates the problem of functional observer design for a class of discrete-time systems with interval time-varying delays. The delay-dependent functional observer for this system is designed to assure the error system is globally α-exponentially stable. We derive the sufficient conditions of the existence of these functional observers. In addition, for the purpose of design, the delay-dependent sufficient conditions, which guarantee the functional observer error system is α-exponentially stable, are established. Furthermore, an internal-delay independent functional observer is proposed for the system. The sufficient conditions of the existence of internal-delay independent functional observer, which ensure estimated error is asymptotically stable, have been obtained. In addition, we give some sufficient conditions to guarantee internal-delay independent functional observer is α-exponentially stable. Finally, three numerical examples are provided to illustrate the results.

This paper is organized as follows. In Section 2, the system description and some preliminaries are given. The design of globally exponentially stable functional observer scheme is presented in Section 3. The internal-delay independent functional observer design method is demonstrated in Section 4. Section 5 gives three simulation examples to show the performances of our method. Finally, Section 6 concludes the paper.

Notations. The symbol Z represents the set of integers, Z^+ represents the set of non-negative integers. In symmetric block matrices, we use an asterisk $*$ to represent a term induced by symmetry. I and 0 represent the identity matrix and null matrix of appropriate dimensions, respectively. X^+ is the pseudo-inverse or the generalized inverse of the matrix X. $X^\perp$ is the right orthogonal of X in a way that $XX^\perp = 0$. $\mathcal{C}$ and $\mathcal{C}^+$ sequentially are the set of complex numbers, and complex numbers with positive real parts, respectively.

2. SYSTEM DESCRIPTION AND PROBLEM FORMULATION

Consider a class of discrete-time systems with time-varying delays:

$$\begin{aligned} x(k+1) &= Ax(k) + A_{d_1}x(k-d_1(k)) + A_{d_2}x(k-d_2(k)) + Bu(k) + Ef(y(k)), \\ y(k) &= Cx(k), \\ z(k) &= Lx(k), \\ x(k) &= \phi(k), \quad k = -d_2, -d_2+1, \cdots, 0, \end{aligned} \tag{1}$$

where $x(k) \in R^n$ is the state vector, $u(k) \in R^m$ is the input vector, $y(k) \in R^p$ is the measurement output, $A, A_{d_1}, A_{d_2}, C, L$ and B are known real constant matrices of appropriate dimensions and matrices C and L are full row rank. Moreover, $z(k) \in R^l$ is the functional to be estimated. The sequence $\phi(k)$ is the initial condition; the time-varying delay $d_i(k)$ satisfies the following condition:

$$0 < d_1 \leq d_i(k) \leq d_2, \quad i = 1, 2,$$

where d_1 and d_2 are non-negative integers.

Definition 1. (i) The delayed discrete-time system (1) is said to be α-exponentially stable if there exist two scalars $\alpha > 0, \beta > 0$ such that the following inequality holds:

$$\|\mu(k)\| \leq \beta e^{-\alpha k} \sup_{k \in Z \cap [-d_2, 0]} \|\mu(k)\|, \quad \forall k \in Z^+, \tag{2}$$

where $\mu(k) = x(k)$.

(ii) Let the estimation error $e(k) = \hat{z}(k) - z(k)$. A minimum-order functional observer error systems for the system (1) is said to be α-exponentially stable if there exist two scalars $\alpha > 0, \beta > 0$ such that (2) holds, where $\mu(k) = e(k)$.

Lemma 1. (Zhu and Yang [16]) For any constant matrix $W \in R^{n \times n}$ with $W = W^T > 0$, integers $n_1 < n_2$, vector function $\omega : \{n_1,\ n_1 + 1, \cdots, n_2\} \to R^q$ such that the sums concerned are well defined, then

$$(n_2 - n_1 + 1) \sum_{i=n_1}^{n_2} \omega^T(i) W \omega(i) \geq \left(\sum_{i=n_1}^{n_2} \omega^T(i) \right)^T W \left(\sum_{i=n_1}^{n_2} \omega^T(i) \right).$$

Our goal is to design the functional observer for a class of discrete-time systems with mixed time-varying delays such that the minimum-order functional observer error system for the system (1) is α-exponentially stable.

3. DELAY-DEPENDENT OBSERVER DESIGN

3.1. Observer structure and stability analysis

Consider the following minimum-order observer:

$$\begin{aligned} s(k+1) &= Ws(k) + W_{d_1} s(k - d_1(k)) + W_{d_2} s(k - d_2(k)) + Gu(k) + \bar{E} f(y(k)) \\ &\quad + Hy(k) + H_{d_1} y(k - d_1(k)) + H_{d_2} y(k - d_2(k)), \\ \hat{z}(k) &= s(k) + Vy(k), \end{aligned} \tag{3}$$

where $s(k) \in R^l$ is the observer's state. $W, W_{d_1}, W_{d_2}, H, H_{d_1}, H_{d_2}, G, \bar{E}$ and V are constant matrices of appropriate dimensions.

Let us define the auxiliary error signal

$$\rho(k) = s(k) - \Gamma x(k),$$

where $\Gamma \in R^{l \times n}$ is a constant matrix.

The corresponding estimation error is

$$e(k) = \hat{z}(k) - z(k). \tag{4}$$

Now, we can state the sufficient conditions of the exponential stability of the observer.

Theorem 1. The functional observer error system for the system (1) and (3) is globally α-exponentially stable if
(a) the discrete-time system

$$\rho(k+1) = W\rho(k) + W_{d_1}\rho(k - d_1(k)) + W_{d_2}\rho(k - d_2(k)), \tag{5}$$

is globally α-exponentially stable;
(b) there exists a matrix Γ , such that the following equations are satisfied:

$$W\Gamma + HC - \Gamma A = 0, \tag{6a}$$

$$\begin{aligned} W_{d_1}\Gamma + H_{d_1}C - \Gamma A_{d_1} &= 0, \\ W_{d_2}\Gamma + H_{d_2}C - \Gamma A_{d_2} &= 0, \end{aligned} \tag{6b}$$

$$\begin{aligned} G - \Gamma B &= 0, \\ \bar{E} - \Gamma E &= 0, \end{aligned} \tag{6c}$$

$$\Gamma - L + VC = 0. \tag{6d}$$

Proof. From (1) and (3), we get

$$\begin{aligned} \rho(k+1) &= s(k+1 - \Gamma x(k+1) \\ &= Ws(k) + W_{d_1}s(k - d_1(k)) + W_{d_2}s(k - d_2(k)) + Gu(k) + \bar{E}f(y(k)) \\ &\quad + Hy(k) + H_{d_1}y(k - d_1(k)) + H_{d_2}y(k - d_2(k)) - \Gamma(Ax(k) \\ &\quad + A_{d_1}(k - d_1(k)) + A_{d_2}(k - d_2(k)) + Bu(k) + Ef(Cx(k))) \\ &= W\rho(k) + W_{d_1}\rho(k - d_1(k)) + W_{d_2}\rho(k - d_2(k)) + (W\Gamma + HC - \Gamma A)x(k) \\ &\quad + (W_{d_1}\Gamma + H_{d_1}C - \Gamma A_{d_1})x(k - d_1(k)) + (G - \Gamma B)u(k) \\ &\quad + (W_{d_2}\Gamma + H_{d_2}C - \Gamma A_{d_2})x(k - d_2(k)) + (\bar{E} - \Gamma E)f(Cx(k)). \end{aligned} \tag{7}$$

Hence, if there exists a matrix Γ , such that conditions (6) as well as Condition (a) hold, then $\rho(k)$ is globally α-exponentially stable.

The calculation of the error signal $e(k)$ gives,

$$\begin{aligned} e(k) &= \hat{z}(k) - z(k) \\ &= s(k) + Vy(k) - Lx(k) \\ &= \rho(k) + (\Gamma - L + VC)x(k). \end{aligned} \tag{8}$$

From (6d), we get $e(k) = \rho(k)$.

So, if condition (6d) is satisfied, then the estimated error $e(k)$ is globally α-exponentially stable. □

For the discrete-time system (5), we give the following delay-dependent criterion of exponential stability.

Theorem 2. For given scalars $\alpha > 0$, d_1 and d_2 with $d_2 > d_1 > 0$, the system (5) is globally α-exponentially stable if there exist matrices $P > 0$, $Z_1 > 0$, $Z_2 > 0$,

$R_1 > 0,\ R_2 > 0,\ Q_1 > 0,\ Q_2 > 0$ such that the following matrix inequality holds:

$$\begin{pmatrix} \bar\Pi_{11} & 0 & 0 & Z_1 & Z_2 & e^{\alpha}W^TP & d_1\tilde{I}Z_1 & d_2\tilde{I}Z_2 \\ * & -R_1 & 0 & 0 & 0 & \chi P & d_1\chi Z_1 & d_2\chi Z_2 \\ * & * & -R_2 & 0 & 0 & \Lambda P & d_1\Lambda Z_1 & d_2\Lambda Z_2 \\ * & * & * & -Q_1-Z_1 & 0 & 0 & 0 & 0 \\ * & * & * & * & -Q_2-Z_2 & 0 & 0 & 0 \\ * & * & * & * & * & -P & 0 & 0 \\ * & * & * & * & * & * & -Z_1 & 0 \\ * & * & * & * & * & * & * & -Z_2 \end{pmatrix} < 0, \quad (9)$$

where $\bar\Pi_{11} = -P + d_{12}(R_1+R_2) + Q_1 + Q_2 + R_1 + R_2 - Z_1 - Z_2,\ d_{12} = d_2 - d_1,\ \tilde{I} = (e^{\alpha}W - I)^T,\ \chi = e^{\alpha}e^{\alpha d_2}W_{d_1}^T,\ \Lambda = e^{\alpha}e^{\alpha d_2}W_{d_2}^T.$

Proof. We introduce the new variable $v(k) = e^{\alpha k}\rho(k)$. The system (5) is reduced to

$$v(k+1) = e^{\alpha}Wv(k) + e^{\alpha}[W_{d_1}e^{\alpha d_1(k)}v(k-d_1(k)) + W_{d_2}e^{\alpha d_2(k)}v(k-d_2(k))] \quad (10)$$

Consider the Lyapunov–Krasovskii functional:

$$V(k) = V_1(k) + V_2(k) + V_3(k) + V_4(k) + V_5(k),$$

where

$$\begin{aligned} V_1(k) &= v^T(k)Pv(k), \\ V_2(k) &= \sum_{i=k-d_1}^{k-1} v^T(i)Q_1v(i) + \sum_{i=k-d_2}^{k-1} v^T(i)Q_2v(i) \\ &\quad + \sum_{i=k-d_1(k)}^{k-1} v^T(i)R_1v(i) + \sum_{i=k-d_2(k)}^{k-1} v^T(i)R_2v(i), \\ V_3(k) &= \sum_{i=-d_2+1}^{-d_1}\sum_{j=k+i}^{k-1} v^T(j)(R_1+R_2)v(j), \\ V_4(k) &= d_1\sum_{i=-d_1}^{-1}\sum_{j=k+i}^{k-1} \eta^T(j)Z_1\eta(j), \\ V_5(k) &= d_2\sum_{i=-d_2}^{-1}\sum_{j=k+i}^{k-1} \eta^T(j)Z_2\eta(j), \quad \eta(j) = v(j+1) - v(j). \end{aligned}$$

Define $\Delta V(k) = V(k+1) - V(k)$. Then along the solution of (10), we have

$$\begin{aligned} \Delta V_1(k) &= v^T(k+1)Pv(k+1) - v^T(k)Pv(k), \\ &= v^T(k)[e^{2\alpha}W^TPW - P]v(k) + 2e^{\alpha}\sum_{i=1}^{2} e^{\alpha d_i(k)}e^{\alpha}v^T(k-d_i(k))W_{d_i}^TPWv(k) \\ &\quad + \sum_{i=1}^{2} e^{2\alpha}e^{2\alpha d_i(k)}v^T(k-d_i(k))W_{d_i}^TPW_{d_i}v(k-d_i(k)) \\ &\quad + 2e^{2\alpha}e^{\alpha d_1(k)}e^{\alpha d_2(k)}v^T(k-d_1(k))W_{d_1}^TPW_{d_2}v(k-d_2(k)), \end{aligned} \quad (11)$$

$$\begin{aligned} \Delta V_2(k) &\le v^T(k)(Q_1+Q_2+R_1+R_2)v(k) - v^T(k-d_1)Q_1v(k-d_1)) \\ &\quad - v^T(k-d_2)Q_2v(k-d_2) - v^T(k-d_1(k))R_1v(k-d_1(k)) \\ &\quad - v^T(k-d_2(k))R_2v(k-d_2(k)) + \sum_{j=k+1-d_2}^{k-d_1} v^T(j)(R_1+R_2)v(j), \end{aligned} \quad (12)$$

$$\begin{aligned}\Delta V_3(k) &= \sum_{i=-d_2+1}^{-d_1}\sum_{j=k+1+i}^{k} v^T(j)(R_1+R_2)v(j) - \sum_{i=-d_2+1}^{-d_1}\sum_{j=k+i}^{k-1} v^T(j)(R_1+R_2)v(j)\\ &= d_{12}v^T(k)(R_1+R_2)v(k) - \sum_{j=k+1-d_2}^{k-d_1} v^T(j)(R_1+R_2)v(j).\end{aligned} \tag{13}$$

Using Lemma 1, we have

$$\begin{aligned}\Delta V_4(k) &\le d_1^2\eta^T(k)Z_1\eta(k) - \left(\sum_{j=k-d_1}^{k-1}\eta(j)\right)^T Z_1\left(\sum_{j=k-d_1}^{k-1}\eta(j)\right)\\ &= d_1^2\eta^T(k)Z_1\eta(k) - (v(k)-v(k-d_1))^T Z_1(v(k)-v(k-d_1)),\end{aligned} \tag{14}$$

$$\begin{aligned}\Delta V_5(k) &= d_2^2\eta^T(k)Z_2\eta(k) - d_2\sum_{j=k-d_2}^{k-1}\eta^T(j)Z_2\eta(j)\\ &\le d_2^2\eta^T(k)Z_2\eta(k) - (v(k)-v(k-d_2))^T Z_2(v(k)-v(k-d_2)).\end{aligned} \tag{15}$$

From (11)–(15), it follows that

$$\begin{aligned}\Delta V(k) \le\ & v^T(k)[e^{2\alpha}W^TPW - P + d_{12}(R_1+R_2) + Q_1 + Q_2 + R_1 + R_2 - Z_1 - Z_2\\ &+(e^{\alpha}W-I)^T\Upsilon(e^{\alpha}W-I)]v(k) + 2e^{\alpha}\sum_{i=1}^{2} e^{\alpha d_2}e^{\alpha}v^T(k-d_i(k))W_{d_i}^TPWv(k)\\ &+2e^{2\alpha}e^{2\alpha d_2}v^T(k-d_1(k))W_{d_1}^TPW_{d_2}v(k-d_2(k))\\ &+e^{2\alpha}e^{2\alpha d_2}v^T(k-d_1(k))W_{d_1}^T\Upsilon W_{d_1}v(k-d_1(k))\\ &+e^{2\alpha}e^{2\alpha d_2}v^T(k-d_2(k))W_{d_2}^T\Upsilon W_{d_2}v(k-d_2(k))\\ &+2e^{2\alpha}e^{2\alpha d_2}v^T(k-d_1(k))W_{d_1}^T\Upsilon W_{d_2}v(k-d_2(k))\\ &-v^T(k-d_1)[Q_1+Z_1]v(k-d_1) - v^T(k-d_2)[Q_2+Z_2]v(k-d_2)\\ &+2v^T(k)Z_1v(k-d_1) + 2v^T(k)Z_2v(k-d_2)\\ &+v^T(k-d_1(k))[-R_1 + e^{2\alpha}e^{2\alpha d_2}W_{d_1}^TPW_{d_1}]v(k-d_1(k))\\ &+v^T(k-d_2(k))[-R_2 + e^{2\alpha}e^{2\alpha d_2}W_{d_2}^TPW_{d_2}]v(k-d_2(k))\\ &+2v^T(k)(e^{\alpha}W-I)\Upsilon e^{\alpha}\sum_{i=1}^{2} W_{d_1}e^{\alpha d_i(k)}v(k-d_i(k))\\ =\ & \varsigma^T(k\Pi\varsigma(k).\end{aligned} \tag{16}$$

where

$$\begin{aligned}\varsigma(k) &= (v^T(k)\ v^T(k-d_1(k))\ v^T(k-d_2(k))\ v^T(k-d_1)\ v^T(k-d_2)^T)^T,\\ \Pi &= \begin{pmatrix}\Pi_{11} & \Pi_{12} & \Pi_{13} & Z_1 & Z_2\\ * & \Pi_{22} & \Pi_{23} & 0 & 0\\ * & * & \Pi_{33} & 0 & 0\\ * & * & * & Q_1-Z_1 & 0\\ * & * & * & * & Q_2-Z_2\end{pmatrix},\\ \Pi_{11} &= e^{2\alpha}W^TPW - P + d_{12}(R_1+R_2) + Q_1 + Q_2 + R_1 + R_2 - Z_1 - Z_2\\ &\quad +(e^{\alpha}W-I)^T\Upsilon(e^{\alpha}W-I),\\ \Pi_{12} &= e^{2\alpha}e^{\alpha d_2}W^TPW_{d_1} + e^{\alpha}e^{\alpha d_2}(e^{\alpha}W-I)^T\Upsilon W_{d_1},\\ \Pi_{13} &= e^{2\alpha}e^{\alpha d_2}W^TPW_{d_2} + e^{\alpha}e^{\alpha d_2}(e^{\alpha}W-I)^T\Upsilon W_{d_2},\\ \Pi_{22} &= e^{2\alpha}e^{2\alpha d_2}W_{d_1}^T\Upsilon W_{d_1} - R_1 + e^{2\alpha}e^{2\alpha d_2}W_{d_1}^TPW_{d_1},\\ \Pi_{23} &= e^{2\alpha}e^{2\alpha d_2}W_{d_1}^TPW_{d_2} + e^{2\alpha}e^{2\alpha d_2}W_{d_1}^T\Upsilon W_{d_2},\\ \Pi_{33} &= -R_2 + e^{2\alpha}e^{2\alpha d_2}W_{d_2}^TPW_{d_2} + e^{2\alpha}e^{2\alpha d_2}W_{d_2}^T\Upsilon W_{d_2},\quad \Upsilon = d_1^2Z_1 + d_2^2Z_2.\end{aligned}$$

Using Schur complement, $\Pi < 0$ if

$$\begin{pmatrix} \bar{\Pi}_{11} & 0 & 0 & Z_1 & Z_2 & e^{\alpha}W^T & d_1\tilde{I} & d_2\tilde{I} \\ * & -R_1 & 0 & 0 & 0 & \chi & d_1\chi & d_2\chi \\ * & * & -R_2 & 0 & 0 & \Lambda & d_1\Lambda & d_2\Lambda \\ * & * & * & -Q_1-Z_1 & 0 & 0 & 0 & 0 \\ * & * & * & * & -Q_2-Z_2 & 0 & 0 & 0 \\ * & * & * & * & * & -P^{-1} & 0 & 0 \\ * & * & * & * & * & * & -Z_1^{-1} & 0 \\ * & * & * & * & * & * & * & -Z_2^{-1} \end{pmatrix} < 0, \quad (17)$$

where $\bar{\Pi}_{11} = -P + d_{12}(R_1 + R_2) + Q_1 + Q_2 + R_1 + R_2 - Z_1 - Z_2,\ \tilde{I} = (e^{\alpha}W - I)^T$, $\chi = e^{\alpha}e^{\alpha d_2}W_{d_1}^T,\ \Lambda = e^{\alpha}e^{\alpha d_2}W_{d_2}^T$.

Pre- and post-multiplied (17) by $diag\{I, I, I, I, I, P, Z_1, Z_2\}$ yield

$$\begin{pmatrix} \bar{\Pi}_{11} & 0 & 0 & Z_1 & Z_2 & e^{\alpha}W^TP & d_1\tilde{I}Z_1 & d_2\tilde{I}Z_2 \\ * & -R_1 & 0 & 0 & 0 & \chi P & d_1\chi Z_1 & d_2\chi Z_2 \\ * & * & -R_2 & 0 & 0 & \Lambda P & d_1\Lambda Z_1 & d_2\Lambda Z_2 \\ * & * & * & -Q_1-Z_1 & 0 & 0 & 0 & 0 \\ * & * & * & * & -Q_2-Z_2 & 0 & 0 & 0 \\ * & * & * & * & * & -P & 0 & 0 \\ * & * & * & * & * & * & -Z_1 & 0 \\ * & * & * & * & * & * & * & -Z_2 \end{pmatrix} < 0,$$

From (9) it follows that $\Delta V(k) \leq 0$, which implies that the function $V(k)$ is decreasing and

$$V(k) \leq V(0), \quad \forall k \in Z^+.$$

We can verify that

$$\lambda_1\|v(k)\|^2 \leq V(k) \leq V(0) \leq \lambda_2\|\bar{\varphi}\|^2. \quad (18)$$

where

$$\begin{aligned} \lambda_1 &= \lambda_{\min}(P), \\ \lambda_2 &= \lambda_{\max}(P) + d_1\lambda_{\max}(Q_1) + d_2\lambda_{\max}(Q_2) + d_2(\lambda_{\max}(R_1) + \lambda_{\max}(R_2)) \\ &\quad \tfrac{1}{2}d_{12}(d_1 + d_2 - 1)(\lambda_{\max}(R_1) + \lambda_{\max}(R_2)) \\ &\quad +2d_1^2(d_1 + 1)\lambda_{\max}(Z_1) + 2d_2^2(d_2 + 1)\lambda_{\max}(Z_2). \end{aligned}$$

Hence from (18) it follows that

$$\|v(k)\| \leq \sqrt{\frac{\lambda_2}{\lambda_1}}\|\bar{\varphi}\|, \quad \forall k \in Z^+.$$

Returning to the variable $v(k) = e^{\alpha k}\rho(k)$, we have

$$\|\rho(k)\| \leq \sqrt{\frac{\lambda_2}{\lambda_1}}e^{-\alpha k}\|\varphi\|, \quad \forall k \in Z^+.$$

which implies that the zero solution is exponentially stable. This completes the proof of the Theorem 2. □

3.2. Observer design

Let $\tilde{C} = [C^+ \;\; C^\perp]$. First, the following parameters are introduced:

$$[\Gamma_1 \;\; \Gamma_2] = \Gamma\tilde{C}, \quad [L_1 \;\; L_2] = L\tilde{C}, \tag{19}$$

$$[C^+ \;\; C^\perp]\begin{bmatrix} A_{11}^0 & A_{12}^0 \\ A_{21}^0 & A_{22}^0 \end{bmatrix} = A[C^+ \;\; C^\perp], \tag{20a}$$

$$[C^+ \;\; C^\perp]\begin{bmatrix} A_{11}^1 & A_{12}^1 \\ A_{21}^1 & A_{22}^1 \end{bmatrix} = A_{d_1}[C^+, C^\perp], \tag{20b}$$

$$[C^+ \;\; C^\perp]\begin{bmatrix} A_{11}^2 & A_{12}^2 \\ A_{21}^2 & A_{22}^2 \end{bmatrix} = A_{d_2}[C^+ \;\; C^\perp]. \tag{20c}$$

From (6d), we have

$$\Gamma[C^+ \;\; C^\perp] + VC[C^+ \;\; C^\perp] - L[C^+ \;\; C^\perp] = 0.$$

From (19), we get

$$\Gamma_2 = L_2, \tag{21}$$

$$\Gamma_1 - L_1 + V = 0. \tag{22}$$

From (6a) and (6b), we have

$$\begin{aligned} &W\Gamma[C^+ \;\; C^\perp] + HC[C^+ \;\; C^\perp] - \Gamma A[C^+ \;\; C^\perp] = 0, \\ &W_{d_1}\Gamma[C^+ \;\; C^\perp] + H_{d_1}C[C^+ \;\; C^\perp] - \Gamma A_{d_1}[C^+ \;\; C^\perp] = 0, \\ &W_{d_2}\Gamma[C^+ \;\; C^\perp] + H_{d_2}C[C^+ \;\; C^\perp] - \Gamma A_{d_2}[C^+ \;\; C^\perp] = 0. \end{aligned} \tag{23}$$

Then, it follows that

$$\begin{aligned} &W\Gamma_1 + H - \Gamma_1 A_{11}^0 - \Gamma_2 A_{21}^0 = 0, \\ &W_{d_1}\Gamma_1 + H_{d_1} - \Gamma_1 A_{11}^1 - \Gamma_2 A_{21}^1 = 0, \\ &W_{d_2}\Gamma_1 + H_{d_2} - \Gamma_1 A_{11}^2 - \Gamma_2 A_{21}^2 = 0, \end{aligned} \tag{24}$$

$$\begin{aligned} &W\Gamma_2 - \Gamma_1 A_{12}^0 - \Gamma_2 A_{22}^0 = 0, \\ &W_{d_1}\Gamma_2 - \Gamma_1 A_{12}^1 - \Gamma_2 A_{22}^1 = 0, \\ &W_{d_2}\Gamma_2 - \Gamma_1 A_{12}^2 - \Gamma_2 A_{22}^2 = 0. \end{aligned} \tag{25}$$

Now, considering (21) and (25), we have

$$[W \;\; W_{d_1} \;\; W_{d_2} \;\; -\Gamma_1]X = \Sigma, \tag{26}$$

where

$$X = \begin{bmatrix} L_2 & 0 & 0 \\ 0 & L_2 & 0 \\ 0 & 0 & L_2 \\ A_{12}^0 & A_{12}^1 & A_{12}^2 \end{bmatrix}, \quad \Sigma = [L_2 A_{22}^0 \;\; L_2 A_{22}^1 \;\; L_2 A_{22}^2].$$

It can be shown that (26) has a solution if and only if the below condition holds [10]:

Condition I:

$$rank\left(\begin{bmatrix} L_2 & 0 & 0 \\ 0 & L_2 & 0 \\ 0 & 0 & L_2 \\ A_{12}^0 & A_{12}^1 & A_{12}^2 \\ L_2A_{22}^0 & L_2A_{22}^1 & L_2A_{22}^2 \end{bmatrix}\right) = rank\left(\begin{bmatrix} L_2 & 0 & 0 \\ 0 & L_2 & 0 \\ 0 & 0 & L_2 \\ A_{12}^0 & A_{12}^1 & A_{12}^2 \end{bmatrix}\right). \tag{27}$$

If Condition I holds, then it is concluded from (26) that

$$[W \quad W_{d_1} \quad W_{d_2} \quad -\Gamma_1] = Y_1 + \bar{Z}Y_2, \tag{28}$$

where $Y_1 = \Sigma X^+,\ Y_2 = I - XX^+$, and $\bar{Z}$ is an arbitrary parameter.
From (28), we have

$$W = Y_{11} + \bar{Z}Y_{21},\ W_{d_1} = Y_{12} + \bar{Z}Y_{22},\ W_{d_2} = Y_{13} + \bar{Z}Y_{23},\ -\Gamma_1 = Y_{14} + \bar{Z}Y_{24}, \tag{29}$$

where Y_{1i} and $Y_{2i},\ i \in \{1,2,3,4\}$, are the partitions of Y_1 and Y_2 with appropriate dimensions, respectively.

Theorem 3. Assume that Condition I holds. For given scalars $\alpha > 0, d_1$ and d_2 with $d_2 > d_1 > 0$, the functional observer error system for the system (1) and (3) is globally α-exponentially stable if

(a) there exist positive scalars κ_1, κ_2 and matrices $P > 0,\ R_1 > 0,\ R_2 > 0,\ Q_1 > 0,\ Q_2 > 0, K_1,\ K_2,\ K_3$with appropriate dimensions such that the following matrix inequality holds:

$$\bar{\Omega} = \begin{pmatrix} \bar{\Omega}_1 & \bar{\Omega}_2 \\ * & \bar{\Omega}_2 \end{pmatrix} < 0, \tag{30}$$

where

$$\bar{\Omega}_1 = \begin{pmatrix} J_1 & 0 & 0 & \kappa_1 P & \kappa_2 P \\ * & -R_1 & 0 & 0 & 0 \\ * & * & -R_2 & 0 & 0 \\ * & * & * & -Q_1 - \kappa_1 P & 0 \\ * & * & * & * & -Q_2 - \kappa_1 P \end{pmatrix},$$

$$\bar{\Omega}_2 = \begin{pmatrix} e^{\alpha}J_2 & d_1(e^{\alpha}\kappa_1 J_2 - \lambda_1 P) & d_2(e^{\alpha}\kappa_2 J_2 - \kappa_2 P) \\ e^{\alpha}e^{\alpha d_2}J_3 & d_1 e^{\alpha}e^{\alpha d_2}\kappa_1 J_3 & d_2 e^{\alpha}e^{\alpha d_2}\kappa_2 J_3 \\ e^{\alpha}e^{\alpha d_2}J_4 & d_1 e^{\alpha}e^{\alpha d_2}\kappa_1 J_4 & d_2 e^{\alpha}e^{\alpha d_2}\kappa_2 J_4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\bar{\Omega}_3 = diag\{-P \quad -\kappa_1 P \quad -\kappa_2 P\},$$

with $J_1 = -(1+\kappa_1+\kappa_2)P + d_{12}(R_1+R_2) + Q_1 + Q_2 + R_1 + R_2, J_2 = Y_{11}^T P + K_1$, $J_3 = Y_{12}^T P + K_2,\quad J_4 = Y_{13}^T P + K_3$.

(b) the below rank condition is satisfied:

Condition II:

$$rank\left(\left[\begin{array}{ccc} PY_{21} & PY_{22} & PY_{23} \\ K_1^T & K_2^T & K_3^T \end{array}\right]\right) = rank\left(\left[\begin{array}{ccc} PY_{21} & PY_{22} & PY_{23} \end{array}\right]\right).$$

In addition, the observer design parameter $\bar{Z}$ can be computed from the below equation

$$\bar{Z} = \bar{\Gamma}\tilde{Y}^{+}, \tag{31}$$

where $\bar{\Gamma} = (P^{-1}K_1^T \;\; P^{-1}K_2^T \;\; P^{-1}K_3^T),\ \tilde{Y} = (Y_{21} \;\; Y_{22} \;\; Y_{23})$.

Proof. (9) can be written

$$\Omega = \begin{pmatrix} \Omega_1 & \Omega_2 \\ * & \Omega_3 \end{pmatrix} < 0, \tag{32}$$

where

$$\Omega_1 = \begin{pmatrix} \bar{\Pi}_{11} & 0 & 0 & Z_1 & Z_2 \\ * & -R_1 & 0 & 0 & 0 \\ * & * & -R_2 & 0 & 0 \\ * & * & * & -Q_1 - Z_1 & 0 \\ * & * & * & * & -Q_2 - Z_2 \end{pmatrix},$$

$$\Omega_2 = \begin{pmatrix} e^{\alpha}W^TP & d_1(e^{\alpha}W - I)^TZ_1 & d_2(e^{\alpha}W - I)^TZ_2 \\ e^{\alpha}e^{\alpha d_2}W_{d_1}^TP & d_1e^{\alpha}e^{\alpha d_2}W_{d_1}^TZ_1 & d_2e^{\alpha}e^{\alpha d_2}W_{d_1}^TZ_2 \\ e^{\alpha}e^{\alpha d_2}W_{d_2}^TP & d_1e^{\alpha}e^{\alpha d_2}W_{d_2}^TZ_1 & d_2e^{\alpha}e^{\alpha d_2}W_{d_2}^TZ_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\Omega_3 = diag\{-P \quad -Z_1 \quad -Z_2\},$$
$$\bar{\Pi} \;= -P + d_{12}(R_1 + R_2) + Q_1 + Q_2 + R_1 + R_2 - Z_1 - Z_2.$$

From (29), we have

$$\Omega_2 = \begin{pmatrix} e^{\alpha}(Y_{11}^TP + K_1) & d_1(e^{\alpha}(Y_{11}^TZ_1 + U_1) - Z_1) & d_2(e^{\alpha}(Y_{11}^TZ_2 + O_1) - Z_2) \\ e^{\alpha}e^{\alpha d_2}(Y_{12}^TP + K_2) & d_1e^{\alpha}e^{\alpha d_2}(Y_{12}^TZ_1 + U_2) & d_2e^{\alpha}e^{\alpha d_2}(Y_{12}^TZ_2 + O_2) \\ e^{\alpha}e^{\alpha d_2}(Y_{13}^TP + K_3) & d_1e^{\alpha}e^{\alpha d_2}(Y_{13}^TZ_1 + U_3) & d_2e^{\alpha}e^{\alpha d_2}(Y_{13}^TZ_2 + O_3) \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

where $K_i = Y_{2i}^T\bar{Z}^TP,\ U_i = Y_{2i}^T\bar{Z}^TZ_1,\ O_i = Y_{2i}^T\bar{Z}^TZ_2,\ i = 1, 2, 3$. In this line, it is assumed that $Z_1 = \kappa_1 P,\ Z_2 = \kappa_2 P$, then $U_i = \kappa_1 K_i,\ O_i = \kappa_2 K_i,\ i = 1, 2, 3$, and $\Omega_i,\ i = 1, 2, 3$, can be written as

$$\Omega_2 = \begin{pmatrix} e^{\alpha}J_2 & d_1(e^{\alpha}\kappa_1J_2 - \kappa_1P) & d_2(e^{\alpha}\kappa_2J_2 - \kappa_2P) \\ e^{\alpha}e^{\alpha d_2}J_3 & d_1e^{\alpha}e^{\alpha d_2}\kappa_1J_3 & d_2e^{\alpha}e^{\alpha d_2}\kappa_2J_3 \\ e^{\alpha}e^{\alpha d_2}J_4 & d_1e^{\alpha}e^{\alpha d_2}\kappa_1J_4 & d_2e^{\alpha}e^{\alpha d_2}\kappa_2J_4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\Omega_1 = \begin{pmatrix} -J_1 & 0 & 0 & \kappa_1 P & \kappa_2 P \\ * & -R_1 & 0 & 0 & 0 \\ * & * & -R_2 & 0 & 0 \\ * & * & * & -Q_1 - \kappa_1 P & 0 \\ * & * & * & * & -Q_2 - \kappa_2 P \end{pmatrix},$$

$$\Omega_3 = diag\{-P \quad -\kappa_1 P \quad -\kappa_2 P\},$$

where $J_1 = -(1+\kappa_1+\kappa_2)P + d_{12}(R_1+R_2) + Q_1 + Q_2 + R_1 + R_2, J_2 = Y_{11}^T P + K_1$, $J_3 = Y_{12}^T P + K_2, \quad J_4 = Y_{13}^T P + K_3$.

We have

$$\bar{Z}(Y_{21} \quad Y_{22}\; Y_{23}) = (P^{-1}K_1^T \quad P^{-1}K_2^T \quad P^{-1}K_3^T).$$

The parameter $\bar{Z}$ has a unique solution as (31), if and only if Condition II is satisfied. As a result, the observer parameters $W, W_{d_1}, W_{d_2}, H, H_{d_1}, H_{d_2}$ and V can be respectively computed from (29), (24), and (22), which implies that Condition (b) of Theorem 1 is also satisfied. This completes the proof of the theorem. □

Remark 1. For given κ_1, κ_2, we note that (30) is linear matrix inequalities which can be solved efficiently by Matlab LMI Toolbox. Computing the parameters of the functional observer has certain complexity. In order to solve inequality (30), we first use (19), (20) to solve $A_{12}^0, A_{12}^1, A_{12}^2, A_{22}^0, A_{22}^1, A_{22}^2$, Using (21), we then get X, Σ. Using X and Σ, we get $Y_1 = (Y_{11} \quad Y_{12} \quad Y_{13})$, $Y_2 = (Y_{21} \quad Y_{22} \quad Y_{23})$,. Then, Matlab LMI Toolbox is used to solve $\bar{Z}$, by which the parameters of the functional observer are computed.

Remark 2. The steps of calculating the observer parameters are as follows:

Step1. Using Matlab LMI Toolbox to solve (30), we can obtain matrices $P, R_1, R_2, Q_1, Q_2, K_1, K_2, K_3$.

Step 2. According to (31), we calculate $\bar{Z}$;

Step 3. According to (29), we calculate $W, W_{d_1}, W_{d_2}, \Gamma_1$;

Step 4. According to (24), we calculateH, H_{d_1}, H_{d_2};

Step 5. According to (22), we calculate V;

Step 6. According to (6c), we calculate G and $\bar{E}$.

4. INTERNAL-DELAY INDEPENDENT OBSERVER DESIGN

We consider the following observer

$$\begin{aligned} s(k+1) &= Ws(k) + Gu(k) + \bar{E}f(y(k)) + Hy(k) + H_{d_1}y(k-d_1(k)) \\ &\quad +H_{d_2}y(k-d_2(k)), \\ \hat{z}(k) &= s(k) + Vy(k), \end{aligned} \tag{33}$$

where $W, H, H_{d_1}, H_{d_2}G, \bar{E}$ and V are constant matrices of appropriate dimensions.

Corollary 1. Consider the functional observer (33) for system (1). The estimated error system is globally asymptotically stable if
(a) the matrix W is stable.
(b) there exists a matrix Γ such that the following equations hold:

$$W\Gamma + HC - \Gamma A = 0, \tag{34a}$$

$$\begin{aligned} -\Gamma A_{d_1} + H_{d_1} C = 0, \\ -\Gamma A_{d_2} + H_{d_2} C = 0, \end{aligned} \tag{34b}$$

$$\begin{aligned} G - TB = 0, \\ \bar{E} - \Gamma E = 0, \end{aligned} \tag{34c}$$

$$\Gamma + VC - L = 0. \tag{34d}$$

Proof. The proof is similar to the proof of Theorem 1. Thus it is omitted. □

Following the same procedure as explained in Section 3.2, for the observer parameters, the following equation is obtained

$$[W \quad -\Gamma_1]X_1 = \Sigma_1 \tag{35}$$

where

$$X_1 = \begin{bmatrix} L_2 & 0 & 0 \\ A_{12}^0 & A_{12}^1 & A_{12}^2 \end{bmatrix}, \quad \Sigma_1 = [L_2 A_{22}^0 \quad L_2 A_{22}^1 \quad L_2 A_{22}^2].$$

It can be shown that (35) has a solution if and only if the below condition holds:

Condition III :

$$rank\left(\begin{bmatrix} L_2 & 0 & 0 \\ A_{12}^0 & A_{12}^1 & A_{12}^2 \\ L_2 A_{22}^0 & L_2 A_{22}^1 & L_2 A_{22}^2 \end{bmatrix}\right) = rank\left(\begin{bmatrix} L_2 & 0 & 0 \\ A_{12}^0 & A_{12}^1 & A_{12}^2 \end{bmatrix}\right). \tag{36}$$

If Condition III is satisfied, then it is concluded from (35) that

$$[W \quad -\Gamma_1] = \bar{Y}_1 + \check{Z}\bar{Y}_2, \tag{37}$$

where $\bar{Y}_1 = \Sigma_1 X_1^+, \bar{Y}_2 = I - X_1 X_1^+$, and $\check{Z}$ is an arbitrary matrix.
From (37), we have

$$W = \bar{Y}_{11} + \check{Z}\bar{Y}_{21}, \ -T_1 = \bar{Y}_{12} + \check{Z}\bar{Y}_{22},$$

where $\bar{Y}_{1i}$ and $\bar{Y}_{2i}, i \in \{1,2\}$, are the partitions of $\bar{Y}_1$ and $\bar{Y}_2$ with appropriate dimensions, respectively. W is stable if and only if the pair $[\bar{Y}_{11} \ \bar{Y}_{21}]$is detectable, or equivalently the below condition is achieved:

Condition IV:

$$rank\left(\begin{bmatrix} sI - \bar{Y}_{11} \\ \bar{Y}_{21} \end{bmatrix}\right) = l, \quad \forall s \in \mathcal{C}^+.$$

Theorem 4. If Conditions III and IV are satisfied, then there exists internal-delay independent functional observer (33) for the system (1) such that the estimated error system is a globally asymptotically stable.

Theorem 5. Assume that Condition III holds. For given scalars $\alpha > 0, d_1$ and d_2 with $d_2 > d_1$, the functional observer error system for the system (1) and (33) is globally α-exponentially stable if

(c) there exist positive scalars κ_1, κ_2, and matrices $P > 0, R_1 > 0,\ R_2 > 0,\ Q_1 > 0,$ $Q_2 > 0,\ K_1$ with appropriate dimensions such that the following matrix inequality holds:

$$\hat{\Omega} = \begin{pmatrix} \hat{\Omega}_1 & \hat{\Omega}_2 \\ * & \hat{\Omega}_3 \end{pmatrix} < 0, \tag{39}$$

where

$$\hat{\Omega}_1 = \begin{pmatrix} J_1 & 0 & 0 & \kappa_1 P & \kappa_2 P \\ * & -R_1 & 0 & 0 & 0 \\ * & * & -R_2 & 0 & 0 \\ * & * & * & -Q_1 - \kappa_1 P & 0 \\ * & * & * & * & -Q_2 - \kappa_2 P \end{pmatrix},$$

$$\hat{\Omega}_2 = \begin{pmatrix} e^{\alpha} J_2 & d_1(e^{\alpha}\kappa_1 J_2 - \lambda_1 P) & d_2(e^{\alpha}\kappa_2 J_2 - \kappa_2 P) \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\hat{\Omega}_3 = diag\{-P \quad -\kappa_1 P \quad -\kappa_2 P\},$$

with $J_1 = -(1+\kappa_1+\kappa_2)P + d_{12}(R_1+R_2) + Q_1 + Q_2 + R_1 + R_2, J_2 = Y_{11}^T P + K_1$.
(d) the below rank condition is satisfied:
Condition V

$$rank\left(\begin{bmatrix} PY_{21} \\ K_1^T \end{bmatrix}\right) = rank(PY_{21}).$$

In addition, the observer design parameter $\check{Z}$ can be computed from the below equation

$$\check{Z} = \tilde{\Gamma}\tilde{Y}^{+}, \tag{40}$$

where $\tilde{\Gamma} = P^{-1}K_1^T,\ \tilde{Y} = Y_{21}$.

P r o o f. The proof of Theorem 5 is similar to that of Theorem 3, and the detailed proof is omitted. □

Remark 3. In [4], Darouach considered the problem of design of linear functional state observers for the linear discrete-time system with constant delay, and derived some sufficient conditions. In this paper, we investigate functional observer design for a class of discrete-time systems with time-varying delays. Compare to [4], the results obtained in this paper have a wider range of applications.

Remark 4. In [9], Mohajerpoor et al. studied the functional observer design of linear time-invariant systems which are continuous-time systems with constant delays and given the design method. Mohajerpoor et al. [9] did not consider the discrete-time systems. In this paper, we consider functional observer design for a class of discrete-time systems with multiple mixed time-varying delays in the states of the systems. The new method of functional observer design for discrete-time systems is proposed.

Remark 5. In [6], the state vector $x(k)$ is estimated, but in this paper, the functional $z(k) = Lx(k)$ is estimated. In addition, the design method of this paper is different from the design method in [6]. When $L = I$, we have $z(k) = x(k)$, and the estimation for $z(k)$ is equivalent to the estimation for $x(k)$. Compared with [6], the design method of this paper has wider application scope.

5. NUMERICAL EXAMPLES

Example 1. Consider system (1) with the following parameters:

$$A = \begin{bmatrix} 0 & -1 & 0 \\ 2 & 3 & -1 \\ -1 & 0 & -1 \end{bmatrix}, \quad A_{d_1} = \begin{bmatrix} 0 & 0.1 & 0 \\ 0 & 0 & -0.1 \\ -1 & 0 & -1 \end{bmatrix}, \quad A_{d_2} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0.8 & 0.01 \\ 0 & 0.2 & 0.2 \end{bmatrix},$$

$$B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, E = \begin{bmatrix} 0 \\ 0.1 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, L = [0.1 \ \ 1 \ \ 0.1], \ f(Cx(k)) = sin(x_1(k)).$$

Firstly, it can be observed that Condition I is satisfied. For given $\kappa_1 = 0.06, \kappa_2 = 0.05, d_2 = 3, d_{12} = 2, \alpha = 0.5, d_1 = 1$, solving the LMIs (30) yields

$$P = 9.1182, \ R_1 = 0.7761, \ R_2 = 0.6259, \ Q_1 = 0.6087, \ Q_2 = 1.0603,$$
$$K_1 = 1.1989, \ K_2 = 8.1309, \ K_3 = -1.7252.$$

The observer parameters are obtained as

$$W = 0.4894, W_{d_1} = -0.8511, W_{d_2} = 0.1815, \ \Gamma_1 = [0 \ -0.1489], G = 0, \ V = [0.1 \ 1.1489]$$
$$H = [-0.3978 \ \ -0.3738], H_{d_1} = [-0.1 \ \ -0.1267], H_{d_2} = [0 \ \ -0.0716], \ \bar{E} = -0.01489.$$

Moreover, it can be seen that these parameters satisfy Condition II. According Theorem 3, the functional observer (3) is globally α-exponentially stable.

Simulations were performed with input signal equal to $u(k) = 0.1sin(x_1(k))$.

Figure 1 shows the response of $z(k)$ and its estimation with $\alpha = 0.5$.

For given $\kappa_1 = 0.06, \kappa_2 = 0.05, d_2 = 4, d_{12} = 2, \alpha = 4, d_1 = 2$, solving the LMIs (30) yields

$$P = 6.0831, \ R_1 = 0.4171, \ R_2 = 0.3299, \ Q_1 = 0.3444, \ Q_2 = 0.6612,$$
$$K_1 = -0.4348, \ K_2 = 5.4261, \ K_3 = -1.1509.$$

The observer parameters are obtained as

$$W = 0.2895, W_{d_1} = -0.8711, W_{d_2} = 0.1871, \ \Gamma_1 = [0 \ -0.1289], G = 0, \ V = [0.1000 \ 1.1289]$$
$$H = [-0.3578 \ \ -0.3494], H_{d_1} = [-0.1 \ \ -0.1123], H_{d_2} = [0 \ \ -0.0590], \ \bar{E} = -0.0129.$$

Figures 2 shows the response of $z(k)$ and its estimation with $u(k) = 0.1sin(x_1(k))$ and $\alpha = 4$. These simulation results demonstrate that our proposed design is very effective.

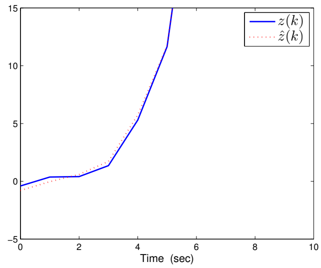

Fig. 1. Responses of $z(k)$ and $\hat{z}(k)$ in Example 1 with $\alpha = 0.5$.

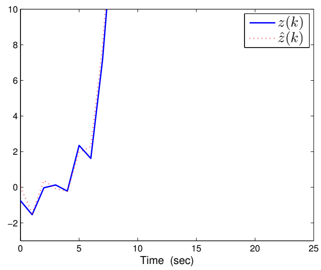

Fig. 2. Responses of $z(k)$ and $\hat{z}(k)$ in Example 1 with $\alpha = 4$.

Example 2. Consider system (1) with the following parameters

$$A = \begin{bmatrix} -1 & 0.1 & -0.2 \\ 0.2 & -0.3 & -0.1 \\ -0.1 & 0 & -1 \end{bmatrix}, \quad A_{d_1} = \begin{bmatrix} 0 & 0.1 & 0.1 \\ 0 & 0.2 & -0.1 \\ 0.1 & 0 & 0.1 \end{bmatrix}, \quad A_{d_2} = \begin{bmatrix} 0 & 0.1 & 0.3 \\ 0 & 0.1 & 0.4 \\ 0 & 0.2 & 0.2 \end{bmatrix},$$

$$B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \; L = [0.1 \;\; 1 \;\; 0], \; E = 0.$$

It can be observed that Condition III is satisfied. For given $\kappa_1 = 0.005, \kappa_2 = 0.05, d_2 = 3, d_{12} = 2, \alpha = 0.15, d_1 = 1$, solving the LMIs (39) yields

$$P = 6.3865, \; R_1 = 0.8026, \; R_2 = 0.2867, \; Q_1 = 0.5661, \; Q_2 = 0.5047, K_1 = 9.3346.$$

The observer parameters were obtained as

$$W = 0.3824, \; \Gamma_1 = [-0.9663 \;\; 0.0426], \; V = [-0.8663 \;\; 1.0426], \; G = 0.9663$$
$$H = [-1.3545 \;\; 0.1001], H_{d_1} = [0.0100 \; 0.0966], H_{d_2} = [0 \;\; 0.0157], \; \bar{E} = 0.$$

Moreover, it can be seen that these parameters satisfy Condition V. According Theorem 5, the functional observer (33) is globally α-exponentially stable.

Simulation were performed with input signal equal to $u(k) = 0.1sin(x_2(k))$. Figures 3 shows the response of $z(k)$and its estimation.

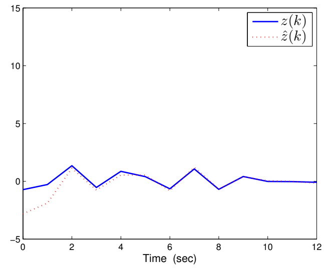

Fig. 3. Responses of $z(k)$ and $\hat{z}(k)$ in Example 2.

Example 3. Consider the discrete-time system which is the model of some electromechanical systems [5]

$$\begin{array}{l} x(k+1) = Ax(k) + Bu(k) + Ef(y(k)), \\ y(k) = Cx(k), \\ z(k) = Lx(k), \end{array} \tag{41}$$

where

$$A = \begin{bmatrix} 1 & h & 0 \\ 0 & 1-ha_2 & hb_1 \\ 0 & -ha_3 & 1-ha_4 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ hb_0 \end{bmatrix}, \quad E = \begin{bmatrix} 0 \\ -ha_1 \\ 0 \end{bmatrix},$$

$$C = [1 \ \ 0 \ \ 0], f(y(k)) = sin(y(k)), \ L = [0 \ \ 1 \ \ 1].$$

We let $b_0 = 40, \ b_1 = 15, a_1 = 35, \ a_2 = 0.25, \ a_3 = 36, \ a_4 = 200$. These values are close to the numerical values given in [5]. Let $h = \frac{1}{200}$.

Firstly, it can be observed that Condition I is satisfied. For given $\kappa_1 = 0.06, \kappa_2 = 0.05, d_2 = 3, d_{12} = 2, \alpha = 0.5, d_1 = 1$, solving the LMIs (30) yields

$$P = 1.5864, \ R_1 = 0.0673, \ R_2 = 0.0673, \ Q_1 = 0.4708, \ Q_2 = 0.4549,$$
$$K_1 = -0.6979, K_2 = 0, K_3 = 0.$$

The observer parameters are obtained as

$$W = 0.0070, W_{d_1} = 0, \quad W_{d_2} = 0, \quad \Gamma_1 = -17.5998, \quad V = 17.5998,$$
$$H = -17.4766, \quad H_{d_1} = 0, \quad H_{d_2} = 0, \quad \bar{E} = -0.1750, \quad G = 0.02.$$

Moreover, it can be seen that these parameters satisfy Condition II. According Theorem 3, the functional observer (3) is globally α-exponentially stable.

Simulations were performed with input signal equal to $u(k) = 0.2sin(x_1(k))$.

Figures 4 shows the response of $z(k)$ and its estimation with $\alpha = 0.5$. The simulation results demonstrate that our proposed design is very effective.

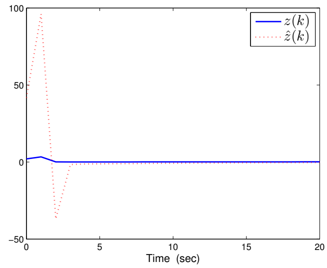

Fig. 4. Responses of $z(k)$ and $\hat{z}(k)$ in Example 3.

6. CONCLUSIONS

Our paper considers the functional observer design for a class of discrete-time systems with time-varying delays. We propose both the delay-dependent and internal-delay independent observer structures. Firstly, the delay-dependent functional observer for a class of discrete-time systems with mixed time-varying delays has been designed to assure the error system is globally exponentially stable. We propose the sufficient conditions of the globally exponential stability of the functional observer error system. In addition, the delay-dependent sufficient conditions to guarantee the functional observer error system is exponentially stable are established, and are expressed in terms of matrix inequalities. Secondly, we also give the sufficient conditions for the existence of an internal-delay independent functional observer. Besides, to guarantee the internal-delay independent functional observer error system is exponentially stable, we establish some sufficient conditions and designed the parameters of the internal-delay independent observer. In the end, three numerical examples are provided to illustrate our approach.

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