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OPTIMAL CONTROL PROBLEM AND MAXIMUM PRINCIPLE FOR FRACTIONAL ORDER COOPERATIVE SYSTEMS

G. M. Bahaa

In this paper, by using the classical control theory, the optimal control problem for fractional order cooperative system governed by Schrödinger operator is considered. The fractional time derivative is considered in a Riemann–Liouville and Caputo senses. The maximum principle for this system is discussed. We first study by using the Lax-Milgram Theorem, the existence and the uniqueness of the solution of the fractional differential system in a Hilbert space. Then we show that the considered optimal control problem has a unique solution. The performance index of a (FOCP) is considered as a function of both state and control variables, and the dynamic constraints are expressed by a Partial Fractional Differential Equation (PFDE). Finally, we impose some constraints on the boundary control. Interpreting the Euler-Lagrange first order optimality condition with an adjoint problem defined by means of right fractional Caputo derivative, we obtain an optimality system for the optimal control. Some examples are analyzed in details.

Keywords: fractional optimal control, cooperative systems;, Schrödinger operator, maximum principle, existence of solution, boundary control, optimality conditions, fractional Caputo derivatives, Riemann–Liouville derivatives

Classification: 26A33, 49J20, 35R11, 49J15, 49K20, 93C20

1. INTRODUCTION

Fractional calculus deals with the generalization of differentiation and integration of non integer orders. In recent years, it has played a significant role in physics, chemistry, biology, economics, control theory, signal and image processing, biophysics, blood flow phenomena, aerodynamics, fitting of experimental data, etc. Extensive treatment and various applications of the fractional calculus are discussed for example in ([14, 16, 23], [37]).

Both fractional calculus of variations and fractional optimal control problems were developed by many authors it is enough to see, for example, works in (see [1] – [3], [11] – [14, 21, 35, 36] and the papers and references therein) similar to a differential equations with integer time derivatives (see [10], [17] – [20], [24] – [26], [38] and the papers and references therein).

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One of the most useful and best known tools employed in the study of integer order partial differential equations is the maximum principle, as it is an useful tool to prove many results such as existence, multiplicity and qualitative properties for their solutions. An excellent overview of the subject up to 1967 can be found in the book by Protter and Weinberger [38]. Several papers have explored maximum principle for integer order different systems (linear, semilinear and nonlinear) involving Laplace and p-Laplace operators. The maximum principle has also been studied for linear elliptic systems. In particular, in [19], the authors proved sufficient and necessary conditions for having the maximum principle and the existence of positive solutions for cooperative linear elliptic systems involving Laplace operator with constant coefficients. In [20], Fleckinger and Serag presented necessary and sufficient conditions for having the maximum principle and for the existence of positive solutions for cooperative semilinear elliptic systems involving Laplace operator with variable coefficients. These results have been extended in [19] to the cooperative nonlinear elliptic system involving the p-Laplacian operators with constant coefficients.

Also one of the recent research topics in this theory is studying the analogies of the maximum principles known for the parabolic and elliptic types of partial differential equations as well as their applications to analysis of solutions to the boundary or initial boundary value problems for the fractional (non integer) partial differential equations. The first publications that should be mentioned in this connection are the papers [15] and [22], where a kind of a maximum principle was employed for analysis of some fractional partial differential equations without an explicit formulation of this principle. In [29], a weak maximum principle for a single-term time-fractional diffusion equation with the Caputo fractional derivative was formulated and proved for the first time. In [31], this principle was applied for an a priori estimate for solutions to the initial-boundary-value problems for a multi-dimensional time-fractional diffusion equation. The weak maximum principles for multi-term time-fractional diffusion equations and time-fractional diffusion equations with the Caputo fractional derivatives of the distributed orders were introduced and applied in [32] and [30], respectively. In [27], a strong maximum principle for time-fractional diffusion equations with the Caputo derivatives was established and applied for proving a uniqueness result for a related inverse source problem of determination of the temporal component of the source equation term.

In [4] – [6], the maximum principles for single, multi-term, and distributed order fractional diffusion equations with the Riemann–Liouville fractional derivatives, respectively, were proved and employed for analysis of solutions to the initial boundary value problems for linear and non-linear time-fractional partial differential equations. A maximum principle for multi-term time-space fractional differential equations with the modified Riesz space-fractional derivative in the Caputo sense was introduced and employed in [39]. In [28], a maximum principle for multi-term time-space variable-order fractional differential equations with the Riesz–Caputo fractional derivatives was proved and applied for analysis of these equations. Finally, we mention a very recent paper [33], where a weak maximum principle for a general time-fractional diffusion equation which was introduced in [23], was derived and employed for proving the uniqueness of both the strong and the weak solutions to the initial-boundary-value problem for this equation. The general time-fractional diffusion equation contains both single-and multi-term time-fractional diffusion equations as well as time-fractional diffusion equation of the distributed order among its particular cases and is a new object in fractional calculus worth to be investigated in detail.

In this paper we consider the following fractional optimal control problem for the following cooperative systems :

$$\left.\begin{aligned} {}_aD^{\beta}y_1(x;t)+(-\Delta+q(x))y_1(x;t) &= ay_1(x;t)+by_2(x;t)+f_1 \text{ in } \Omega,\\ {}_aD^{\beta}y_2(x;t)+(-\Delta+q(x))y_2(x;t) &= cy_1(x;t)+dy_2(x;t)+f_2 \text{ in } \Omega,\\ y_1(x;t) &= g_1 \quad \text{as} \quad |x|\to\infty,\\ y_2(x;t) &= g_2 \quad \text{as} \quad |x|\to\infty,\\ {}_aI^{1-\beta}y_1(x,0^+) &= y_{1,0}(x) \quad \text{in } \Omega,\\ {}_aI^{1-\beta}y_2(x,0^+) &= y_{2,0}(x) \quad \text{in } \Omega, \end{aligned}\right\} \tag{1}$$

where:

(1) a, b, c and d are given numbers such that $b, c > 0$ (in this case, we say that the system (1) is cooperative),

(2) $q(x)$ is a positive function and tending to ∞ at infinity,

(3) The fractional integral ${}_aI^{1-\beta}$ and fractional derivative ${}_aD^{\beta}$ are understood here in the Riemann–Liouville sense, ${}_aI^{1-\beta}y(x,0^+)=\lim_{t\to0^+}{}_aI^{1-\beta}y(x,t)$. As usual, the Shrödinger operator takes the form $A_{q(x)}=-\Delta+q(x)$ in $L^2(\Omega)$ where Δ and $q(x)$, respectively, denote the self-adjoint Laplace operator and the pointwise multiplication operator by the potential $q(x)$ in $L^2(\Omega)$.

A strong motivation for studying and investigating the solution and the properties for fractional cooperative systems governed by Shrödinger operators comes from the fact that they describe efficiently anomalous of fractals physical objects of fractional dimension like some amorphous semiconductors or strongly porous materials and fractional random walk. The existence and uniqueness of solutions for such equations were proved. Fractional optimal control is characterized by the adjoint problem. By using this characterization, particular properties of fractional optimal control are proved.

The rest of the paper is organized as follows. In Section 2, we introduce some definitions and preliminary results. In Section 3, we discuss the maximum principle and existence theorem of the system (1). In Section 4, we formulate the fractional control problem for the cooperative system (1). In section 5, we give the fractional control problem for the scalar case. In section 6, we state some mathematical examples and applications. We formulate an equivalent system to system (1) with fractional derivative in Caputo sense and with different control constraints. Also we generalize our results to n-dimensional coupled fractional system. Finally, some conclusions are formulated in section 7.

2. PRELIMINARIES

In this section we introduce some basic definitions related to fractional derivatives. Let $n \in N^*$ and Ω be a bounded open subset of $\mathbb{R}^n$ with a smooth boundary $\partial\Omega$ of class C^2.

For a time $T > 0$, we set $Q = \Omega \times (0,T)$ and $\Sigma = \partial\Omega \times (0,T)$, lateral boundary of Q.

Definition 2.1. *The Left Riemann–Liouville Fractional Integral (left RLFI)* and *the Right Riemann–Liouville Fractional Integral (right RLFI)* are presented respectively by

$$ {}_aI^{\beta} f(t) = \frac{1}{\Gamma(\beta)} \int_a^t (t-\tau)^{\beta-1} f(\tau)\,\mathrm{d}\tau, \tag{2}$$

$$ I_b^{\beta} f(t) = \frac{1}{\Gamma(\beta)} \int_t^b (\tau-t)^{\beta-1} f(\tau)\,\mathrm{d}\tau, \tag{3}$$

where $\beta > 0,\ n-1 < \beta < n$. From now on, $\Gamma(\beta)$ represents the Gamma function.

The Left Riemann–Liouville Fractional Derivative (left RLFD) is given by

$$ {}_aD^{\beta} f(t) = \frac{1}{\Gamma(n-\beta)} \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^n \int_a^t (t-\tau)^{n-\beta-1} f(\tau)\,\mathrm{d}\tau. \tag{4}$$

The Right Riemann–Liouville Fractional Derivative (right RLFD) is defined by

$$ D_b^{\beta} f(t) = \frac{1}{\Gamma(n-\beta)} \left(-\frac{\mathrm{d}}{\mathrm{d}t}\right)^n \int_t^b (\tau-t)^{n-\beta-1} f(\tau)\,\mathrm{d}\tau. \tag{5}$$

The fractional derivative of a constant takes the form

$$ {}_aD^{\beta} C = C\ \frac{(t-a)^{-\beta}}{\Gamma(1-\beta)}, \tag{6}$$

and the fractional derivative of a power of t has the following form

$$ {}_aD^{\beta} (t-a)^{\alpha} = \frac{\Gamma(\beta+1)(t-a)^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}, \tag{7}$$

for $\alpha > -1, \beta \geq 0$.

Definition 2.2. The Caputo's fractional derivatives are defined as follows:

The Left Caputo Fractional Derivative (left CFD)

$$ {}_a^CD^{\beta} f(t) = \frac{1}{\Gamma(n-\beta)} \int_a^t (t-\tau)^{n-\beta-1} \left(\frac{\mathrm{d}}{\mathrm{d}\tau}\right)^n f(\tau)\,\mathrm{d}\tau, \tag{8}$$

and

The Right Caputo Fractional Derivative (right CFD)

$$ {}^CD_b^{\beta} f(t) = \frac{1}{\Gamma(n-\beta)} \int_t^b (\tau-t)^{n-\beta-1} \left(-\frac{\mathrm{d}}{\mathrm{d}\tau}\right)^n f(\tau)\,\mathrm{d}\tau, \tag{9}$$

where β represents the order of the derivative such that $n-1 < \beta < n$. Further, it holds

$$ {}_0^CD^{\beta}\, c = 0, \text{ where c is a constant,} $$

and

$${}_0^C D^\beta t^n = \begin{cases} 0, \text{ for } n \in N_0 \text{ and } n < [\beta], \\ \frac{\Gamma(n+1)}{\Gamma(n+1-\beta)} t^{n-\beta}, \text{ for } n \in N_0 \text{ and } n \geq [\beta], \end{cases}$$

where $N_0 = 0, 1, 2, \ldots$. We recall that for $\beta \in N$, the Caputo differential operator coincides with the usual differential operator of integer order. Note also that when $T = +\infty$, ${}^C D_b^\beta f(t)$ is the Weyl fractional integral of order β of f. The Caputo fractional derivative is a sort of regularization in the time origin for the Riemann–Liouville fractional derivative.

Lemma 2.3. (See Bahaa [9, 10]) Let $T > 0, u \in C^m([0,T]), \beta \in (m-1, m), m \in N$ and $v \in C^1([0,T])$. Then for $t \in [0,T]$ the following properties hold:

$${}_0 D^\beta v(t) = \frac{\mathrm{d}}{\mathrm{d}t}\, {}_0 I^{1-\beta} v(t), \quad m = 1, \tag{10}$$

$${}_0 D^\beta\, {}_0 I^\beta v(t) = v(t), \tag{11}$$

$${}_0 I^\beta\, {}_0^C D^\beta u(t) = u(t) - \sum_{k=0}^{m-1} \frac{t^k}{k!} u^{(k)}(0), \tag{12}$$

$$\lim_{t\to 0^+} {}_0^C D^\beta u(t) = \lim_{t\to 0^+} {}_0 I^\beta u(t) = 0. \tag{13}$$

Definition 2.4. Relation between RLFD and the CFD

The relation between the right RLFD and the right CFD is as follows:

$${}^C D_b^\beta x(t) = D_b^\beta x(t) - \sum_{k=0}^{n-1} \frac{x^{(k)}(b)}{\Gamma(k-\beta+1)} (b-t)^{(k-\beta)}, \tag{14}$$

The relation between the left RLFD and the left CFD is as follows:

$${}_a^C D^\beta x(t) = {}_a D^\beta x(t) - \sum_{k=0}^{n-1} \frac{x^{(k)}(a)}{\Gamma(k-\beta+1)} (t-a)^{k-\beta}. \tag{15}$$

If x and $x^{(i)}, i = 1, \ldots, n-1$, vanish at $t = a$, then ${}_a D^\beta x(t) = {}_a^C D^\beta x(t)$, and if they vanish at $t = b$, then $D_b^\beta x(t) = {}^C D_b^\beta x(t)$.

INTEGRATION BY PARTS

In [1] – [3], a formula for the fractional integration by parts on the whole interval $[a, b]$ was given by the following lemma

Lemma 2.5. Let $\beta > 0$, $p, q \geq 1$, and $\frac{1}{p} + \frac{1}{q} \leq 1 + \beta$ ($p \neq 1$ and $q \neq 1$ in the case when $\frac{1}{p} + \frac{1}{q} = 1 + \beta$)

(a) If $\varphi \in L_p(a, b)$ and $\psi \in L_q(a, b)$, then

$$\int_a^b \varphi(t)({}_a I^\beta \psi)(t)\,\mathrm{d}t = \int_a^b \psi(t)(I_b^\beta \varphi)(t)\,\mathrm{d}t. \tag{16}$$

(b) If $g \in I_b^\beta(L_p)$ and $f \in {}_aI^\beta(L_q)$, then

$$\int_a^b g(t)({}_aD^\beta f)(t)\,\mathrm{d}t = \int_a^b f(t)(D_b^\beta g)(t)\,\mathrm{d}t, \tag{17}$$

where ${}_aI^\beta(L_p) := \{f : f = {}_aI^\beta g,\ g \in L_p(a,b)\}$ and $I_b^\beta(L_p) := \{f : f = I_b^\beta g,\ g \in L_p(a,b)\}$.

In [1] − [3], other formulas for the fractional integration by parts on the subintervals $[a,r]$ and $[r,b]$ were given by the following lemmas.

Lemma 2.6. Let $\beta > 0$, $p, q \geq 1$, $r \in (a,b)$ and $\frac{1}{p} + \frac{1}{q} \leq 1 + \beta$ ($p \neq 1$ and $q \neq 1$ in the case when $\frac{1}{p} + \frac{1}{q} = 1 + \beta$).

(a) If $\varphi \in L_p(a,b)$ and $\psi \in L_q(a,b)$, then

$$\int_a^r \varphi(t)({}_aI^\beta \psi)(t)\,\mathrm{d}t = \int_a^r \psi(t)(I_r^\beta \varphi)(t)\,\mathrm{d}t, \tag{18}$$

and thus if $g \in I_r^\beta(L_p)$ and $f \in {}_aI^\beta(L_q)$, then

$$\int_a^r g(t)(\,{}_aD^\beta f)(t)\,\mathrm{d}t = \int_a^r f(t)(D_r^\beta g)(t)\,\mathrm{d}t, \tag{19}$$

(b) If $\varphi \in L_p(a,b)$ and $\psi \in L_q(a,b)$, then

$$\int_r^b \varphi(t)({}_aI^\beta \psi)(t)\,\mathrm{d}t = \int_r^b \psi(t)(\,I_b^\beta \varphi)(t)\,\mathrm{d}t$$

$$+ \frac{1}{\Gamma(\beta)} \int_a^r \psi(t) \left(\int_r^b \varphi(s)(s-t)^{\beta-1}\,\mathrm{d}s \right) \mathrm{d}t, \tag{20}$$

and hence if $g \in I_b^\beta(L_p)$ and $f \in {}_aI^\beta(L_q)$, then

$$\int_r^b g(t)(\,{}_aD^\beta f)(t)\,\mathrm{d}t = \int_r^b f(t)(D_b^\beta g)(t)\,\mathrm{d}t$$

$$- \frac{1}{\Gamma(\beta)} \int_a^r (\,{}_aD^\beta f)(t)(\int_r^b (D_b^\beta g)(s)(s-t)^{\beta-1}\,\mathrm{d}s)\,\mathrm{d}t. \tag{21}$$

That is

$$\int_r^b g(t)(\,{}_aD^\beta f)(t)\,\mathrm{d}t = \int_r^b f(t)(D_b^\beta g)(t)\,\mathrm{d}t$$

$$- \frac{1}{\Gamma(\beta)} \int_a^r f(t) D_r^\beta \left(\int_r^b (D_b^\beta g)(s)(s-t)^{\beta-1}\,\mathrm{d}s \right) \mathrm{d}t. \tag{22}$$

Lemma 2.7. Let $\beta > 0$, $p, q \geq 1$, $r \in (a, b)$ and $\frac{1}{p} + \frac{1}{q} \leq 1 + \beta$ ($p \neq 1$ and $q \neq 1$ in the case when $\frac{1}{p} + \frac{1}{q} = 1 + \beta$).

(a) If $\varphi \in L_p(a, b)$ and $\psi \in L_q(a, b)$, then

$$\int_r^b \varphi(t)(I_b^\beta \psi)(t)\,\mathrm{d}t = \int_r^b \psi(t)({}_rI^\beta \varphi)(t)\,\mathrm{d}t, \tag{23}$$

and thus if $g \in {}_rI^\beta(L_p)$ and $f \in I_b^\beta(L_q)$, then

$$\int_r^b g(t)(D_b^\beta f)(t)\,\mathrm{d}t = \int_r^b f(t)({}_rD^\beta g)(t)\,\mathrm{d}t. \tag{24}$$

(b) If $\varphi \in L_p(a, b)$ and $\psi \in L_q(a, b)$, then

$$\begin{aligned}\int_a^r \varphi(t)({}_bI^\beta \psi)(t)\,\mathrm{d}t &= \int_a^r \psi(t)(I_a^\beta \varphi)(t)\,\mathrm{d}t \\ &+ \frac{1}{\Gamma(\beta)} \int_r^b \psi(t) \left(\int_a^r \varphi(s)(t-s)^{\beta-1}\,\mathrm{d}s \right) \mathrm{d}t\end{aligned} \tag{25}$$

and hence if $g \in {}_aI^\beta(L_p)$ and $f \in I_b^\beta(L_q)$, then

$$\begin{aligned}\int_a^r g(t)(D_b^\beta f)(t)\,\mathrm{d}t &= \int_a^r f(t)({}_aD^\beta g)(t)\,\mathrm{d}t \\ &- \frac{1}{\Gamma(\beta)} \int_r^b (D_b^\beta f)(t)(\int_a^r ({}_aD^\beta g)(s)(t-s)^{\beta-1}\,\mathrm{d}s)\,\mathrm{d}t.\end{aligned} \tag{26}$$

That is

$$\begin{aligned}\int_a^r g(t)(D_b^\beta f)(t)\,\mathrm{d}t &= \int_a^r f(t)({}_aD^\beta g)(t)\,\mathrm{d}t \\ &- \frac{1}{\Gamma(\beta)} \int_r^b f(t)_r D^\beta (\int_a^r ({}_aD^\beta g)(s)(t-s)^{\beta-1}\,\mathrm{d}s)\,\mathrm{d}t.\end{aligned} \tag{27}$$

Remark 2.8. Let $0 < \beta < 1$. Then for any $\phi \in C^\infty(\overline{Q})$, where $\overline{Q}$ is the closure set of Q, we have:

$$\begin{aligned}&\int_0^T \int_\Omega ({}_aD^\beta y(x,t) + Ay(x,t))\phi(x,t)\,\mathrm{d}x\mathrm{d}t \\ &= \int_\Omega \phi(x,T)_a I^{1-\beta} y(x,T)\,\mathrm{d}x - \int_\Omega \phi(x,0)_a I^{1-\beta} y(x,0^+)\,\mathrm{d}x \\ &+ \int_0^T \int_{\partial\Omega} y \frac{\partial \phi}{\partial \nu_A}\,\mathrm{d}\partial\Omega\mathrm{d}t - \int_0^T \int_{\partial\Omega} \frac{\partial y}{\partial \nu_A} \phi\,\mathrm{d}\sigma\mathrm{d}t \\ &+ \int_0^T \int_\Omega y(x,t)({}^C D_b^\beta \phi(x,t) + A^* \phi(x,t))\,\mathrm{d}x\mathrm{d}t.\end{aligned} \tag{28}$$

where A^* is conjugate of the operator A; which given in the next section and

$$\frac{\partial y}{\partial \nu_A} = \sum_{i,j=1}^{n} a_{ij} \frac{\partial y}{\partial x_j} cos(n, x_j) \quad \text{on } \partial\Omega,$$

$cos(n, x_j)$ is the ith direction cosine of n,n being the normal at $\partial\Omega$ exterior to Ω.

Proof. (See Mophou [35, 36]) Since the left integration in (26) can be written as

$$\begin{aligned} &\int_0^T \int_\Omega ({}_aD^\beta y(x,t) + Ay(x,t))\phi(x,t)\,\mathrm{d}x\mathrm{d}t \\ &= \int_0^T \int_\Omega {}_aD^\beta y(x,t)\phi(x,t)\,\mathrm{d}x\mathrm{d}t + \int_0^T \int_\Omega Ay(x,t)\phi(x,t)\,\mathrm{d}x\mathrm{d}t, \end{aligned} \tag{29}$$

the last integration in (29) is equivalent to

$$\begin{aligned} &\int_0^T \int_\Omega Ay(x,t)\phi(x,t)\,\mathrm{d}x\mathrm{d}t \\ &= \int_0^T \int_\Omega y(x,t)A^*\phi(x,t)\,\mathrm{d}x\mathrm{d}t + \int_0^T \int_{\partial\Omega} y\frac{\partial \phi}{\partial \nu_A}\,\mathrm{d}\sigma\mathrm{d}t - \int_0^T \int_{\partial\Omega} \frac{\partial y}{\partial \nu_A}\phi\,\mathrm{d}\sigma\mathrm{d}t, \end{aligned} \tag{30}$$

the second integration in (29) is equivalent to

$$\begin{aligned} &\int_0^T \int_\Omega {}_aD^\beta y(x,t)\phi(x,t)\,\mathrm{d}x\mathrm{d}t \\ &= \int_\Omega \Big[\int_0^T \phi(x,t)(\frac{\mathrm{d}}{\mathrm{d}t}\,{}_aI^{1-\beta}y(x,t))\,\mathrm{d}t\Big]\mathrm{d}x \\ &= \int_\Omega \phi(x,T)_aI^{1-\beta}y(x,T))\,\mathrm{d}x - \int_\Omega \phi(x,0)\,{}_aI^{1-\beta}y(x,0))\,\mathrm{d}x \\ &- \int_\Omega \Big[\int_0^T \phi'(x,t)_aI^{1-\beta}y(x,t)\,\mathrm{d}t\Big]\mathrm{d}x \end{aligned} \tag{31}$$

the last integration in (31) is equivalent to

$$\begin{aligned} &- \int_\Omega \Big[\int_0^T \phi'(x,t)_aI^{1-\beta}y(x,t)\,\mathrm{d}t\Big]\mathrm{d}x \\ &= - \int_\Omega \Big[\int_0^T \phi'(x,t)\Big(\frac{1}{\Gamma(1-\beta)}\int_0^t (t-s)^{-\beta}y(x,s)\,\mathrm{d}s\Big)\mathrm{d}t\Big]\mathrm{d}x \\ &= - \int_\Omega \Big[\int_0^T y(x,s)\Big(\frac{1}{\Gamma(1-\beta)}\int_s^T (t-s)^{-\beta}\phi'(x,t)\,\mathrm{d}t\Big)\mathrm{d}s\Big]\mathrm{d}x \\ &= \int_\Omega \Big[\int_0^T y(x,s)^C D_b^\beta \phi(x,s)\,\mathrm{d}s\Big]\mathrm{d}x, \end{aligned} \tag{32}$$

we deduce that (31) is given by

$$\begin{aligned}&\int_0^T \int_\Omega {}_aD^\beta y(x,t)\phi(x,t)\,\mathrm{d}x\mathrm{d}t\\&= \int_\Omega \phi(x,T)_a I^{1-\beta} y(x,T)\,\mathrm{d}x - \int_\Omega \phi(x,0)_a I^{1-\beta} y(x,0)\,\mathrm{d}x\\&+ \int_\Omega \left[\int_0^T y(x,s)^C D_b^\beta \phi(x,s)\,\mathrm{d}s\right]\mathrm{d}x.\end{aligned} \tag{33}$$

Hence adding (33) to (30), we obtain (28). □

Theorem 2.9. Lax Milgram Theorem (Lions [25], Lions and Magenes [26])
Let H be a real Hilbert space. Let $(\cdot,\cdot)_H$ be its inner product, and $||\cdot||_H$ the associated norm. Let H' be its topological dual (i. e. the space of continuous linear forms on H). Let π be a bilinear form on H, and let $f \in H'$ be a continuous linear form on H. Let H_h be a closed vector subspace of H (in practice, H_h is finite dimensional). The Lax–Milgram theorem states existence and uniqueness of the solution to the following general problems:

$$\text{find } u \in H \text{ such that}: \ \forall v \in H, \pi(u,v) = f(v) \tag{34}$$

$$\text{find } u_h \in H_h \text{ such that}: \forall v_h \in H_h, \pi(u_h,v_h) = f(v_h) \tag{35}$$

The main statement is the following:

Lax–Milgram theorem. Assume that π is bounded with continuity constant $C \le 0$ and coercive with constant $\alpha > 0$. Then, there exists a unique $u \in H$ solution to Problem (13) and there exists a unique $u_h \in H_h$ solution to Problem (14). Moreover, $||u||_H \le \frac{1}{c}||f||_{H'}$ and for all $v_h \in H_h$, $||u - u_h||_H \le \frac{C}{\alpha}||u - v_h||_H$

3. MAXIMUM PRINCIPLE AND EXISTENCE THEOREM

We are concerned with the following form of the Maximum Principle: The hypotheses $f_1 \ge 0$ and $f_2 \ge 0$ on Ω implies $y_1 \ge 0$ and $y_2 \ge 0$ for any solution $y = (y_1, y_2)$ of (1).

We first study the Maximum Principle for system (1) and then we prove the existence of positive weak solutions for this system by using Lax–Milgram method. In [17, 18], Fleckinger obtained the necessary and sufficient conditions for having the maximum principle and the existence of positive solutions for cooperative integer order optimal control for the system:

$$\left.\begin{aligned}\frac{\partial y_1}{\partial t} + (-\Delta + q(x))y_1 &= ay_1 + by_2 + f_1 \quad \text{in}\,\mathbb{R}^n,\\ \frac{\partial y_2}{\partial t} + (-\Delta + q(x))y_2 &= cy_1 + dy_2 + f_2 \quad \text{in}\,\mathbb{R}^n,\\ y_1 &= g_1 \quad \text{as} \quad |x| \to \infty,\\ y_2 &= g_2 \quad \text{as} \quad |x| \to \infty,\\ y_1(x,0) &= y_{1,0}(x) \quad \text{in}\,\mathbb{R}^n,\\ y_2(x,0) &= y_{2,0}(x) \quad \text{in}\,\mathbb{R}^n,\end{aligned}\right\} \tag{36}$$

which are:

$$\left.\begin{array}{l} a < \lambda(q(x)), \quad d < \lambda(q(x)) \\ (\lambda(q(x)) - a)(\lambda(q(x)) - d) > bc, \end{array}\right\} \tag{37}$$

where $\lambda(q(x))$ is defined later.

Here, we shall use the same conditions (37) to prove the existence of the state of our system (1); then we study the existence of fractional optimal control for system (1).

To prove the existence of the state $y = (y_1, y_2)$ of system (1), we state briefly some results introduced in [10] concerning the eigenvalue problem:

$$\left.\begin{array}{l} (-\Delta + q(x))\phi = \lambda(q(x))\phi \quad \text{in} \quad \Omega \\ \phi(x) \to 0 \quad \text{as} \quad |x| \to \infty, \quad \phi > 0. \end{array}\right\} \tag{38}$$

The associated variational space is $V_{q(x)}(\Omega)$, the completion of $D(\Omega)$, with respect to the norm:

$$||y||_{q(x)} = \left(\int_\Omega |\Delta y|^2 + q(x)|y|^2 \,\mathrm{d}x \right)^{\frac{1}{2}}. \tag{39}$$

Since the imbedding of $V_{q(x)}(\Omega)$ into $L^2(\Omega)$ is compact. Then the operator $(-\Delta + q(x))$ considered as an operator in $L^2(\Omega)$ is positive self-adjoint with compact inverse. Hence its spectrum consists of an infinite sequence of positive eigenvalues tending to infinity; moreover the smallest one which is called the principle eigenvalue denoted by $\lambda(q(x))$ is simple and is associated with an eigenfunction which does not change sign in $\mathbb{R}^n$. It is characterized by:

$$\lambda(q(x)) \int_\Omega |y|^2 dx \le \int_\Omega |\Delta y|^2 + q(x)|y|^2 \,\mathrm{d}x \quad \forall y \in V_{q(x)}(\Omega). \tag{40}$$

Now, to study our system (1), we have the embedding

$$V_{q(x)}(\Omega) \times V_{q(x)}(\Omega) \to L^2(\Omega) \times L^2(\Omega)$$

is continuous and compact then, we define a bilinear form

$$\pi : (V_{q(x)}(\Omega))^2 \times (V_{q(x)}(\Omega))^2 \to \mathbb{R}$$

by

$$\begin{aligned} \pi((y_1, y_2), (\phi_1, \phi_2)) = {} & \frac{1}{b} \int_\Omega [\Delta y_1 \Delta \phi_1 + q(x) y_1 \phi_1] \,\mathrm{d}x + \frac{1}{c} \int_\Omega [\Delta y_2 \Delta \phi_2 + q(x) y_2 \phi_2] \,\mathrm{d}x \\ & - \int_\Omega y_1 \phi_2 \,\mathrm{d}x - \frac{d}{c} \int_\Omega y_2 \phi_2 \,\mathrm{d}x - \frac{a}{b} \int_\Omega y_1 \phi_1 \,\mathrm{d}x - \int_\Omega y_2 \phi_1 \,\mathrm{d}x. \end{aligned} \tag{41}$$

It is easy to check that π is a continuous bilinear form; and then by Lax–Milgram Theorem (Theorem (2.9)), we have the following theorem:

Theorem 3.1. For $f_1, f_2 \in L^2(\partial\Omega)$, there exists a unique solution $y = (y_1, y_2) \in (V_{q(x)}(\Omega))^2$ of system (1) if conditions (37) are satisfied.

P r o o f.

We choose m large enough such that $a+m > 0$ and $d+m > 0$ and define on $V_{q(x)}(\Omega)$ the equivalent norm

$$||y||^2_{q(x),m} = \int_\Omega [|\Delta y|^2 + (m + q(x))|y|^2]\,\mathrm{d}x$$

to norm (39), and we write (41) as:

$$\begin{aligned}
&\pi((y_1, y_2), (\phi_1, \phi_2)) \\
&= \frac{1}{b}\int_\Omega [\Delta y_1 \Delta \phi_1 + (q(x) + m)y_1\phi_1]\,\mathrm{d}x - \frac{a+m}{b}\int_\Omega y_1\phi_1\,\mathrm{d}x - \int_\Omega y_2\phi_1 dx \\
&+ \frac{1}{c}\int_\Omega [\Delta y_2 \Delta \phi_2 + (q(x) + m)y_2\phi_2] dx - \frac{d+m}{c}\int_\Omega y_2\phi_2\,\mathrm{d}x - \int_\Omega y_1\phi_2\,\mathrm{d}x.
\end{aligned}$$

Then

$$\begin{aligned}
\pi((y_1, y_2), (y_1, y_2)) = &\frac{1}{b}\int_\Omega [|\Delta y_1|^2 + (q(x) + m)|y_1|^2]\,\mathrm{d}x - \frac{a+m}{b}\int_{\mathbb{R}^n} |y_1|^2\,\mathrm{d}x - \int_\Omega y_1 y_2\,\mathrm{d}x \\
&+ \frac{1}{c}\int_\Omega [|\Delta y_2|^2 + (q(x) + m)|y_2|^2]\,\mathrm{d}x - \frac{d+m}{c}\int_\Omega |y_2|^2\,\mathrm{d}x - \int_\Omega y_1 y_2\,\mathrm{d}x.
\end{aligned}$$

By Cauchy Schwartz inequality, we have

$$\begin{aligned}
\pi((y_1, y_2), (y_1, y_2)) \geq &\frac{1}{b}\int_\Omega [|\Delta y_1|^2 + (q(x) + m)|y_1|^2]\,\mathrm{d}x - \frac{a+m}{b}\int_\Omega |y_1|^2 dx \\
&+ \frac{1}{c}\int_\Omega [|\Delta y_2|^2 + (q(x) + m)|y_2|^2]\,\mathrm{d}x - \frac{d+m}{c}\int_\Omega |y_2|^2\,\mathrm{d}x \\
&- 2\Big(\int_\Omega |y_1|^2\,\mathrm{d}x\Big)^{\frac{1}{2}}\Big(\int_\Omega |y_2|^2\,\mathrm{d}x\Big)^{\frac{1}{2}},
\end{aligned}$$

from (40), we deduce

$$\begin{aligned}
\pi((y_1, y_2), (y_1, y_2)) \geq &\frac{1}{b}\left(1 - \frac{a+m}{\lambda(q(x)) + m}\right)||y_1||^2_{q(x),m} + \frac{1}{c}\left(1 - \frac{d+m}{\lambda(q(x)) + m}\right)||y_2||^2_{q(x),m} \\
&\frac{2}{\lambda + m}||y_1||_{q(x),m}||y_2||_{q(x),m}.
\end{aligned}$$

If (37) holds, then

$$\pi((y_1, y_2), (y_1, y_2)) \geq C(||y_1||^2_{q(x),m} + ||y_2||^2_{q(x),m}) \tag{42}$$

which prove the coerciveness of the bilinear form π. Then by using Lax–Milgram Theorem (Theorem 2.9), for $f_1, f_2 \in L^2(\Omega)$ system (1) has a unique solution $y = (y_1, y_2) \in (V_{q(x)}(\Omega))^2$ if conditions (37) are satisfied. □

4. FORMULATION OF THE FRACTIONAL CONTROL PROBLEM

The space $L^2(\partial\Omega) \times L^2(\partial\Omega)$ is the space of controls. For a control $u = (u_1, u_2) \in (L^2(\partial\Omega))^2$, the state $y(u) = (y_1(u), y_2(u))$ of the system is given by the solution of

$$\left.\begin{aligned} {}_aD^{\beta} y_1(u) + (-\Delta + q(x))y_1(u) = ay_1(u) + by_2(u) + f_1 \quad \text{in}\,\Omega, \\ {}_aD^{\beta} y_2(u) + (-\Delta + q(x))y_2(u) = cy_1(u) + dy_2(u) + f_2 \quad \text{in}\,\Omega, \\ y_1 = u_1 \quad \text{as} \quad |x| \to \infty, \\ y_2 = u_2 \quad \text{as} \quad |x| \to \infty, \\ {}_aI^{1-\beta} y_1(x, 0^+) = 0 \quad \text{in}\,\Omega, \\ {}_aI^{1-\beta} y_2(x, 0^+) = 0 \quad \text{in}\,\Omega. \end{aligned}\right\} \tag{43}$$

The observation equations are given by

$$z_i(u) = y_i(u, t), \text{ for each } i = 1, 2. \tag{44}$$

For given $z_d = (z_{d1}, z_{d2})$ in $(L^2(\Omega))^2$; the cost function is given by:

$$J(v) = \int_{\Omega} (y_1(v) - z_{\mathrm{d}1})^2 + (y_2(v) - z_{\mathrm{d}2})^2 \,\mathrm{d}x + (Nv, v)_{(L^2(\partial\Omega))^2} \tag{45}$$

where $N \in \mathcal{L}((L^2(\partial\Omega))^2, (L^2(\partial\Omega))^2)$ is hermitian positive definite operator:

$$(Nu, u) \geq \eta ||u||^2_{(L^2(\Omega))^2}. \tag{46}$$

The control problem then is to find

$$\left.\begin{aligned} &u = (u_1, u_2) \in U_{ad} \quad \text{such that} \\ &J(u) \leq J(v) \end{aligned}\right\} \tag{47}$$

where U_{ad} is a closed convex subset of $(L^2(\partial\Omega))^2$.

Under the given consideration, we may apply the theorem of Lions [25, 26] to obtain the following result:

Theorem 4.1. Assume that (41) and (46) hold. If the cost function is given by (45), then there exists an optimal control $u = (u_1, u_2)$; Moreover it is characterized by the following equations and inequalities:

$$\left.\begin{aligned} {}^{C}D^{\beta}_{b} p_1(u) + (-\Delta + q(x))p_1(u) - ap_2(u) - cp_2(u) = y_1(u) - z_{1d} \quad \text{in} \quad \Omega \\ {}^{C}D^{\beta}_{b} p_2(u) + (-\Delta + q(x))p_2(u) - bp_1(u) - dp_2(u) = y_2(u) - z_{2d} \quad \text{in} \quad \Omega \\ p_1(u) = 0 \quad p_2(u) = 0 \quad \text{on} \quad \partial\Omega \end{aligned}\right\} \tag{48}$$

$$\int_{\partial\Omega} \frac{\partial p_1(u)}{\partial \nu_A}(v_1 - u_1) + \frac{\partial p_2(u)}{\partial \nu_A}(v_2 - u_2)\,\mathrm{d}\sigma + (Nu, v - u)_{(L^2(\partial\Omega))^2} \geq 0 \quad \forall v \in U_{ad}$$

together with (43), where $p(u) = (p_1(u), p_2(u))$ is the adjoint state.

P r o o f. The control u is characterized by

$$J'(u)(v-u) \geq 0 \quad \forall u \in U_{ad}$$

which is equivalent to

$$(y(u) - z_d, y(v) - y(u))_{(L^2(\mathbb{R}^n))^2} + (Nu, v-u)_{(L^2(\partial\Omega))^2} \geq 0$$

i. e.,

$$\begin{aligned}(y_1(u) - z_{1d}, y_1(v) - y_1(u))_{L^2(\mathbb{R}^n)} + (y_2(u) - z_{2d}, y_2(v) - y_2(u))_{L^2(\mathbb{R}^n)} \\ +(Nu, v-u)_{(L^2(\partial\Omega))^2} \geq 0.\end{aligned} \tag{49}$$

Since for $p = (p_1, p_2), y = (y_1, y_2)$,

$$A(\phi = \{\phi_1, \phi_2\}) \to A\phi := \{(-\Delta + q(x))\phi_1 - a\phi_1 - b\phi_2, (-\Delta + q(x))\phi_2 - c\phi_1 - d\phi_2\} \quad \text{for} \quad \phi \in (V_q'(\Omega))^2$$

we have $(A^*p, y) = (p, Ay)$. Indeed

$$\begin{aligned}(p, Ay) &= (p_1, (-\Delta + q(x))y_1 - ay_1 - by_2) + (p_2, (-\Delta + q(x))y_2 - cy_1 - dy_2) \\ &= (p_1, (-\Delta + q(x))y_1) - a(p_1, y_1) - b(p_1, y_2) + (p_2, (-\Delta + q(x))y_2) - c(p_2, y_1) \\ &\quad - d(p_2, y_2) \\ &= ((-\Delta + q(x))p_1, y_1) - a(p_1, y_1) - c(p_2, y_1) + ((-\Delta + q(x))p_2, y_2) - d(p_2, y_2) \\ &\quad - b(p_1, y_2) \\ &= ((-\Delta + q(x))p_1 - ap_1 - cp_2, y_1) + ((-\Delta + q(x))p_2 - bp_1 - dp_2, y_2) \\ &= (A^*p, y)\end{aligned}$$

where

$$A^*(p = (p_1, p_2)) \to \{(-\Delta + q(x))p_1 - ap_1 - cp_2, (-\Delta + q(x))p_2 - bp_1 - dp_2\}$$

where A^* is the adjoint for A, $p = (p_1, p_2)$ is the adjoint state. Then $A^*p = y(u) - z_d$ can be written as

$$\begin{gathered}(-\Delta + q(x))p_1 - ap_1 - cp_2 = y_1(u) - z_{1d} \\ (-\Delta + q(x))p_2 - bp_1 - dp_2 = y_2(u) - z_{2d} \\ p_1(u) = p_2(u) = 0.\end{gathered}$$

Now, multiplying the first two equations in (48) by $(y_1(v) - y_1(u))$ and $(y_2(v) - y_2(u))$

respectively and adding the two equations, and applying Green's formula, we obtain:

$$\begin{aligned}
&(y_1(u)-z_{1d}, y_1(v)-y_1(u))_{L^2(\Omega} + (y_2(u)-z_{2d}, y_2(v)-y_2(u))_{L^2(\Omega)}\\
&= ({}^C D_b^\beta p_1(u) + (-\Delta + q(x))p_1 - ap_1 - cp_2, y_1(v)-y_1(u))\\
&\quad + ({}^C D_b^\beta p_2(u) + (-\Delta + q(x))p_2 - bp_1 - dp_2, y_2(v)-y_2(u))\\
&= (p_1(u),{}_a D^\beta (y_1(v)-y_1(u)) + (p_1(x,0),{}_a I^{1-\beta}(y_1(v;x,0^+)\\
&\qquad - y_1(u;x,0^+)))_{L^2(\Omega)} + (-\Delta + q(x))(y_1(v)-y_1(u))_{L^2(\Omega)}\\
&\qquad - \left(\frac{\partial p_1(u)}{\partial \nu_A}, y_1(v)-y_1(u))_{L^2(\partial\Omega)} + (p_1(u), \frac{\partial}{\partial \nu_A}(y_1(v)-y_1(u)\right)_{L^2(\partial\Omega)}\\
&\qquad - a(p_1(u), y_1(v)-y_1(u)) - b(p_1(u), y_2(v)-y_2(u))_{L^2(\Omega)}\\
&\qquad + (p_2(u),{}_a D^\beta (y_2(v)-y_2(u)) + (p_2(x,0),{}_a I^{1-\beta}(y_2(v;x,0^+) - y_2(u;x,0^+)))_{L^2(\Omega)}\\
&\qquad + (-\Delta + q(x))(y_2(v)-y_2(u))_{L^2(\Omega)} - (\frac{\partial p_2(u)}{\partial \nu_A}, y_2(v)\\
&\qquad - y_2(u))_{L^2(\partial\Omega)} + (p_2(u), \frac{\partial}{\partial \nu_A}(y_2(v)-y_2(u))_{L^2(\partial\Omega)}\\
&\qquad - c(p_2(u), y_1(v)-y_1(u))_{L^2(\Omega)} - d(p_2(u), y_2(v)-y_2(u))_{L^2(\Omega)}.
\end{aligned}$$

From (43), we obtain

$$\begin{aligned}
&(y_1(u)-z_{1d}, y_1(v)-y_1(u))_{L^2(\Omega)} + (y_2(u)-z_{2d}, y_2(v)-y_2(u))_{L^2(\Omega)}\\
&= (p_1(u), a(y_1(v)-y_1(u)) + b(y_2(v)-y_2(u)) + f_1 - f_1 - a(y_1(v)-y_1(u)))_{L^2(\Omega)}\\
&\quad + \left(\frac{\partial p_1(u)}{\partial \nu_A}, v_1 - u_1\right)_{L^2(\partial\Omega)}\\
&\quad + \left(0, \frac{\partial}{\partial \nu_A}(y_1(v)-y_1(u))_{L^2(\partial\Omega)} - c(p_2(u), y_1(v)-y_1(u)\right)_{L^2(\Omega)}\\
&\quad (p_2(u), c(y_1(v)-y_1(u)) + d(y_2(v)-y_2(u)) + f_2 - f_2 - c(y_1 - y_1(u)))_{L^2(\Omega)}\\
&\quad + \left(\frac{\partial p_2(u)}{\partial \nu_A}, v_2 - u_2\right)_{L^2(\partial\Omega)}\\
&\quad + (0, \frac{\partial}{\partial \nu_A}(y_2(v)-y_2(u))_{L^2(\partial\Omega)} - d(p_2(u), y_2(v)-y_2(u))_{L^2(\Omega)}.
\end{aligned}$$

Then (49) is equivalent to:

$$\left(\frac{\partial p_1(u)}{\partial \nu_A}, v_1 - u_1)_{L^2(\partial\Omega)} + \left(\frac{\partial p_2(u)}{\partial \nu_A}, v_2 - u_2\right)_{L^2(\partial\Omega)} + (Nu, v-u)_{(L^2(\partial\Omega)^2)}\right) \geq 0.$$

i. e.,

$$\int_{\partial\Omega}\left(\frac{\partial p_1(u)}{\partial \nu_A}(v_1 - u_1) + \frac{\partial p_2(u)}{\partial \nu_A}(v_2 - u_2))\,\mathrm{d}\sigma + (Nu, v-u)_{(L^2(\partial\Omega^2)}\right) \geq 0$$

$\forall u \in U_{ad}, v \in U_{ad}$. Which completes the proof of the theorem. □

5. FORMULATION OF THE FRACTIONAL OPTIMAL CONTROL PROBLEM FOR THE SCALAR CASE

To study the fractional optimal control for the scalar case:

$$\left.\begin{aligned} {}_aD^\beta y + (-\Delta + q(x))y = ay + f \quad \text{in } \Omega,\\ y(x) = g \quad \text{in } \partial\Omega,\\ {}_aI^{1-\beta} y(x, 0^+) = 0 \end{aligned}\right\} \tag{50}$$

we define a bilinear form $\pi : V_{q(x)}(\Omega) \times V_{q(x)}(\Omega) \to \mathbb{R}$ by

$$\pi(y, \phi) = \int_\Omega (\nabla y \nabla \phi + q(x) y \phi)\, \mathrm{d}x - a \int_\Omega y\phi\, \mathrm{d}x.$$

As in Theorem (3.1), we can prove π is coercive if $a < \lambda(q(x))$ and then there exists a unique solution of (50) for $f \in L^2(\Omega)$. Therefore, the state of the system is given by the solution of:

$$\left.\begin{aligned} {}_aD^\beta y(u) + (-\Delta + q(x))y(u) = ay(u) + f \quad \text{in } \Omega,\\ y(u) = u \quad \text{in } \partial\Omega,\\ {}_aI^{1-\beta} y(x, 0^+) = 0 \end{aligned}\right\} \tag{51}$$

where u is given in the space $U = L^2(\partial\Omega)$ of controls. For given z_d in $L^2(\Omega)$, the cost function is given by

$$J(v) = \int_\Omega |y(v) - z_d|^2\, \mathrm{d}x + \int_{\partial\Omega} (Nv)v\, \mathrm{d}\sigma \tag{52}$$

where N is a given hermitian positive definite operator. Then we have the following characterization of optimal control for this system:

$$\left.\begin{aligned} {}^CD_b^\beta p(u) + (-\Delta + q(x))p(u) - ap(u) = y(u) - z_d \quad \text{in } \Omega,\\ p(u) = 0 \quad \text{in } \partial\Omega, \end{aligned}\right\} \tag{53}$$

$$\int_{\partial\Omega} \frac{\partial p(u)}{\partial \nu_A}(v - u)\, \mathrm{d}\sigma + (Nu, v - u)_{L^2(\partial\Omega)} \geq 0, \quad \forall \quad v \in U_{ad}, \tag{54}$$

together with (51), where $p(u)$ is the adjoint state.

6. MATHEMATICAL EXAMPLES AND APPLICATIONS

This section is devoted to introduce some mathematical examples and applications to illustrate the control problem in the paper.

Example 6.1. Neumann problem with boundary control.

We consider an example of time-fractional cooperative system governed by Schrödinger operator which is analogous to that considered in section 4 but with Neumann boundary condition and boundary control.

In this example we consider the space

$$\mathcal{W}(0,T) := \{y : y \in L^2(0,T;V_{q(x)}(\Omega)), D_b^{\beta} y(t) \in L^2(0,T;(V_{q(x)}(\Omega))^{'})\}$$

in which a solution of a fractional differential systems is contained. Let $y(u) = (y_1(u), y_2(u)) \in \mathcal{W}(0,T)$ be the state of the system which is given by,

$$\left.\begin{aligned} {}_aD^{\beta} y_1(u) + (-\Delta + q(x))y_1(u) &= ay_1(u) + by_2(u) + f_1 \quad \text{in}\,\Omega,\\ {}_aD^{\beta} y_2(u) + (-\Delta + q(x))y_2(u) &= cy_1(u) + dy_2(u) + f_2 \quad \text{in}\,\Omega,\\ \frac{\partial y_1(x,t)}{\partial \nu_A}|_{\Sigma} &= u_1 \quad \text{as} \quad |x| \to \infty,\\ \frac{\partial y_2(x,t)}{\partial \nu_A}|_{\Sigma} &= u_2 \quad \text{as} \quad |x| \to \infty,\\ {}_aI^{1-\beta} y_1(x,0^+) &= 0 \quad \text{in}\,\Omega,\\ {}_aI^{1-\beta} y_2(x,0^+) &= 0, \quad \text{in}\,\Omega. \end{aligned}\right\} \tag{55}$$

The control $u = (u_1, u_2)$ is taken in $U = L^2(\Sigma) \times L^2(\Sigma)$. Let us consider the case where we have partial observation of the final state

$$z(v) = y_1(x,T;v),$$

and the cost function $J(v)$ for $v = (v_1, v_2)$ is given by

$$J(v) = \int_{\Omega} (y_1(x,T;v) - z_d)^2 \, \mathrm{d}x + (Nv, v)_{(L^2(\Sigma))^2}, \quad z_d \in L^2(\Omega),$$

where $N = (N_1, N_2) \in \mathcal{L}(L^2(\Sigma), L^2(\Sigma))$ is hermitian positive definite operator:

$$(Nu, u) \geq c||u||^2_{L^2(\Sigma)}, \; c > 0. \tag{56}$$

Control Constraints: We define U_{ad}(set of admissible controls) is closed, convex subset of $U = L^2(\Sigma) \times L^2(\Sigma)$.

Control Problem: We want to minimize J over U_{ad} i. e. find $u = (u_1, u_2)$ such that

$$J(u) = \inf_{v=(v_1,v_2)\in U_{ad}} J(v). \tag{57}$$

Problem (57) admits a unique solution and the optimal control is characterized by the state system (55) with the adjoint state is given by

$$\left.\begin{aligned} {}^CD_b^{\beta} p_1(u) + (-\Delta + q(x))p_1 - ap_1 - cp_2 &= y_1(u) - z_{1d},, \quad \text{in } \Omega,\\ {}^CD_b^{\beta} p_2(u) + (-\Delta + q(x))p_2 - bp_1 - dp_2 &= y_2(u) - z_{2d}, \quad \text{in } \Omega,\\ \frac{\partial p_1(u)}{\partial \nu_{A^*}} &= 0 \quad \text{on}\,\Sigma,\\ \frac{\partial p_2(u)}{\partial \nu_{A^*}} &= 0 \quad \text{on}\,\Sigma,\\ p_1(x,T;u) &= y_1(u) - z_d, \quad \text{in } \Omega,\\ p_2(x,T;u) &= 0, \quad \text{in } \Omega. \end{aligned}\right\} \tag{58}$$

The optimality condition is

$$\int_{\partial\Omega}(y_1(u)-z_d, v_1-u_1)\,\mathrm{d}\sigma+(Nu, v-u)_{(L^2(\partial\Omega))^2}\geq 0 \quad \forall u=(u_1,u_2), v=(v_1,v_2)\in U_{ad}. \tag{59}$$

Example 6.2. No constraints problem.
In the case of no constraint on the control $U = U_{ad}$ and $N = (N_1, N_2)$ is a diagonal matrix of operators. Then (59) reduces to

$$p_1 + N_1u_1 = 0 \quad \text{on } \Sigma, \quad p_2 + N_2u_2 = 0 \quad \text{on } \Sigma,$$

which equivalent to

$$u_1 = -N_1^{-1}(p_1(u)|_{\Sigma})), \quad u_2 = -N_2^{-1}(p_2(u)|_{\Sigma}). \tag{60}$$

The fractional optimal control is obtained by the simultaneous solving (55) and (58) (where we eliminate u_1, u_2 with the aid of (60)) and then utilizing (60).

Example 6.3. Constraint problem.
If we take

$$\mathcal{U}_{ad}=\Big\{u_i|u_i\in L^2(\Sigma), u_i\geq 0 \quad \text{almost everywhere on} \quad \Sigma, i=1,2\Big\},$$

and $N=\nu\times$Identity, (59) gives

$$u_1\geq 0, \quad p_1(u)+\nu_1u_1\geq 0, \quad u_1(p_1(u)+\nu_1u_1)=0 \qquad \text{on } \Sigma,$$
$$u_2\geq 0, \quad p_2(u)+\nu_2u_2\geq 0, \quad u_2(p_2(u)+\nu_2u_2)=0 \qquad \text{on } \Sigma.$$

The fractional optimal control is obtained by the solution of the fractional problem

$$\begin{aligned}
&{}_aD^{\beta}y_1(x,t;u)+(-\Delta+q(x))y_1(x,t;u)=ay_1(x,t;u)+by_2(x,t;u)+f_1\\
&\qquad \text{in } \Omega \quad a.e.\, t\in]0,T[,\ f_1\in L^2(\Omega),\\
&{}_aD^{\beta}y_2(x,t;u)+(-\Delta+q(x))y_2(x,t;u)=cy_1(x,t;u)+dy_2(x,t;u)+f_2\\
&\qquad \text{in } \Omega, \quad a.e.\, t\in]0,T[,\ f_2\in L^2(\Omega),
\end{aligned}$$

$${}^CD_b^{\beta}p_1(x,t;u)+(-\Delta+q(x))p_1(x,t;u)-ap_1(x,t;u)-cp_2(x,t;u)=y_1(x,t;u)-z_{1d}, \quad \text{in}\,\Omega,$$
$${}^CD_b^{\beta}p_2(x,t;u)+(-\Delta+q(x))p_2(x,t;u)-bp_1(x,t;u)-dp_2(x,t;u)=y_2(x,t;u)-z_{2d}, \quad \text{in}\,\Omega,$$

$$\begin{aligned}
&{}_aI^{1-\beta}y_1(x,0^+)=0, \quad \text{in } \Omega, && {}_aI^{1-\beta}y_2(x,0^+)=0 \quad \text{in } \Omega,\\
&p_1(x,T;u)=y_1(u)-z_d, \quad \text{in}\,\Omega, && p_2(x,T;u)=0, \quad \text{in}\,\Omega,\\
&\frac{\partial y_1(x,t)}{\partial\nu_A}|_{\Sigma}, \geq 0,\ x\in\partial\Omega, t\in(0,T), && \frac{\partial y_2(x,t)}{\partial\nu_A}|_{\Sigma}, \geq 0,\ x\in\partial\Omega, t\in(0,T),\\
&\frac{\partial p_1(u)}{\partial\nu_{A^*}}=0, \quad \text{on}\,\Sigma, && \frac{\partial p_2(u)}{\partial\nu_{A^*}}=0, \quad \text{on}\,\Sigma,\\
&p_1+\nu_1\frac{\partial y_1}{\partial\nu_A}\geq 0, \quad \text{on } \Sigma, && \frac{\partial y_1}{\partial\nu_A}[p_1+\nu_1\frac{\partial y_1}{\partial\nu_A}]=0, \quad \text{on } \Sigma,\\
&p_2+\nu_2\frac{\partial y_2}{\partial\nu_A}\geq 0, \quad \text{on } \Sigma, && \frac{\partial y_2}{\partial\nu_A}[p_2+\nu_2\frac{\partial y_2}{\partial\nu_A}]=0, \quad \text{on } \Sigma,
\end{aligned}$$

hence

$$u_1 = \frac{\partial y_1}{\partial \nu_A}|_\Sigma, \quad u_2 = \frac{\partial y_2}{\partial \nu_A}|_\Sigma.$$

Example 6.4. Coupled fractional control problem in a Caputo sense

We consider an example analogous to that considered in example 6.1. but the fractional time derivative is considered in a Caputo sense. The optimality systems is given by:

The state equations is given by:

$$\left.\begin{array}{r} {}_a^C D^\beta y_1(x,t;u) + (-\Delta + q(x))y_1(x,t;u) = ay_1(x,t;u) + by_2(x,t;u) + f_1 \quad \text{in} \quad \Omega, \\ {}_a^C D^\beta y_2(x,t;u) + (-\Delta + q(x))y_2(x,t;u) = cy_1(x,t;u) + dy_2(x,t;u) + f_2 \quad \text{in} \quad \Omega, \\ y_1(x,0;u) = 0, \quad x \in \Omega, \\ y_2(x,0;u) = 0, \quad x \in \Omega, \\ \frac{\partial y_1(x,t)}{\partial \nu_A}|_\Sigma = u_1,\ x \in \partial\Omega, t \in (0,T), \\ \frac{\partial y_2(x,t)}{\partial \nu_A}|_\Sigma = u_2\ , x \in \partial\Omega, t \in (0,T). \end{array}\right\} \tag{61}$$

The adjoint state is given by:

$$\left.\begin{array}{r} {}^C D_b^\beta p_1(x,t;u) + (-\Delta + q(x))p_1(x,t;u) - ap_1(x,t;u) - cp_2(x,t;u) = 0, \quad \text{in } \Omega, \\ {}^C D_b p_2(x,t;u) + (-\Delta + q(x))p_2(x,t;u) - bp_1(x,t;u) - dp_2(x,t;u) = 0, \quad \text{in } \Omega, \\ \frac{\partial p_1(u)}{\partial \nu_{A^*}} = 0, \quad \text{on } \Sigma, \\ \frac{\partial p_2(u)}{\partial \nu_{A^*}} = 0 \text{ on } \Sigma, \\ p_1(x,T;u) = y_1(u) - z_d, \text{ in } \Omega, \\ p_2(x,T;u) = 0 \text{ in } \Omega. \end{array}\right\} \tag{62}$$

The optimality condition is given by:

$$\int_{\partial\Omega} (y_1(u) - z_d, v_1 - u_1)\, d\sigma + (Nu, v - u)_{(L^2(\partial\Omega))^2} \ge 0 \quad \forall u = (u_1, u_2), v = (v_1, v_2) \in U_{ad}. \tag{63}$$

P r o o f. Now, multiplying the first two equations in (62) by $(y_1(v) - y_1(u))$ and $(y_2(v) - y_2(u))$ respectively and adding the two equations, and applying Green's formula, we obtain:

$$\begin{aligned} 0 &= ({}^C D_b^\beta p_1(u) + (-\Delta + q(x))p_1(u) - ap_1(u) - cp_2(u), y_1(v) - y_1(u)) \\ &= (p_1(u), ({}_a^C D^\beta + (-\Delta + q(x))(y_1(v) \\ &\quad - y_1(u))_{L^2(\Omega)} + (p_1(x,0), (y_1(v;x,0) - y_1(u;x,0)))_{L^2(\Omega)} \\ &\quad - (\frac{\partial p_1(u)}{\partial \nu_A}, y_1(v) - y_1(u))_{L^2(\partial\Omega)} + (p_1(u), \frac{\partial}{\partial \nu_A}(y_1(v) - y_1(u))_{L^2(\partial\Omega)} \\ &\quad - a(p_1(u), y_1(v) - y_1(u)) - c(p_2(u), y_1(v) - y_1(u))_{L^2(\Omega)} \end{aligned} \tag{64}$$

$$
\begin{aligned}
0 &= ({}^C D_b^{\beta} p_2(u) + (-\Delta + q(x)) p_2(u) - b p_1 - d p_2(u), y_2(v) - y_2(u)) \\
&= (p_2(u), {}_a^C D^{\beta} + (-\Delta + q(x))(y_2(v) - y_2(u))_{L^2(\Omega)} + (p_2(x,0), (y_2(v;x,0) - y_2(u;x,0)))_{L^2(\Omega)} \\
&- (\frac{\partial p_2(u)}{\partial \nu_A}, y_2(v) - y_2(u))_{L^2(\partial\Omega)} + (p_2(u), \frac{\partial}{\partial \nu_A}(y_2(v) - y_2(u))_{L^2(\partial\Omega)} \\
&- b(p_1(u), y_2(v) - y_2(u))_{L^2(\Omega)} - d(p_2(u), y_2(v) - y_2(u))_{L^2(\Omega)}
\end{aligned} \tag{65}
$$

From (61),(62), we obtain

$$
\begin{aligned}
0 = (p_1(u), a(y_1(v) - y_1(u)) + b(y_2(v) - y_2(u)) + f_1 - f_1 - a(y_1(v) \\
- y_1(u)))_{L^2(\Omega)} - (0, y_1(v) - y_1(u))_{L^2(\partial\Omega)} \\
+ (y_1(u) - z_d, v_1 - u_1)_{L^2(\partial\Omega)} - c(p_2(u), y_1(v) - y_1(u))_{L^2(\Omega)}
\end{aligned} \tag{66}
$$

$$
\begin{aligned}
0 = (p_2(u), c(y_1(v) - y_1(u)) + d(y_2(v) - y_2(u)) + f_2 - f_2 - c(y_1 - y_1(u)))_{L^2(\Omega)} \\
+ (0, y_2(v) - y_2(u))_{L^2(\partial\Omega)} \\
+ (0, v_2 - u_2))_{L^2(\partial\Omega)} - b(p_1(u), y_2(v) - y_2(u))_{L^2(\Omega)}.
\end{aligned} \tag{67}
$$

Then if we add (66),(67), we obtain

$$
(p_1(u), v_1 - u_1)_{L^2(\partial\Omega)} = 0
$$

i. e.,

$$
(y_1(u) - z_d, v_1 - u_1)_{L^2(\partial\Omega)} = 0.
$$

$$
\int_{\partial\Omega} (y_1(u) - z_d, v_1 - u_1)\, \mathrm{d}\sigma + (Nu, v - u)_{(L^2(\partial\Omega))^2} \geq 0 \quad \forall u = (u_1, u_2), v = (v_1, v_2) \in U_{ad}.
$$

Which completes the proof of the theorem. □

Example 6.5. n-coupled fractional system.

We can generalize our results to $n-$dimensional coupled fractional system as follows. The state of the system is given, for each $i = 1, 2, \ldots, n$, by

$$
{}_a^C D^{\beta} y_i(u) + (-\Delta + q(x)) y_i(u) = \sum_{j=1}^{n} a_j y_j(u) + f_i(t), \text{ in } \Omega, \quad a.e.\, t \in]0, T[, \ f_1 \in L^2(\Omega),
$$

$$
y_i(x, t; u) = u_i, \quad \text{as } |x| \to \infty,
$$

$$
y_i(x, 0) = 0, \quad x \in \Omega.
$$

The adjoint state is given by

$$
{}^C D_b^{\beta} p_i(u) + (-\Delta + q(x)) p_i(u) + \sum_{j=1}^{n} b_j p_j(u) = y_i(u) - z_{id}, \quad \text{in } \Omega,
$$

$$
p_i(x, t; u) = 0, \quad \text{on } \partial\Omega,
$$

where b_i are the transpose of a_i. The optimality condition is

$$
\int_{\partial\Omega} \sum_{i=1}^{n} \frac{\partial p_i(u)}{\partial \nu_{A^*}} (v_i - u_i)\, \mathrm{d}\sigma + (Nu, v - u)_{(L^2(\partial\Omega))^n} \geq 0,
$$

$$
\forall v = (v_1, v_2, \ldots, v_n) \in U_{ad},\ u = (u_1, u_2, \ldots, u_n) \in U_{ad}.
$$

Remark 6.6. If we take $\beta = 1$, in the previews sections we obtain the classical results in the optimal control with integer derivatives see [25, 26] .

7. CONCLUSIONS

In this paper, we proved a weak maximum principle for the weak solution to an initial boundary value problem for a single-order fractional order cooperative systems involving Schrödinger operator. Also we studied the fractional optimal control of problem for this system. The fractional derivatives was defined in the Riemann–Liouville and Caputo senses. The analytical results were given in terms of Euler–Lagrange equations for the fractional optimal control problems. The formulation presented and the resulting equations are very similar to those for classical optimal control problems. The optimization problem presented in this paper constitutes a generalization of the optimal control problem of parabolic systems with Dirichlet and Neumann boundary conditions considered in [25] to fractional optimal control problem for cooperative systems involving Schrödinger operator. Many infinity of variations on the above problem are possible to study with the help of Lions formalism in [25] and Dubovitskii–Milyutin formalisms see [24]. Also some numerical and graphical results can be studied for system (1). Those problems need further investigations form tasks for future research.

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REFERENCES

[1] O. P. Agrawal: Formulation of Euler–Lagrange equations for fractional variational problems. Math. Anal. Appl. *272* (2002), 368–379.

[2] O P. Agrawal: A general formulation and solution scheme for fractional optimal control problems. Nonlinear Dynamics *38* (2004), 323–337. DOI:10.1007/s11071-004-3764-6

[3] O. P. Agrawal and D. A. Baleanu: Hamiltonian formulation and direct numerical scheme for fractional optimal control problems. J. Vibration Control *13* (2007), 9–10, 1269–1281. DOI:10.1177/1077546307077467

[4] M. Al-Refai and Yu. Luchko: Maximum principles for the fractional diffusion equations with the Riemann–Liouville fractional derivative and their applications. Fract. Calc. Appl. Anal. *17* (2014), 2, 483–498. DOI:10.2478/s13540-014-0181-5

[5] M. Al-Refai and Yu. Luchko: Maximum principle for the multi-term time fractional diffusion equations with the Riemann–Liouville fractional derivatives. Appl. Math. Comput. *257* (2015), 40–51. DOI:10.1016/j.amc.2014.12.127

[6] M. Al-Refai and Yu. Luchko: Analysis of fractional diffusion equations of distributed order: Maximum principles and their applications. Analysis *36* (2016), 123–133. DOI:10.1515/anly-2015-5011

[7] G. M. Bahaa: Fractional optimal control problem for variational inequalities with control constraints. IMA J. Math. Control Inform. *35* (2018), 1, 107–122. DOI:10.1186/s13662-016-0976-2

[8] G. M. Bahaa: Fractional optimal control problem for differential system with control constraints. Filomat J. *30* (2016), 8, 2177–2189.

[9] G. M. Bahaa: Fractional optimal control problem for differential system with control constraints with delay argument. Advances Difference Equations *2017* (2017), 69, 1–19. DOI:10.1186/s13662-017-1121-6

[10] G. M. Bahaa: Optimal control for cooperative parabolic systems governed by Schrödinger operator with control constraints. IMA J. Math. Control Inform. *24* (2007), 1–12. DOI:10.1093/imamci/dnl001

[11] G. M. Bahaa and A. Hamiaz: Optimality conditions for fractional differential inclusions with non-singular Mittag–Leffler Kernel. Adv. Difference Equations (2018), 257.

[12] D. A. Baleanu and O. P. Agrawal: Fractional Hamilton formalism within Caputo's derivative. Czechosl. J. Phys. *56* (2006), 10/11 1087–1092. DOI:10.1007/s10582-006-0406-x

[13] N. R. O. Bastos, D. Mozyrska, and D. F. M. Torres: Fractional derivatives and integrals on time scales via the inverse generalized Laplace transform. Int. J. Math. Comput. *11* (2011), J11, 1–9.

[14] N. R. O. Bastos, R. A. C. Ferreira, and D. F. M. Torres: Necessary optimality conditions for fractional difference problems of the calculus of variations. Discrete Cont. Dyn. Syst. *29* (2011), 2, 417–437. DOI:10.3934/dcds.2011.29.417

[15] S. D. Eidelman and A. N. Kochubei: Cauchy problem for fractional diffusion equations. J. Diff. Equat. *199*(2004), 211–255. DOI:10.1016/j.jde.2003.12.002

[16] A. M. A. El-Sayed: Fractional differential equations. Kyungpook Math. J. *28* (1988), 2, 18–22.

[17] J. Fleckinger: Estimates of the number of eigenvalues for an operator of Schrödinger type. Proc. Royal Soc. Edinburg *89A* (1981), 355–361. DOI:10.1017/s0308210500020357

[18] J. Fleckinger: Method of sub-super solutions for some elliptic systems defined on Ω. Preprint UMR MIP, Universite Toulouse *3* (1994).

[19] J. Fleckinger, J. Hernándes, and F. de Thélin: On maximum principle and existence of positive solutions for cooperative elliptic systems. Diff. Int. Eqns. *8* (1995), 69–85.

[20] J. Fleckinger and H. Serag: Semilinear cooperative elliptic systems on R^n. Rend. di Mat. *15* (1995), VII, 89–108.

[21] S. F. Frederico Gastao and D. F. M. Torres: Fractional optimal control in the sense of Caputo and the fractional Noether's theorem. Int. Math. Forum *3* (2008), 10, 479–493.

[22] A. N. Kochubei: Fractional order diffusion. Diff. Equations *26* (1990), 485–492.

[23] A. N. Kochubei: General fractional calculus, evolution equations, and renewal processes. Integr. Equat. Oper. Theory *71* (2011), 583–600. DOI:10.1007/s00020-011-1918-8

[24] W. Kotarski, H. A. El-Saify, and G. M. Bahaa: Optimal control of parabolic equation with an infinite number of variables for non-standard functional and time delay. IMA J. Math. Control Inform. *19* (2002), 4, 461–476. DOI:10.1093/imamci/19.4.461

[25] J. L. Lions: Optimal Control Of Systems Governed By Partial Differential Equations. Springer-Verlag, Band *170* (1971).

[26] J. L. Lions and E. Magenes: Non-Homogeneous Boundary Value Problem and Applications. Springer-Verlag, New York 1972. DOI:10.1007/978-3-642-65217-2

[27] Y. Liu, W. Rundell, and M. Yamamoto: Strong maximum principle for fractional diffusion equations and an application to an inverse source problem. Fract. Calc. Appl. Anal. *19* (2016), 4, 888–906. DOI:10.1515/fca-2016-0048

[28] Z. Liu, Sh. Zeng, and Y. Bai: Maximum principles for multi-term spacetime variable order fractional diffusion equations and their applications. Fract. Calc. Appl. Anal. *19* (2016), 1, 188–211. DOI:10.1515/fca-2016-0011

[29] Yu. Luchko: Maximum principle for the generalized time fractional diffusion equation. J. Math. Anal. Appl. *351* (2009), 218–223. DOI:10.1016/j.jmaa.2008.10.018

[30] Yu. Luchko: Boundary value problems for the generalized time fractional diffusion equation of distributed order. Fract. Calc. Appl. Anal. *12* (2009), 409–422.

[31] Yu. Luchko: Some uniqueness and existence results for the initial boundary value problems for the generalized time fractional diffusion equation. Comput. Math. Appl. *59* (2010), 1766–1772. DOI:10.1016/j.camwa.2009.08.015

[32] Yu. Luchko: Initial boundary value problems for the generalized multiterm time fractional diffusion equation. J. Math. Anal. Appl. *374* (2011), 538–548. DOI:10.1016/j.jmaa.2010.08.048

[33] Yu. Luchko and M. Yamamoto: General time fractional diffusion equation: Some uniqueness and existence results for the initial boundary value problems. Fract. Calc. Appl. Anal. *19* (2016), 3, 676–695. DOI:10.1515/fca-2016-0036

[34] I. Matychyna and V. Onyshchenkob: On time-optimal control of fractional-order systems. J. Comput. Appl. Math. *339* (2018), 245–257. DOI:10.1016/j.cam.2017.10.016

[35] G. M. Mophou: Optimal control of fractional diffusion equation. Comput. Math. Appl. *61* (2011), 68–78. DOI:10.1016/j.camwa.2010.10.030

[36] G. M. Mophou: Optimal control of fractional diffusion equation with state constraints. Comput. Math. Appl. *62* (2011), 1413–1426. DOI:10.1016/j.camwa.2011.04.044

[37] K. B. Oldham and J. Spanier: The Fractional Calculus. Academic Press, New York 1974.

[38] M. Protter and H. Weinberger: Maximum Principles in Differential Equations. Prentice Hall, Englewood Clifs, 1967. DOI:10.1007/978-1-4612-5282-5

[39] H. Ye, F. Liu, V. Anh, and I. Turner: Maximum principle and numerical method for the multi-term time-space Riesz-Caputo fractional differential equations. Appl. Math. Comput. *227* (2014), 531–540. DOI:10.1016/j.amc.2013.11.015

G. M. Bahaa, Current address: Department of Mathematics, College of Sciences, Taibah University, Al-Madinah Al-Munawarah. Saudi Arabia. Permanent address: Department of Mathematics and Computer Science, Faculty of Science, Beni-Suef University, Beni-Suef. Egypt.

e-mail: Bahaa_gm@yahoo.com