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ON CONTINUITY OF THE ENTROPY-BASED DIFFERENTLY IMPLICATIONAL ALGORITHM

Yiming Tang and Witold Pedrycz

Aiming at the previously-proposed entropy-based differently implicational algorithm of fuzzy inference, this study analyzes its continuity. To begin with, for the FMP (fuzzy modus ponens) and FMT (fuzzy modus tollens) problems, the continuous as well as uniformly continuous properties of the entropy-based differently implicational algorithm are demonstrated for the Tchebyshev and Hamming metrics, in which the R-implications derived from left-continuous t-norms are employed. Furthermore, four numerical fuzzy inference examples are provided, and it is found that the entropy-based differently implicational algorithm can obtain more reasonable solution in contrast with the fuzzy entropy full implication algorithm. Finally, in the entropy-based differently implicational algorithm, we point out that the first fuzzy implication reflects the effect of rule base, and that the second fuzzy implication embodies the inference mechanism.

Keywords: fuzzy inference, fuzzy entropy, compositional rule of inference, continuity

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1. INTRODUCTION

Fuzzy inference plays a key role in fuzzy control, pattern recognition, image processing, data mining and many others [6, 7, 23, 30, 44]. In order to solve the basic problems of fuzzy inference [4, 21], i. e., FMP (fuzzy modus ponens) and FMT (fuzzy modus tollens) as follows:

$$\text{FMP: from given rule } A \to B \text{ and input } A^*, \text{compute the output } B^*, \tag{1}$$

$$\text{FMT: from given rule } A \to B \text{ and input } B^*, \text{compute the output } A^*, \tag{2}$$

where $A, A^* \in F(U)$ (representing the set of fuzzy subsets of U) and $B, B^* \in F(V)$, the CRI (compositional rule of inference) method proposed by Zadeh is the commonly used strategy [13, 45]. In the CRI method, a single fuzzy implication was used. Then the method was extended to triple fuzzy implications, and the full implication algorithm was proposed by Wang in 1999 [41]. The main mode of the full implication algorithm is to find the smallest $B^* \in F(V)$ (or the largest $A^* \in F(U)$) such that

$$(A(u) \to B(v)) \to (A^*(u) \to B^*(v)) \tag{3}$$

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attains its maximum for any $u \in U$, $v \in V$ (here $\to$ denotes a fuzzy implication on $[0, 1]$). As its follow-up, the fuzzy entropy full implication algorithm [11] was proposed being guided by the principle of maximum entropy (established by E. T. Jaynes in 1975), which showed how to choose the optimal solution to the fuzzy inference problem.

The full implication algorithm is nowadays a point of intensive researches, which includes related logic fundamentals, reversibility, pointwise optimization (see [5, 20, 28, 38]). However it is not ideal from the perspective of fuzzy controller (see [18]), which exhibits inferior response ability and practicality. To deal with such problem, we proposed the differently implicational algorithm in [33]. In this new method, (3) is changed into

$$(A(u) \to_1 B(v)) \to_2 (A^*(u) \to_2 B^*(v)), \tag{4}$$

in which $\to_1$ and $\to_2$ stand for different fuzzy implications. It is noted that the differently implicational algorithm is a generalization of both the full implication algorithm and the CRI method. In [34], the reversibility property of the differently implicational algorithm was analyzed when using the expansion, reduction and other type operators, which demonstrated that its reversibility property was satisfied. Moreover, we established fuzzy controllers based on the differently implicational algorithm, and found that their response abilities were good. In [36], we found that 190 fuzzy controllers via the differently implicational algorithm could be used in practical systems; meanwhile only 19 fuzzy controllers via the CRI method and 2 ones via the full implication algorithm were found in. Here fuzzy implications used in three algorithms were in the same scope, while the singleton fuzzier together with the combination of the centroid defuzzier and the defuzzier of average from the maximum were employed in these fuzzy controllers. Therefore, the differently implicational algorithm has larger effective choosing space for usable fuzzy controllers, which can obtain practically sound fuzzy controllers in contrast with the full implication algorithm and the CRI method. More studies completed with regard to the differently implicational algorithm [32, 35, 37], confirmed the advantages of the approach.

By virtue of the principle of maximum entropy, we put forward the differently implicational algorithm with maximum fuzzy entropy in [39], which referred to as the entropy-based differently implicational algorithm. Its optimal solution is the fuzzy set B^* (or A^*) with maximum fuzzy entropy such that (4) assumes its maximum.

For (1) of fuzzy inference, when the input A^* is close to A, if the inference result B^* is totally different from B, then such inference mechanism can not be acceptant in applications. It is natural to anticipate that small deviation of input will not lead to a huge differences in the inference result. This is regarded as the continuity problem. It is pointed out in [16] that the continuity is important for fuzzy inference. As a result, we investigate the continuity of the entropy-based differently implicational algorithm.

Section 2 covers some preliminaries, which includes related definitions and some previous works on the entropy-based differently implicational algorithm. In Section 3, the continuous as well as uniformly continuous properties of the entropy-based differently implicational algorithm are analyzed for the FMP problem. In Section 4, the continuity of the entropy-based differently implicational algorithm is researched for the FMT problem. In Section 5, some examples are shown, demonstrating that the entropy-based differently implicational algorithm can obtain better solution than the fuzzy entropy full implication algorithm. Section 6 provides some discussions of the results. Section 7 concludes the paper.

2. PRELIMINARIES

Definition 2.1. (Baczyński and Jayaram [1, 2], Mas et al. [21]) A function $I : [0,1]^2 \to [0,1]$ is called a fuzzy implication if it satisfies the following conditions:

(Q1): I is decreasing w.r.t. the first variable,

(Q2): I is increasing w.r.t. the second variable,

(Q3): $I(0,0) = 1, I(1,1) = 1, I(1,0) = 0$.

$I(a,b)$ is also written as $a \to b$ $(a, b \in [0,1])$.

It is easy to note that

(LB): $I(0,b) = 1,\ b \in [0,1]$ (the left boundary condition),

(RB): $I(a,1) = 1,\ a \in [0,1]$ (the right boundary condition),

hold for any fuzzy implication I, thus the following condition holds:

(NC): $I(0,1) = 1$ (the normality condition).

Definition 2.2. (Klement et al. [17]) A function $T : [0,1]^2 \to [0,1]$ is said to be a t-norm if T is associative, commutative, increasing and satisfies $T(1,a) = a$ $(a \in [0,1])$.

Definition 2.3. (Mas et al. [21]) A fuzzy implication $\to$ is called an R-implication if there exists a left-continuous t-norm T satisfying the following formula (where $\vee$ denotes supremum):

$$a \to b = \vee\{x \in [0,1] |\ T(a,x) \leq b\}, \quad a, b \in [0,1]. \tag{5}$$

As for Definition 2.3, in general for any t-norm T we have supremum in the formula (5), but when T is left-continuous, then we have maximum in (5) (see details in [2, 10]).

Definition 2.4. (Novák et al. [22]) Suppose that $T, \to$ are two $[0,1]^2 \to [0,1]$ functions. $\to$ is called a residual of T, if the following condition satisfies:

$$T(a,b) \leq c \Longleftrightarrow b \leq a \to c \quad (a, b, c \in [0,1]). \tag{6}$$

Proposition 2.5. (Fodor and Roubens [9]) Let T be a t-norm. T is left-continuous if and only if $\to$ is a residual of T, where $\to$ is achieved from (5).

Lemma 2.6. (Wang and Fu [42]) Let I be an R-implication, and T a left-continuous t-norm, and I a residual of T, then I satisfies (Q1), (Q2), (Q3) and the following conditions:

(OP): $a \leq b \Longleftrightarrow I(a,b) = 1$ (the ordering property),

(EP): $I(a, I(b,c)) = I(b, I(a,c))$ (the exchange principle),

(NP): $I(1,a) = a$ (the left neutrality property),

(Q4): $a \leq I(b,c) \Longleftrightarrow b \leq I(a,c)$,

(Q5): $I(\sup_{x\in X} x,\ a) = \inf_{x\in X} I(x,a)$,

(Q6): $I(a, \inf_{x\in X} x) = \inf_{x\in X} I(a,x)$,

(Q7): $I(T(a,b),c) = I(a, I(b,c))$,

(Q8): I is left-continuous w.r.t. the first variable,

(Q9): I is right-continuous w.r.t. the second variable,

where $a, b, c, x \in [0,1]$ and $X \subset [0,1],\ X \neq \varnothing$.

Here we mainly consider the following R-implications, including Lukasiewicz implication I_L, I_0 implication [27] (which is also called I_{FD}, see [1]), Gödel implication I_G, Goguen implication I_{Go}, and $I_{ep}, I_{y-0.5}$ (see [33, 40]).

$$I_L(a,b) = \begin{cases} 1, & a \le b \\ 1-a+b, & a > b \end{cases},$$

$$I_0(a,b) = \begin{cases} 1, & a \le b \\ (1-a) \vee b, & a > b \end{cases},$$

$$I_G(a,b) = \begin{cases} 1, & a \le b \\ b, & a > b \end{cases},$$

$$I_{Go}(a,b) = \begin{cases} 1, & a \le b \\ b/a, & a > b \end{cases},$$

$$I_{ep}(a,b) = \begin{cases} 1, & a \le b \\ (2b-ab)/(a+b-ab), & a > b \end{cases},$$

$$I_{y-0.5}(a,b) = \begin{cases} 1, & a \le b \\ 1-(\sqrt{1-b}-\sqrt{1-a}\ )^2, & a > b \end{cases}.$$

Proposition 2.7. (Tang and Liu [33]) The t-norms of which $I_L, I_0, I_G, I_{Go}, I_{ep}, I_{y-0.5}$ are the corresponding residuals, are:

$$T_L(a,b) = \begin{cases} a+b-1, & a+b > 1 \\ 0, & a+b \le 1 \end{cases},$$

$$T_0(a,b) = \begin{cases} a \wedge b, & a+b > 1 \\ 0, & a+b \le 1 \end{cases},$$

$$T_G(a,b) = a \wedge b,$$

$$T_{Go}(a,b) = a \times b,$$

$$T_{ep}(a,b) = ab/(2-a-b+ab),$$

$$T_{y-0.5}(a,b) = \begin{cases} 1-(r(a,b))^2, & r(a,b) \le 1 \\ 0, & r(a,b) > 1 \end{cases}, \quad \text{(where } r(a,b) = \sqrt{1-a}+\sqrt{1-b}\ \text{)}.$$

Definition 2.8. Suppose that Z is any non-empty set, and that $F(Z)$ represents the set of fuzzy subsets of Z. Then a partial order relation $\le_F$ on $F(Z)$ is defined as follows:

$$A \le_F B \Longleftrightarrow A(z_0) \le B(z_0) \quad (\forall z_0 \in Z),$$

in which $A, B \in F(Z)$.

Lemma 2.9. (Wang and Zhou [43]) $\langle F(Z), \le_F \rangle$ is a complete lattice.

It is easy to prove Lemma 2.10.

Lemma 2.10. $|a \wedge c - b \wedge c| \le |a-b|$, $|a \vee c - b \vee c| \le |a-b|$, where $a, b, c \in [0,1]$.

Lemma 2.11. (Hong and Hwang [12]) If $U \to R$ mappings f, g are bounded, in which U is a nonempty set and R is the set of real number, then for any $u \in U$, we have

(i) $|\sup_{u\in U} f(u) - \sup_{u\in U} g(u)| \le \sup_{u\in U} |f(u)-g(u)|$;

(ii) $|\inf_{u\in U} f(u) - \inf_{u\in U} g(u)| \le \sup_{u\in U} |f(u)-g(u)|$.

Definition 2.12. (Szmidt and Kacprzyk [31]) Let E be a set-to-point mapping $E : F(X) \to [0,1]$. E is a fuzzy entropy measure if it satisfies the four De Luca and Termini axioms:

(i) $E(A) = 0$ if and only if A is non-fuzzy,
(ii) $E(A) = 1$ if and only if $A(x) = 0.5$ for all $x \in X$,
(iii) $E(A) \leq E(B)$ if A is less fuzzy than B, i. e.,
if $A(x) \leq B(x)$ when $B(x) \leq 0.5$, and $A(x) \geq B(x)$ when $B(x) \geq 0.5$,
(iv) $E(A) = E(A^c)$.

In this section, we briefly review the solutions produced by the entropy-based differently implicational algorithm [39].

For convenience, we denote $R_1(u, v) = A(u) \to_1 B(v)$. Aiming at FMP (1), from the viewpoint of entropy-based differently implicational algorithm, the following principle is given:

Entropy-based differently implicational principle for FMP: The solution B^* of FMP problem (1) is the fuzzy set (in $< F(V), \leq_F >$) with maximum fuzzy entropy letting (4) get its maximum.

Definition 2.13. (Tang et al. [39]) Let $A, A^* \in F(U)$, $B \in F(V)$, if B^* (in $< F(V), \leq_F >$) makes (4) obtain its maximum for any $u \in U, v \in V$, then B^* is said to be an initial entropy-inference solution of FMP.

Definition 2.14. (Tang et al. [39]) If $A, A^* \in F(U)$, $B \in F(V)$, and non-empty set $\mathbb{E}$ is the set of all initial entropy-inference solutions of FMP, and lastly D^* is the fuzzy set with maximum fuzzy entropy in $\mathbb{E}$, then D^* is said to be a formal entropy-inference solution of FMP.

Assume that the maximum of (4) for FMP at every point (u, v) is $M(u, v)$.

Proposition 2.15. (i) If D_1 is an initial entropy-inference solution of FMP, and $D_1 \leq_F D_2$ (in which $D_1, D_2 \in < F(V), \leq_F >$). Then D_2 is an initial entropy-inference solution of FMP. (ii) $M(u, v) = R_1(u, v) \to_2 (A^*(u) \to_2 1) = 1$.

Proof. (i) Since D_1 is an initial entropy-inference solution of FMP, $R_1(u, v) \to_2 (A^*(u) \to_2 D_1(v)) = M(u, v)$ $(u \in U, v \in V)$. Because $D_1 \leq_F D_2$ and $\to_2$ satisfies (Q2), it follows that $A^*(u) \to_2 D_1(v) \leq A^*(u) \to_2 D_2(v)$ and

$$\begin{aligned} M(u, v) &= R_1(u, v) \to_2 (A^*(u) \to_2 D_1(v)) \\ &\leq R_1(u, v) \to_2 (A^*(u) \to_2 D_2(v)). \end{aligned}$$

Note that $R_1(u, v) \to_2 (A^*(u) \to_2 D_2(v)) \leq M(u, v)$, thus $R_1(u, v) \to_2 (A^*(u) \to_2 D_2(v)) = M(u, v)$ $(u \in U, v \in V)$. Therefore D_2 is also an initial entropy-inference solution of FMP. (ii) Similar to (i), it is easy to verify that $M(u, v) = R_1(u, v) \to_2 (A^*(u) \to_2 1)$. Moreover, since (RB) holds for $\to_2$, one has $M(u, v) = 1$. □

Here we interpret the definition of the formal entropy-inference solution of FMP. From Proposition 2.15, it is easy to find that for an initial entropy-inference solution D_1, any fuzzy set D_2 which satisfies $D_1 \leq_F D_2$ is also an initial entropy-inference solution. It is obvious that there are many initial entropy-inference solutions B^*, that is, B^* is not unique. Then we need to determine which one is the best solution. In the light of the principle of maximum entropy proposed by E. T. Jaynes [14, 15], we choose the fuzzy set with maximum fuzzy entropy as the best solution, which is the formal entropy-inference solution of FMP.

Theorem 2.16. Suppose that $\rightarrow_2$ is an R-implication, and that T is a left-continuous t-norm, and that $\rightarrow_2$ is a residual of T, then the formal entropy-inference solution of FMP can be computed with the aid of the following formula:

$$B^*(v) = 0.5 \vee \sup_{u\in U}\{T(A^*(u), R_1(u,v))\},\ v \in V. \tag{7}$$

Proof. Denote $D^*(v) = \sup_{u\in U}\{T(A^*(u), R_1(u,v))\}$, $v \in V$. First, we show that B^* makes (4) take its maximum $M(u,v)$ for any $u \in U, v \in V$. It follows from the expression of D^* that

$$T(A^*(u), R_1(u,v)) \leq D^*(v),\ u \in U, v \in V. \tag{8}$$

Since $\rightarrow_2$ is an R-implication, it follows from Lemma 2.6 that $M(u,v) = 1$ and $\rightarrow_2$ satisfies (OP), (Q9). Noting that $\rightarrow_2$ is a residual of T, we obtain from (8) that $R_1(u,v) \leq A^*(u) \rightarrow_2 D^*(v)$ and

$$R_1(u,v) \rightarrow_2 (A^*(u) \rightarrow_2 D^*(v)) = 1 = M(u,v)$$

hold for any $u \in U, v \in V$. Thus $D^* \in \mathbb{E}$ and D^* is an initial entropy-inference solution of FMP.

Second, we verify that D^* is the minimum of $\mathbb{E}$. Suppose that any fuzzy set $D \in \mathbb{E}$. Thus $(u \in U, v \in V)$

$$R_1(u,v) \rightarrow_2 (A^*(u) \rightarrow_2 D(v)) = M(u,v) = 1.$$

Taking into account that $\rightarrow_2$ is a residual of T and that $\rightarrow_2$ satisfies (OP), we obtain $R_1(u,v) \leq A^*(u) \rightarrow_2 D(v)$ and $T(A^*(u), R_1(u,v)) \leq D(v)$ $(u \in U, v \in V)$. Thus $D(v)$ is an upper bound of $\{T(A^*(u), R_1(u,v)) \mid u \in U\}$, $v \in V$. Hence it follows from the expression of D^* that $D^* \leq_F D$. These imply that D^* is the minimum of $\mathbb{E}$.

Denote $B^* = 0.5 \vee D^*$. Then $D^* \leq_F B^*$. It follows from Proposition 2.15 that B^* is also an initial entropy-inference solution of FMP, and thus $B^* \in \mathbb{E}$.

Then we prove that B^* is the fuzzy set with maximum entropy in $\mathbb{E}$. In fact, let B_k be any fuzzy set in $\mathbb{E}$. Obviously $D^* \leq_F B_k$. If $D^*(v) \leq B_k(v) \leq 0.5$ or $D^*(v) \leq 0.5 \leq B_k(v)$, then $B^*(v) = 0.5$; if $0.5 \leq D^*(v) \leq B_k(v)$, then $B^*(v) = D^*(v)$ $(v \in V)$. Together we get $B_k(v) \leq B^*(v) \leq 0.5$ or $0.5 \leq B^*(v) \leq B_k(v)$, and hence we get from Definition 2.12 that $E(B^*) \geq E(B_k)$.

Therefore, B^* expressed as (7) is the formal entropy-inference solution of FMP in the light of Definition 2.14.

□

Example 2.17. The formal entropy-inference solution of FMP is expressed as follows $(v \in V)$:

(i) If $\to_2$ takes I_L, then $B^*(v) = 0.5 \vee \sup_{u\in U}\{A^*(u) + R_1(u,v) - 1\}$.

(ii) If $\to_2$ takes I_0, then $B^*(v) = 0.5 \vee \sup_{u\in E_v}\{A^*(u) \wedge R_1(u,v)\}$, where $E_v = \{u \in U|\ A^*(u) + R_1(u,v) > 1\}$.

(iii) If $\to_2$ takes I_G, then $B^*(v) = 0.5 \vee \sup_{u\in U}\{A^*(u) \wedge R_1(u,v)\}$.

(iv) If $\to_2$ takes I_{Go}, then $B^*(v) = 0.5 \vee \sup_{u\in U}\{A^*(u) \times R_1(u,v)\}$.

(v) If $\to_2$ takes I_{ep}, then $B^*(v) = 0.5 \vee \sup_{u\in U}\{A^*(u) \times R_1(u,v)/[2 - A^*(u) - R_1(u,v) + A^*(u) \times R_1(u,v)]\}$.

(vi) If $\to_2$ takes $I_{y-0.5}$, then $B^*(v) = 0.5 \vee \sup_{u\in E_v}\{1 - [\sqrt{1 - A^*(u)} + \sqrt{1 - R_1(u,v)}\,]^2\}$, where $E_v = \{u \in U|\ \sqrt{1 - A^*(u)} + \sqrt{1 - R_1(u,v)} \leq 1\}$.

For the FMT (2), from the viewpoint of the entropy-based differently implicational algorithm, the following principle is formulated:

Entropy-based differently implicational principle for FMT: The solution A^* of FMT problem (2) is the fuzzy set (in $< F(U), \leq_F >$) with maximum fuzzy entropy letting (4) achieve its maximum.

Definition 2.18. (Tang et al. [39]) Let $A \in F(U)$, $B, B^* \in F(V)$, if A^* (in $< F(U), \leq_F >$) lets (4) get its maximum for any $u \in U, v \in V$, then A^* is said to be an initial entropy-inference solution of FMT.

Definition 2.19. (Tang et al. [39]) If $A \in F(U)$, $B, B^* \in F(V)$, and non-empty set $\mathbb{F}$ is the set of all initial entropy-inference solutions of FMT, and lastly C^* is the fuzzy set with maximum fuzzy entropy in $\mathbb{F}$, then C^* is called a formal entropy-inference solution of FMT.

Assume that the maximum of (4) for FMT at every point (u,v) is $N(u,v)$.

Proposition 2.20. (i) If C_1 is an initial entropy-inference solution of FMT, and $C_2 \leq_F C_1$ (in which $C_1, C_2 \in < F(U), \leq_F >$). Then C_2 is an initial entropy-inference solution of FMT. (ii) $N(u,v) = 1$.

P r o o f. (i) Since C_1 is an initial entropy-inference solution of FMT, one has $R_1(u,v) \to_2 (C_1(u) \to_2 B^*(v)) = N(u,v)$ $(u \in U, v \in V)$. Because $C_2 \leq_F C_1$ and $\to_2$ satisfies (Q1), (Q2), it follows that $C_1(u) \to_2 B^*(v) \leq C_2(u) \to_2 B^*(v)$ and

$$N(u,v) \leq R_1(u,v) \to_2 (C_2(u) \to_2 B^*(v))\ (u \in U,\ v \in V).$$

It is similar to Proposition 2.15 that D_2 is also an initial entropy-inference solution of FMT.

(ii) Similar to Proposition 2.15, it is easy to find

$$N(u,v) = R_1(u,v) \to_2 (0 \to_2 B^*(v))\ (u \in U,\ v \in V)..$$

Since $\to_2$ is a fuzzy implication, it follows that (LB) and (RB) hold for $\to_2$, thus $N(u,v) = 1$. □

Theorem 2.21. If $\to_2$ is an R-implication, then the formal entropy-inference solution of FMT comes as follows:

$$A^*(u) = 0.5 \wedge \inf_{v \in V} \{R_1(u,v) \to_2 B^*(v)\}, \ u \in U. \tag{9}$$

Proof. Denote $C^*(u) = \inf_{v \in V}\{R_1(u,v) \to_2 B^*(v)\}, \ v \in V$. First, we verify that C^* makes (4) take its maximum $N(u,v)$ for any $u \in U, v \in V$. It follows from the expression of C^* that

$$C^*(u) \leq R_1(u,v) \to_2 B^*(v), \ u \in U, v \in V. \tag{10}$$

Since $\to_2$ is an R-implication, we get from Lemma 2.6 and Proposition 2.20 that $N(u,v) = 1$ and $\to_2$ satisfies (OP), (Q4). Then it follows from (10) that $R_1(u,v) \leq C^*(u) \to_2 B^*(v)$ and

$$R_1(u,v) \to_2 (C^*(u) \to_2 B^*(v)) = 1 = N(u,v)$$

hold for any $u \in U, v \in V$. Thus $C^* \in \mathbb{F}$ and C^* is an initial entropy-inference solution of FMT.

Second, we prove that C^* is the maximum of $\mathbb{F}$. Suppose that any fuzzy set $C \in \mathbb{F}$. Thus

$$R_1(u,v) \to_2 (C(u) \to_2 B^*(v)) = N(u,v) = 1$$

holds for any $u \in U, v \in V$. Noting that $\to_2$ satisfies (OP) and (Q4), we get $R_1(u,v) \leq C(u) \to_2 B^*(v)$ and $C(u) \leq R_1(u,v) \to_2 B^*(v)$ hold for any $u \in U, v \in V$. Thus $C(u)$ is a lower bound of

$$\{R_1(u,v) \to_2 B^*(v) \mid v \in V\}, \ u \in U.$$

So it follows from the expression of C^* that $C \leq_F C^*$. These imply that C^* is the maximum of $\mathbb{F}$.

Denote $A^* = 0.5 \wedge C^*$. Then $A^* \leq_F C^*$. It follows from Proposition 2.20 that A^* is also an initial entropy-inference solution of FMT, and thus $A^* \in \mathbb{F}$.

Then we prove that A^* is the fuzzy set with maximum entropy in $\mathbb{F}$. In fact, let A_k be any fuzzy set in $\mathbb{F}$. Obviously $A_k \leq_F C^*$. If $A_k(u) \leq C^*(u) \leq 0.5$, then $A^*(u) = C^*(u)$; if $A_k(u) \leq 0.5 \leq C^*(u)$ or $0.5 \leq A_k(u) \leq C^*(u)$, then $A^*(u) = 0.5$ $(u \in U)$. Together we get $A_k(u) \leq A^*(u) \leq 0.5$ or $0.5 \leq A^*(u) \leq A_k(u)$, and hence it from Definition 2.12 that $E(A^*) \geq E(A_k)$.

Therefore, A^* expressed as (9) is the formal entropy-inference solution of FMT in the light of Definition 2.19. □

Example 2.22. The formal entropy-inference solution of FMT is computed as follows (where $F_u = \{v \in V | \ R_1(u,v) > B^*(v)\}$):

(i) If $\to_2$ takes I_L, then $A^*(u) = 0.5 \wedge \inf_{v \in F_u}\{1 - R_1(u,v) + B^*(v)\}, \ u \in U$.

(ii) If $\to_2$ takes I_0, then $A^*(u) = 0.5 \wedge \inf_{v \in F_u}\{(1 - R_1(u,v)) \vee B^*(v)\}, \ u \in U$.

(iii) If $\to_2$ takes I_G, then $A^*(u) = 0.5 \wedge \inf_{v \in F_u}\{B^*(v)\}, \ u \in U$.

(iv) If $\to_2$ takes I_{Go}, then $A^*(u) = 0.5 \wedge \inf_{v \in F_u}\{B^*(v)/R_1(u,v)\}, \ u \in U$.

(v) If $\to_2$ takes I_{ep}, then $A^*(u) = 0.5 \wedge \inf_{v\in F_u}\{(2B^*(v) - R_1(u,v)\times B^*(v))/(R_1(u,v) + B^*(v) - R_1(u,v)\times B^*(v))\}$, $u \in U$.

(vi) If $\to_2$ takes $I_{y-0.5}$, then $A^*(u) = 0.5 \wedge \inf_{v\in F_u}\{1 - (\sqrt{1 - B^*(v)} - \sqrt{1 - R_1(u,v)}\,)^2\}$, $u \in U$.

3. CONTINUITY OF ENTROPY-BASED DIFFERENTLY IMPLICATIONAL ALGORITHM FOR FMP

A distance function d is regard as a metric if it is positive definite ($d(a,b) \geq 0$, and $d(a,b) = 0$ if and only if $a = b$), symmetric ($d(a,b) = d(b,a)$), and possesses the triangle inequality ($d(a,c) \leq d(a,b) + d(b,c)$) for any points a, b, c. The concept of distance has been developed to fuzzy set. Here suppose that d is a distance between fuzzy sets [3, 29], which is a metric.

Definition 3.1. A fuzzy inference algorithm for FMP (1) is a mapping $f : F(U) \to F(V)$, i.e., there exists an output $B^* = f(A^*) \in F(V)$ for any input $A^* \in F(U)$.

(i) For any $\varepsilon > 0$, if there exists $\delta > 0$ making $d(f(A_1^*), f(A_2^*)) < \varepsilon$ whenever $d(A_1^*, A_2^*) < \delta$ for any $A_1^*, A_2^* \in F(U)$, then f is said to be uniformly continuous in metric d;

(ii) For any $\varepsilon > 0$, if there exists $\delta > 0$ making $d(f(A^*), f(A)) < \varepsilon$ whenever $d(A^*, A) < \delta$ for any $A^* \in F(U)$, then f is said to be continuous at $A \in F(U)$ in metric d.

Definition 3.2. A fuzzy inference algorithm for FMT (2) is a mapping $g : F(V) \to F(U)$.

(i) For any $\varepsilon > 0$, if there exists $\delta > 0$ making $d(g(B_1^*), g(B_2^*)) < \varepsilon$ whenever $d(B_1^*, B_2^*) < \delta$ for any $B_1^*, B_2^* \in F(V)$, then g is said to be uniformly continuous in metric d;

(ii) For any $\varepsilon > 0$, if there exists $\delta > 0$ making $d(g(B^*), g(B)) < \varepsilon$ whenever $d(B^*, B) < \delta$ for any $B^* \in F(V)$, then g is said to be continuous at $B \in F(V)$ in metric d.

The practical problems often concern a finite number of elements. For example, in the areas of natural language processing and computing with words, all works are always done on the basis of a word corpus which only includes a finite number of words. Moreover, nowadays the computer is only able to store and deal with finite numbers. As a result, here we suppose that the universes U, V are finite sets, i.e., $U = \{u_1, u_2, \ldots, u_m\}$, $V = \{v_1, v_2, \ldots, v_n\}$. Besides, the universe which includes infinite number of elements, belongs to another situation, and it is beyond the scope of this paper. Here we mainly consider the two frequently used metrics, i.e., the following Tchebyshev metric d_T and Hamming metric d_H (where $A, B \in F(U)$):

$$d_T(A,B) = \max_{u\in U} |A(u) - B(u)|,$$

$$d_H(A,B) = \frac{1}{m}\sum_{u\in U} |A(u) - B(u)|.$$

Theorem 3.3. The entropy-based differently implicational algorithm for FMP expressed as (7) is uniformly continuous in metric $d \in \{d_T, d_H\}$, if the t-norm T is continuous.

Proof. For any inputs $A_1^*, A_2^* \in F(U)$, we analyze the continuity property of the entropy-based differently implicational algorithm for FMP. Taking into account that the t-norm T is continuous, it follows that T is uniformly continuous w.r.t. its first variable in [0,1]. As a result, for any $\varepsilon > 0$, there exists $\delta_1 > 0$ such that

$$|T(A_1^*(u), R_1(u,v)) - T(A_2^*(u), R_1(u,v))| < \varepsilon \tag{11}$$

hold if $|A_1^*(u) - A_2^*(u)| < \delta_1 \ (u \in U)$.

(i) For the case of $d = d_T$, we employ

$$\delta = \delta_1.$$

Suppose that

$$d_T(A_1^*, A_2^*) < \delta.$$

It implies that

$$\max_{u \in U} |A_1^*(u) - A_2^*(u)| < \delta$$

and that

$$|A_1^*(u) - A_2^*(u)| < \delta = \delta_1 \qquad (u \in U).$$

Thus, there exists $\delta > 0$ such that (11) holds. Note that U, V are finite, then the meaning of sup is the same as max. In virtue of Lemma 2.10 as well as Lemma 2.11, we have

$$\begin{aligned}
& d_T(B_1^*, B_2^*) \\
& = \max_{v \in V} \Big| B_1^*(v) - B_2^*(v) \Big| \\
& = \max_{v \in V} \Big| 0.5 \vee \max_{u \in U} \{T(A_1^*(u), R_1(u,v))\} - 0.5 \vee \max_{u \in U} \{T(A_2^*(u), R_1(u,v))\} \Big| \\
& \leq \max_{v \in V} \Big| \max_{u \in U} \{T(A_1^*(u), R_1(u,v))\} - \max_{u \in U} \{T(A_2^*(u), R_1(u,v))\} \Big| \\
& \leq \max_{v \in V} \max_{u \in U} \Big| T(A_1^*(u), R_1(u,v)) - T(A_2^*(u), R_1(u,v)) \Big| \\
& < \max_{v \in V} \max_{u \in U} \varepsilon \\
& = \varepsilon.
\end{aligned}$$

In this process mentioned above, it is noted that U, V are finite sets, which ensures that (11) implies

$$\max_{v \in V} \max_{u \in U} \Big| T(A_1^*(u), R_1(u,v)) - T(A_2^*(u), R_1(u,v)) \Big| < \max_{v \in V} \max_{u \in U} \varepsilon.$$

As a result, there exists $\delta > 0$ such that $d_T(B_1^*, B_2^*) < \varepsilon$ if $d_T(A_1^*, A_2^*) < \delta$, therefore the entropy-based differently implicational algorithm for FMP expressed as (7) is uniformly continuous in d_T.

(ii) For the case of $d = d_H$, we choose

$$\delta = \delta_1/m.$$

Assume that $d_H(A_1^*, A_2^*) < \delta$. It implies that

$$\sum_{u\in U} |A_1^*(u) - A_2^*(u)|/m < \delta$$

and that

$$|A_1^*(u) - A_2^*(u)| < m\delta = \delta_1 \qquad (u \in U).$$

So there exists $\delta > 0$ such that (11) holds. Moreover, we have from Lemma 2.10 and Lemma 2.11 that

$$\begin{aligned}
&d_H(B_1^*, B_2^*)\\
&= \frac{1}{n}\sum_{v\in V}\Big|B_1^*(v) - B_2^*(v)\Big|\\
&= \frac{1}{n}\sum_{v\in V}\Big|0.5 \vee \max_{u\in U}\{T(A_1^*(u), R_1(u,v))\} - 0.5 \vee \max_{u\in U}\{T(A_2^*(u), R_1(u,v))\}\Big|\\
&\le \frac{1}{n}\sum_{v\in V}\Big|\max_{u\in U}\{T(A_1^*(u), R_1(u,v))\} - \max_{u\in U}\{T(A_2^*(u), R_1(u,v))\}\Big|\\
&\le \frac{1}{n}\sum_{v\in V}\max_{u\in U}\Big|T(A_1^*(u), R_1(u,v)) - T(A_2^*(u), R_1(u,v))\Big|\\
&< \frac{1}{n}\sum_{v\in V}\max_{u\in U}\varepsilon\\
&= \varepsilon.
\end{aligned}$$

Therefore, there exists $\delta > 0$ such that $d_H(B_1^*, B_2^*) < \varepsilon$ if $d_H(A_1^*, A_2^*) < \delta$, which implies that the entropy-based differently implicational algorithm for FMP computed by (7) is uniformly continuous in d_H. □

It is easy to note that if f is uniformly continuous then it is continuous, thus we derive Theorem 3.4 from Theorem 3.3.

Theorem 3.4. The entropy-based differently implicational algorithm for FMP expressed as (7) is continuous in metric $d \in \{d_T, d_H\}$, if the t-norm T is continuous.

For $\rightarrow_2 \in \{I_L, I_G, I_{Go}, I_{ep}, I_{y-0.5}\}$, its corresponding t-norm is continuous, then we have Corollary 3.5.

Corollary 3.5. If $\rightarrow_2 \in \{I_L, I_G, I_{Go}, I_{ep}, I_{y-0.5}\}$, then the entropy-based differently implicational algorithm for FMP is uniformly continuous in $d \in \{d_T, d_H\}$, and thus continuous in $d \in \{d_T, d_H\}$.

It is noted that T_0 (which is called the nilpotent minimum) is an important left-continuous t-norm and has many desirable characteristics (see [8]), then we investigate the corresponding case that $\to_2$ takes I_0.

Proposition 3.6. For any $A_1^*, A_2^* \in F(U)$, there exists $\delta_0 > 0$ such that if $d(A_1^*, A_2^*) < \delta_0$ where $d \in \{d_T, d_H\}$, then

$$E_{1v} = E_{2v} \quad (v \in V),$$

in which

$$E_{1v} = \{u \in U \mid A_1^*(u) + R_1(u,v) > 1\}, \quad E_{2v} = \{u \in U \mid A_2^*(u) + R_1(u,v) > 1\}.$$

P r o o f. For any $v \in V$, we analyze the relationship between E_{1v} and E_{2v}.

(i) For the case of $d = d_T$, we employ

$$\delta_1 = \min_{u \in E_{1v}} [A_1^*(u) + R_1(u,v) - 1].$$

Obviously

$$A_1^*(u) + R_1(u,v) - 1 \geq \delta_1 > 0 \quad (u \in E_{1v}, v \in V). \tag{12}$$

Suppose that $d_T(A_1^*, A_2^*) < \delta_1$. Then

$$\max_{u \in U} |A_1^*(u) - A_2^*(u)| < \delta_1 \ \ (u \in U),$$

and thus we get

$$A_1^*(u) - \delta_1 < A_2^*(u) < A_1^*(u) + \delta_1 \quad (u \in U). \tag{13}$$

For any $u_0 \in E_{1v}$, we get

$$A_1^*(u_0) + R_1(u_0, v) > 1,$$

and it follows from (12), (13) that

$$A_2^*(u_0) > A_1^*(u_0) - \delta_1 \geq A_1^*(u_0) - [A_1^*(u_0) + R_1(u_0,v) - 1] = 1 - R_1(u_0,v),$$

thus $u_0 \in E_{2v}$, hence we get $E_{1v} \subset E_{2v}$.

Take

$$\delta_2 = \min_{u \in E_{2v}} [A_2^*(u) + R_1(u,v) - 1].$$

Similarly we can obtain $E_{2v} \subset E_{1v}$ if $d_T(A_1^*, A_2^*) < \delta_2$.

Choose

$$\delta_0 = \min\{\delta_1, \delta_2\},$$

thus $E_{1v} \subset E_{2v}$ and $E_{2v} \subset E_{1v}$ if $d_T(A_1^*, A_2^*) < \delta_0$. Consequently, we obtain that if $d_T(A_1^*, A_2^*) < \delta_0$ then $E_{1v} = E_{2v}$.

(ii) For the case of $d = d_H$, we employ

$$\delta_3 = \min_{u \in E_{1v}} [A_1^*(u) + R_1(u,v) - 1]/(2m+1).$$

Obviously

$$[A_1^*(u) + R_1(u,v) - 1]/(2m+1) \geq \delta_3 > 0 \quad (u \in E_{1v}, v \in V). \tag{14}$$

Suppose that $d_H(A_1^*, A_2^*) < \delta_3$. Then

$$\sum_{u \in U} |A_1^*(u) - A_2^*(u)|/m < \delta_3$$

and thus

$$|A_1^*(u) - A_2^*(u)| < m\delta_3 \quad (u \in U),$$

so we get

$$A_1^*(u) - m\delta_3 < A_2^*(u) < A_1^*(u) + m\delta_3 \quad (u \in U). \tag{15}$$

For any $u_0 \in E_{1v}$, we get $A_1^*(u_0) + R_1(u_0, v) > 1$, and it follows from (14), (15) that

$$\begin{array}{rcl} A_2^*(u_0) & > & A_1^*(u_0) - m\delta_3 \\ & \geq & A_1^*(u_0) - m[A_1^*(u_0) + R_1(u_0, v) - 1]/(2m + 1) \\ & = & [(m + 1)A_1^*(u_0) - mR_1(u_0, v) + m]/(2m + 1) \\ & > & [(m + 1)(1 - R_1(u_0, v)) - mR_1(u_0, v) + m]/(2m + 1) \\ & = & 1 - R_1(u_0, v). \end{array}$$

Thus $u_0 \in E_{2v}$, hence we get $E_{1v} \subset E_{2v}$.

Take

$$\delta_4 = \min_{u \in E_{2v}} [A_2^*(u) + R_1(u, v) - 1]/(2m + 1).$$

Similarly we can obtain $E_{2v} \subset E_{1v}$ if $d_H(A_1^*, A_2^*) < \delta_4$.

Choose

$$\delta_0 = \min\{\delta_3, \delta_4\},$$

thus we obtain that if $d_H(A_1^*, A_2^*) < \delta_0$ then $E_{1v} = E_{2v}$. □

Theorem 3.7. If $\to_2$ takes I_0, then the entropy-based differently implicational algorithm for FMP is uniformly continuous in $d \in \{d_T, d_H\}$, and thus continuous in $d \in \{d_T, d_H\}$.

Proof. (i) It follows from Proposition 3.6 that there exists $\delta_0 > 0$ such that $E_{1v} = E_{2v}$ if $d_T(A_1^*, A_2^*) < \delta_0$ $(u \in U)$. For any $\varepsilon > 0$, take

$$\delta = \min\{\delta_0, \varepsilon\}.$$

Suppose that $d_T(A_1^*, A_2^*) < \delta$. Then

$$E_{1v} = E_{2v} \quad (v \in V),$$

and

$$|A_1^*(u) - A_2^*(u)| < \delta \quad (u \in U).$$

Thus from Example 2.17(ii), Lemma 2.10 and Lemma 2.11 we get

$$
\begin{aligned}
&d_T(B_1^*, B_2^*) \\
&\quad = \max_{v\in V}\left|B_1^*(v) - B_2^*(v)\right| \\
&\quad = \max_{v\in V}\left|0.5 \vee \sup_{u\in E_{1v}}\{A_1^*(u)\wedge R_1(u,v)\} - 0.5 \vee \sup_{u\in E_{2v}}\{A_2^*(u)\wedge R_1(u,v)\}\right| \\
&\quad = \max_{v\in V}\left|0.5 \vee \sup_{u\in E_{1v}}\{A_1^*(u)\wedge R_1(u,v)\} - 0.5 \vee \sup_{u\in E_{1v}}\{A_2^*(u)\wedge R_1(u,v)\}\right| \\
&\quad \le \max_{v\in V}\left|\sup_{u\in E_{1v}}\{A_1^*(u)\wedge R_1(u,v)\} - \sup_{u\in E_{1v}}\{A_2^*(u)\wedge R_1(u,v)\}\right| \\
&\quad \le \max_{v\in V}\sup_{u\in E_{1v}}\left|(A_1^*(u)\wedge R_1(u,v)) - (A_2^*(u)\wedge R_1(u,v))\right| \\
&\quad \le \max_{v\in V}\sup_{u\in E_{1v}}\left|A_1^*(u) - A_2^*(u)\right| \\
&\quad < \max_{v\in V}\sup_{u\in E_{1v}}\delta \\
&\quad = \delta \\
&\quad \le \varepsilon.
\end{aligned}
$$

There exists $\delta > 0$ such that $d_T(B_1^*, B_2^*) < \varepsilon$ if $d_T(A_1^*, A_2^*) < \delta$, therefore the entropy-based differently implicational algorithm for FMP is uniformly continuous in d_T, and thus it is also continuous in d_T.

(ii) We get from Proposition 3.6 that there exists $\delta_0 > 0$ such that $E_{1v} = E_{2v}$ if $d_H(A_1^*, A_2^*) < \delta_0$ $(u \in U)$. For any $\varepsilon > 0$, take

$$\delta = \min\{\delta_0/(m+1),\ \varepsilon/(m+1)\}.$$

Suppose that $d_H(A_1^*, A_2^*) < \delta$. Then

$$E_{1v} = E_{2v} \quad (v \in V),$$

and

$$\sum_{u\in U} |A_1^*(u) - A_2^*(u)|/m < \delta$$

and thus

$$|A_1^*(u) - A_2^*(u)| < m\delta \quad (u \in U).$$

Thus it follows from Example 2.17(ii), Lemma 2.10 and Lemma 2.11 that

$$\begin{aligned}
&d_H(B_1^*, B_2^*)\\
&= \frac{1}{n}\sum_{v\in V}\left|0.5 \vee \sup_{u\in E_{1v}}\{A_1^*(u)\wedge R_1(u,v)\} - 0.5 \vee \sup_{u\in E_{2v}}\{A_2^*(u)\wedge R_1(u,v)\}\right|\\
&\le \frac{1}{n}\sum_{v\in V}\left|\sup_{u\in E_{1v}}\{A_1^*(u)\wedge R_1(u,v)\} - \sup_{u\in E_{1v}}\{A_2^*(u)\wedge R_1(u,v)\}\right|\\
&\le \frac{1}{n}\sum_{v\in V}\sup_{u\in E_{1v}}\left|(A_1^*(u)\wedge R_1(u,v)) - (A_2^*(u)\wedge R_1(u,v))\right|\\
&\le \frac{1}{n}\sum_{v\in V}\sup_{u\in E_{1v}}\left|A_1^*(u) - A_2^*(u)\right|\\
&< \frac{1}{n}\sum_{v\in V}\sup_{u\in E_{1v}} m\delta\\
&= m\delta\\
&\le m\varepsilon/(m+1)\\
&< \varepsilon.
\end{aligned}$$

As a result, the entropy-based differently implicational algorithm for FMP is uniformly continuous in d_H, and thus it is also continuous in d_H. □

4. CONTINUITY OF ENTROPY-BASED DIFFERENTLY IMPLICATIONAL ALGORITHM FOR FMT

Theorem 4.1. Assume that the R-implication $\rightarrow_2$ satisfies

(Q10) I is continuous w.r.t. the second variable,

then the entropy-based differently implicational algorithm for FMT expressed as (8) is uniformly continuous in $d \in \{d_T, d_H\}$, and thus continuous in $d \in \{d_T, d_H\}$.

Proof. For any inputs $B_1^*, B_2^* \in F(V)$, we verify the continuous property of the entropy-based differently implicational algorithm for FMT. Since $\rightarrow_2$ satisfies (Q10), it follows that $\rightarrow_2$ is uniformly continuous w.r.t. its second variable on [0,1]. Therefore, for any $\varepsilon > 0$, there exists $\delta_1 > 0$ making the relationship

$$|(R_1(u,v) \rightarrow_2 B_1^*(v)) - (R_1(u,v) \rightarrow_2 B_2^*(v))| < \varepsilon \tag{16}$$

holds for any $u \in U$ if $|B_1^*(v) - B_2^*(v)| < \delta_1$ $(v \in V)$.

(i) Here we prove the case of $d = d_T$. We set

$$\delta = \delta_1.$$

Suppose that

$$d_T(B_1^*, B_2^*) < \delta.$$

Then

$$\max_{v\in V} |B_1^*(v) - B_2^*(v)| < \delta$$

and

$$|B_1^*(v) - B_2^*(v)| < \delta = \delta_1 \quad (v \in V).$$

So (16) holds, and according to Lemma 2.10 and Lemma 2.11, we get

$$\begin{aligned}
& d_T(A_1^*, A_2^*) \\
= & \max_{u\in U} \left| A_1^*(u) - A_2^*(u) \right| \\
= & \max_{u\in U} \left| 0.5 \wedge \inf_{v\in V} \{R_1(u,v) \to_2 B_1^*(v)\} - 0.5 \wedge \inf_{v\in V} \{R_1(u,v) \to_2 B_2^*(v)\} \right| \\
\leq & \max_{u\in U} \left| \inf_{v\in V} \{R_1(u,v) \to_2 B_1^*(v)\} - \inf_{v\in V} \{R_1(u,v) \to_2 B_2^*(v)\} \right| \\
\leq & \max_{u\in U} \sup_{v\in V} \left| (R_1(u,v) \to_2 B_1^*(v)) - (R_1(u,v) \to_2 B_2^*(v)) \right| \\
< & \max_{u\in U} \sup_{v\in V} \varepsilon \\
= & \varepsilon.
\end{aligned}$$

That is, there exists $\delta > 0$ such that $d_T(A_1^*, A_2^*) < \varepsilon$ if $d_T(B_1^*, B_2^*) < \delta$, therefore the entropy-based differently implicational algorithm for FMT expressed as (8) is uniformly continuous in d_T. And thus it is also continuous in d_T.

(ii) Furthermore, we verify the case of $d = d_H$. We employ

$$\delta = \delta_1/n.$$

Suppose that

$$d_H(B_1^*, B_2^*) < \delta.$$

Then

$$\sum_{v\in V} |B_1^*(v) - B_2^*(v)|/n < \delta$$

and

$$|B_1^*(v) - B_2^*(v)| < n\delta = \delta_1 \quad (v \in V).$$

So (16) holds, and according to Lemma 2.10 and Lemma 2.11, we get

$$\begin{aligned}
&d_H(A_1^*, A_2^*)\\
&= \frac{1}{m}\sum_{u\in U}\Big|A_1^*(u) - A_2^*(u)\Big|\\
&= \frac{1}{m}\sum_{u\in U}\Big|0.5 \wedge \inf_{v\in V}\{R_1(u,v) \to_2 B_1^*(v)\} - 0.5 \wedge \inf_{v\in V}\{R_1(u,v) \to_2 B_2^*(v)\}\Big|\\
&\le \frac{1}{m}\sum_{u\in U}\Big|\inf_{v\in V}\{R_1(u,v) \to_2 B_1^*(v)\} - \inf_{v\in V}\{R_1(u,v) \to_2 B_2^*(v)\}\Big|\\
&\le \frac{1}{m}\sum_{u\in U}\sup_{v\in V}\Big|(R_1(u,v) \to_2 B_1^*(v)) - (R_1(u,v) \to_2 B_2^*(v))\Big|\\
&< \frac{1}{m}\sum_{u\in U}\sup_{v\in V}\varepsilon\\
&= \varepsilon.
\end{aligned}$$

That is, there exists $\delta > 0$ such that $d_H(A_1^*, A_2^*) < \varepsilon$ if $d_H(B_1^*, B_2^*) < \delta$, so the entropy-based differently implicational algorithm for FMT computed as (8) is uniformly continuous in d_H. And then it is also continuous in d_H. □

For $\to_2 \in \{I_L, I_{Go}, I_{ep}, I_{y-0.5}\}$, it is easy to find that $\to_2$ satisfies (Q10). We can get Corollary 4.2 from Theorem 4.1.

Corollary 4.2. If $\to_2 \in \{I_L, I_{Go}, I_{ep}, I_{y-0.5}\}$, then the entropy-based differently implicational algorithm for FMT is uniformly continuous in $d \in \{d_T, d_H\}$, and thus continuous in $d \in \{d_T, d_H\}$.

Moreover, when $\to_2$ is only right-continuous w.r.t. the second variable, whether is the entropy-based differently implicational algorithm for FMT continuous? Here we analyze the typical case of $\to_2 \in \{I_0, I_G\}$.

Proposition 4.3. For any $B_1^*, B_2^* \in F(V)$, there exists $\delta_0 > 0$ such that if $d(B_1^*, B_2^*) < \delta_0$ where $d \in \{d_T, d_H\}$, then $F_{1u} = F_{2u}$ $(u \in U)$, in which

$$F_{1u} = \{v \in V \mid R_1(u,v) > B_1^*(v)\}, \quad F_{2u} = \{v \in V \mid R_1(u,v) > B_2^*(v)\}.$$

P r o o f. For any $u \in U$, we research the relationship between F_{1u} and F_{2u}.

(i) For the case of $d = d_T$, we choose

$$\delta_1 = \min_{v\in F_{1u}} [R_1(u,v) - B_1^*(v)].$$

Evidently

$$R_1(u,v) - B_1^*(v) \ge \delta_1 > 0 \quad (u \in U, v \in F_{1u}). \tag{17}$$

Suppose that $d_T(B_1^*, B_2^*) < \delta_1$. Then

$$\max_{v\in V} |B_1^*(v) - B_2^*(v)| < \delta_1,$$

and so we have

$$B_1^*(v) - \delta_1 < B_2^*(v) < B_1^*(v) + \delta_1 \quad (v \in V). \tag{18}$$

For any $v_0 \in F_{1u}$, we obtain

$$R_1(u, v_0) > B_1^*(v_0),$$

and it follows from (17), (18) that

$$B_2^*(v_0) < B_1^*(v_0) + \delta_1 \leq B_1^*(v_0) + R_1(u, v_0) - B_1^*(v_0) = R_1(u, v_0),$$

thus $v_0 \in F_{2u}$, hence we have $F_{1u} \subset F_{2u}$.

Take

$$\delta_2 = \min_{v \in F_{2u}} [R_1(u, v) - B_2^*(v)].$$

Similarly we can achieve $F_{2u} \subset F_{1u}$ if $d_T(B_1^*, B_2^*) < \delta_2$.

Take

$$\delta_0 = \min\{\delta_1, \delta_2\},$$

thus $F_{1u} \subset F_{2u}$ and $F_{2u} \subset F_{1u}$ if $d_T(B_1^*, B_2^*) < \delta_0$. As a result, we obtain that if $d_T(B_1^*, B_2^*) < \delta_0$ then $F_{1u} = F_{2u}$.

(ii) For the case of $d = d_H$, we take

$$\delta_3 = \min_{v \in F_{1u}} [R_1(u, v) - B_1^*(v)]/(2n + 1).$$

Obviously

$$[R_1(u, v) - B_1^*(v)]/(2n + 1) \geq \delta_3 > 0 \quad (u \in U, v \in F_{1u}). \tag{19}$$

Suppose that $d_H(B_1^*, B_2^*) < \delta_3$. Then

$$\sum_{v \in V} |B_1^*(v) - B_2^*(v)|/n < \delta_3$$

and thus

$$|B_1^*(v) - B_2^*(v)| < n\delta_3 \quad (v \in V),$$

so we get

$$B_1^*(v) - n\delta_3 < B_2^*(v) < B_1^*(v) + n\delta_3 \quad (v \in V). \tag{20}$$

For any $v_0 \in F_{1u}$, we obtain

$$R_1(u, v_0) > B_1^*(v_0),$$

and it follows from (19), (20) that

$$\begin{aligned}
B_2^*(v_0) &< B_1^*(v_0) + n\delta_3 \\
&\leq B_1^*(v_0) + \frac{n[R_1(u, v_0) - B_1^*(v_0)]}{2n + 1} \\
&= \frac{1}{2n + 1}[(n + 1)B_1^*(v_0) + nR_1(u, v_0)] \\
&< \frac{1}{2n + 1}[(n + 1)R_1(u, v_0) + nR_1(u, v_0)] \\
&= R_1(u, v_0).
\end{aligned}$$

Thus $v_0 \in F_{2u}$, hence we get $F_{1u} \subset F_{2u}$.

Take

$$\delta_4 = \min_{v \in F_{2u}} [R_1(u,v) - B_2^*(v)]/(2n+1).$$

Similarly we can obtain $F_{2u} \subset F_{1u}$ if $d_H(B_1^*, B_2^*) < \delta_4$.

Choose

$$\delta_0 = \min\{\delta_3, \delta_4\},$$

thus we can get that if $d_H(B_1^*, B_2^*) < \delta_0$ then $F_{1u} = F_{2u}$. □

Theorem 4.4. If $\rightarrow_2 \in \{I_0, I_G\}$, then the entropy-based differently implicational algorithm for FMT is uniformly continuous in $d \in \{d_T, d_H\}$, and thus continuous in $d \in \{d_T, d_H\}$.

Proof. Here we only prove the case of I_0 while the case of I_G can be obtained similarly.

(i) We get from Proposition 4.3 that there exists $\delta_0 > 0$ such that $F_{1u} = F_{2u}$ if $d_T(B_1^*, B_2^*) < \delta_0$ $(u \in U)$. For any $\varepsilon > 0$, take

$$\delta = \min\{\delta_0, \varepsilon\}.$$

Suppose that $d_T(B_1^*, B_2^*) < \delta$. Then

$$F_{1u} = F_{2u} \quad (u \in U),$$

and

$$|B_1^*(v) - B_2^*(v)| < \delta \quad (v \in V).$$

Thus from Example 2.22(ii), Lemma 2.10 and Lemma 2.11 one has

$$\begin{aligned}
&d_T(A_1^*, A_2^*)\\
=\ & \max_{u\in U} \Big| A_1^*(u) - A_2^*(u) \Big| \\
=\ & \max_{u\in U} \Big| 0.5 \wedge \inf_{v\in F_{1u}} \{(1 - R_1(u,v)) \vee B_1^*(v)\} - 0.5 \wedge \inf_{v\in F_{2u}} \{(1 - R_1(u,v)) \vee B_2^*(v)\} \Big| \\
=\ & \max_{u\in U} \Big| 0.5 \wedge \inf_{v\in F_{1u}} \{(1 - R_1(u,v)) \vee B_1^*(v)\} - 0.5 \wedge \inf_{v\in F_{1u}} \{(1 - R_1(u,v)) \vee B_2^*(v)\} \Big| \\
\leq\ & \max_{u\in U} \Big| \inf_{v\in F_{1u}} \{(1 - R_1(u,v)) \vee B_1^*(v)\} - \inf_{v\in F_{1u}} \{(1 - R_1(u,v)) \vee B_2^*(v)\} \Big| \\
\leq\ & \max_{u\in U} \sup_{v\in F_{1u}} \Big| [(1 - R_1(u,v)) \vee B_1^*(v)] - [(1 - R_1(u,v)) \vee B_2^*(v)] \Big| \\
\leq\ & \max_{u\in U} \sup_{v\in F_{1u}} \Big| B_1^*(v) - B_2^*(v) \Big| \\
<\ & \max_{u\in U} \sup_{v\in F_{1u}} \delta \\
=\ & \delta \\
\leq\ & \varepsilon.
\end{aligned}$$

As a result, there exists $\delta > 0$ such that $d_T(A_1^*, A_2^*) < \varepsilon$ if $d_T(B_1^*, B_2^*) < \delta$. In the sequel the entropy-based differently implicational algorithm for FMT is uniformly continuous in d_T. And thus it is also continuous in d_T.

(ii) It follows from Proposition 4.3 that there exists $\delta_0 > 0$ such that $F_{1u} = F_{2u}$ if $d_H(B_1^*, B_2^*) < \delta_0$ $(u \in U)$. For any $\varepsilon > 0$, choose

$$\delta = \min\{\delta_0/(n+1),\ \varepsilon/(n+1)\}.$$

Suppose that $d_H(B_1^*, B_2^*) < \delta$. Then

$$F_{1u} = F_{2u} \quad (u \in U),$$

and

$$\sum_{v \in V} |B_1^*(v) - B_2^*(v)|/n < \delta$$

and thus

$$|B_1^*(v) - B_2^*(v)| < n\delta \quad (v \in V).$$

Thus it follows from Example 2.22(ii), Lemma 2.10 and Lemma 2.11 that we get

$d_H(A_1^*, A_2^*)$

$$\begin{aligned}
&= \frac{1}{m}\sum_{u \in U}\left|0.5 \wedge \inf_{v \in F_{1u}}\{(1 - R_1(u,v)) \vee B_1^*(v)\} - 0.5 \wedge \inf_{v \in F_{2u}}\{(1 - R_1(u,v)) \vee B_2^*(v)\}\right| \\
&= \frac{1}{m}\sum_{u \in U}\left|0.5 \wedge \inf_{v \in F_{1u}}\{(1 - R_1(u,v)) \vee B_1^*(v)\} - 0.5 \wedge \inf_{v \in F_{1u}}\{(1 - R_1(u,v)) \vee B_2^*(v)\}\right| \\
&\le \frac{1}{m}\sum_{u \in U}\left|\inf_{v \in F_{1u}}\{(1 - R_1(u,v)) \vee B_1^*(v)\} - \inf_{v \in F_{1u}}\{(1 - R_1(u,v)) \vee B_2^*(v)\}\right| \\
&\le \frac{1}{m}\sum_{u \in U}\sup_{v \in F_{1u}}\left|[(1 - R_1(u,v)) \vee B_1^*(v)] - [(1 - R_1(u,v)) \vee B_2^*(v)]\right| \\
&\le \frac{1}{m}\sum_{u \in U}\sup_{v \in F_{1u}}\left|B_1^*(v) - B_2^*(v)\right| \\
&< \frac{1}{m}\sum_{u \in U}\sup_{v \in F_{1u}} n\delta \\
&\le n\varepsilon/(n+1) \\
&< \varepsilon
\end{aligned}$$

Therefore, the entropy-based differently implicational algorithm for FMT is uniformly continuous in d_H. And thus it is also continuous in d_H. □

5. EXAMPLES

The entropy-based differently implicational algorithm was proposed in [26], which included its solving process and analysis of reversibility property. However, in [26], we did not show any specific computing example of the entropy-based differently implicational algorithm. As a result, in order to help reader to understand this algorithm in a deeper level, we add some examples here. Besides, these examples also verifies that the entropy-based differently implicational algorithm is better than the fuzzy entropy full implication algorithm (see what follows).

Example 5.1. Suppose $U = \{u_1, u_2, \cdots, u_7\}$, $V = \{v_1, v_2, \cdots, v_6\}$, and

$$A = \frac{0.3}{u_1} + \frac{0.9}{u_2} + \frac{0.3}{u_3} + \frac{0.5}{u_4} + \frac{0.8}{u_5} + \frac{0}{u_6} + \frac{0.6}{u_7},$$
$$B = \frac{0.6}{v_1} + \frac{0.8}{v_2} + \frac{0.4}{v_3} + \frac{0.7}{v_4} + \frac{0.2}{v_5} + \frac{0.85}{v_6},$$
$$A^* = \frac{0.2}{u_1} + \frac{1.0}{u_2} + \frac{0.6}{u_3} + \frac{0}{u_4} + \frac{0.75}{u_5} + \frac{0.3}{u_6} + \frac{0.5}{u_7}.$$

Here $\to_2$ takes I_{Go}, $\to_1$ takes I_G. Then it follows from Example 2.17 that the formal entropy-inference solution of FMP is as follows:

$$B_1^*(v) = 0.5 \vee \sup_{u \in U}\{A^*(u) \times (A(u) \to_1 B(v))\}.$$

Thus we obtain:

$$\begin{aligned} B_1^*(v_1) &= 0.5 \vee \sup_{u \in U}\{A^*(u) \times (A(u) \to_1 B(v_1))\}, \\ &= 0.5 \vee [A^*(u_1) \times (A(u_1) \to_1 B(v_1))] \vee [A^*(u_2) \times (A(u_2) \to_1 B(v_1))] \vee \\ &\quad \cdots \vee [A^*(u_7) \times (A(u_7) \to_1 B(v_1))], \\ &= 0.5 \vee [0.2 \times (0.3 \to_1 0.6)] \vee [1.0 \times (0.9 \to_1 0.6)] \vee [0.6 \times (0.3 \to_1 0.6)] \\ &\quad \vee [0 \times (0.5 \to_1 0.6)] \vee [0.75 \times (0.8 \to_1 0.6)] \vee [0.3 \times (0 \to_1 0.6)] \\ &\quad \vee [0.5 \times (0.6 \to_1 0.6)] \\ &= 0.5 \vee 0.2 \vee 0.6 \vee 0.6 \vee 0 \vee 0.45 \vee 0.3 \vee 0.5 = 0.6. \end{aligned}$$

Similarly, we obtain:

$$\begin{aligned} B_1^*(v_2) &= 0.5 \vee 0.2 \vee 0.8 \vee 0.6 \vee 0 \vee 0.75 \vee 0.3 \vee 0.5 = 0.8. \\ B_1^*(v_3) &= 0.5 \vee 0.2 \vee 0.4 \vee 0.6 \vee 0 \vee 0.3 \vee 0.3 \vee 0.2 = 0.6. \\ B_1^*(v_4) &= 0.5 \vee 0.2 \vee 0.7 \vee 0.6 \vee 0 \vee 0.525 \vee 0.3 \vee 0.5 = 0.7. \\ B_1^*(v_5) &= 0.5 \vee 0.04 \vee 0.2 \vee 0.12 \vee 0 \vee 0.15 \vee 0.3 \vee 0.1 = 0.5. \\ B_1^*(v_6) &= 0.5 \vee 0.2 \vee 0.85 \vee 0.6 \vee 0 \vee 0.75 \vee 0.3 \vee 0.5 = 0.85. \end{aligned}$$

Together we get that the formal entropy-inference solution of FMP is as follows:

$$B_1^* = \frac{0.6}{v_1} + \frac{0.8}{v_2} + \frac{0.6}{v_3} + \frac{0.7}{v_4} + \frac{0.5}{v_5} + \frac{0.85}{v_6}.$$

Example 5.2. Aiming at the same U, V, A, B, A^* as Example 5.1, $\to_2$ takes I_{Go}, and $\to_1$ employs I_{Go}. According to Example 2.17, we obtain the formal entropy-inference solution of FMP as follows:

$$\begin{aligned}
B_2^*(v_1) &= 0.5 \vee \sup_{u\in U}\{A^*(u) \times (A(u) \to_1 B(v_1))\}, \\
&= 0.5 \vee [A^*(u_1) \times (A(u_1) \to_1 B(v_1))] \vee [A^*(u_2) \times (A(u_2) \to_1 B(v_1))] \vee \\
&\quad \cdots \vee [A^*(u_7) \times (A(u_7) \to_1 B(v_1))], \\
&= 0.5 \vee [0.2 \times (0.3 \to_1 0.6)] \vee [1.0 \times (0.9 \to_1 0.6)] \vee [0.6 \times (0.3 \to_1 0.6)] \\
&\quad \vee [0 \times (0.5 \to_1 0.6)] \vee [0.75 \times (0.8 \to_1 0.6)] \vee [0.3 \times (0 \to_1 0.6)] \\
&\quad \vee [0.5 \times (0.6 \to_1 0.6)] \\
&= 0.5 \vee 0.2 \vee \frac{2}{3} \vee 0.6 \vee 0 \vee \frac{9}{16} \vee 0.3 \vee 0.5 = \frac{2}{3}.
\end{aligned}$$

Furthermore, we have:

$$\begin{aligned}
B_2^*(v_2) &= 0.5 \vee 0.2 \vee \frac{8}{9} \vee 0.6 \vee 0 \vee 0.75 \vee 0.3 \vee 0.5 = \frac{8}{9}. \\
B_2^*(v_3) &= 0.5 \vee 0.2 \vee \frac{4}{9} \vee 0.6 \vee 0 \vee \frac{3}{8} \vee 0.3 \vee \frac{1}{3} = 0.6. \\
B_2^*(v_4) &= 0.5 \vee 0.2 \vee \frac{7}{9} \vee 0.6 \vee 0 \vee \frac{21}{32} \vee 0.3 \vee 0.5 = \frac{7}{9}. \\
B_2^*(v_5) &= 0.5 \vee \frac{2}{15} \vee \frac{2}{9} \vee 0.4 \vee 0 \vee \frac{3}{16} \vee 0.3 \vee \frac{1}{6} = 0.5. \\
B_2^*(v_6) &= 0.5 \vee 0.2 \vee \frac{17}{18} \vee 0.6 \vee 0 \vee 0.75 \vee 0.3 \vee 0.5 = \frac{17}{18}.
\end{aligned}$$

Together we have

$$B_2^* = \frac{2/3}{v_1} + \frac{8/9}{v_2} + \frac{0.6}{v_3} + \frac{7/9}{v_4} + \frac{0.5}{v_5} + \frac{17/18}{v_6}.$$

Remark 5.3. In Example 5.2, $\to_1, \to_2$ employ the same fuzzy implication, then the entropy-based differently implicational algorithm degenerates into the fuzzy entropy full implication algorithm. From Example 5.1, Example 5.2, it is easy to find $B_1^* \leq_F B_2^*$, then it follows from Definition 2.12 that (noting that $B_1^*(v) \wedge B_2^*(v) \geq 0.5$, $v \in V$)

$$E(B_2^*) \leq E(B_1^*),$$

that is, the fuzzy entropy of B_2^* is smaller than the one of B_1^*. Therefore, according to the entropy-based differently implicational principle for FMP, the entropy-based differently implicational algorithm can get better solution than the fuzzy entropy full implication algorithm.

Example 5.4. Suppose $U = \{u_1, u_2, \cdots, u_6\}$, $V = \{v_1, v_2, \cdots, v_7\}$, and

$$A = \frac{0.9}{u_1} + \frac{0.3}{u_2} + \frac{0.95}{u_3} + \frac{0.5}{u_4} + \frac{0.85}{u_5} + \frac{0.6}{u_6},$$
$$B = \frac{0.6}{v_1} + \frac{0.8}{v_2} + \frac{0.85}{v_3} + \frac{0.5}{v_4} + \frac{0.4}{v_5} + \frac{0.55}{v_6} + \frac{0.3}{v_7},$$
$$B^* = \frac{0.75}{v_1} + \frac{0.2}{v_2} + \frac{0.3}{v_3} + \frac{0.9}{v_4} + \frac{0.05}{v_5} + \frac{0.1}{v_6} + \frac{0.4}{v_7}.$$

Here $\to_2$ takes I_L, $\to_1$ takes I_{Go}. Then we get from Example 2.22 that the formal entropy-inference solution of FMT is as follows:

$$A_1^*(u) = 0.5 \wedge \inf_{v \in F_u} \{1 - (A(u) \to_1 B(v)) + B^*(v)\},\ u \in U,$$

where $F_u = \{v \in V |\ A(u) \to_1 B(v) > B^*(v)\}$.

For u_1, we can get

$$F_{u_1} = \{v \in V |\ A(u_1) \to_1 B(v) > B^*(v)\} = \{v_2, v_3, v_5, v_6\}.$$

Then it follows that

$$\begin{aligned} A_1^*(u_1) &= 0.5 \wedge \inf_{v \in F_{u_1}} \{1 - (A(u_1) \to_1 B(v)) + B^*(v)\} \\ &= 0.5 \wedge [1 - (0.9 \to_1 0.8) + 0.2] \wedge [1 - (0.9 \to_1 0.85) + 0.3] \\ &\quad \wedge [1 - (0.9 \to_1 0.4) + 0.05] \wedge [1 - (0.9 \to_1 0.55) + 0.1] \\ &= 0.5 \wedge \frac{14}{45} \wedge \frac{16}{45} \wedge \frac{109}{180} \wedge \frac{22}{45} = \frac{14}{45}. \end{aligned}$$

Similarly, we can obtain

$$A_1^*(u_2) = 0.5 \wedge 0.75 \wedge 0.2 \wedge 0.3 \wedge 0.9 \wedge 0.05 \wedge 0.1 \wedge 0.4 = 0.05.$$
$$A_1^*(u_3) = 0.5 \wedge \frac{34}{95} \wedge \frac{77}{190} \wedge \frac{239}{380} \wedge \frac{99}{190} = \frac{34}{95}.$$
$$A_1^*(u_4) = 0.5 \wedge 0.75 \wedge 0.2 \wedge 0.3 \wedge 0.9 \wedge 0.25 \wedge 0.1 \wedge 0.8 = 0.1.$$
$$A_1^*(u_5) = 0.5 \wedge \frac{22}{85} \wedge 0.3 \wedge \frac{197}{340} \wedge \frac{77}{170} = \frac{22}{85}.$$
$$A_1^*(u_6) = 0.5 \wedge 0.75 \wedge 0.2 \wedge 0.3 \wedge \frac{23}{60} \wedge \frac{11}{60} \wedge 0.9 = \frac{11}{60}.$$

The formal entropy-inference solution of FMT is as follows:

$$A_1^* = \frac{14/45}{u_1} + \frac{0.05}{u_2} + \frac{34/95}{u_3} + \frac{0.1}{u_4} + \frac{22/85}{u_5} + \frac{11/60}{u_6}.$$

Example 5.5. Aiming at the same U, V, A, B, B^* as Example 5.4, $\to_2$ takes I_{Go}, and $\to_1$ employs I_{Go}. According to Example 2.22, we obtain the formal entropy-inference solution of FMT as follows:

$$A_2^*(u) = 0.5 \wedge \inf_{v \in F_u} \{B^*(v)/(A(u) \to_1 B(v))\},\ u \in U$$

where $F_u = \{v \in V|\ A(u) \to_1 B(v) > B^*(v)\}$.

For u_1, we can get

$$F_{u_1} = \{v \in V|\ A(u_1) \to_1 B(v) > B^*(v)\} = \{v_2, v_3, v_5, v_6\}.$$

Then

$$\begin{aligned} A_2^*(u_1) &= 0.5 \wedge \inf_{v \in F_{u_1}} \{B^*(v)/(A(u_1) \to_1 B(v))\} \\ &= 0.5 \wedge [0.2/(0.9 \to_1 0.8)] \wedge [0.3/(0.9 \to_1 0.85)] \\ &\quad \wedge [0.05/(0.9 \to_1 0.4)] \wedge [0.1/(0.9 \to_1 0.55)] \\ &= 0.5 \wedge \frac{9}{40} \wedge \frac{27}{85} \wedge \frac{9}{80} \wedge \frac{9}{55} = \frac{9}{80}. \end{aligned}$$

Similarly, we can obtain

$$\begin{aligned} A_2^*(u_2) &= 0.5 \wedge 0.75 \wedge 0.2 \wedge 0.3 \wedge 0.9 \wedge 0.05 \wedge 0.1 \wedge 0.4 = 0.05. \\ A_2^*(u_3) &= 0.5 \wedge \frac{19}{80} \wedge \frac{57}{170} \wedge \frac{19}{160} \wedge \frac{19}{110} = \frac{19}{160}. \\ A_2^*(u_4) &= 0.5 \wedge 0.75 \wedge 0.2 \wedge 0.3 \wedge 0.9 \wedge \frac{1}{16} \wedge 0.1 \wedge \frac{2}{3} = \frac{1}{16}. \\ A_2^*(u_5) &= 0.5 \wedge \frac{17}{80} \wedge 0.3 \wedge \frac{17}{160} \wedge \frac{17}{110} = \frac{17}{160}. \\ A_2^*(u_6) &= 0.5 \wedge 0.75 \wedge 0.2 \wedge 0.3 \wedge \frac{3}{40} \wedge \frac{6}{55} \wedge 0.8 = \frac{3}{40}. \end{aligned}$$

Together we get

$$A_2^* = \frac{9/80}{u_1} + \frac{0.05}{u_2} + \frac{19/160}{u_3} + \frac{1/16}{u_4} + \frac{17/160}{u_5} + \frac{3/40}{u_6}.$$

Remark 5.6. In Example 5.5, $\to_1, \to_2$ both take I_{Go}, then the entropy-based differently implicational algorithm degenerates into the fuzzy entropy full implication algorithm. It follows from Example 5.4 and Example 5.5 that $A_1^* \geq_F A_2^*$, then we get from Definition 2.12 that (noting that $A_1^*(u) \vee A_2^*(u) \leq 0.5$, $u \in U$)

$$E(A_2^*) \leq E(A_1^*).$$

As a result, in the light of the entropy-based differently implicational principle for FMT, the entropy-based differently implicational algorithm can obtain better solution than the fuzzy entropy full implication algorithm.

6. DISCUSSION

Here we should point out that the results obtained above can be generalized for the case of p rules and q inputs. We give the interpretation for the case of FMP (while the case of FMT can be similarly verified).

When there are p rules, (1) is changed into

$$\text{FMP: from } A_i \to B_i \ (i = 1, 2, \ldots, p) \text{ and input } A^*, \text{compute } B^*, \tag{21}$$

where $A_i, A^* \in F(U), B_i, B^* \in F(V)$ $(i = 1, 2, \ldots, p)$.

Similar to [19, 36], the inference relation of the ith rule can be considered as a fuzzy relation from U to V $(i = 1, \cdots, p)$, which is represented by $A_i(u) \to_1 B_i(v)$. And such p rules can be connected by "or" relation, so the whole rule should be

$$\rho_1(u, v) \triangleq \vee_{i=1}^{p} (A_i(u) \to_1 B_i(v)).$$

Given $A^* \in F(U)$, the inference conclusion $B^* \in F(V)$ can be achieved by the entropy-based differently implicational algorithm. Thus (4) should be transformed into:

$$\rho_1(u, v) \to_2 (A^*(u) \to_2 B^*(v)). \tag{22}$$

Obviously, the previous $R_1(u, v)$ for (1) is changed into $\rho_1(u, v)$ for (21). Then it is similar to Theorem 2.16, we can obtain the following theorem:

Theorem 6.1. Suppose that $\to_2$ is an R-implication, and that T is a left-continuous t-norm, and that $\to_2$ is a residual of T, then the formal entropy-inference solution for (22) of FMP can be computed as follows:

$$B^*(v) = 0.5 \vee \sup_{u \in U} \{T(A^*(u), \rho_1(u, v))\}, \ v \in V. \tag{23}$$

For (4) and (22), the proving process for continuity is same for $R_1(u, v)$ and $\rho_1(u, v)$, so the change from $R_1(u, v)$ to $\rho_1(u, v)$ does not influence the continuity. Then, it is obvious to know that all the results for continuity obtained above (for one rule) are also correct for the case of p rules.

Furthermore, for the case of p rules and q inputs, here we use $q = 2$ to explain the problem (noting that the conclusion is similar when q employs other value). Then (1) is changed into

$$\text{FMP: from } A_i, B_i \to C_i \ (i = 1, 2, \ldots, p) \text{ and input } A^*, B^*, \text{compute } C^*, \tag{24}$$

where $A_i, A^* \in F(U), B_i, B^* \in F(V), C_i, C^* \in F(W)$ $(i = 1, 2, \ldots, p)$.

Similarly the inference relation of $i-$th rule can be transformed into $(A_i(u) \wedge B_i(v)) \to_1 C_i(w)$, and we obtain the whole rule

$$\varrho_1(u, v, w) \triangleq \vee_{i=1}^{p} ((A_i(u) \wedge B_i(v)) \to_1 C_i(w)).$$

Thus (4) should be changed into:

$$\varrho_1(u, v, w) \to_2 ((A^*(u) \wedge B^*(v)) \to_2 C^*(w)). \tag{25}$$

Obviously, the previous $R_1(u, v)$ for (1) is changed into $\varrho_1(u, v, w)$ for (24). Then it is similar to Theorem 2.16, we can obtain the following theorem:

Theorem 6.2. Suppose that $\rightarrow_2$ is an R-implication, and that T is a left-continuous t-norm, and that I is a residual of T, then the formal entropy-inference solution for (25) of FMP can be computed as follows:

$$B^*(v) = 0.5 \vee \sup_{u \in U}\{T((A^*(u) \wedge B^*(v)),\ \varrho_1(u,v,w))\},\ v \in V. \tag{26}$$

Due to the input is changed, then Definition 3.1 is adjusted into the following Definition 6.3:

Definition 6.3. A fuzzy inference algorithm for FMP (24) is a mapping $f : F(U) \times F(V) \rightarrow F(W)$, i.e., there exists an output $C^* = f(A^*, B^*) \in F(W)$ for any input $A^* \in F(U), B^* \in F(V)$.

(i) For any $\varepsilon > 0$, if there exists $\delta > 0$ making $d(f(A_1^*, B_1^*), f(A_2^*, B_2^*)) < \varepsilon$ whenever $d(A_1^*, A_2^*) < \delta$ and $d(B_1^*, B_2^*) < \delta$ for any $A_1^*, A_2^* \in F(U), B_1^*, B_2^* \in F(V)$, then f is said to be uniformly continuous in metric d;

(ii) For any $\varepsilon > 0$, if there exists $\delta > 0$ making $d(f(A^*, B^*), f(A, B)) < \varepsilon$ whenever $d(A^*, A) < \delta$ and $d(B^*, B) < \delta$ for any $A^* \in F(U), B^* \in F(V)$, then f is said to be continuous at $A \in F(U), B \in F(V)$ in metric d.

Here we compare (4) with (25). The verifying process for continuity is same for $R_1(u,v)$ and $\varrho_1(u,v,w)$. Moreover, the operation $\wedge$ keeps the continuity, then the alternation from $A^*(u)$ to $A^*(u) \wedge B^*(v)$ does not affect the continuity. Consequently, it is evident to find that all the results for continuity mentioned above are also correct for the case of p rules and q inputs.

Finally, we discuss the duty of first fuzzy implication $\rightarrow_1$ and second fuzzy implication $\rightarrow_2$ in the entropy-based differently implicational algorithm. On the one hand, it is easy to find that the form of the solution of the entropy-based differently implicational algorithm is basically determined only if $\rightarrow_2$ employs a fuzzy implication, and thus $\rightarrow_2$ determines the inference mechanism (see Theorem 2.16, Theorem 6.1, Theorem 6.2). On the other hand, $\rightarrow_1$ frequently exists as the form of $R_1(u,v)$ (or $\rho_1(u,v)$, $\varrho_1(u,v,w)$), which reflects the effect of rule base. Moreover, $\rightarrow_2$ has leading status for the entropy-based differently implicational algorithm due to its effect on direction of inference.

To sum up, $\rightarrow_2$ and $\rightarrow_1$ respectively reflects the inference mechanism and effect of rule base. As a result, the mode which makes $\rightarrow_1, \rightarrow_2$ employ different fuzzy implications, indicates separating of the rule base and reasoning mechanism, which further shows the reasonability of the entropy-based differently implicational algorithm.

7. CONCLUSIONS

We previously proposed the entropy-based differently implicational algorithm of fuzzy inference, and obtained some preliminary results. Following this, here we investigated its continuity for the FMP and FMT problems. In detail, the continuous together with uniformly continuous properties of the entropy-based differently implicational algorithm are verified for six typical R-implications in the Tchebyshev metric and Hamming metric. Moreover, some numerical fuzzy inference examples are shown, which demonstrate that the entropy-based differently implicational algorithm can achieve better solution than the fuzzy entropy full implication algorithm. Lastly, it is pointed out that $\to_2$ and $\to_1$ respectively reflects the inference mechanism and effect of rule base. These works would accelerate the development of the fields of fuzzy inference, fuzzy logic, fuzzy controller as well as related applications. In the future, it is worth studying the entropy-based differently implicational algorithm and other fuzzy inference strategies in the frame of Granular Computing (see [24, 25, 26]).

In [38], we proposed the symmetric implicational method which was derived from

$$(A(u) \to_1 B(v)) \to_2 (A^*(u) \to_1 B^*(v)). \tag{27}$$

The main advantage of the symmetric implicational method is to consider the factors of both the logic system and the reasoning model, thus it is better than the differently implicational algorithm from this angle. However, from another viewpoint (e. g. fuzzy controllers), the differently implicational algorithm may have better performance than the symmetric implicational method. Then a question arise: Whether does the symmetric implicational method perform better than the differently implicational algorithm? A direct conclusion about this cannot be easily reached, and it is our next topic to be discovered.

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REFERENCES

[1] M. Baczyński and B. Jayaram: (S,N)- and R-implications: A state-of-the-art survey. Fuzzy Set Syst. *159* (2008), 1836–1859. DOI:10.1016/j.fss.2007.11.015

[2] M. Baczyński and B. Jayaram: Fuzzy implications (Studies in Fuzziness and Soft Computing, Vol. 231. Springer, Berlin Heidelberg 2008.

[3] B. B. Chaudhuria and A. Rosenfeldb: A modified Hausdorff distance between fuzzy sets. Inform. Sci. *118* (1999), 159–171. DOI:10.1016/s0020-0255(99)00037-7

[4] S. S. Dai, D. W. Pei, and S. M. Wang: Perturbation of fuzzy sets and fuzzy reasoning based on normalized Minkowski distances. Fuzzy Set Syst. *189* (2012), 63–73. DOI:10.1016/j.fss.2011.07.012

[5] S. S. Dai, D. W. Pei, and D. H. Guo: Robustness analysis of full implication inference method. Int. J. Approx. Reason. *54* (2013), 653–666. DOI:10.1016/j.ijar.2012.11.007

[6] F. Liu, W. G. Zhang, and J. H. Fu: A new method of obtaining the priority weights from an interval fuzzy preference relation. Inform. Sci. *185* (2012), 32–42. DOI:10.1016/j.ins.2011.09.019

[7] F. Liu, W. G. Zhang, and Z. X. Wang: A goal programming model for incomplete interval multiplicative preference relations and its application in group decision-making Eur. J. Oper. Res. *218* (2012), 747–754. DOI:10.1016/j.ejor.2011.11.042

[8] J. C. Fodor: Contrapositive symmetry of fuzzy implications. Fuzzy Set Syst. *69* (1995), 141–156. DOI:10.1016/0165-0114(94)00210-x

[9] J. Fodor and M. Roubens: Fuzzy Preference Modeling and Multicriteria Decision Support. Kluwer Academic Publishers, Dordrecht, 1994. DOI:10.1007/978-94-017-1648-2

[10] S. Gottwald: A Treatise on Many-Valued Logics. Research Studies, Studies in Logic and Computation *9*, Baldock 2001.

[11] F. F. Guo, T. Y. Chen, and Z. Q. Xia: Triple I methods for fuzzy reasoning based on maximum fuzzy entropy principle. Fuzzy Syst. Math. *17* (2003), 55–59.

[12] D. H. Hong and S. Y. Hwang: A note on the value similarity of fuzzy systems variable. Fuzzy Set Syst. *66* (1994), 383–386. DOI:10.1016/0165-0114(94)90107-4

[13] B. Jayaram: On the law of importation $(x \wedge y) \to z \equiv (x \to (y \to z))$ in fuzzy logic. IEEE Trans. Fuzzy Syst. *16* (2008), 130–144. DOI:10.1109/tfuzz.2007.895969

[14] E. T. Jaynes: Where do we stand on maximum entropy? In: The Maximum Entropy Formalism (R. .D. Levine and M. Tribus, eds.), MIT Press, Cambridge 1978, pp. 15–118.

[15] E. T. Jaynes: On the rationale of maximum-entropy methods. Proc. IEEE, *70* (1982), 939–952. DOI:10.1109/proc.1982.12425

[16] S. Jenei: Continuity on Zadeh's compositional rule of inference. Fuzzy Set Syst. *104* (1999), 333–339. DOI:10.1016/s0165-0114(97)00198-x

[17] E. P. Klement, R. Mesiar, and E. Pap: Triangular Norms. Kluwer Academic Publishers, Dordrecht 2000. DOI:10.1007/978-94-015-9540-7

[18] H. X. Li: Probability representations of fuzzy systems. Sci. China Ser. F-Inf. Sci. *49* (2006), 339–363. DOI:10.1007/s11432-006-0339-9

[19] H. Li and E. S. Lee: Interpolation representations of fuzzy logic systems. Comput. Math. Appl. *45* (2003), 1683–1693. DOI:10.1016/s0898-1221(03)00147-0

[20] M. X. Luo and B. Liu: Robustness of interval-valued fuzzy inference triple I algorithms based on normalized Minkowski distance. J. Log. Algebr. Methods *86* (2017), 298–307. DOI:10.1016/j.jlamp.2016.09.006

[21] M. Mas, M. Monserrat, J. Torrens, and E. Trillas: A survey on fuzzy implication functions. IEEE Trans. Fuzzy Syst. *15* (2007), 1107–1121. DOI:10.1109/tfuzz.2007.896304

[22] V. Novák, I. Perfilieva, and J. Močkoř: Mathematical Principles of Fuzzy Logic. Kluwer Academic Publishes, Boston, Dordrecht 1999. DOI:10.1007/978-1-4615-5217-8

[23] L. M. Pang, K. M. Tay, and C. P. Lim: Monotone fuzzy rule relabeling for the zero-order TSK fuzzy inference system. IEEE Trans. Fuzzy Syst. *24* (2016), 1455–1463. DOI:10.1109/tfuzz.2016.2540059

[24] W. Pedrycz: Granular Computing: Analysis and Design of Intelligent Systems. CRC Press/Francis and Taylor, Boca Raton 2013.

[25] W. Pedrycz: From fuzzy data analysis and fuzzy regression to granular fuzzy data analysis. Fuzzy Set Syst. *274* (2015), 12–17. DOI:10.1201/9781315216737

[26] W. Pedrycz and X. M. Wang: Designing fuzzy sets with the use of the parametric principle of justifiable granularity. IEEE Trans. Fuzzy Syst. *24* (2016), 489–496. DOI:10.1109/tfuzz.2015.2453393

[27] D. W. Pei: R_0 implication: characteristics and applications. Fuzzy Set Syst. *131* (2002), 297–302. DOI:10.1016/s0165-0114(02)00053-2

[28] D. W. Pei: Unified full implication algorithms of fuzzy reasoning. Inform. Sci. *178* (2008), 520–530. DOI:10.1016/j.ins.2007.09.003

[29] A. Rosenfeld: Distances between fuzzy sets. Pattern Recogn. Lett. *3* (1985), 229–233. DOI:10.1016/0167-8655(85)90002-9

[30] P. Sarkoci and M. Šabo: Information boundedness principle in fuzzy inference process. Kybernetika *38* (2002), 327–338.

[31] E. Szmidt and J. Kacprzyk: Entropy for intuitionistic fuzzy sets. Fuzzy Set Syst. *118* (2001), 467–477. DOI:10.1016/s0165-0114(98)00402-3

[32] Y. M. Tang: Differently implicational hierarchical inference algorithm under interval-valued fuzzy environment. In: Proc. 2015 IEEE International Conference on Fuzzy Systems (FUZZ-IEEE 2015), Istanbul, pp. 1–8. DOI:10.1109/fuzz-ieee.2015.7337944

[33] Y. M. Tang and X. P. Liu: Differently implicational universal triple I method of (1, 2, 2) type. Comput. Math. Appl. *59* (2010), 1965–1984. DOI:10.1016/j.camwa.2009.11.016

[34] Y. M. Tang and F. J. Ren: Universal triple I method for fuzzy reasoning and fuzzy controller. Iran. J. Fuzzy Syst. *10* (2013), 1–24. DOI:10.1109/fskd.2013.6816179

[35] Y. M. Tang and F. J. Ren: Variable differently implicational algorithm of fuzzy inference. J. Intell. Fuzzy Syst. *28* (2015), 1885–1897.

[36] Y. M. Tang and F. J. Ren: Fuzzy systems based on universal triple I method and their response functions. Int. J. Inf. Tech. Decis. *16* (2017), 443–471. DOI:10.1142/s0219622014500746

[37] Y. M. Tang, F. J. Ren, and Y. X. Chen: Differently implicational α-universal triple I restriction method of (1, 2, 2) type. J. Syst. Eng. Electron. *23* (2012), 560–573. DOI:10.1109/jsee.2012.00070

[38] Y. M. Tang and X. Z. Yang: Symmetric implicational method of fuzzy reasoning. Int. J. Approx. Reason. *54* (2013), 1034–1048. DOI:10.1016/j.ijar.2013.04.012

[39] Y. M. Tang, X. Z. Yang, and F. Yue: Universal triple I method with maximum fuzzy entropy employing R-implications. In: Proc. the 10th International Conference on Fuzzy Systems and Knowledge Discovery (FSKD 2013), pp. 125–129. DOI:10.1109/fskd.2013.6816179

[40] L. X. Wang: A Course in Fuzzy Systems and Control. Prentice-Hall, Englewood Cliffs, NJ, 1997.

[41] G. J. Wang: On the logic foundation of fuzzy reasoning. Inform. Sci. *117* (1999), 47–88. DOI:10.1016/s0020-0255(98)10103-2

[42] G. J. Wang and L. Fu: Unified forms of triple I method. Comput. Math. Appl. *49* (2005), 923–932. DOI:10.1016/j.camwa.2004.01.019

[43] G. J. Wang and H. J. Zhou: Introduction to Mathematical Logic and Resolution Principle. Co-published by Science Press and Alpha International Science Ltd., 2009.

[44] X. Y. Yang, F. S. Yu, and W. Pedrycz: Long-term forecasting of time series based on linear fuzzy information granules and fuzzy inference system. Int. J. Approx. Reasoning *81* (2017), 1–27. DOI:10.1016/j.ijar.2016.10.010

[45] L. A. Zadeh: Outline of a new approach to the analysis of complex systems and decision processes. IEEE Trans. Syst. Man Cyber. *3* (1973), 28–44. DOI:10.1109/tsmc.1973.5408575

Yiming Tang, School of Computer and Information, Hefei University of Technology, Hefei, 230009, P.R.China; and Department of Electrical and Computer Engineering, University of Alberta, Edmonton, AB T6R 2V4. Canada.
e-mail: tym608@163.com

Witold Pedrycz, Department of Electrical and Computer Engineering, University of Alberta, Edmonton, AB T6R 2V4, Canada; and Department of Electrical and Computer Engineering, Faculty of Engineering, King Abdulaziz University, Jeddah, 21589, Saudi Arabia; and Systems Research Institute, Polish Academy of Sciences, Warsaw 01-447. Poland.
e-mail: wpedrycz@ualberta.ca