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SOME METHODS TO OBTAIN T-NORMS AND T-CONORMS ON BOUNDED LATTICES

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In this study, we introduce new methods for constructing t-norms and t-conorms on a bounded lattice L based on a priori given t-norm acting on $[a,1]$ and t-conorm acting on $[0,a]$ for an arbitrary element $a \in L\backslash\{0,1\}$. We provide an illustrative example to show that our construction methods differ from the known approaches and investigate the relationship between them. Furthermore, these methods are generalized by iteration to an ordinal sum construction for t-norms and t-conorms on a bounded lattice.

Keywords: bounded lattice, t-norm, t-conorm, ordinal sum

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1. INTRODUCTION

The concepts of triangular norms (briefly t-norms) with 1 as the neutral element and triangular conorms (briefly t-conorms) with 0 as the neutral element were introduced by Schweizer and Sklar [26] in the framework of probabilistic metric spaces and extensively studied by Klement, Mesiar and Pap [23]. Following the definition of t-norms and t-conorms on the real unit interval $[0,1]$, these operators were studied on some more general structures [18], for example, a bounded lattice, and investigated in topology [19, 21] and logic [16, 22]. They are extensively used in many applications in fuzzy set theory, fuzzy logic, multicriteria decision support and several branches of information sciences [5, 20]. Therefore, the knowledge of the structure of t-norms and t-conorms is very important from the theoretical point of view.

One of the construction methods for t-norms and t-conorms on the real unit interval $[0,1]$ is the ordinal sum construction based on ordinal sums of lattices of Birkhoff [4] and ordinal sums of semigroups [6]. There are many initiatives for this construction method extended for bounded lattices based on Goguen's proposal [18] that considers fuzzy sets with membership values from bounded lattices. In [13, 14], t-norms and t-conorms defined as operations on a bounded poset or lattice were investigated. In the meantime, a class of t-norms and t-conorms on any bounded lattice was generated by use of interior operators and closure operators, correspondingly. Some constructions were presented with lattice-valued t-norms and t-conorms which generalize most of the

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developed techniques. In [25], an ordinal sum construction of t-norms and t-conorms was introduced on some bounded lattices which are not necessarily a chain or an ordinal sum of posets. In this construction, some necessary and sufficient conditions are presented for the ordinal sum on a bounded lattice of arbitrary t-norms (t-conorms) to yield a t-norm (t-conorm). The other investigations of ordinal sum constructions for obtaining t-norms and t-conorms on bounded lattices can be found in [24], where also some additional conditions are required in order to ensure that an ordinal sum of arbitrary t-norms (t-conorms) is a t-norm (t-conorm). Uninorm-like operations as a generalization of t-norms and t-conorms with the underlying operations given by ordinal sums were studied in [15]. In [17], it was proposed a modification of ordinal sums of t-norms (t-conorms) resulting to a t-norm (t-conorm) valid on an arbitrary bounded lattice. In [10], considering a bounded lattice L, based on a priori given t-norm and t-conorm on a subinterval of L, some new construction methods for t-norms and t-conorms on L were introduced. In the meantime, a complete generalization of the ordinal sums of t-norms and t-conorms was demonstrated on an arbitrary bounded lattice.

Following the demonstration of the above constructions for t-norms and t-conorms valid on bounded lattices, the other researches were promoting this area. The main aim of this study is to introduce the further methods to generate t-norms and t-conorms on bounded lattices different from the known methods. For this purpose, for an arbitrary element $a \in L\backslash\{0,1\}$, based on the existence a t-norm V on the subinterval $[a,1]$ and a t-conorm W on the subinterval $[0,a]$, we present new construction methods for t-norms and t-conorms on bounded lattices. The present paper is organized as follows. In Section 2, after the known constructions and their resulting elements are briefly discussed, we propose new methods for generating t-norms and t-conorms on a bounded lattice L. In addition, we provide some results and illustrative examples to show the relationships between our methods and the existing approaches. Our constructions exploit a t-norm V acting on a subintreval $[a,1]$ and a t-conorm W acting on a subinterval $[0,a]$ for an arbitrary element $a \in L\backslash\{0,1\}$. In Section 3, we present new classes of t-norms and t-conorms constructed by iteration on bounded lattices. Finally, some concluding remarks are added.

2. CONSTRUCTIONS OF T-NORMS AND T-CONORMS

First, we provide a brief examination concerning bounded lattices and t-norms and t-conorms on them.

Definition 2.1. (Birkhoff [4]) A lattice $(L,\leq)$ is called bounded if it has the top and bottom elements, which are written as 1 and 0, respectively, that is, there exist two elements $1,0 \in L$ such that $0 \leq x \leq 1$, for all $x \in L$.

Definition 2.2. (Birkhoff [4]) Let $(L,\leq,0,1)$ be a bounded lattice and $a,b \in L$. If a and b are incomparable, we use the notation $a \parallel b$. We denote the set of all incomparable elements with a by I_a, that is, $I_a = \{x \in L \mid x \parallel a\}$.

Definition 2.3. (Birkhoff [4]) Let $(L,\leq,0,1)$ be a bounded lattice and $a,b \in L$ such that $a \leq b$. Then a subinterval $[a,b]$ of L is defined as follows:

$$[a,b] = \{x \in L \mid a \leq x \leq b\}.$$

Similarly, it is defined $]a,b] = \{x \in L \mid a < x \leq b\}$, $[a,b[= \{x \in L \mid a \leq x < b\}$ and $]a,b[= \{x \in L \mid a < x < b\}$.

Definition 2.4. (Çaylı et al. [7], Çaylı [9, 11]) Let $(L, \leq, 0, 1)$ be a bounded lattice. An operation $T : L^2 \to L$ is called a t-norm on L if it is commutative, associative, increasing with respect to both variables and has the neutral element 1 such that $T(x,1) = x$, for all $x \in L$.

Definition 2.5. (Aşıcı and Karaçal [1], Aşıcı [2, 3], Çaylı and Karaçal [8]) Let $(L, \leq, 0, 1)$ be a bounded lattice. An operation $S : L^2 \to L$ is called a t-conorm on L if it is commutative, associative, increasing with respect to both variables and has the neutral element 0 such that $S(x,0) = x$, for all $x \in L$.

Consider the lattice $L^* = \left\{(x_1,x_2) \mid (x_1,x_2) \in [0,1]^2 \text{ and } x_1 + x_2 \leq 1\right\}$ with the following order:

$$(x_1,x_2) \leq_{L^*} (y_1,y_2) \Leftrightarrow x_1 \leq y_1 \text{ and } x_2 \geq y_2$$

$(L^*, \leq_{L^*})$ is a complete lattice. Its bottom and top elements are denoted by $0_{L^*} = (0,1)$ and $1_{L^*} = (1,0)$, respectively.
Given the elements $(x_1,x_2), (y_1,y_2) \in L^*$, if they are incomparable, we use the notation $(x_1,x_2) \parallel_{L^*} (y_1,y_2)$.

Definition 2.6. (Deschrijver and Kerre [12]) An operation $T : (L^*)^2 \to L$ is called a t-norm on L^* if it is commutative, associative, increasing with respect to both variables and satisfies $T(1_{L^*}, x) = x$, for all $x \in L^*$.

Definition 2.7. (Deschrijver and Kerre [12]) An operation $S : (L^*)^2 \to L$ is called a t-conorm on L^* if it is commutative, associative, increasing with respect to both variables and satisfies $S(0_{L^*}, x) = x$, for all $x \in L^*$.

Definition 2.8. (Drossos and Navara [13], Drossos [14]) Let $(L, \leq, 0, 1)$ be a bounded lattice. A mapping $cl : L \to L$ is called a closure operator on L if it satisfies the following properties:

i) $x \leq cl(x)$

ii) $cl(x) = cl(cl(x))$

iii) $cl(x \vee y) = cl(x) \vee cl(y)$

for all $x, y \in L$.

Definition 2.9. (Drossos and Navara [13], Drossos [14]) Let $(L, \leq, 0, 1)$ be a bounded lattice. A mapping $int : L \to L$ is called an interior operator on L if it satisfies the following properties:

i) $int(x) \leq x$

ii) $int(x) = int(int(x))$

iii) $int\,(x \wedge y) = int\,(x) \wedge int\,(y)$

for all $x, y \in L$.

Based on the constructions of t-norms and t-conorms on the unit interval $[0,1]$, the study of the constructions of t-norms and t-conorms defined on bounded lattices has recently become important. In the papers [13, 14, 10, 17, 24, 25], t-norms and t-conorms defined on bounded lattices were investigated and several methods for constructing these operators were introduced. Furthermore, by using the existence of a priori given t-norm and t-conorm, some ordinal sum constructions for t-norms and t-conorms on bounded lattices were presented in [10, 17, 24, 25]. However, in [24, 25], several necessary and sufficient conditions are required in order to ensure that an ordinal sum of any t-norms (t-conorm) is a t-norm (t-conorm). We aim to enhance these constructions by introducing the different construction methods for t-norms and t-conorms valid on any bounded lattice. For this purpose, for an arbitrary element $a \in L\backslash\{0,1\}$, based on the existence of a t-norm V acting on the subinterval $[a,1]$ and a t-conorm W acting on the subinterval $[0,a]$, we introduce new construction methods for obtaining t-norms and t-conorms on a bounded lattice L. Note that our constructions have different characteristics compared with the construction methods described above, and they are presented in Theorems 2.15 and 2.21.

Now, let us recall the construction methods presented in [13, 14, 10, 17, 25] in Theorems 2.10, 2.11, 2.13, 2.14.

A large class of lattice-valued t-norms and t-conorms can be described by means of interior operators and closure operators, respectively. In the following Theorem 2.10, it is proposed a method for generating t-norms and t-conorms, applicable on any bounded lattice based on interior operators and closure operators.

Theorem 2.10. (Drossos and Navara [13], Drossos [14]) Let $(L, \leq, 0, 1)$ be a bounded lattice, $cl : L \to L$ be a closure operator on L and $int : L \to L$ be an interior operator on L. Then the functions $T^{(1)} : L^2 \to L$ and $S^{(1)} : L^2 \to L$ are, respectively, a t-norm and a t-conorm on L, where

$$T^{(1)}\,(x,y) = \begin{cases} x \wedge y & \text{if } 1 \in \{x,y\}\,, \\ int\,(x) \wedge int\,(y) & \text{otherwise} \end{cases} \tag{1}$$

and

$$S^{(1)}\,(x,y) = \begin{cases} x \vee y & \text{if } 0 \in \{x,y\}\,, \\ cl\,(x) \vee cl\,(y) & \text{otherwise.} \end{cases} \tag{2}$$

In the following Theorem 2.11, an ordinal sum construction of t-norms and t-conorms is described on a bounded lattice, where several necessary and sufficient conditions are required for ensuring whether an ordinal sum on a bounded lattice of arbitrary t-norms (t-conorms) is, in fact, a t-norm (t-conorm).

Theorem 2.11. (Saminger [25]) Let $(L, \leq, 0, 1)$ be a bounded lattice, $a \in L\backslash\{0,1\}$, $V : [a,1]^2 \to [a,1]$ be a t-norm and $W : [0,a]^2 \to [0,a]$ be a t-conorm. Ordinal sum extensions $T^{(2)}$ of V to L and $S^{(2)}$ of W to L are given by

$$T^{(2)}\,(x,y) = \begin{cases} V\,(x,y) & \text{if } (x,y) \in [a,1]^2\,, \\ x \wedge y & \text{otherwise} \end{cases} \tag{3}$$

and

$$S^{(2)}(x,y)=\begin{cases} W(x,y) & \text{if } (x,y)\in[0,a]^2, \\ x\vee y & \text{otherwise.}\end{cases} \tag{4}$$

However, the above-defined function $T^{(2)}$ needs not be a t-norm on L, in general. Similarly, $S^{(2)}$ needs not be a t-conorm on L, in general.

Example 2.12. Consider the lattice $(L^*,\leq_{L^*})$ and $a=(0.75,0.25)$. Define the t- norm $V:[a,1]^2\to[a,1]$ by

$$V((x_1,x_2),(y_1,y_2))=(4x_1y_1-3x_1-3y_1+3,x_2+y_2-4x_2y_2)$$

If we construct the function $T:(L^*)^2\to L^*$ by using the formula (3) in Theorem 2.11, in that case T is not monotone on L^*, since $T((0.8,0.2),(0.8,0.2)) = (0.76,0.24) \parallel_{L^*} (0.6,\ 0.2)=T((0.8,0.2),(0.6,0.2))$ while $(0.6,0.2)\leq_{L^*}(0.8,0.2)$.
In addition, T is not associative on L^*, since

$$T((0.6,0.2)\,T((0.8,0.2),(0.8,0.2)))=T((0.6,0.2),(0,76,0.24))=(0.6,0.24)$$

and

$$T(T((0.6,0.2)\,(0.8,0.2)),(0.8,0.2))=T((0.6,\ 0.2),(0.8,0.2))=(0.6,0.2)$$

Therefore, the function $T:(L^*)^2\to L^*$ is not a t-norm on L^*.

In order to avoid this problem, some modified versions of the above mentioned ordinal sum construction were proposed in [10, 17]. Considering an arbitrary bounded lattice L, for any element $a\in L\backslash\{0,1\}$, based on a t-norm V acting on the subinterval $[a,1]$, the constructions given by the formulas (5) and (7), respectively, in Theorems 2.13 and 2.14 yield a t-norm on L. Similarly, based on a t-norm W acting on the subinterval $[0,a]$, the constructions given by the formulas (6) and (8), respectively, in Theorems 2.13 and 2.14 yield a t-conorm on L.

Theorem 2.13. (Çaylı [9, 10]) Let $(L,\leq,0,1)$ be a bounded lattice and $a\in L\backslash\{0,1\}$. If V is a t-norm on $[a,1]$ and W is a t-conorm on $[0,a]$, then the functions $T^{(3)}:L^2\to L$ and $S^{(3)}:L^2\to L$ are, respectively, a t-norm and a t-conorm on L, where

$$T^{(3)}(x,y)=\begin{cases} V(x,y) & \text{if } (x,y)\in[a,1[^2, \\ x\wedge y & \text{if } 1\in\{x,y\}, \\ 0 & \text{otherwise}\end{cases} \tag{5}$$

and

$$S^{(3)}(x,y)=\begin{cases} W(x,y) & \text{if } (x,y)\in]0,a]^2, \\ x\vee y & \text{if } 0\in\{x,y\}, \\ 1 & \text{otherwise.}\end{cases} \tag{6}$$

Theorem 2.14. (Ertuğrul et al. [17]) Let $(L, \leq, 0, 1)$ be a bounded lattice and $a \in L\backslash\{0,1\}$. If V is a t-norm on $[a,1]$ and W is a t-conorm on $[0,a]$, then the functions $T^{(4)} : L^2 \to L$ and $S^{(4)} : L^2 \to L$ are, respectively, a t-norm and a t-conorm on L, where

$$T^{(4)}(x,y) = \begin{cases} V(x,y) & \text{if } (x,y) \in [a,1[^2, \\ x \wedge y & \text{if } 1 \in \{x,y\}, \\ x \wedge y \wedge a & \text{otherwise} \end{cases} \tag{7}$$

and

$$S^{(4)}(x,y) = \begin{cases} W(x,y) & \text{if } (x,y) \in]0,a]^2, \\ x \vee y & \text{if } 0 \in \{x,y\}, \\ x \vee y \vee a & \text{otherwise.} \end{cases} \tag{8}$$

Now, in the following Theorem 2.15, considering any bounded lattice L, we introduce a construction method for generating t-norms on L by means of a t-norm V acting on $[a,1]$ for an element $a \in L\backslash\{0,1\}$.

Theorem 2.15. Let $(L, \leq, 0, 1)$ be a bounded lattice and $a \in L\backslash\{0,1\}$. If V is a t-norm on $[a,1]$, then the function $T^{(5)} : L^2 \to L$ is a t-norm on L, where

$$T^{(5)}(x,y) = \begin{cases} V(x,y) & \text{if } (x,y) \in [a,1[^2, \\ 0 & \text{if } (x,y) \in [0,a[^2 \cup [0,a[\times I_a \cup I_a \times [0,a[\cup I_a \times I_a, \\ x \wedge y & \text{if } 1 \in \{x,y\}, \\ x \wedge y \wedge a & \text{otherwise.} \end{cases} \tag{9}$$

Proof. We have $T^{(5)}(x,1) = x \wedge 1 = x$, for all $x \in L$. So, $1 \in L$ is a neutral element of T. It is easy to see commutativity of $T^{(5)}$.

i) Monotonicity: We prove that if $x \leq y$, then $T^{(5)}(x,z) \leq T^{(5)}(y,z)$, for all $z \in L$. The proof is split into all possible cases.

If $z = 1$, then we have $T^{(5)}(x,z) = T^{(5)}(x,1) = x \leq y = T^{(5)}(y,1) = T^{(5)}(y,z)$.

1. Let $x < a$.

1.1. $y < a$,

1.1.1. $z < a$ and $z \in I_a$,

$$T^{(5)}(x,z) = 0 = T^{(5)}(y,z)$$

1.1.2. $1 > z \geq a$,

$$T^{(5)}(x,z) = x \wedge z \wedge a \leq y \wedge z \wedge a = T^{(5)}(y,z)$$

1.2. $1 > y \geq a$,

1.2.1. $z < a$ or $z \in I_a$,

$$T^{(5)}(x,z) = 0 \leq y \wedge z \wedge a = T^{(5)}(y,z)$$

1.2.2. $1 > z \geq a$,

$$T^{(5)}(x,z) = x \wedge z \wedge a \leq V(y,z) = T^{(5)}(y,z)$$

1.3. $y \in I_a$,

1.3.1. $z < a$ or $z \in I_a$,

$$T^{(5)}(x,z) = 0 = T^{(5)}(y,z)$$

1.3.2. $1 > z \geq a$,

$$T^{(5)}(x,z) = x \wedge z \wedge a \leq y \wedge z \wedge a = T^{(5)}(y,z)$$

1.4. $y = 1$,

1.4.1. $z < a$ or $z \in I_a$,

$$T^{(5)}(x,z) = 0 \leq z = T^{(5)}(y,z)$$

1.4.2. $1 > z \geq a$,

$$T^{(5)}(x,z) = x \wedge z \wedge a \leq z = T^{(5)}(y,z)$$

2. Let $1 > x \geq a$.

2.1. $1 > y \geq a$,

2.1.1. $z < a$ or $z \in I_a$,

$$T^{(5)}(x,z) = x \wedge z \wedge a \leq y \wedge z \wedge a = T^{(5)}(y,z)$$

2.1.2. $1 > z \geq a$,

$$T^{(5)}(x,z) = V(x,z) \leq V(y,z) = T^{(5)}(y,z)$$

2.2. $y = 1$,

2.2.1. $z < a$ or $z \in I_a$,

$$T^{(5)}(x,z) = x \wedge z \wedge a \leq z = T^{(5)}(y,z)$$

2.2.2. $1 > z \geq a$,

$$T^{(5)}(x,z) = V(x,z) \leq V(1,z) = z = T^{(5)}(y,z)$$

3. Let $x \in I_a$.

3.1. $1 > y \geq a$,

3.1.1. $z < a$ or $z \in I_a$,

$$T^{(5)}(x,z) = 0 \leq y \wedge z \wedge a = T^{(5)}(y,z)$$

3.1.2. $1 > z \geq a$,

$$T^{(5)}(x,z) = x \wedge z \wedge a \leq V(y,z) = T^{(5)}(y,z)$$

3.2. $y \in I_a$,

3.2.1. $z < a$ or $z \in I_a$,

$$T^{(5)}(x,z) = 0 = T^{(5)}(y,z)$$

3.2.2. $1 > z \geq a$,

$$T^{(5)}(x,z) = x \wedge z \wedge a \leq y \wedge z \wedge a = T^{(5)}(y,z)$$

3.3. $y = 1$,

3.3.1. $z < a$ or $z \in I_a$,

$$T^{(5)}(x,z) = 0 \leq z = T^{(5)}(y,z)$$

3.3.2. $1 > z \geq a$,

$$T^{(5)}(x,z) = x \wedge z \wedge a \leq z = T^{(5)}(y,z)$$

ii) Associativity: We demonstrate that $T^{(5)}(x,T^{(5)}(y,z)) = T^{(5)}(T^{(5)}(x,y),z)$ for all $x,y,z \in L$. Again the proof is split into all possible cases considering the relationships of the elements x,y,z and a.

If at least one of the elements x,y,z is equal to 1, then $T^{(5)}$ is associative. Taking into account this fact, it is enough to check only those cases in which the elements x,y,z are not equal to 1.

1. Let $x < a$,

1.1. $y < a$,

1.1.1. $z < a$ or $z \in I_a$,

$$\begin{aligned} T^{(5)}(x,T^{(5)}(y,z)) &= T^{(5)}(x,0) = 0 \\ &= T^{(5)}(0,z) \\ &= T^{(5)}(T^{(5)}(x,y),z) \end{aligned}$$

1.1.2. $1 > z \geq a$,

$$\begin{aligned} T^{(5)}(x,T^{(5)}(y,z)) &= T^{(5)}(x, y \wedge z \wedge a) = 0 \\ &= 0 \wedge z \wedge a \\ &= T^{(5)}(0,z) \\ &= T^{(5)}(T^{(5)}(x,y),z) \end{aligned}$$

1.2. $1 > y \geq a$,

1.2.1. $z < a$ or $z \in I_a$,

$$\begin{aligned} T^{(5)}(x,T^{(5)}(y,z)) &= T^{(5)}(x, y \wedge z \wedge a) = 0 \\ &= T^{(5)}(x \wedge y \wedge a, z) \\ &= T^{(5)}(T^{(5)}(x,y),z) \end{aligned}$$

1.2.2. $1 > z \geq a$,

$$\begin{aligned} T^{(5)}(x,T^{(5)}(y,z)) &= T^{(5)}(x, V(y,z)) = x \wedge V(y,z) \wedge a = x \\ &= x \wedge z \wedge a \\ &= T^{(5)}(x,z) \\ &= T^{(5)}(x \wedge y \wedge a, z) \\ &= T^{(5)}(T^{(5)}(x,y),z) \end{aligned}$$

1.3. $y \in I_a$,

1.3.1. $z < a$ or $z \in I_a$,

$$\begin{aligned} T^{(5)}(x, T^{(5)}(y,z)) &= T^{(5)}(x,0) = 0 \\ &= T^{(5)}(0,z) \\ &= T^{(5)}(T^{(5)}(x,y),z) \end{aligned}$$

1.3.2. $1 > z \geq a$,

$$\begin{aligned} T^{(5)}(x, T^{(5)}(y,z)) &= T^{(5)}(x, y \wedge z \wedge a) = 0 \\ &= 0 \wedge z \wedge a \\ &= T^{(5)}(0,z) \\ &= T^{(5)}(T^{(5)}(x,y),z) \end{aligned}$$

2. Let $1 > x \geq a$.

2.1. $y < a$,

2.1.1. $z < a$ or $z \in I_a$,

$$\begin{aligned} T^{(5)}(x, T^{(5)}(y,z)) &= T^{(5)}(x,0) = x \wedge 0 \wedge a \\ &= T^{(5)}(y,z) \\ &= T^{(5)}(x \wedge y \wedge a, z) \\ &= T^{(5)}(T^{(5)}(x,y),z) \end{aligned}$$

2.1.2. $1 > z \geq a$,

$$\begin{aligned} T^{(5)}(x, T^{(5)}(y,z)) &= T^{(5)}(x, y \wedge z \wedge a) = T^{(5)}(x,y) = x \wedge y \wedge a = y \\ &= y \wedge z \wedge a \\ &= T^{(5)}(y,z) \\ &= T^{(5)}(x \wedge y \wedge a, z) \\ &= T^{(5)}(T^{(5)}(x,y),z) \end{aligned}$$

2.2. $1 > y \geq a$,

2.2.1. $z < a$,

$$\begin{aligned} T^{(5)}(x, T^{(5)}(y,z)) &= T^{(5)}(x, y \wedge z \wedge a) = T^{(5)}(x,z) = x \wedge z \wedge a = z \\ &= V(x,y) \wedge z \wedge a \\ &= T^{(5)}(V(x,y), z) \\ &= T^{(5)}(T^{(5)}(x,y),z) \end{aligned}$$

2.2.2. $z \in I_a$,

$$\begin{aligned} T^{(5)}(x, T^{(5)}(y,z)) &= T^{(5)}(x, y \wedge z \wedge a) = T^{(5)}(x, z \wedge a) = x \wedge z \wedge a = z \wedge a \\ &= V(x,y) \wedge z \wedge a \\ &= T^{(5)}(V(x,y), z) \\ &= T^{(5)}(T^{(5)}(x,y),z) \end{aligned}$$

2.2.3. $1 > z \geq a$,

$$\begin{aligned} T^{(5)}(x, T^{(5)}(y,z)) &= T^{(5)}(x, V(y,z)) = V(x, V(y,z)) \\ &= V(V(x,y), z) \\ &= T^{(5)}(V(x,y), z) \\ &= T^{(5)}(T^{(5)}(x,y),z) \end{aligned}$$

2.3. $y \in I_a$,

2.3.1. $z < a$ or $z \in I_a$,

$$\begin{aligned} T^{(5)}(x, T^{(5)}(y,z)) &= T^{(5)}(x,0) = 0 \\ &= T^{(5)}(x \wedge y \wedge a, z) \\ &= T^{(5)}(T^{(5)}(x,y), z) \end{aligned}$$

2.3.2. $1 > z \geq a$,

$$\begin{aligned} T^{(5)}(x, T^{(5)}(y,z)) &= T^{(5)}(x, y \wedge z \wedge a) = T^{(5)}(x, y \wedge a) = x \wedge y \wedge a = y \wedge a \\ &= y \wedge z \wedge a \\ &= T^{(5)}(y \wedge a, z) \\ &= T^{(5)}(x \wedge y \wedge a, z) \\ &= T^{(5)}(T^{(5)}(x,y), z) \end{aligned}$$

3. Let $x \in I_a$.

3.1. $y < a$,

3.1.1. $z < a$ or $z \in I_a$,

$$\begin{aligned} T^{(5)}(x, T^{(5)}(y,z)) &= T^{(5)}(x,0) = 0 \\ &= T^{(5)}(0, z) \\ &= T^{(5)}(T^{(5)}(x,y), z) \end{aligned}$$

3.1.2. $1 > z \geq a$,

$$\begin{aligned} T^{(5)}(x, T^{(5)}(y,z)) &= T^{(5)}(x, y \wedge z \wedge a) = T^{(5)}(x, y) = 0 \\ &= 0 \wedge z \wedge a \\ &= T^{(5)}(0, z) \\ &= T^{(5)}(T^{(5)}(x,y), z) \end{aligned}$$

3.2. $1 > y \geq a$,

3.2.1. $z < a$,

$$\begin{aligned} T^{(5)}(x, T^{(5)}(y,z)) &= T^{(5)}(x, y \wedge z \wedge a) \ = T^{(5)}(x, z) = 0 \\ &= T^{(5)}(x \wedge a, z) \\ &= T^{(5)}(x \wedge y \wedge a, z) \\ &= T^{(5)}(T^{(5)}(x,y), z) \end{aligned}$$

3.2.2. $z \in I_a$,

$$\begin{aligned} T^{(5)}(x, T^{(5)}(y,z)) &= T^{(5)}(x, y \wedge z \wedge a) \ = T^{(5)}(x, z \wedge a) = 0 \\ &= T^{(5)}(x \wedge a, z) \\ &= T^{(5)}(x \wedge y \wedge a, z) \\ &= T^{(5)}(T^{(5)}(x,y), z) \end{aligned}$$

3.2.3. $1 > z \geq a$,

$$\begin{aligned} T^{(5)}(x, T^{(5)}(y,z)) &= T^{(5)}(x, V(y,z)) = x \wedge V(y,z) \wedge a = x \wedge a \\ &= T^{(5)}(x \wedge a, z) \\ &= T^{(5)}(x \wedge y \wedge a, z) \\ &= T^{(5)}(T^{(5)}(x,y), z) \end{aligned}$$

3.3. $y \in I_a$,

3.3.1. $z \in I_a$ or $z < a$,

$$\begin{aligned} T^{(5)}(x, T^{(5)}(y,z)) &= T^{(5)}(x,0) = 0 \\ &= T^{(5)}(0,z) \\ &= T^{(5)}(T^{(5)}(x,y),z) \end{aligned}$$

3.3.2. $1 > z \geq a$,

$$\begin{aligned} T^{(5)}(x, T^{(5)}(y,z)) &= T^{(5)}(x, y \wedge z \wedge a) = T^{(5)}(x, y \wedge a) = 0 \\ &= 0 \wedge z \wedge a \\ &= T^{(5)}(0,z) \\ &= T^{(5)}(T^{(5)}(x,y),z) \end{aligned}$$

So, we have that $T^{(5)}$ is a t-norm on L. □

Remark 2.16. Observe that the t-norm $T^{(5)}$ considered in Theorem 2.15 can be described alternatively as

$$T^{(5)}(x,y) = \begin{cases} V(x,y) & \text{if } (x,y) \in [a,1[^2, \\ 0 & \text{if } (x,y) \in [0,a[^2 \cup [0,a[\times I_a \cup I_a \times [0,a[\cup I_a \times I_a, \\ y \wedge a & \text{if } (x,y) \in [a,1[\times I_a, \\ x \wedge a & \text{if } (x,y) \in I_a \times [a,1[, \\ x & \text{if } (x,y) \in [0,a] \times [a,1[, \\ y & \text{if } (x,y) \in [a,1[\times [0,a], \\ x \wedge y & \text{if } 1 \in \{x,y\}. \end{cases}$$

Remark 2.17. From Remark 2.16, we get the t-norm $T^{(5)}$ on a bounded lattice L as shown in Figure 1.

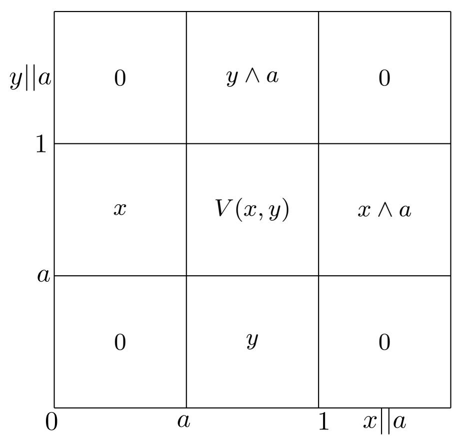

Fig. 1: t-norm $T^{(5)}$ on L given in Theorem 2.15.

Corollary 2.18. Let $(L, \leq, 0, 1)$ be a bounded lattice and $a \in L \backslash \{0, 1\}$. If a is an atom of L and V is a t-norm on $[a, 1]$, then the t-norm $T^{(5)}$ is given as follows:

$$T^{(5)}(x, y) = \begin{cases} V(x, y) & \text{if } (x, y) \in [a, 1[^2, \\ x \wedge y & \text{if } 1 \in \{x, y\}. \\ 0 & \text{otherwise.} \end{cases}$$

Remark 2.19. Let $(L, \leq, 0, 1)$ be a bounded lattice, $a \in L \backslash \{0, 1\}$ and V be a t-norm on $[a, 1]$. Consider t-norms $T^{(1)}$, $T^{(3)}$, $T^{(4)}$, $T^{(5)}$ on L defined in Theorems 2.10, 2.13, 2.14, 2.15, respectively. Then the following relationships hold:
i) $T^{(3)} \leq T^{(5)} \leq T^{(4)}$.
ii) If a is an atom of L, then $T^{(3)} = T^{(4)} = T^{(5)}$.
iii) If in Theorem 2.10 the interior operator $int : L \to L$ is defined by $int(x) = x \wedge a$ for all $x \in L$, then we have $T^{(4)} \geq T^{(1)}$.
iv) If the function $T^{(2)}$ defined in Theorem 2.11 is a t-norm on L and in Theorem 2.10 the interior operator $int : L \to L$ is defined by $int(x) = x \wedge a$ for all $x \in L$, then we have $T^{(2)} \geq T^{(1)}$.
v) If a is an atom of L and in Theorem 2.10, the interior operator $int : L \to L$ is defined by $int(x) = x \wedge a$ for all $x \in L$, then we have $T^{(3)} = T^{(4)} = T^{(5)} \geq T^{(1)}$.
vi) If in Theorem 2.10 the interior operator $int : L \to L$ is defined by $int(x) = x \wedge a$ for all $x \in L$ and in Theorem 2.14 the t-norm $V : [a, 1]^2 \to [a, 1]$ is defined by $V(x, y) = \begin{cases} x \wedge y & \text{if } 1 \in \{x, y\}, \\ a & \text{otherwise} \end{cases}$, then we have $T^{(1)} = T^{(4)}$.
vii) Note that if every element of L is comparable with a, i. e., $I_a = \emptyset$, then the function $T^{(2)}$ defined in Theorem 2.11 is a t-norm on L. In that case, we have $T^{(2)} = T^{(4)}$.
viii) If $I_a = \emptyset$ and in Theorem 2.14 the t-norm $V : [a, 1]^2 \to [a, 1]$ is defined by $V(x, y) = x \wedge y$, then we have $T^{(1)} \leq T^{(4)}$.
ix) If $I_a = \emptyset$ and a is an atom of L, then we have $T^{(2)} = T^{(3)} = T^{(4)} = T^{(5)}$.
x) If $I_a = \emptyset$, a is an atom of L and in Theorems 2.11 − 2.15, the t-norm $V : [a, 1]^2 \to [a, 1]$ is defined by $V(x, y) = x \wedge y$, then we have $T^{(1)} \leq T^{(2)} = T^{(3)} = T^{(4)} = T^{(5)}$.
xi) If $I_a = \emptyset$, a is an atom of L and in Theorem 2.10 the interior operator $int : L \to L$ is defined by $int(x) = x$ for all $x \in L$, then we have $T^{(2)} = T^{(3)} = T^{(4)} = T^{(5)} \leq T^{(1)}$.
xii) If $I_a = \emptyset$, a is an atom of L and in Theorem 2.10 the interior operator $int : L \to L$ is defined by $int(x) = x \wedge a$ for all $x \in L$, then we have $T^{(2)} = T^{(3)} = T^{(4)} = T^{(5)} \geq T^{(1)}$.
xiii) If the function $T^{(2)}$ defined in Theorem 2.11 is a t-norm on L, then we have $T^{(3)} \leq T^{(5)} \leq T^{(4)} \leq T^{(2)}$.
xiv) Note that if in Theorem 2.11 the t-norm $V : [a, 1]^2 \to [a, 1]$ is defined by $V(x, y) = x \wedge y$, the function $T^{(2)}$ is a t-norm on L. In that case, we have $T^{(1)} \leq T^{(2)}$.
xv) If the function $T^{(2)}$ defined in Theorem 2.11 is a t-norm on L and in Theorem 2.10 the interior operator $int : L \to L$ is defined by $int(x) = x$ for all $x \in L$, then we have $T^{(2)} \leq T^{(1)}$.
xvi) If in Theorem 2.11 the t-norm $V : [a, 1]^2 \to [a, 1]$ is defined by $V(x, y) = x \wedge y$ and in Theorem 2.10 the interior operator $int : L \to L$ is defined by $int(x) = x$ for all $x \in L$, then we have $T^{(1)} = T^{(2)}$.
xvii) If in Theorem 2.10 the interior operator $int : L \to L$ is defined by $int(x) = x$ for all $x \in L$, then we have $T^{(3)} \leq T^{(5)} \leq T^{(4)} \leq T^{(1)}$.

xviii) $T^{(1)}$, $T^{(2)}$, $T^{(3)}$, $T^{(4)}$, $T^{(5)}$ on L do not have to coincide on any bounded lattice unless we choose specify some special conditions. Let us demonstrate this argument by the following example.

Example 2.20. Consider the bounded lattice $L = \{0, b, c, d, f, a, g, h, 1\}$ with the lattice diagram shown in Figure 2. Take a t-norm $V : [a, 1]^2 \to [a, 1]$ defined by $V(x, y) = x \wedge y$ for all $x, y \in [a, 1]$ and a interior operator $int : L \to L$ defined by $int\,(0) = 0$, $int\,(b) = int\,(d) = int\,(c) = int\,(f) = b$, $int\,(a) = a$, $int\,(g) = int\,(h) = g$, $int\,(1) = 1$. By applying Theorems 2.10, 2.11, 2.13, 2.14, 2.15, respectively, the corresponding t-norms $T^{(1)} : L^2 \to L$, $T^{(2)} : L^2 \to L$, $T^{(3)} : L^2 \to L$, $T^{(4)} : L^2 \to L$ and $T^{(5)} : L^2 \to L$ are given by Tables 1−5, respectively.

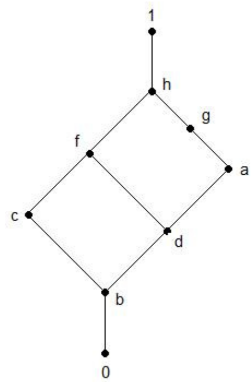

Fig. 2: The lattice L.

$T^{(1)}$	0	b	d	c	f	a	g	h	1
0	0	0	0	0	0	0	0	0	0
b	0	b	b	b	b	b	b	b	b
d	0	b	b	b	b	b	b	b	d
c	0	b	b	b	b	b	b	b	c
f	0	b	b	b	b	b	b	b	f
a	0	b	b	b	b	a	a	a	a
g	0	b	b	b	b	a	g	g	g
h	0	b	b	b	b	a	g	g	h
1	0	b	d	c	f	a	g	h	1

Tab. 1: The t-norm $T^{(1)}$ on L.

$T^{(2)}$	0	b	d	c	f	a	g	h	1
0	0	0	0	0	0	0	0	0	0
b	0	b	b	b	b	b	b	b	b
d	0	b	d	b	d	d	d	d	d
c	0	b	b	c	c	b	b	c	c
f	0	b	d	c	f	d	d	f	f
a	0	b	d	b	d	a	a	a	a
g	0	b	d	b	d	a	g	g	g
h	0	b	d	c	f	a	g	h	h
1	0	b	d	c	f	a	g	h	1

Tab. 2: The t-norm $T^{(2)}$ on L.

$T^{(3)}$	0	b	d	c	f	a	g	h	1
0	0	0	0	0	0	0	0	0	0
b	0	0	0	0	0	0	0	0	b
d	0	0	0	0	0	0	0	0	d
c	0	0	0	0	0	0	0	0	c
f	0	0	0	0	0	0	0	0	f
a	0	0	0	0	0	a	a	a	a
g	0	0	0	0	0	a	g	g	g
h	0	0	0	0	0	a	g	h	h
1	0	b	d	c	f	a	g	h	1

Tab. 3: The t-norm $T^{(3)}$ on L.

It is easy to see that the t-norms $T^{(1)}$, $T^{(2)}$, $T^{(3)}$, $T^{(4)}$ and $T^{(5)}$ are different from each other.

Now, in the following Theorem 2.21, we propose a construction method to obtain t-conorms on a bounded lattice L. This method considers the existence of a t-conorm defined on a subinterval $[0, a]$ for an arbitrary element $a \in L \backslash \{0, 1\}$.

Theorem 2.21. Let $(L, \leq, 0, 1)$ be a bounded lattice and $a \in L \backslash \{0, 1\}$. If W is a t-conorm on $[0, a]$, then the function $S^{(5)} : L^2 \to L$ is a t-conorm on L, where

$$S^{(5)}(x,y) = \begin{cases} W(x,y) & \text{if } (x,y) \in]0,a]^2, \\ 1 & \text{if } (x,y) \in]a,1]^2 \cup]a,1] \times I_a \cup I_a \times]a,1] \cup I_a \times I_a, \\ x \vee y & \text{if } 0 \in \{x,y\}, \\ x \vee y \vee a & \text{otherwise.} \end{cases} \tag{10}$$

The result is proved similarly as Theorem 2.15.

$T^{(4)}$	0	b	d	c	f	a	g	h	1
0	0	0	0	0	0	0	0	0	0
b	0	b	b	b	b	b	b	b	b
d	0	b	d	b	d	d	d	d	d
c	0	b	b	b	b	b	b	b	c
f	0	b	d	b	d	d	d	d	f
a	0	b	d	b	d	a	a	a	a
g	0	b	d	b	d	a	g	g	g
h	0	b	d	b	d	a	g	h	h
1	0	b	d	c	f	a	g	h	1

Tab. 4: The t-norm $T^{(4)}$ on L.

$T^{(5)}$	0	b	d	c	f	a	g	h	1
0	0	0	0	0	0	0	0	0	0
b	0	0	0	0	0	b	b	b	b
d	0	0	0	0	0	d	d	d	d
c	0	0	0	0	0	b	b	b	c
f	0	0	0	0	0	d	d	d	f
a	0	b	d	b	d	a	a	a	a
g	0	b	d	b	d	a	g	g	g
h	0	b	d	b	d	a	g	h	h
1	0	b	d	c	f	a	g	h	1

Tab. 5: The t-norm $T^{(5)}$ on L.

Remark 2.22. Observe that the t-conorm $S^{(5)}$ considered in Theorem 2.21 can be described alternatively as

$$S^{(5)}(x,y) = \begin{cases} W(x,y) & \text{if } (x,y) \in \left]0,a\right]^2, \\ 1 & \text{if } (x,y) \in \left]a,1\right]^2 \cup \left]a,1\right] \times I_a \cup I_a \times \left]a,1\right] \cup I_a \times I_a, \\ y \vee a & \text{if } (x,y) \in \left]0,a\right] \times I_a, \\ x \vee a & \text{if } (x,y) \in I_a \times \left]0,a\right], \\ x & \text{if } (x,y) \in [a,1] \times \left]0,a\right], \\ y & \text{if } (x,y) \in \left]0,a\right] \times [a,1], \\ x \vee y & \text{if } 0 \in \{x,y\}. \end{cases}$$

Remark 2.23. From Remark 2.22, we get the t-conorm $S^{(5)}$ on a bounded lattice L as shown in Figure 3.

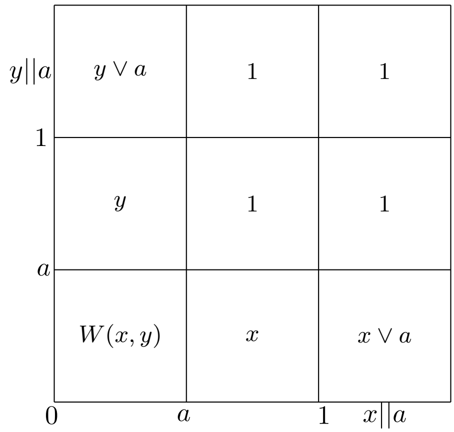

Fig. 3: t-conorm $S^{(5)}$ on L given in Theorem 2.21.

Corollary 2.24. Let $(L, \leq, 0, 1)$ be a bounded lattice and $a \in L \backslash \{0, 1\}$. If a is a coatom of L and W is a t-conorm on $[0, a]$, then the t-conorm $S^{(5)}$ is given as follows:

$$S^{(5)}(x, y) = \begin{cases} W(x, y) & \text{if } (x, y) \in]0, a]^2, \\ x \vee y & \text{if } 0 \in \{x, y\}, \\ 1 & \text{otherwise.} \end{cases}$$

Remark 2.25. Let $(L, \leq, 0, 1)$ be a bounded lattice, $a \in L \backslash \{0, 1\}$ and W be a t-conorm on $[0, a]$. Consider t-conorms $S^{(1)}$, $S^{(3)}$, $S^{(4)}$, $S^{(5)}$ on L defined in Theorems 2.10, 2.13, 2.14, 2.21, respectively. Then the following relationships hold:
i) $S^{(3)} \geq S^{(5)} \geq S^{(4)}$.
ii) If a is a coatom of L, then $S^{(3)} = S^{(4)} = S^{(5)}$.
iii) If in Theorem 2.10 the closure operator $cl : L \to L$ is defined by $cl(x) = x \vee a$ for all $x \in L$, then we have $S^{(4)} \leq S^{(1)}$.
iv) If the function $S^{(2)}$ defined in Theorem 2.11 is a t-conorm on L and in Theorem 2.10 the closure operator $cl : L \to L$ is defined by $cl(x) = x \vee a$ for all $x \in L$, then we have $S^{(2)} \leq S^{(1)}$.
v) If a is a coatom of L and in Theorem 2.10 the closure operator $cl : L \to L$ is defined by $cl(x) = x \vee a$ for all $x \in L$, then we have $S^{(3)} = S^{(4)} = S^{(5)} \leq S^{(1)}$.
vi) If in Theorem 2.10 the closure operator $cl : L \to L$ is defined by $cl(x) = x \vee a$ for all $x \in L$ and in Theorem 2.14 the t-conorm $W : [0, a]^2 \to [0, a]$ is defined by $W(x, y) = \begin{cases} x \vee y & \text{if } 0 \in \{x, y\}, \\ a & \text{otherwise} \end{cases}$, then we have $S^{(1)} = S^{(4)}$.
vii) Note that if every element of L is comparable with a, i. e., $I_a = \emptyset$, then the function $S^{(2)}$ defined in Theorem 2.11 is a t-conorm on L. In that case, we have $S^{(2)} = S^{(4)}$.
viii) If $I_a = \emptyset$ and in Theorem 2.14 the t-conorm $W : [0, a]^2 \to [0, a]$ is defined by $W(x, y) = x \vee y$, then we have $S^{(1)} \geq S^{(4)}$.
ix) If $I_a = \emptyset$ and a is a coatom of L, then we have $S^{(2)} = S^{(3)} = S^{(4)} = S^{(5)}$.
x) If $I_a = \emptyset$, a is a coatom of L and in Theorems 2.11–2.21 the t-conorm $W : [0, a]^2 \to$

$[0,a]$ is defined by $W(x,y) = x \vee y$, then we have $S^{(1)} \geq S^{(2)} = S^{(3)} = S^{(4)} = S^{(5)}$.
xi) If $I_a = \emptyset$, a is a coatom of L and in Theorem 2.10 the closure operator $cl : L \to L$ is defined by $cl\,(x) = x$ for all $x \in L$, then we have $S^{(2)} = S^{(3)} = S^{(4)} = S^{(5)} \geq S^{(1)}$.
xii) If $I_a = \emptyset$, a is a coatom of L and in Theorem 2.10 the closure operator $cl : L \to L$ is defined by $cl\,(x) = x \vee a$ for all $x \in L$, then we have $S^{(2)} = S^{(3)} = S^{(4)} = S^{(5)} \leq S^{(1)}$.
xiii) If the function $S^{(2)}$ defined in Theorem 2.11 is a t-conorm on L, then we have $S^{(3)} \geq S^{(5)} \geq S^{(4)} \geq S^{(2)}$.
xiv) Note that if in Theorem 2.11 the t-conorm $W : [0,a]^2 \to [0,a]$ is defined by $W(x,y) = x \vee y$, the function $S^{(2)}$ is a t-conorm on L. In that case, we have $S^{(1)} \geq S^{(2)}$.
xv) If the function $S^{(2)}$ defined in Theorem 2.11 is a t-conorm on L and in Theorem 2.10 the closure operator $cl : L \to L$ is defined by $cl\,(x) = x$ for all $x \in L$, then we have $S^{(2)} \geq S^{(1)}$.
xvi) If in Theorem 2.11 the t-conorm $W : [0,a]^2 \to [0,a]$ is defined by $W(x,y) = x \vee y$ and in Theorem 2.10 the closure operator $cl : L \to L$ is defined by $cl\,(x) = x$ for all $x \in L$, then we have $S^{(1)} = S^{(2)}$.
xvii) If in Theorem 2.10 the closure operator $cl : L \to L$ is defined by $cl\,(x) = x$ for all $x \in L$, then we have $S^{(3)} \geq S^{(5)} \geq S^{(4)} \geq S^{(1)}$.
xviii) $S^{(1)}$, $S^{(2)}$, $S^{(3)}$, $S^{(4)}$, $S^{(5)}$ on L do not have to coincide on any bounded lattice unless we choose specify some special conditions. Let us demonstrate this argument by the following example.

Example 2.26. Consider the bounded lattice $L = \{0, t, n, s, k, m, a, r, p, q, 1\}$ with the lattice diagram shown in Figure 4. Take a t-conorm $W : [a,a]^2 \to [0,a]$ defined by $W(x,y) = x \vee y$ for all $x, y \in [0,a]$ and a closure operator $cl : L \to L$ defined by $cl\,(0) = 0$, $cl\,(s) = cl\,(t) = cl\,(n) = s$, $cl\,(a) = a$, $cl\,(k) = cl\,(r) = cl\,(p) = p$, $cl\,(m) = cl\,(q) = cl\,(1) = 1$. By applying Theorems 2.10, 2.11, 2.13, 2.14, 2.21, respectively, the corresponding t-conorms $S^{(1)} : L^2 \to L$, $S^{(2)} : L^2 \to L$, $S^{(3)} : L^2 \to L$, $S^{(4)} : L^2 \to L$ and $S^{(5)} : L^2 \to L$ are given by Tables 6 − 10, respectively.

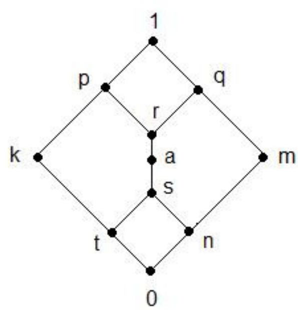

Fig. 4: The lattice L.

$S^{(1)}$	0	t	s	n	k	m	a	p	q	r	1
0	0	t	s	n	k	m	a	p	q	r	1
t	t	s	s	s	p	1	a	p	1	p	1
s	s	s	s	s	p	1	a	p	1	p	1
n	n	s	s	s	p	1	a	p	1	p	1
k	k	p	p	p	p	1	p	p	1	p	1
m	m	1	1	1	1	1	1	1	1	1	1
a	a	a	a	a	p	1	a	p	1	p	1
p	p	p	p	p	p	1	p	p	1	p	1
q	q	1	1	1	1	1	1	1	1	1	1
r	r	p	p	p	p	1	p	p	1	p	1
1	1	1	1	1	1	1	1	1	1	1	1

Tab. 6: The t-conorm $S^{(1)}$ on L.

$S^{(2)}$	0	t	s	n	k	m	a	p	q	r	1
0	0	t	s	n	k	m	a	p	q	r	1
t	t	t	s	s	k	q	a	p	q	r	1
s	s	s	s	s	p	q	a	p	q	r	1
n	n	s	s	n	p	m	a	p	q	r	1
k	k	k	p	p	k	1	p	p	1	p	1
m	m	q	q	m	1	m	q	1	q	q	1
a	a	a	a	a	p	q	a	p	q	r	1
p	p	p	p	p	p	1	p	p	1	p	1
q	q	q	q	q	1	q	q	1	q	q	1
r	r	r	r	r	p	q	r	p	q	r	1
1	1	1	1	1	1	1	1	1	1	1	1

Tab. 7: The t-conorm $S^{(2)}$ on L.

$S^{(3)}$	0	t	s	n	k	m	a	p	q	r	1
0	0	t	s	n	k	m	a	p	q	r	1
t	t	t	s	s	1	1	a	1	1	1	1
s	s	s	s	s	1	1	a	1	1	1	1
n	n	s	s	n	1	1	a	1	1	1	1
k	k	1	1	1	1	1	1	1	1	1	1
m	m	1	1	1	1	1	1	1	1	1	1
a	a	a	a	a	1	1	a	1	1	1	1
p	p	1	1	1	1	1	1	1	1	1	1
q	q	1	1	1	1	1	1	1	1	1	1
r	r	1	1	1	1	1	1	1	1	1	1
1	1	1	1	1	1	1	1	1	1	1	1

Tab. 8: The t-conorm $S^{(3)}$ on L.

$S^{(4)}$	0	t	s	n	k	m	a	p	q	r	1
0	0	t	s	n	k	m	a	p	q	r	1
t	t	t	s	s	p	q	a	p	q	r	1
s	s	s	s	s	p	q	a	p	q	r	1
n	n	s	s	n	p	q	a	p	q	r	1
k	k	p	p	p	p	1	p	p	1	p	1
m	m	q	q	q	1	q	q	1	q	q	1
a	a	a	a	a	p	q	a	p	q	r	1
p	p	p	p	p	p	1	p	p	1	p	1
q	q	q	q	q	1	q	q	1	q	q	1
r	r	r	r	r	p	q	r	p	q	r	1
1	1	1	1	1	1	1	1	1	1	1	1

Tab. 9: The t-conorm $S^{(4)}$ on L.

$S^{(5)}$	0	t	s	n	k	m	a	p	q	r	1
0	0	t	s	n	k	m	a	p	q	r	1
t	t	t	s	s	p	q	a	p	q	r	1
s	s	s	s	s	p	q	a	p	q	r	1
n	n	s	s	n	p	q	a	p	q	r	1
k	k	p	p	p	1	1	p	1	1	1	1
m	m	q	q	q	1	1	q	1	1	1	1
a	a	a	a	a	p	q	a	p	q	r	1
p	p	p	p	p	1	1	p	1	1	1	1
q	q	q	q	q	1	1	q	1	1	1	1
r	r	r	r	r	1	1	r	1	1	1	1
1	1	1	1	1	1	1	1	1	1	1	1

Tab. 10: The t-norm $S^{(5)}$ on L.

It is easy to see that the t-conorms $S^{(1)}$, $S^{(2)}$, $S^{(3)}$, $S^{(4)}$ and $S^{(5)}$ are different from each other.

3. T-NORMS AND T-CONORMS CONSTRUCTED BY ITERATION

Theorem 3.1. Let $(L, \leq, 0, 1)$ be a bounded lattice and $\{a_0, a_1, a_2, ..., a_n\}$ be a finite chain in L such that $a_0 = 1 > a_1 > a_2 > ... > a_n = 0$. Let $V : [a_1, 1]^2 \to [a_1, 1]$ be a t-norm on the sublattice $[a_1, 1]$. Then the operation $T = T_n : L^2 \to L$ defined recursively as follows is a t-norm, where $V = T_1$ and for $i \in \{2, 3, ..., n\}$ the operation

$T_i : [a_i, 1]^2 \to [a_i, 1]$ is given by

$$T_i(x,y) = \begin{cases} T_{i-1}(x,y) & \text{if } (x,y) \in [a_{i-1}, 1[^2, \\ a_i & \text{if } (x,y) \in [a_i, a_{i-1}[^2 \cup [a_i, a_{i-1}[\times I_{a_{i-1}} \\ & \cup I_{a_{i-1}} \times [a_i, a_{i-1}[\cup I_{a_{i-1}} \times I_{a_{i-1}}, \\ x \wedge y & \text{if } 1 \in \{x,y\}, \\ x \wedge y \wedge a_{i-1} & \text{otherwise.} \end{cases} \tag{11}$$

The proof follows easily from Theorem 2.15 by induction and therefore it is omitted. The construction described inductively by formula (11) can be considered as a ordinal sum construction for t-norms. Obviously, in Theorem 3.1, if L is a chain then the formula (11) reduces to

$$T_i(x,y) = \begin{cases} T_{i-1}(x,y) & \text{if } (x,y) \in [a_{i-1}, 1[^2, \\ a_i & \text{if } (x,y) \in [a_i, a_{i-1}[^2, \\ x \wedge y \wedge a_{i-1} & \text{if } (x,y) \in [a_i, a_{i-1}[\times [a_{i-1}, 1[\cup [a_{i-1}, 1[\times [a_i, a_{i-1}[, \\ x \wedge y & \text{otherwise.} \end{cases}$$

Theorem 3.2. Let $(L, \leq, 0, 1)$ be a bounded lattice and $\{b_0, b_1, b_2, ..., b_n\}$ be a finite chain in L such that $b_0 = 0 < b_1 < b_2 < ... < b_n = 1$. Let $W : [0, b_1]^2 \to [0, b_1]$ be a t-conorm on the sublattice $[0, b_1]$. Then the operation $S = S_n : L^2 \to L$ defined recursively as follows is a t-conorm, where $W = S_1$ and for $i \in \{2, 3, ..., n\}$ the operation $S_i : [0, b_i]^2 \to [0, b_i]$ is given by

$$S_i(x,y) = \begin{cases} S_{i-1}(x,y) & \text{if } (x,y) \in]0, b_{i-1}]^2, \\ b_i & \text{if } (x,y) \in]b_{i-1}, b_i]^2 \cup]b_{i-1}, b_i] \times I_{b_{i-1}} \\ & \cup I_{b_{i-1}} \times]b_{i-1}, b_i] \cup I_{b_{i-1}} \times I_{b_{i-1}}, \\ x \vee y & \text{if } 0 \in \{x,y\}, \\ x \vee y \vee b_{i-1} & \text{otherwise.} \end{cases} \tag{12}$$

The proof follows easily from Theorem 2.21 by induction and therefore it is omitted. The construction described inductively by formula (12) can be considered as a ordinal sum construction for t-conorms. Obviously, in Theorem 3.2, if L is a chain then the formula (12) reduces to

$$S_i(x,y) = \begin{cases} S_{i-1}(x,y) & \text{if } (x,y) \in]0, b_{i-1}]^2, \\ b_i & \text{if } (x,y) \in]b_{i-1}, b_i]^2, \\ x \vee y \vee b_{i-1} & \text{if } (x,y) \in]0, b_{i-1}] \times]b_{i-1}, b_i] \cup]b_{i-1}, b_i] \times]0, b_{i-1}] \\ x \vee y & \text{otherwise.} \end{cases}$$

4. CONCLUDING REMARKS

In this study, we introduced new methods for constructing t-norms and t-conorms on an arbitrary bounded lattice L by means of a t-norm V acting on a subinterval $[a, 1]$ and a t-conorm W acting on a subinterval $[0, a]$ for an arbitrary element $a \in L \backslash \{0, 1\}$. Note that the t-norm $T^{(5)}$ defined by the formula (9) in Theorem 2.15 needs not coincide with a predescribed t-norm K acting on the subinterval $[0, a]$. In order to show this argument, consider the lattice characterized by Figure 2 in Example 2.20 and force $T^{(5)} \mid [0, a]^2$ to be infimum t-norm on $[0, a]$, i. e., $T^{(5)}(x, y) = x \wedge y$ for all $x, y \in [0, a]$. In that case, we have $T^{(5)}\left(f, T^{(5)}(g, b)\right) = T^{(5)}(f, b) = 0$ and $T^{(5)}\left(T^{(5)}(f, g), b\right) = T^{(5)}(d, b) = b$. So, the associativity of $T^{(5)}$ is violated. Therefore, we can not force $T^{(5)}$ to coincide with a predescribed t-norm K acting on the subinterval $[0, a]$. Similarly, t-conorm $S^{(5)}$ defined by the formula (10) in Theorem 2.21 needs not coincide with a predescribed t-conorm M acting on the subinterval $[a, 1]$. In addition, we provided some illustrative examples (see Examples 2.20, 2.26) in order to show that our construction methods for t-norms and t-conorms on an arbitrary bounded lattice do not have to coincide with the known approaches. We also showed that they can be generalized by iteration to an ordinal sum construction for t-norms and t-conorms, applicable on any bounded lattice. Our results allow to construct new types of t-norms and t-conorms in the framework of lattices frequently considered in the information systems areas, including intuitionistic fuzzy sets, fuzzy sets type 2 and interval lattice and interval-valued fuzzy sets.

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