KYBERNETIKA — VOLUME 55 (2019), NUMBER 1, PAGES 44–62

TOWARDS THE PROPERTIES OF FUZZY MULTIPLICATION FOR FUZZY NUMBERS

Alexandru Mihai Bica, Dorina Fechete and Ioan Fechete

In this paper, by using a new representation of fuzzy numbers, namely the ecart-representation, we investigate the possibility to consider such multiplication between fuzzy numbers that is fully distributive. The algebraic and topological properties of the obtained semiring are studied making a comparison with the properties of the existing fuzzy multiplication operations. The properties of the generated fuzzy power are investigated.

Keywords: fuzzy number, semiring, fuzzy product distributivity

Classification: 03E72

1. INTRODUCTION

The binary operations with fuzzy numbers were introduced by using the Zadeh's extension principle (see [12, 13] and [19]). In order to simplify the computation of these operations, the use of the level sets is more appropriate generating the LU-representation (see [11, 16, 36, 3]). The interest for operations with fuzzy numbers is motivated by their applications in decision making (see [5] and [40]) and there are several ways to define the sum and the product of two fuzzy numbers. It is well-known that any binary operation with fuzzy numbers can be obtained by applying the Zadeh's extension principle which for fuzzy numbers described in LU-representation keeps the expression presented in [3]. Another approach in constructing binary operations with fuzzy numbers is based on the use of T-norms (see [20] and [38]) having applications in Possibility Theory. Algebraic properties of operations with interactive fuzzy numbers are presented in [6] and [7]. Arithmetic operations for fuzzy numbers with shape preserving properties are introduced in [18]. As told Markov in [27] and [28] there is no distributivity in the context of interval arithmetic excepting very special cases, that has consequences for fuzzy number arithmetic relating the distributivity properties. A special kind of multiplication of two fuzzy numbers with the support not containing zero is the cross-product introduced in [2] and investigated in [4] for its trapezoidal approximation with applications in Geology.

Generally, the addition and the multiplication determine a structure of commutative monoid with cancellation and identity laws on the set of fuzzy numbers. But, the distributivity law holds only in some particular cases. For instance, we can state that

DOI: 10.14736/kyb-2019-1-0044

$(A+B)\cdot C = A\cdot C + B\cdot C$, only if A and B are both positive or both negative, which is true for the existing variants of fuzzy multiplication (see [22, 25, 30, 31] for the classical fuzzy product, [2, 4] for the fuzzy cross-product, and [8] and [35] for the fuzzy product using the end-points of the level sets in the case of continuous fuzzy numbers with positive support). Even for the simplest case of triangular and trapezoidal fuzzy numbers (see [23] and [39]), the distributivity laws are valid as strong equalities only under restrictions. As it is well-known, the distributivity laws are valid as additive equivalences (see [25]).

Concerning the validity of the distributivity laws as strong equalities, there is only one exception offered by the operations proposed in [21]. These operations are defined by using another representation of fuzzy numbers, those described by the middle of the core and by the left and right spread of each level set. In this context, the arithmetic type operations are involved for operating with the middle of the core, while for the left spread and right spread the latticeal operation $\vee$ is used. The product and the substraction are defined everywhere, the distributivity laws hold without restriction, but since the operation $\vee$ preserves only the greatest spread of the involved operands, some information about the other operands is lost. This is the price paid for a rich set of algebraic properties. So, the problem to construct algebraic operations with fuzzy numbers in such a manner that keep all information in operands and to enrich the set of algebraic properties is not solved in [21]. In that follows, we show that could exist fuzzy multiplications for which the distributivity laws are fully preserved.

In the purpose to obtain such fuzzy multiplication, based on the results from [21] and [10], we use the new representation of fuzzy numbers proposed in [21], which will be called by us the MCE-representation. In this context, by using this representation, we define two products (the fuzzy direct product and the fuzzy MCE type cross-product). These products can be defined for any fuzzy numbers (not depending by the sign of the support for the operands) and are fully distributive. The distributivity laws does not depend by the positivity of the support of the involved fuzzy numbers. Moreover, the important class of symmetric fuzzy numbers is invariant under such fuzzy multiplication. This property is not valid for the classical multiplication generated by the Zadeh's extension principle. As it is well-known, symmetric fuzzy numbers have various applications in mechanical engineering (see [19]), fuzzy control systems (see [3]), fuzzy linear programming (see [33]), and in some applications there are represented by Gaussian type fuzzy numbers (see [3] and [19]). Elaborating algebraic structures for the set of fuzzy numbers, the symmetric fuzzy numbers are involved in the construction of additive and multiplicative equivalences caused by the gap of the invertibility properties (see [24, 25], and [34]).

The paper is organized as follows: in Section 2 we remember the MCE-representation of fuzzy numbers introduced in [21] and in Section 3, by using the MCE-representation, we define two multiplication operations between fuzzy numbers that not depend by the "ign" of the involved fuzzy numbers. The main result of Section 3 is Theorem 3.2, which proves the fully distributivity laws for these two multiplications. In Section 4 we investigate the topological structure of the set of fuzzy numbers that is compatible with the two introduced fuzzy multiplications, by defining two types of norms for fuzzy numbers that are consistent with the metric defined in [21]. These norms have two supplementary interesting properties (see statements 3 and 4 of Proposition 4.1). Of special interest is Theorem 5.3 which extends for fuzzy numbers the known situation of the existence of the isomorphism pair $(\exp, \ln)$ that maps the groups $(\mathbb{R}, +)$ and $\left(\mathbb{R}_+^*, \cdot\right)$ on the real axis. Section 5 is devoted to comparing the properties of the fuzzy power, generated by the fuzzy direct product, with the properties of the fuzzy power from [4, 22], and [25].

2. PRELIMINARIES

By a fuzzy number [3], as usually, we mean a function $A : \mathbb{R} \to [0, 1]$ which is normal (i. e., there exists $x_0 \in \mathbb{R}$, such that $A(x_0) = 1$), convex (i. e., $A(\lambda x + (1 - \lambda) y) \geqslant \min\{A(x), A(y)\}$, for all $x, y \in \mathbb{R}$ and $\lambda \in [0, 1]$), upper semicontinuous on $\mathbb{R}$ (i. e., for all $x_0 \in \mathbb{R}$ and for all $\varepsilon > 0$ there exists a neighborhood V_0 of x_0 such that $A(x) - A(x_0) \leqslant \varepsilon$, for all $x \in V_0$) and has compact support (i. e., the closure of the set $\{x \in \mathbb{R} : A(x) > 0\}$ is a compact interval of $\mathbb{R}$).

If $A : \mathbb{R} \to [0, 1]$ is a fuzzy number, the corresponding $t-$level sets $[A]_t$, defined by

$$[A]_t = \begin{cases} \overline{\{x \in \mathbb{R} : A(x) > 0\}}, & \text{if} \quad t = 0 \\ \{x \in \mathbb{R} : A(x) \geqslant t\}, & \text{if} \quad 0 < t \leqslant 1 \end{cases}$$

are compact intervals including $suppA = [A]_0$ (the support of A), and respectively, core$A = [A]_1$ (the core of A). It is known that (see [3]), if $[A]_t = \left[x_A^-(t), x_A^+(t)\right]$, for each $t \in [0, 1]$, then the functions $x_A^-, x_A^+ : [0, 1] \to \mathbb{R}$ (defining the endpoints of the $t-$level sets) are bounded, left-continuous on $(0, 1]$ and continuous at 0, x_A^- is increasing, x_A^+ is decreasing and $x_A^-(t) \leqslant x_A^+(t)$, for all $t \in [0, 1]$. Moreover, a fuzzy number A is completely determined by a pair $x_A = \left(x_A^-, x_A^+\right)$ of functions $x_A^-, x_A^+ : [0, 1] \to \mathbb{R}$ satisfying these conditions (see [3], [16]), obtaining in this way the LU-representation of a fuzzy number. The addition and the scalar product defined for fuzzy numbers in [16] and [3] organize the set of fuzzy numbers as a cancellative quasilinear space over $\mathbb{R}$. Such linear structure was presented in [26].

We will denote by $\mathfrak{F}$ the set of all fuzzy numbers.

Following the construction from [21], if a fuzzy number $A : \mathbb{R} \to [0, 1]$ has the $t-$level sets $[A]_t = \left[x_A^-(t), x_A^+(t)\right]$, the functions $\Theta_A^-, \Theta_A^+, \Delta_A : [0, 1] \to \mathbb{R}_+$ (where $\mathbb{R}_+ = [0, +\infty)$), defined by

$$\begin{cases} \Theta_A^-(t) = a - x_A^-(t) \\ \Theta_A^+(t) = x_A^+(t) - a \end{cases}, \quad \text{for each} \quad t \in [0, 1]$$

are bounded, decreasing, left-continuous on $(0, 1]$ and continuous at 0. Here, $\Delta_A = x_A^+ - x_A^- = \Theta_A^- + \Theta_A^+$ denotes the spread of the fuzzy number A and $a = \frac{1}{2}\left(x_A^-(1) + x_A^+(1)\right)$ is the middle point of the core$A$.

We call Θ_A^-, Θ_A^+ be the left and the right deviations relatively to the middle point of the core of A, and Δ_A is the width function of the fuzzy number A. If A is a unimodal fuzzy number (i. e. $x_A^-(1) = x_A^+(1)$), then the significance of the left and right deviations Θ_A^-, Θ_A^+ can be expressed as follows. The right deviation function Θ_A^+ indicates the degree of possibility of numbers greater than "a" to be considered relevant for the description

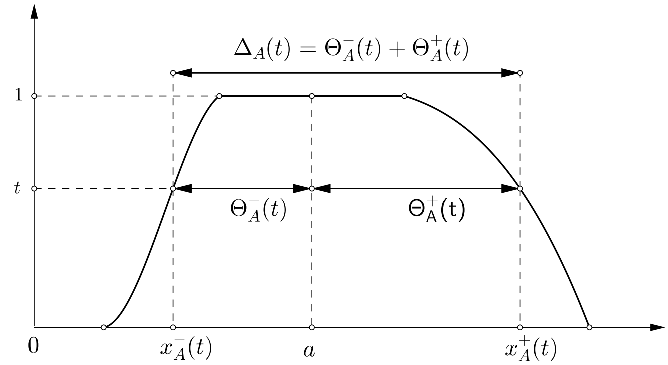

Fig. 1.

of the uncertain data represented by the fuzzy number A. A similar significance can be considered for the left deviation Θ_A^- related to the degree of possibility for numbers smaller than "a" to be relevant for the description of A. The discussion could be extended to general fuzzy numbers, not only for unimodal, when $x_A^-(1) \leqslant x_A^+(1)$. So, a good characterization of a fuzzy number can be made by specifying the spread to the left and to the right by its middle point. Consequently, a natural requirement for operations with fuzzy numbers is to specify the evolution of the left and right deviations after operating.

Thus, a fuzzy number $A \in \mathfrak{F}$ can be also represented by a system $A = \left(a; \Theta_A^-, \Theta_A^+\right)$, where $a \in \mathbb{R}$ and the functions $\Theta_A^-, \Theta_A^+ : [0,1] \to [0,+\infty)$ are bounded, nonincreasing, left-continuous on $(0;1]$, continuous at 0 and $\Theta_A^-(1) = \Theta_A^+(1)$.

Triplet representation of fuzzy numbers was proposed even by L. Stefanini and M. L. Guerra in [37], where the decomposition in this triplet consist of a crisp component (that is the core rectangle set), a middle $\alpha-$level function and a function on $[0,1]$ describing the symmetry by 0 of the fuzzy number. The idea to consider in [37] the crisp component as a core rectangle will allow us to introduce the fuzzy exponential and the fuzzy logarithm considering in Theorem 5.3 the subset $\mathfrak{F}^*$ and obtaining the isomorphism between $(\mathfrak{F},+)$ and $(\mathfrak{F}^*,\odot)$.

A fuzzy number $A \in \mathfrak{F}$ is symmetric iff $\Theta_A^- = \Theta_A^+$, and we denote by $S\mathfrak{F}$ the set of symmetric fuzzy numbers.

Definition 2.1. We call this representation $A = \left(a; \Theta_A^-, \Theta_A^+\right)$ be the MCE-representation (middle-core-ecart) of a fuzzy number.

The MCE-representation allow us to define multiplications between fuzzy numbers that are fully distributive related to the addition. Pointwise, we can represent a fuzzy number A, both by $A = \left(a; \Theta_A^-(t), \Theta_A^+(t)\right)_{t\in[0,1]}$ or by its level sets, $A = \left[x_A^-(t), x_A^+(t)\right]_{t\in[0,1]}$ (see Figure 1).

The triplet $\left(a; \Theta_A^-, \Theta_A^+\right)$ could represent the fuzzy number $x_A = \left(x_A^-, x_A^+\right)$ (which in initial stage is written in LU-form) and therefore this triplet $\left(a; \Theta_A^-, \Theta_A^+\right)$ can be considered instead of the pair $\left(x_A^-, x_A^+\right)$. The functions are conversely recuperated by setting $x_A^-(t) = a - \Theta_A^-(t)$ and $x_A^+(t) = a + \Theta_A^+(t)$, for $t \in [0,1]$.

Denote by $\mathfrak{F}_c$ the subset of $\mathfrak{F}$, which contains only those fuzzy numbers for which the functions that define the endpoints of the level sets are continuous. We also consider the set Ω defined by

$$\Omega = \{(f_1, f_2) \mid f_1, f_2 : [0,1] \to [0,+\infty) \text{ bounded, nonincreasing,}$$

$$\text{left continuous on } (0;1]\text{, continuous at } t = 0 \text{ and } f_1(1) = f_2(1)\}$$

and the subset Ω_C of Ω with continuous f_1 and f_2 components. With these notations, we can identify the set $\mathfrak{F}_c$ with the Cartesian product $\mathbb{R} \times \Omega_C$ and for any fuzzy number $A \in \mathfrak{F}$ we observe that $\left(\Theta_A^-, \Theta_A^+\right) \in \Omega$.

3. SEMIRING STRUCTURES ON $\mathfrak{F}_C$

The proper structure describing fuzzy arithmetic is a commutative semiring as was pointed out in [29].

Definition 3.1. (see Golan [17]) A commutative semiring is an algebraic structure $(S, +, \cdot, 0)$, such that $(S, +, 0)$ is a commutative monoid, $(S, \cdot)$ is a commutative semigroup, the distributivity law is fulfilled and $0 \cdot a = 0$ for all $a \in S$. If $(S, \cdot, 1)$ is a monoid, the semiring is said to be with identity.

Whenever "$\star$" is an operation on the set $\mathbb{R}$ and $f, g : [0,1] \to \mathbb{R}$ are two functions, we define the function $f \star g : [0,1] \to \mathbb{R}$ by $(f \star g)(t) = f(t) \star g(t)$, for each $t \in [0,1]$.

It is easy to see that $(\Omega, +, \cdot)$ is a commutative semiring with identity "(+" and "$\cdot$" denotes the usual pointwise addition, respectively, multiplication). If we put $\theta : x \mapsto 0$ and $\epsilon : x \mapsto 1$, for each $x \in [0,1]$, then the null element of the semiring Ω is the pair (θ, θ) and the identity of this semiring is (ϵ, ϵ).

Now, considering two fuzzy numbers $A = \left(a; \Theta_A^-, \Theta_A^+\right)$ and $B = \left(b; \Theta_B^-, \Theta_B^+\right)$ we define the following operations:

- the usual sum
$$A + B = \left(a + b;\ \Theta_A^- + \Theta_B^-,\ \Theta_A^+ + \Theta_B^+\right)$$
- the fuzzy direct product
$$A \odot B = \left(a \cdot b;\ \Theta_A^- \cdot \Theta_B^-,\ \Theta_A^+ \cdot \Theta_B^+\right)$$
- the fuzzy MCE type cross-product
$$A \otimes B = \left(a \cdot b;\ \Theta_A^- \cdot \Theta_B^- + \Theta_A^+ \cdot \Theta_B^+,\ \Theta_A^- \cdot \Theta_B^+ + \Theta_A^+ \cdot \Theta_B^-\right).$$

Since $\Theta_A^-, \Theta_A^+, \Theta_B^-$ and Θ_B^+ are positive valued decreasing functions, then so are $\Theta_{A\odot B}^- = \Theta_A^- \cdot \Theta_B^-$, $\Theta_{A\odot B}^+ = \Theta_A^+ \cdot \Theta_B^+$, $\Theta_{A\otimes B}^- = \Theta_A^- \cdot \Theta_B^- + \Theta_A^+ \cdot \Theta_B^+$ and $\Theta_{A\otimes B}^+ = \Theta_A^- \cdot \Theta_B^+ + \Theta_A^+ \cdot \Theta_B^-$. Since $\Theta_A^-(1) = \Theta_A^+(1)$ and $\Theta_B^-(1) = \Theta_B^+(1)$, then $\Theta_{A\odot B}^-(1) = \Theta_{A\odot B}^+(1)$ and $\Theta_{A\otimes B}^-(1) = \Theta_{A\otimes B}^+(1)$. So the above introduced products are well defined. Now, since $\Theta^-_{A+B}=\Theta^-_A+\Theta^-_B$, $\Theta^+_{A+B}=\Theta^+_A+\Theta^+_B$ and $\Theta^-_{A\odot B}=\Theta^-_A\cdot\Theta^-_B$, $\Theta^+_{A\odot B}=\Theta^+_A\cdot\Theta^+_B$, the construction of the operations "+" and "$\odot$" occur naturally. The behavior of the left and right deviations (Θ^-_A and Θ^+_A) after applying the fuzzy multiplication motivates the construction of the fuzzy product $\odot$.

The meaning of the product "$\otimes$" is revealed in Theorem 3.2 in connection with the semigroup morphism property of $\Delta:(\mathfrak{F},\otimes)\to(B([0,1],\mathbb{R}_+),\cdot)$, where $B([0,1],\mathbb{R}_+)=\{f:[0,1]\to\mathbb{R}_+\mid f \text{ bounded}\}$. The fuzzy MCE type cross-product differs by the cross product introduced in [2] and [4]. Since the operations defined in [21] are described by $A\star B=\left(a\star b,\Theta^-_A\vee\Theta^-_B,\ \Theta^+_A\vee\Theta^+_B\right)$, where $\star$ stands for $+,\cdot,-,\div$, the products $\odot$ and $\otimes$ differ by $\star$, the representation of fuzzy numbers being the same $A=\left(a;\Theta^-_A,\Theta^+_A\right)$.

The function $\mathrm{d}:\mathfrak{F}\times\mathfrak{F}\to[0,+\infty)$, defined by

$$\mathrm{d}(A,B)=|a-b|+\sup_{t\in[0,1]}\max\left(\left|\Theta^-_A(t)-\Theta^-_B(t)\right|,\left|\Theta^+_A(t)-\Theta^+_B(t)\right|\right)$$

is a metric on $\mathfrak{F}$ (see [21]).

Theorem 3.2. $(\mathfrak{F},+,\odot)$ and $(\mathfrak{F},+,\otimes)$ are commutative semirings with non-zero identity. If $A,B\in\mathfrak{F}$, then $\Delta_{A+B}=\Delta_A+\Delta_B$ and $\Delta_{A\otimes B}=\Delta_A\cdot\Delta_B$. The semiring $(\mathfrak{F},+,\odot)$ is cancellative and each of the subsets $L^1_*=\left\{A=\left(a;\Theta^-_A,\Theta^+_A\right)\in\mathfrak{F}:\Theta^+_A=\theta\right\}$ and $R^1_*=\left\{A=\left(a;\Theta^-_A,\Theta^+_A\right)\in\mathfrak{F}:\Theta^-_A=\theta\right\}$ of the unimodal left-sided and right-sided fuzzy numbers (see 1 and 2) are proper prime subtractive ideals in $(\mathfrak{F},+,\odot)$ not containing the identity $\overline{1}=(1,\epsilon,\epsilon)$. Moreover, $(S\mathfrak{F},+,\odot)$ and $(S\mathfrak{F},+,\otimes)$ are subsemirings in $(\mathfrak{F},+,\odot)$ and $(\mathfrak{F},+,\otimes)$, respectively.

Proof. It is easy to see that $(\mathfrak{F},+)$ and $(\mathfrak{F},\odot)$ are commutative monoids with the neutral elements $\overline{0}=(0,\theta,\theta)$ and $\overline{1}=(1,\epsilon,\epsilon)$, respectively. Obviously, $A\odot\overline{0}=\overline{0}$ and $A\otimes\overline{0}=\overline{0}$, $\forall A\in\mathfrak{F}$. Since the sum and the product on L are continuous we infer that "+", "$\odot$", "$\otimes$" are continuous operations. So, $(\mathfrak{F},+)$ and $(\mathfrak{F},\odot)$ are topological monoids and $(\mathfrak{F},\otimes)$ is a topological semigroup in the metric topology generated by d. Moreover, since $A\otimes\widehat{1}=\left(a\cdot 1,\Theta^-_A\cdot\epsilon+\Theta^+_A\cdot\theta,\Theta^-_A\cdot\theta+\Theta^+_A\cdot\epsilon\right)=\left(a;\Theta^-_A,\Theta^+_A\right)=A$, $\forall A\in\mathfrak{F}$, we infer that $\widehat{1}=(1,\epsilon,\theta)$ is the neutral element in $(\mathfrak{F},\otimes)$.

In order to prove the associativity of $\otimes$ we consider now, $A=\left(a;\theta^-_A,\theta^+_A\right)$, $B=\left(b;\theta^-_B,\theta^+_B\right)$ and $C=\left(c;\theta^-_C,\theta^+_C\right)$, three arbitrary fuzzy numbers. Since $(ab)c=a(bc)$ and

$$\begin{aligned}\Theta^-_{(A\otimes B)\otimes C}&=\Theta^-_{A\otimes B}\cdot\Theta^-_C+\Theta^+_{A\otimes B}\cdot\Theta^+_C\\&=\left(\Theta^-_A\cdot\Theta^-_B+\Theta^+_A\cdot\Theta^+_B\right)\cdot\Theta^-_C+\left(\Theta^-_A\cdot\Theta^+_B+\Theta^+_A\cdot\Theta^-_B\right)\cdot\Theta^+_C\\&=\Theta^-_A\cdot\left(\Theta^-_B\cdot\Theta^-_C+\Theta^+_B\cdot\Theta^+_C\right)+\Theta^+_A\cdot\left(\Theta^+_B\cdot\Theta^-_C+\Theta^-_B\cdot\Theta^+_C\right)\\&=\Theta^-_A\cdot\Theta^-_{B\otimes C}+\Theta^+_A\cdot\Theta^+_{B\otimes C}=\Theta^-_{A\otimes(B\otimes C)}\end{aligned}$$

$$\begin{aligned}\Theta^+_{(A\otimes B)\otimes C}&=\Theta^-_{A\otimes B}\cdot\Theta^+_C+\Theta^+_{A\otimes B}\cdot\Theta^-_C\\&=\left(\Theta^-_A\cdot\Theta^-_B+\Theta^+_A\cdot\Theta^+_B\right)\cdot\Theta^+_C+\left(\Theta^-_A\cdot\Theta^+_B+\Theta^+_A\cdot\Theta^-_B\right)\cdot\Theta^-_C\\&=\Theta^-_A\cdot\left(\Theta^-_B\cdot\Theta^+_C+\Theta^+_B\cdot\Theta^-_C\right)+\Theta^+_A\cdot\left(\Theta^+_B\cdot\Theta^+_C+\Theta^-_B\cdot\Theta^-_C\right)\\&=\Theta^-_A\cdot\Theta^+_{B\otimes C}+\Theta^+_A\cdot\Theta^-_{B\otimes C}=\Theta^+_{A\otimes(B\otimes C)}\end{aligned}$$

we obtain that $(A \otimes B) \otimes C = A \otimes (B \otimes C)$, i. e., the product "$\otimes$" is associative. Also, since $(a+b)\,c = ac + bc$ and

$$\begin{aligned} \Theta^-_{(A+B)\otimes C} &= \Theta^-_{A+B}\cdot\Theta^-_C + \Theta^+_{A+B}\cdot\Theta^+_C \\ &= \Theta^-_A\cdot\Theta^-_C + \Theta^-_B\cdot\Theta^-_C + \Theta^+_A\cdot\Theta^+_C + \Theta^+_B\cdot\Theta^+_C \\ &= \Theta^-_{A\otimes C} + \Theta^-_{B\otimes C} \end{aligned}$$

$$\begin{aligned} \Theta^+_{(A+B)\otimes C} &= \Theta^-_{A+B}\cdot\Theta^+_C + \Theta^+_{A+B}\cdot\Theta^-_C \\ &= \Theta^-_A\cdot\Theta^+_C + \Theta^-_B\cdot\Theta^+_C + \Theta^+_A\cdot\Theta^-_C + \Theta^+_B\cdot\Theta^-_C \\ &= \Theta^+_{A\otimes C} + \Theta^+_{B\otimes C} \end{aligned}$$

the distributivity of "$\otimes$" is satisfied. Obviously, the commutativity of "$\otimes$" is satisfied too. The distributivity of the product "$\odot$" is also obvious. Indeed,

$$\begin{aligned} \Theta^-_{(A+B)\odot C} &= \Theta^-_{A+B}\cdot\Theta^-_C = \left(\Theta^-_A + \Theta^-_B\right)\cdot\Theta^-_C = \Theta^-_A\cdot\Theta^-_C + \Theta^-_B\cdot\Theta^-_C \\ \Theta^+_{(A+B)\odot C} &= \Theta^+_{A+B}\cdot\Theta^+_C = \left(\Theta^+_A + \Theta^+_B\right)\cdot\Theta^+_C = \Theta^+_A\cdot\Theta^+_C + \Theta^+_B\cdot\Theta^+_C. \end{aligned}$$

We can see that $\Delta_{A+B} = \Theta^-_{A+B} + \Theta^+_{A+B} = \left(\Theta^-_A + \Theta^+_A\right) + \left(\Theta^-_B + \Theta^+_B\right) = \Delta_A + \Delta_B$ and $\Delta_{A\otimes B} = \Theta^-_{A\otimes B} + \Theta^+_{A\otimes B} = \left(\Theta^-_A + \Theta^+_A\right)\cdot\left(\Theta^-_B + \Theta^+_B\right) = \Delta_A\cdot\Delta_B$. Moreover,

$$\begin{aligned} A\odot\overline{0} &= \left(a\cdot 0, \Theta^-_A\cdot\theta, \Theta^+_A\cdot\theta\right) = (0,\theta,\theta) = \overline{0}, \quad \forall A\in\mathfrak{F} \\ A\otimes\overline{0} &= \left(a\cdot 0, \Theta^-_A\cdot\theta + \Theta^+_A\cdot\theta, \Theta^+_A\cdot\theta + \Theta^-_A\cdot\theta\right) = (0,\theta,\theta) = \overline{0}, \quad \forall A\in\mathfrak{F} \end{aligned}$$

and consequently, $(\mathfrak{F},+,\odot)$ and $(\mathfrak{F},+,\otimes)$ are commutative semirings. Since $(1,\epsilon,\epsilon)\neq(0,\theta,\theta)$ and $(1,\epsilon,\theta)\neq(0,\theta,\theta)$, we see that the semirings $(\mathfrak{F},+,\odot)$ and $(\mathfrak{F},+,\otimes)$ have non-zero identity. The usefulness of the product $\otimes$ is done by the homomorphism properties $\Delta_{A+B} = \Delta_A + \Delta_B$, $\Delta_{A\otimes B} = \Delta_A\cdot\Delta_B$ of the map $\Delta : \mathfrak{F}\to B\left([0,1],\mathbb{R}_+\right)$ between the semirings $(\mathfrak{F},+,\otimes)$ and $\left(B\left([0,1],\mathbb{R}_+\right),+,\cdot\right)$ (for the notion of semiring homomorphism, the reader can consult [1]). It is interesting to observe that $\Delta\left((0,\theta,\theta)\right) = 0$ and $\Delta\left((1,\epsilon,\theta)\right) = 1$.

The subsets L^1 and R^1 of the left-sided and right-sided fuzzy numbers are introduced and studied in [9]. Here we consider their subsets of unimodal fuzzy numbers

$$L^1_* = \{u\in L^1 : u^1_- = u^1_+\} = \left\{A = \left(a;\Theta^-_A,\Theta^+_A\right)\in\mathfrak{F} : x^-_A(1) = x^+_A(1),\ \Theta^+_A = \theta\right\} \quad (1)$$

and

$$R^1_* = \{u\in R^1 : u^1_- = u^1_+\} = \left\{A = \left(a;\Theta^-_A,\Theta^+_A\right)\in\mathfrak{F} : x^-_A(1) = x^+_A(1),\ \Theta^-_A = \theta\right\} \quad (2)$$

observing that the MCE representation of $A\in L^1_*$ is $A = \left(a;\Theta^-_A,\theta\right)$ and of $A\in R^1_*$ is $A = \left(a;\theta,\Theta^+_A\right)$, respectively.

It is obvious that for $A,B\in L^1_*$, $A = \left(a;\Theta^-_A,\theta\right)$ and $B = \left(b;\Theta^-_B,\theta\right)$, we have

$$\begin{array}{rcl} A+B & = & \left(a+b;\Theta_A^- + \Theta_B^-, \theta\right) \in L_*^1 \\ A\odot B & = & \left(a\cdot b;\ \Theta_A^- \cdot \Theta_B^-,\ \theta\right) \in L_*^1 \end{array}$$

and respectively, for $A, B \in R_*^1$, $A = \left(a;\theta,\Theta_A^+\right)$ and $B = \left(b;\theta,\Theta_B^+\right)$, we have

$$\begin{array}{rcl} A+B & = & \left(a+b;\theta,\Theta_A^+ + \Theta_B^+\right) \in R_*^1 \\ A\odot B & = & \left(a\cdot b;\theta,\Theta_A^+ \cdot \Theta_B^+\right) \in R_*^1. \end{array}$$

So, $\left(L_*^1,+,\odot\right)$ and $\left(R_*^1,+,\odot\right)$ are subsemirings in $(\mathfrak{F},+,\odot)$.
Moreover, for $A = \left(a;\Theta_A^-,\Theta_A^+\right) \in \mathfrak{F}$, $B = \left(b;\Theta_B^-,\theta\right) \in L_*^1$ and $C = \left(c;\theta,\Theta_C^+\right) \in R_*^1$, we have

$$A\odot B = \left(a\cdot b;\Theta_A^-\cdot\Theta_B^-,\Theta_A^+\cdot\theta\right) = \left(a\cdot b;\Theta_A^-\cdot\Theta_B^-,\theta\right) \in L_*^1$$

and

$$A\odot C = \left(a\cdot c;\Theta_A^-\cdot\theta,\Theta_A^+\cdot\Theta_C^+\right) = \left(a\cdot c;\theta,\Theta_A^+\cdot\Theta_C^+\right) \in R_*^1.$$

Therefore $\left(L_*^1,+,\odot\right)$ and $\left(R_*^1,+,\odot\right)$ are ideals in $(\mathfrak{F},+,\odot)$.
Now let $A = \left(a;\Theta_A^-,0\right) \in L_*^1$ and $B = \left(b;\Theta_B^-,\Theta_B^+\right) \in \mathfrak{F}$ such that $A+B \in L_*^1$, which means $\Theta_B^+ + \theta = \theta$. Then $\Theta_B^+ = \theta$ and we infer that $B \in L_*^1$. Consequently, L_*^1 is a subtractive ideal in $(\mathfrak{F},+,\odot)$ (or k-ideal, according to [14] and [15]). Similarly it can be proven that R_*^1 is a subtractive ideal in $(\mathfrak{F},+,\odot)$.

Finally, we consider $A = \left(a;\Theta_A^-,\Theta_A^+\right)$, $B = \left(b;\Theta_B^-,\Theta_B^+\right) \in \mathfrak{F}$ such that $A\odot B \in L_*^1$. Then, $\Theta_A^+(t)\cdot\Theta_B^+(t) = \theta$ for all $t \in [0,1]$ and, according to the left-continuity on $(0;1]$ and continuity in $t=0$ of Θ_A^+ and Θ_B^+, we infer that the situation $\Theta_A^+(t_1) = 0$, $\Theta_B^+(t_1) \neq 0$, $\Theta_A^+(t_2) \neq 0$, $\Theta_B^+(t_2) = 0$, for $t_1, t_2 \in [0,1]$, $t_1 \neq t_2$, is not possible. So, $\Theta_A^+(t) = \theta$ for all $t \in [0,1]$, or $\Theta_B^+(t) = 0$ for all $t \in [0,1]$, which means $A \in L_*^1$, or $B \in L_*^1$. Thus, we conclude that L_*^1 is a proper prime ideal in $(\mathfrak{F},+,\odot)$. Similarly, it can be proven that R_*^1 is a proper prime ideal in $(\mathfrak{F},+,\odot)$. It is interesting to observe that $L_*^1 \cap R_*^1 = \mathbb{R}$ and obviously $(\mathbb{R},+,\odot)$ is an ideal in $(\mathfrak{F},+,\odot)$, but not a strongly irreducible ideal.

For arbitrary $A = \left(a;\Theta_A^-,\Theta_A^+\right)$, $B = \left(b;\Theta_B^-,\Theta_B^+\right) \in S\mathfrak{F}$ we have $\Theta_A^- = \Theta_A^+$ and $\Theta_B^- = \Theta_B^+$, and thus $A+B$, $A\odot B \in S\mathfrak{F}$. Since in this case $\Theta_A^-\cdot\Theta_B^- + \Theta_A^+\cdot\Theta_B^+ = \Theta_A^-\cdot\Theta_B^+ + \Theta_A^+\cdot\Theta_B^-$, we infer that $A\otimes B \in S\mathfrak{F}$, obtaining that $(S\mathfrak{F},+,\odot)$ and $(S\mathfrak{F},+,\otimes)$ are subsemirings in $(\mathfrak{F},+,\odot)$ and $(\mathfrak{F},+,\otimes)$, respectively. We can see that $\overline{1} = (1,\epsilon,\epsilon) \in S\mathfrak{F}$, but $\widehat{1} = (1,\epsilon,\theta) \notin S\mathfrak{F}$, and the classical multiplication generated by the Zadeh's extension principle doesn't preserves the symmetric fuzzy numbers. Indeed, let us consider the triangular fuzzy number $A \in S\mathfrak{F}$ with $A_t = \left[x_A^-(t), x_A^+(t)\right]$, $x_A^-(t) = \frac{t+1}{2}$, $x_A^+(t) = \frac{3-t}{2}$, $t \in [0,1]$. We see that $a = 1$, $\Theta_A^-(t) = a - x_A^-(t) = \frac{1-t}{2} = x_A^+(t) - a = \Theta_A^+(t)$, and considering the classical multiplication $[A\cdot A]_t = \left[x_A^-(t)\cdot x_A^-(t), x_A^+(t)\cdot x_A^+(t)\right] = \left[\frac{t^2+2t+1}{4}, \frac{9-6t+t^2}{4}\right]$ we see that $\Theta_{A\cdot A}^-(t) = 1 - \frac{t^2+2t+1}{4} = \frac{3-2t-t^2}{4} \neq \frac{5-6t+t^2}{4} = \frac{9-6t+t^2}{4} - 1 = \Theta_{A\cdot A}^+(t)$, $\forall t \in [0,1)$, and therefore $A\cdot A \notin S\mathfrak{F}$. This is a gap of the classical multiplication and working in applications with symmetric fuzzy numbers, the use of the multiplication $\odot$ or $\otimes$ seems to be more adequate. □

It is obvious that $(\mathfrak{F}_c,+,\odot)$ and $(\mathfrak{F}_c,+,\otimes)$ are subsemirings in $(\mathfrak{F},+,\odot)$ and $(\mathfrak{F},+,\otimes)$, respectively. After elementary computation it can be observed that the product "$\otimes$" is not generally cancellative.

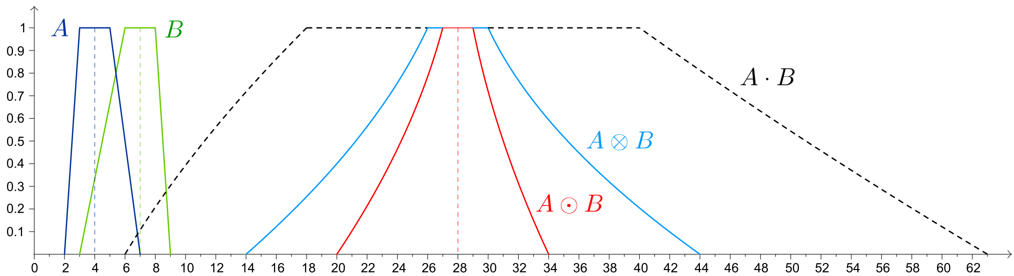

Fig. 2.

Remark 3.3. If $a \in \mathbb{R}$, the crisp number $\widetilde{a}$ has the MCE representation $(a;\theta,\theta)$. Since $\widetilde{a}+\widetilde{b}=\widetilde{a+b}$ and $\widetilde{a}\odot\widetilde{b}=\widetilde{a}\otimes\widetilde{b}=\widetilde{a\cdot b}$, for each $a,b\in\mathbb{R}$, we conclude that the field of real numbers is embedded in both semirings $(\mathfrak{F},+,\odot)$ and $(\mathfrak{F},+,\otimes)$ as a subsemiring, but the identity of $\mathbb{R}$ differs from the identities of these semirings.

Also, the group of units of the semiring $(\mathfrak{F},+,\odot)$ consist of the non-trivial intervals of the form $[a-x,a+x]=(a;x,x)$, with $a\in\mathbb{R}-\{0\}$ and fixed $x>0$. Obviously, the inverse of $(a;x,x)$ is $\left(\frac{1}{a};\frac{1}{x},\frac{1}{x}\right)=\left[\frac{1}{a}-\frac{1}{x},\frac{1}{a}+\frac{1}{x}\right]$. We can see that this group of units is identical to $L^1\cap R^1$.

Example 3.4. If

$$\begin{array}{rcl} A &=& [t+2,\ 7-2t]=(4;\ 2-t,\ 3-2t)\\ B &=& [3t+3,\ 9-t]=(7;\ 4-3t,\ 2-t) \end{array}$$

are two fuzzy numbers, then

$$\begin{array}{rcl} A\cdot B &=& \left[(t+2)(3t+3),(7-2t)(9-t)\right]\\ &=& \left(29;\ \ -3t^2-9t+23,2t^2-25t+34\right) \end{array}$$

$$\begin{array}{rcl} A\odot B &=& (28;\ (2-t)(4-3t),(3-2t)(2-t))\\ &=& \left[-3t^2+10t+20,2t^2-7t+34\right] \end{array}$$

$$\begin{array}{rcl} A\otimes B &=& \left(28;\ (2-t)(4-3t)+(3-2t)(2-t),(2-t)^2+(3-2t)(4-3t)\right)\\ &=& \left[-5t^2+17t+14,7t^2-21t+44\right] \end{array}$$

where $A\cdot B$ is the usual product (based on the Zadeh's extension principle, defined by $A\cdot B=\left[x_A^-\cdot x_B^-,\ x_A^+\cdot x_B^+\right]$, since A and B has positive support) and $A\odot B$ and $A\otimes B$ are the two products introduced above. These are represented in Figure 2.

Remark 3.5. If $A=\left[x_A^-,x_A^+\right]$ and $B=\left[x_B^-,x_B^+\right]$ in LU-representation, then we can write the result after applying the above presented multiplications in LU-representation, too, as follows:

1. $\begin{cases} x^-_{A\odot B} &= ab - \Theta^-_{A\odot B} = a\cdot x^-_B + b\cdot x^-_A - x^-_A\cdot x^-_B \\ x^+_{A\odot B} &= ab + \Theta^+_{A\odot B} = 2ab - a\cdot x^+_B - b\cdot x^+_A + x^+_A\cdot x^+_B \end{cases}$

2. $\begin{cases} x^-_{A\otimes B} &= ab - \Theta^-_{A\otimes B} \\ &= a\left(x^-_B + x^+_B\right) + b\left(x^-_A + x^+_A\right) - x^-_A\cdot x^-_B - x^+_A\cdot x^+_B - ab \\ x^+_{A\otimes B} &= ab + \Theta^+_{A\otimes B} \\ &= a\left(x^-_B + x^+_B\right) + b\left(x^-_A + x^+_A\right) - x^-_A\cdot x^+_B - x^+_A\cdot x^-_B - ab \end{cases}$

where $a = \frac{1}{2}\left(x^-_A(1) + x^+_A(1)\right)$ and $b = \frac{1}{2}\left(x^-_B(1) + x^+_B(1)\right)$.

Remark 3.6. By Theorem 3.2, the products $\odot$ and $\otimes$ are fully distributive on $\mathfrak{F}$, $A\otimes(B+C) = A\otimes B + A\otimes C$, $A\odot(B+C) = A\odot B + A\odot C$, for all $A,B,C\in\mathfrak{F}$, and by comparison with the results from [22, 25, 23],

$$A \circledast (B+C) = A\circledast B + A\circledast C, \text{ iff } B, C \text{ have the same sign, or } B, C\in\mathbb{S}_0$$

and [4],

$$A \circledast (B+C) = A\circledast B + A\circledast C, \text{ iff } B, C \text{ have the same sign,}$$

we observe an improvement of the distributivity in Theorem 3.2. We have used here a modified notation $\circledast$ for the multiplications from [22, 25, 23], and [4], in order to avoid confusion with the direct product $\odot$ introduced here.

4. THE TOPOLOGICAL STRUCTURE OF $\mathfrak{F}_C$

For each fuzzy number $A = \left(a;\Theta^-_A,\Theta^+_A\right)\in\mathfrak{F}$, we define $\langle A\rangle$ by

$$\langle A\rangle = \sup_{t\in[0,1]} \max\left(\Theta^-_A(t),\Theta^+_A(t)\right).$$

Since Θ^-_A and Θ^+_A are positive valued and decreasing functions, it follows that

$$\langle A\rangle = \max\left(\Theta^-_A(0),\Theta^+_A(0)\right)\geqslant 0.$$

Defining, for each $n\in\{1,2\}$, the functions $\|\cdot\|_n : \mathfrak{F}\to[0,+\infty)$ by:

$$\begin{aligned} \|A\|_1 &= |a| + \langle A\rangle \\ \|A\|_2 &= |a| + 2\langle A\rangle \end{aligned}$$

we investigate the properties of $\|\cdot\|_1$ and $\|\cdot\|_2$ relating the operations $+,\odot,\otimes$.

Proposition 4.1. (i) For each $A,B\in\mathfrak{F}$ and $\lambda\in\mathbb{R}$, the functions $\|\cdot\|_n$ satisfy the following properties:

1. $\|A\|_n\geqslant 0$, $\forall A\in\mathfrak{F}$ and $\|A\|_n = 0 \Leftrightarrow A = \overline{0}$, for $n\in\{1,2\}$;
2. $\|A+B\|_n \leqslant \|A\|_n + \|B\|_n$, for $n\in\{1,2\}$;
3. $\|A\odot B\|_1 \leqslant \|A\|_1\cdot\|B\|_1$;

4. $\|A \otimes B\|_2 \leqslant \|A\|_2 \cdot \|B\|_2$.

(ii) The metric d on $\mathfrak{F}$ is complete and it satisfies the following properties:

1. $\mathrm{d}(A+C, B+C) = \mathrm{d}(A,B)$;
2. $\mathrm{d}(A+C, B+D) \leqslant \mathrm{d}(A,B) + \mathrm{d}(C,D)$;
3. $\mathrm{d}(A \odot C, B \odot C) \leqslant \|C\|_1 \cdot \mathrm{d}(A,B)$;
4. $\mathrm{d}(A \otimes C, B \otimes C) \leqslant \|C\|_2 \cdot \mathrm{d}(A,B)$;

for all $A,B,C,D \in \mathfrak{F}$.

Proof. (i) An easy computation shows that the statements 1 and 2 are satisfied.
If $A = \left(a; \Theta_A^-, \Theta_A^+\right)$ and $B = \left(b; \Theta_B^-, \Theta_B^+\right)$ then

$$\begin{aligned}
\|A \odot B\|_1 &= |ab| + \langle A \odot B \rangle \\
&= |a| \cdot |b| + \max\left(\Theta_A^-(0) \cdot \Theta_B^-(0), \Theta_A^+(0) \cdot \Theta_B^+(0)\right) \\
&\leqslant \left(|a| + \max\left(\Theta_A^-(0), \Theta_A^+(0)\right)\right) \cdot \left(|b| + \max\left(\Theta_B^-(0), \Theta_B^+(0)\right)\right) \\
&= \|A\|_1 \cdot \|B\|_1
\end{aligned}$$

$$\begin{aligned}
\|A \otimes B\|_2 &= |ab| + 2\langle A \otimes B \rangle \\
&= |a| \cdot |b| + 2 \cdot \max\left(\Theta_{A\otimes B}^-(0), \Theta_{A\otimes B}^+(0)\right) \\
&\leqslant \left(|a| + 2 \cdot \max\left(\Theta_A^-(0), \Theta_A^+(0)\right)\right) \cdot \left(|b| + 2 \cdot \max\left(\Theta_B^-(0), \Theta_B^+(0)\right)\right) \\
&= \|A\|_2 \cdot \|B\|_2
\end{aligned}$$

which completes the proof.

(ii) Since this metric is equivalent with the metric D_∞ (see [3]) having $D_\infty(u,v) \leqslant \mathrm{d}(u,v) \leqslant 3D_\infty(u,v)$, $\forall u,v \in \mathfrak{F}$ (see [21]), it follows that d is complete. A trivial verification shows that the statements 1 and 2 are satisfied. We consider now $A = \left(a; \Theta_A^-, \Theta_A^+\right)$, $B = \left(b; \Theta_B^-, \Theta_B^+\right)$, $C = \left(c; \Theta_C^-, \Theta_C^+\right)$ and we prove that the statements 3 and 4 are also true. Indeed,

$$\begin{aligned}
&\mathrm{d}(A \odot C, B \odot C) \\
&= |ac - bc| + \sup_{t\in[0,1]} \max\left(\left|\Theta_{A\odot C}^-(t) - \Theta_{B\odot C}^-(t)\right|, \left|\Theta_{A\odot C}^+(t) - \Theta_{B\odot C}^+(t)\right|\right) \\
&= |c| \cdot |a-b| + \sup_{t\in[0,1]} \max\left(\left|\Theta_A^-(t) - \Theta_B^-(t)\right| \cdot \left|\Theta_C^-(t)\right|, \left|\Theta_A^+(t) - \Theta_B^+(t)\right| \cdot \left|\Theta_C^+(t)\right|\right) \\
&\leqslant \max\left(|c|, \sup_{t\in[0,1]} \max\left(\left|\Theta_C^-(t)\right|, \left|\Theta_C^+(t)\right|\right)\right) \cdot \\
&\cdot \left(|a-b| + \sup_{t\in[0,1]} \max\left(\left|\Theta_A^-(t) - \Theta_B^-(t)\right|, \left|\Theta_A^+(t) - \Theta_B^+(t)\right|\right)\right) \\
&\leqslant \left(|c| + \sup_{t\in[0,1]} \max\left(\left|\Theta_C^-(t)\right|, \left|\Theta_C^+(t)\right|\right)\right) \cdot \mathrm{d}(A,B) \\
&\leqslant \|C\|_1 \cdot \mathrm{d}(A,B)
\end{aligned}$$

and since

$$\begin{aligned}&\left|\Theta_{A\otimes C}^{-}(t)-\Theta_{B\otimes C}^{-}(t)\right|\\&=\left|\Theta_{A}^{-}(t)\cdot\Theta_{C}^{-}(t)+\Theta_{A}^{+}(t)\cdot\Theta_{C}^{+}(t)-\Theta_{B}^{-}(t)\cdot\Theta_{C}^{-}(t)-\Theta_{B}^{+}(t)\cdot\Theta_{C}^{+}(t)\right|\\&\leqslant\Theta_{C}^{-}(t)\cdot\left|\Theta_{A}^{-}(t)-\Theta_{B}^{-}(t)\right|+\Theta_{C}^{+}(t)\cdot\left|\Theta_{A}^{+}(t)-\Theta_{B}^{+}(t)\right|\\&\leqslant 2\max\left(\Theta_{C}^{-}(t),\Theta_{C}^{+}(t)\right)\cdot\max\left(\left|\Theta_{A}^{-}(t)-\Theta_{B}^{-}(t)\right|,\left|\Theta_{A}^{+}(t)-\Theta_{B}^{+}(t)\right|\right)\end{aligned}$$

$$\begin{aligned}&\left|\Theta_{A\otimes C}^{+}(t)-\Theta_{B\otimes C}^{+}(t)\right|\\&=\left|\Theta_{A}^{-}(t)\cdot\Theta_{C}^{+}(t)+\Theta_{A}^{+}(t)\cdot\Theta_{C}^{-}(t)-\Theta_{B}^{-}(t)\cdot\Theta_{C}^{+}(t)-\Theta_{B}^{+}(t)\cdot\Theta_{C}^{-}(t)\right|\\&\leqslant\Theta_{C}^{+}(t)\cdot\left|\Theta_{A}^{-}(t)-\Theta_{B}^{-}(t)\right|+\Theta_{C}^{-}(t)\cdot\left|\Theta_{A}^{+}(t)-\Theta_{B}^{+}(t)\right|\\&\leqslant 2\max\left(\Theta_{C}^{-}(t),\Theta_{C}^{+}(t)\right)\cdot\max\left(\left|\Theta_{A}^{-}(t)-\Theta_{B}^{-}(t)\right|,\left|\Theta_{A}^{+}(t)-\Theta_{B}^{+}(t)\right|\right)\end{aligned}$$

it follows that

$$\begin{aligned}&|ac-bc|+\max\left(\left|\Theta_{A\otimes C}^{-}(t)-\Theta_{B\otimes C}^{-}(t)\right|,\left|\Theta_{A\otimes C}^{+}(t)-\Theta_{B\otimes C}^{+}(t)\right|\right)\\&\leqslant|c|\cdot|a-b|+2\max\left(\Theta_{C}^{-}(t),\Theta_{C}^{+}(t)\right)\cdot\max\left(\left|\Theta_{A}^{-}(t)-\Theta_{B}^{-}(t)\right|,\left|\Theta_{A}^{+}(t)-\Theta_{B}^{+}(t)\right|\right)\\&\leqslant\max\left(|c|,2\max\left(\Theta_{C}^{-}(t),\Theta_{C}^{+}(t)\right)\right)\cdot\\&\cdot\left(|a-b|+\max\left(\left|\Theta_{A}^{-}(t)-\Theta_{B}^{-}(t)\right|,\left|\Theta_{A}^{+}(t)-\Theta_{B}^{+}(t)\right|\right)\right).\end{aligned}$$

Therefore

$$\mathrm{d}(A\otimes C,B\otimes C)\leqslant\left(|c|+2\max\left(\Theta_{C}^{-}(t),\Theta_{C}^{+}(t)\right)\right)\cdot\mathrm{d}(A,B)\leqslant\|C\|_2\cdot\mathrm{d}(A,B).$$

□

Definition 4.2. We say that the sequence $(A_n)_{n\geqslant 1}\subset\mathfrak{F}$ converges to $A\in\mathfrak{F}$ iff $\lim_{n\to\infty}\mathrm{d}(A_n,A)=0$ and we will use, in this case, the notation $\lim_{n\to\infty}A_n=A$.

Remark 4.3. By Proposition 4.1 (ii), the convergence given by the metric $\mathrm{d}(A,B)$ becomes equivalent with the Pompeiu–Hausdorff convergence. Moreover, from properties 3 and 4 of Proposition 4.1 (ii), we deduce that the operations $\odot$ and $\otimes$ are continuous in the sense that, by $\lim_{n\to\infty}A_n=A$, we have that $\lim_{n\to\infty}(A_n\odot B)=A\odot B$ and $\lim_{n\to\infty}(A_n\otimes B)=A\otimes B$, for each $B\in\mathfrak{F}$.

Remark 4.4. If $A_n=\left[x_{A_n}^{-},x_{A_n}^{+}\right]=\left(a_n;\Theta_{A_n}^{-},\Theta_{A_n}^{+}\right)$ and $A=\left[x_A^{-},x_A^{+}\right]=\left(a;\Theta_A^{-},\Theta_A^{+}\right)$, then the following statements are equivalent:

1. $\lim_{n\to\infty}A_n=A$;
2. $x_{A_n}^{-}\rightrightarrows x_A^{-}$ and $x_{A_n}^{+}\rightrightarrows x_A^{+}$ on $[0,1]$;
3. $\lim_{n\to\infty}a_n=a$, $\Theta_{A_n}^{-}\rightrightarrows\Theta_A^{-}$ and $\Theta_{A_n}^{+}\rightrightarrows\Theta_A^{+}$ on $[0,1]$.
   Here, the symbol "$\rightrightarrows$" denotes the uniform convergence on a compact interval (on $[0,1]$ in this case).

5. THE FUZZY EXPONENTIAL AND POWER OF A FUZZY NUMBER

We define here an extended variant for fuzzy exponential and fuzzy logarithm.

Let us denote by $\mathfrak{F}^*$ the set

$$\left\{A=\left(a;\Theta_A^-,\Theta_A^+\right)\in\mathfrak{F}: a>0,\ \text{and}\ \Theta_A^-(t),\Theta_A^+(t)\geqslant 1,\ \forall\, t\in[0,1]\right\}$$

and $\mathfrak{F}_{\mathrm{c}}^*=\mathfrak{F}^*\cap\mathfrak{F}_{\mathrm{c}}$. Since $A,B\in\mathfrak{F}^*$ implies that $A\odot B$, $A\otimes B\in\mathfrak{F}^*$, it follows that $(\mathfrak{F}^*,\odot)$ and $(\mathfrak{F}^*,\otimes)$ are submonoids in $(\mathfrak{F},\odot)$ and $(\mathfrak{F},\otimes)$, respectively. We define the exponential function $\exp:\mathfrak{F}\to\mathfrak{F}^*$, by $A\mapsto e^A=(e^a;\ e^{\Theta_A^-},e^{\Theta_A^+}$ and the logarithmic function $\ln:\mathfrak{F}^*\to\mathfrak{F}$, by $A\mapsto\ln A=(\ln a;\ \ln\circ\Theta_A^-,\ln\circ\Theta_A^+)$, where $A=(a;\Theta_A^-,\Theta_A^+)$ and "$\circ$" denotes the composition of functions.

Remark 5.1. In the context of MCE-representation, in order to have a good definition of the fuzzy logarithm, that is $\ln(A)\in\mathfrak{F}$ for a fuzzy number A, we need to define $\ln:\mathfrak{F}^*\to\mathfrak{F}$. This lead us to the situation that when A is crisp, having $\Theta_A^-=\Theta_A^+=0$, we arrive to $e^A=\left(e^a;\ e^0,e^0\right)=(e^a;\ 1,1)$, which means the interval $[e^a-1,e^a+1]$ (that is a core rectangle set). This is on accordance with the decomposition proposed in [37] by L. Stefanini and M. L. Guerra, containing the crisp component as a core rectangle set.

Example 5.2. If $A=\left(a;\Theta_A^-,\Theta_A^+\right)=\left[x_A^-,x_A^+\right]$ is a fuzzy number, the standard exponential function (from extension principle) is defined by $\operatorname{Exp}(A)=\left[e^{x_A^-},e^{x_A^+}\right]$.

If $A=[t+2,\ 7-2t]=(4;\ 2-t,\ 3-2t)$, then

$$\begin{aligned}\operatorname{Exp}(A)&=\left[e^{t+2},\ e^{7-2t}\right]=\left(\frac{e^3+e^5}{2};\frac{e^3+e^5}{2}-e^{t+2}\ ,e^{7-2t}-\frac{e^3+e^5}{2}\right)\\ \exp(A)&=\left(e^4;\ e^{2-t},e^{3-2t}\right)=\left[e^4-e^{2-t},e^4+e^{3-2t}\right]\end{aligned}$$

and

$$\begin{aligned}\operatorname{supp}(\operatorname{Exp}(A))&=\left[e^2,e^7\right]\cong[7.3,\ 1096.6]\\ \operatorname{core}(\operatorname{Exp}(A))&=\left[e^3,e^5\right]\cong[20.1,\ 148.4]\\ \operatorname{supp}(\exp(A))&=\left[e^4-e^2,e^4+e^3\right]\cong[47.2,\ 74.6]\\ \operatorname{core}(\exp(A))&=\left[e^4-e,e^4+e\right]\cong[51.8,\ 57.3].\end{aligned}$$

We see that in the case of classical fuzzy exponential, the support of $\operatorname{Exp}(A)$ could be overly large in comparison with the support of extended exponential $\exp(A)$, that has a reasonable length and therefore becomes suitable for applications.

Theorem 5.3. The functions exp and ln, defined above, are monoid algebraic isomorphisms between $(\mathfrak{F},+)$ and $(\mathfrak{F}^*,\odot)$, and continuous monoid isomorphisms between $(\mathfrak{F}_{\mathrm{c}},+,\mathrm{d})$ and $(\mathfrak{F}_{\mathrm{c}}^*,\odot,\mathrm{d})$ with $\ln=\exp^{-1}$.

P r o o f. Obviously, $e^{\overline{0}} = \overline{1}$ and $\ln \overline{1} = \overline{0}$. If $A = \left(a; \Theta_A^-, \Theta_A^+\right)$ and $B = \left(b; \Theta_B^-, \Theta_B^+\right)$, then

$$e^{A+B} = \left(e^{a+b}; e^{\Theta_A^- + \Theta_B^-}, e^{\Theta_A^+ + \Theta_B^+}\right) = \left(e^a; e^{\Theta_A^-}, e^{\Theta_A^+}\right) \odot \left(e^b; e^{\Theta_B^-}, e^{\Theta_B^+}\right) = e^A \odot e^B$$

respectively,

$$\begin{aligned} \ln(A \odot B) &= \left(\ln(a \cdot b);\ \ln\left(\Theta_A^- \cdot \Theta_B^-\right),\ \ln\left(\Theta_A^+ \cdot \Theta_B^+\right)\right) \\ &= \left(\ln a;\ \ln \circ \Theta_A^-, \ln \circ \Theta_A^+\right) + \left(\ln b;\ \ln \circ \Theta_B^-, \ln \circ \Theta_B^+\right) \\ &= \ln(A) + \ln(B) \end{aligned}$$

and so, the functions exp and ln are monoid homomorphisms. Elementary calculus lead to $(\exp \circ \ln)(A) = A$, $\forall A \in \mathfrak{F}^*$ and $(\ln \circ \exp)(A) = A$, $\forall A \in \mathfrak{F}$, which means that ln is the inverse of the exponential function exp and conversely. Now, let $(A_n)_{n\in\mathbb{N}} = \left(\left(a_n; \Theta_{A_n}^-, \Theta_{A_n}^+\right)\right)_{n\in\mathbb{N}} \subset \mathfrak{F}_c^*$ and $A = \left(a; \Theta_A^-, \Theta_A^+\right) \in \mathfrak{F}_c^*$ such that $\lim_{n\to\infty} A_n = A$. So, according to Remark 4.4, we have $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} \Theta_{A_n}^-(t) = \Theta_A^-(t)$, $\lim_{n\to\infty} \Theta_{A_n}^+(t) = \Theta_A^+(t)$, uniformly for all $t \in [0,1]$. We see that $\ln(A_n) = \left(\ln(a_n); \ln \circ \Theta_{A_n}^-, \ln \circ \Theta_{A_n}^+\right)$ and $\ln A = \left(\ln a; \ln \circ \Theta_A^-, \ln \circ \Theta_A^+\right)$. According to the continuity of $\ln : (0,\infty) \to \mathbb{R}$, we have

$$\begin{aligned} \lim_{n\to\infty} \ln(a_n) &= \ln\left(\lim_{n\to\infty} a_n\right) = \ln a \\ \lim_{n\to\infty} \left(\ln \circ \Theta_{A_n}^-\right)(t) &= \lim_{n\to\infty} \ln\left(\Theta_{A_n}^-\right)(t) = \ln\left(\lim_{n\to\infty} \Theta_{A_n}^-(t)\right) = \ln\left(\Theta_A^-(t)\right) \\ &= \left(\ln \circ \Theta_A^-\right)(t) \\ \lim_{n\to\infty} \left(\ln \circ \Theta_{A_n}^+\right)(t) &= \lim_{n\to\infty} \ln\left(\Theta_{A_n}^+\right)(t) = \ln\left(\lim_{n\to\infty} \Theta_{A_n}^+(t)\right) = \ln\left(\Theta_A^+(t)\right) \\ &= \left(\ln \circ \Theta_A^+\right)(t) \end{aligned}$$

uniformly for $t \in [0,1]$. By Remark 4.4 we conclude that the sequence $(\ln(A_n))_{n\in\mathbb{N}}$ is convergent in $\mathfrak{F}_c$ in the sense of Definition 4.2 and $\lim_{n\to\infty} \ln(A_n) = \ln A$. This is the continuity of $\ln : \mathfrak{F}_c^* \to \mathfrak{F}_c$. The continuity of the function $\exp : \mathfrak{F}_c \to \mathfrak{F}_c^*$ can be proved similarly. □

Remark 5.4. As in the case of real numbers, where the exponential and logarithmic functions establish the isomorphism between (the Abelian groups) $(\mathbb{R}, +)$ and $\left(\mathbb{R}_+^*, \cdot\right)$, the functions exp and ln constructed above, establish the isomorphism between the monoids $(\mathfrak{F}, +)$ and $(\mathfrak{F}^*, \odot)$. This result extends for fuzzy numbers the known result from the real axis, and it is similar to the result from [8]. In [8] the result was obtained on $\mathfrak{F}_c^+$, where $\mathfrak{F}_c^+ = \{A \in \mathfrak{F}_c : supp(A) \subset (0,\infty)\}$, and in Theorem 5.3 the isomorphism is established on the subset $\mathfrak{F}^*$ of $\mathfrak{F}$, with $A \in \mathfrak{F}$ under the restriction $\Theta_A^-(t), \Theta_A^+(t) \geqslant 1$, $\forall t \in [0,1]$, that can be considered as the price paid here for the fully distributivity property. Of course, in $\mathfrak{F}^*$, the restriction $supp(A) \subset (0,\infty)$ is not imposed including the possibility to have $x_A^-(t) \leqslant 0$ for $t \in (0,1)$.

The fuzzy direct product permit us to introduce a new kind of fuzzy power with fuzzy exponent, as follows.

Definition 5.5. If $A = \left(a; \Theta_A^-, \Theta_A^+\right)$ and $B = \left(b; \Theta_B^-, \Theta_B^+\right)$ are two fuzzy numbers such that:

1. a^b is defined (in $\mathbb{R}$);
2. $\left(\Theta_A^-(t)\right)^{\Theta_B^-(t)}$ and $\left(\Theta_A^+(t)\right)^{\Theta_B^+(t)}$ are defined for each $t \in [0,1]$;
3. the functions $\left(\Theta_A^-\right)^{\Theta_B^-}$ and $\left(\Theta_A^+\right)^{\Theta_B^+}$ are decreasing;

then we define the $B-$power of A by $A^B = \left(a^b; \left(\Theta_A^-\right)^{\Theta_B^-}, \left(\Theta_A^+\right)^{\Theta_B^+}\right)$.

Remark 5.6. For instance, if $A \in \mathfrak{F}^*$, then A^B can be constructed for any $B \in \mathfrak{F}$, and we have $A^B \in \mathfrak{F}^*$.

Theorem 5.7. The fuzzy-power defined above, satisfies the following properties:

1. $A^{\overline{0}} = \overline{1}$ and $A^{\overline{1}} = A$, for all $A \in \mathfrak{F}^*$;
2. $A^{B \odot C} = \left(A^B\right)^C$, for all $A \in \mathfrak{F}^*$ and $B, C \in \mathfrak{F}$;
3. $(A \odot B)^C = A^C \odot B^C$, for all $A, B \in \mathfrak{F}^*$ and $C \in \mathfrak{F}$;
4. $A^B \odot A^C = A^{B+C}$, for all $A \in \mathfrak{F}^*$ and $B, C \in \mathfrak{F}$.

Proof. These statements results from the properties of the powers that involve operations such as $\left(\Theta_A^-\right)^{\Theta_B^-}$ and $\left(\Theta_A^+\right)^{\Theta_B^+}$. Indeed,

$$\begin{aligned}
A^{\overline{0}} &= \left(a^0; \left(\Theta_A^-\right)^0, \left(\Theta_A^+\right)^0\right) = (1; \epsilon, \epsilon) = \overline{1}, \\
A^{\overline{1}} &= \left(a; \left(\Theta_A^-\right)^\epsilon, \left(\Theta_A^+\right)^\epsilon\right) = A, \\
A^{B \odot C} &= \left(a^{b \cdot c}; \left(\Theta_A^-\right)^{\Theta_B^- \cdot \Theta_C^-}, \left(\Theta_A^+\right)^{\Theta_B^+ \cdot \Theta_C^+}\right) \\
&= \left(\left(a^b\right)^c; \left(\left(\Theta_A^-\right)^{\Theta_B^-}\right)^{\Theta_C^-}, \left(\left(\Theta_A^+\right)^{\Theta_B^+}\right)^{\Theta_C^+}\right) = \left(A^B\right)^C, \\
A^{B+C} &= \left(a^{b+c}; \left(\Theta_A^-\right)^{\Theta_B^- + \Theta_C^-}, \left(\Theta_A^+\right)^{\Theta_B^+ + \Theta_C^+}\right) \\
&= \left(a^b \cdot a^c; \left(\Theta_A^-\right)^{\Theta_B^-} \cdot \left(\Theta_A^-\right)^{\Theta_C^-}, \left(\Theta_A^+\right)^{\Theta_B^+} \cdot \left(\Theta_A^+\right)^{\Theta_C^+}\right) \\
&= \left(a^b; \left(\Theta_A^-\right)^{\Theta_B^-}, \left(\Theta_A^+\right)^{\Theta_B^+}\right) \odot \left(a^c; \left(\Theta_A^-\right)^{\Theta_C^-}, \left(\Theta_A^+\right)^{\Theta_C^+}\right) = A^B \odot A^C, \\
(A \odot B)^C &= \left((a \cdot b)^c; \left(\Theta_A^- \cdot \Theta_B^-\right)^{\Theta_C^-}, \left(\Theta_A^+ \cdot \Theta_B^+\right)^{\Theta_C^+}\right) \\
&= \left(a^c \cdot b^c; \left(\Theta_A^-\right)^{\Theta_C^-} \cdot \left(\Theta_B^-\right)^{\Theta_C^-}, \left(\Theta_A^+\right)^{\Theta_C^+} \cdot \left(\Theta_B^+\right)^{\Theta_C^+}\right) = A^C \odot B^C.
\end{aligned}$$

□

Remark 5.8. We see that the fourth property from Theorem 5.7 is not valid with equality in [25] (being valid only with multiplicative equivalence), but for the fuzzy direct product is valid. Moreover, the third property is valid in [25] for nonzero crisp exponent and with fuzzy numbers having positive support as basis. Here, this property is valid with any fuzzy exponent, but the basis are taken in the subset $\mathfrak{F}^*$. The second property is valid in [25] for exponents with support not containing zero and fuzzy numbers having positive support as basis, while in Theorem 5.7 the basis should belongs to $\mathfrak{F}^*$ and the exponents are arbitrary in $\mathfrak{F}$. So, the set of the algebraic properties is enriched for this type of fuzzy power, containing the fourth supplementary property. But here, the property $A^{-B} = 1/\left(A^B\right) = (1/A)^B$ cannot be considered because $1/A$ is not defined. An alternative for this property is considered in [10] for unimodal symmetric fuzzy numbers A, B, with A having positive midpoint.

CONCLUSIONS

The possibility to consider fuzzy multiplications preserving the distributivity laws regarding the addition is investigated obtaining two examples $\odot$ and $\otimes$ of such fuzzy multiplication. According to Theorem 3.2, the class of symmetric fuzzy numbers is invariant to any of the multiplications $\odot$ and $\otimes$, that in some applications, can be an advantage over the classical multiplication generated by the Zadeh's extension principle. Moreover, the uncertainty included in the two operands is not lost as it is suggested in Figure 2 (this failure it happens for the operations introduced in [21] with the operand having smaller spread). Interesting connections of the semiring $(\mathfrak{F}, +, \odot)$ with the sets of left-sided and right-sided fuzzy numbers, introduced and studied in [9], are pointed out in the framework of subtractive ideals, in Theorem 3.2.

Of course, the group properties $A + (-A) = \overline{0}$ and $A \odot A^{-1} = \overline{1}$ cannot be obtained because of deep reasonings that take in consideration the origin of the notion of fuzzy numbers. Such properties could be valid only with suitable equivalences, as was observed in [24].

Considering the subset $\mathfrak{F}^*$ of $\mathfrak{F}$, the semirings $(\mathfrak{F}, +)$ and $(\mathfrak{F}^*, \odot)$ are isomorphic and $(\mathfrak{F}_{\mathrm{c}}, +, \mathrm{d})$ and $(\mathfrak{F}^*_{\mathrm{c}}, \odot, \mathrm{d})$ are isomorphic and homeomorphic, too, with d be the complete metric defined in [21]. In addition to this, the fuzzy power defined in Section 5 has a supplementary attribute, $A^B \odot A^C = A^{B+C}$, for $A \in \mathfrak{F}^*$ and $B, C \in \mathfrak{F}$, that is additional to the properties of the fuzzy power presented in [25].

ACKNOWLEDGEMENT

The authors thank the anonymous reviewers for their valuable comments that greatly improved the manuscript.

(Received December 19, 2017)

REFERENCES

[1] P. J. Allen: A fundamental theorem of homomorphisms for semirings. Proc. Amer. Math. Soc. *21* (1969), 412–416. DOI:10.1090/s0002-9939-1969-0237575-4

[2] A. I. Ban and B. Bede: Properties of the cross product of fuzzy numbers. J. Fuzzy Math. *14* (2006), 513–531.

[3] B. Bede: Mathematics of Fuzzy Sets and Fuzzy Logic. Springer-Verlag, Berlin, Heidelberg 2013.

[4] B. Bede and J. Fodor: Product type operations between fuzzy numbers and their applications in geology. Acta Polytechn. Hungar. *3* (2006), 123–139.

[5] Ch.-Ch. Chou: The canonical representation of multiplication operation on triangular fuzzy numbers. Comput. Math. Appl. *45* (2003), 1601–1610. DOI:10.1016/s0898-1221(03)00139-1

[6] L. Coroianu: Necessary and sufficient conditions for the equality of the interactive and non-interactive sums of two fuzzy numbers. Fuzzy Sets Syst. *283* (2016), 40–55. DOI:10.1016/j.fss.2014.10.026

[7] L. Coroianu and R. Fuller: Necessary and sufficient conditions for the equality of interactive and non-interactive extensions of continuous functions. Fuzzy Sets Syst. *331* (2018), 116–130. DOI:10.1016/j.fss.2017.07.023

[8] A. M. Bica: Algebraic structures for fuzzy numbers from categorial point of view. Soft Computing *11* (2007), 1099–1105. DOI:10.1007/s00500-007-0167-x

[9] A. M. Bica: One-sided fuzzy numbers and applications to integral equations from epidemiology. Fuzzy Sets Syst. *219* (2013), 27–48. DOI:10.1016/j.fss.2012.08.002

[10] A. M. Bica: The middle-parametric representation of fuzzy numbers and applications to fuzzy interpolation. Int. J. Approximate Reasoning *68* (2016), 27–44. DOI:10.1016/j.ijar.2015.10.001

[11] M. Delgado, M. A. Vila, and W. Voxman: On a canonical representation of fuzzy numbers. Fuzzy Sets Syst. *93* (1998), 125–135. DOI:10.1016/s0165-0114(96)00144-3

[12] D. Dubois and H. Prade: Operations on fuzzy numbers. Int. J. Syst. Sci. *9* (1978), 613–626. DOI:10.1080/00207727808941724

[13] D. Dubois and H. Prade: Fuzzy Sets and Systems: Theory and Applications. Academic Press, New York 1980.

[14] R. Ebrahimi Atani and S. Ebrahimi Atani: Ideal theory in commutative semirings. Bul. Acad. Ştiinţe Repub. Mold. Mat. *57* (2008), 2, 14–23.

[15] R. Ebrahimi Atani: The ideal theory in quotients of commutative semirings. Glasnik Matematicki *42* (2007), 301–308. DOI:10.3336/gm.42.2.05

[16] R. Goetschel and W. Voxman: Elementary fuzzy calculus. Fuzzy Sets Syst. *18* (1986), 31–43. DOI:10.1016/0165-0114(86)90026-6

[17] J. S. Golan: Semirings and their Applications. Kluwer Academic Publishers, Dordrecht 1999. DOI:10.1007/978-94-015-9333-5_21

[18] M. L. Guerra and L. Stefanini: Crisp profile symmetric decomposition of fuzzy numbers. Appl. Math. Sci. *10* (2016), 1373–1389. DOI:10.12988/ams.2016.59598

[19] M. Hanss: Applied Fuzzy Arithmetic – An Introduction with Engineering Applications. Springer-Verlag, Berlin 2005. DOI:10.1007/b138914

[20] A. Kolesárová and D. Vivona: Entropy of T-sums and T-products of L-R fuzzy numbers. Kybernetika *37* (2001), 2, 127–145.

[21] M. Ma, M. Friedman, and A. Kandel: A new fuzzy arithmetic. Fuzzy Sets Syst. *108* (1999), 83–90. DOI:10.1016/s0165-0114(97)00310-2

[22] M. Mareš: Multiplication of fuzzy quantities. Kybernetika *28* (1992), 5, 337–356.

[23] M. Mareš: Brief note on distributivity of triangular fuzzy numbers. Kybernetika *31* (1995), 5, 451–457.

[24] M. Mareš: Fuzzy zero, algebraic equivalence: yes or no? Kybernetika *32* (1996), 4, 343–351.

[25] M. Mareš: Weak arithmetics of fuzzy numbers. Fuzzy Sets Syst. *91* (1997), 143–153. DOI:10.1016/s0165-0114(97)00136-x

[26] S. Markov: On quasilinear spaces of convex bodies and intervals. J. Comput. Appl. Math. *162* (2004), 93–112. DOI:10.1016/j.cam.2003.08.016

[27] S. Markov: On directed interval arithmetic and its applications. J. Universal Computer Sci. *7* (1995), 514–526. DOI:10.1007/978-3-642-80350-5_43

[28] S. Markov: On the algebraic properties of intervals and some applications. Reliable Computing *7* (2001), 113–127. DOI:10.1023/a:1011418014248

[29] R. Mesiar and J. Ribarik: Pan operations structure. Fuzzy Sets Syst. *74* (1995), 365–369. DOI:10.1016/0165-0114(94)00314-w

[30] M. Mizumoto and K. Tanaka: The four operations of arithmetic on fuzzy numbers. Systems Comput. Controls *7* (1976), 5, 73–81.

[31] M. Mizumoto and K. Tanaka: Some properties of fuzzy numbers. In: Advances in Fuzzy Set Theory and Applications (M. H. Gupta, R. K. Ragade, and R. R. Yager, eds.), North-Holland, Amsterdam, 1979, pp. 156–164.

[32] J. N. Mordeson and P. S. Nair: Fuzzy Mathematics: An Introduction for Engineers and Scientists. Studies in Fuzziness and Soft Computing, Physica-Verlag, Heidelberg, New York 2001.

[33] S. H. Nasseri and N. Mahdavi-Amiri: Some duality results on linear programming problems with symmetric fuzzy numbers. Fuzzy Inf. Eng. *1* (2009), 1, 59–66. DOI:10.1007/s12543-009-0004-2

[34] D. Qiu and W. Zhang: Symmetric fuzzy numbers and additive equivalence of fuzzy numbers. Soft Comput. *17* (2013), 1471–1477. DOI:10.1007/s00500-013-1000-3

[35] J. Schneider: Arithmetic of fuzzy numbers and intervals-a new perspective with examples. arXiv: 1310.5604 [math.GM] (2016).

[36] L. Stefanini, L. Sorini, and M. L. Guerra: Parametric representation of fuzzy numbers and application to fuzzy calculus. Fuzzy Sets Syst. *157* (2006), 2423–2455. DOI:10.1016/j.fss.2006.02.002

[37] L. Stefanini and M. L. Guerra: On fuzzy arithmetic operations: some properties and distributive approximations. Int. J. Appl. Math. *19* (2006), 171–199.

[38] A. Stupňanová: A probabilistic approach to the arithmetics of fuzzy numbers. Fuzzy Sets Syst. *264* (2015), 64–75. DOI:10.1016/j.fss.2014.08.013

[39] A. Taleshian and S. Rezvani: Multiplication operation on trapezoidal fuzzy numbers. J. Phys. Sci. *15* (2011), 17–26.

[40] E. K. Zavadskas, J. Antucheviciene, S. H. Razavi Hajiagha, and S. Sadat Hashemi: Extension of weighted aggregated sum product assessment with interval-valued intuitionistic fuzzy numbers (WASPAS-IVIF). Appl. Soft Comput. *24* (2014), 1013–1021. DOI:10.1016/j.asoc.2014.08.031

Alexandru Mihai Bica, Department of Mathematics and Informatics, University of Oradea. Romania.
e-mail: abica@uoradea.ro

Dorina Fechete, Department of Mathematics and Informatics, University of Oradea. Romania.
e-mail: dfechete@uoradea.ro

Ioan Fechete, Department of Mathematics and Informatics, University of Oradea. Romania.
e-mail: ifechete@uoradea.ro