KYBERNETIKA — VOLUME *55* (2019), NUMBER 1, PAGES 12–23

A BOUND FOR THE RANK-ONE TRANSIENT OF INHOMOGENEOUS MATRIX PRODUCTS IN SPECIAL CASE

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We consider inhomogeneous matrix products over max-plus algebra, where the matrices in the product satisfy certain assumptions under which the matrix products of sufficient length are rank-one, as it was shown in [6] (*Shue, Anderson, Dey 1998*). We establish a bound on the transient after which any product of matrices whose length exceeds that bound becomes rank-one.

Keywords: max-plus algebra, matrix product, rank-one, walk, Trellis digraph

Classification: 15A80, 68R99, 16Y60, 05C20, 05C22, 05C25

1. INTRODUCTION

By max-plus algebra we mean the linear algebra developed over the max-plus semiring $\mathbb{R}_{\max}$, which is the set $\mathbb{R}_{\max} = \mathbb{R} \cup \{-\infty\}$ equipped the additive operator $a \oplus b = \max\{a, b\}$ and the multiplicative operator $a \otimes b = a + b$. We will be mostly interested in the max-plus matrix multiplication $A \otimes B$ defined for any two matrices $A = (a_{i,j})$ and $B = (b_{i,j})$ with entries in $\mathbb{R}_{\max}$ of appropriate sizes by the rule

$$(A \otimes B)_{i,j} = \bigoplus_{1 \le k \le n} a_{i,k} \otimes b_{k,j} \;= \max_{1 \le k \le n} a_{i,k} + b_{k,j}.$$

In particular, the kth max-plus power of a square matrix A is defined as

$$A^{\otimes k} = \underbrace{A \otimes A \otimes \ldots \otimes A}_{(k \text{ times})}.$$

A lot of work has been done on max-plus powers of a single matrix. Main results of the present paper are in some relation to the bounds on the ultimate periodicity of the sequence of max-plus matrix powers $\{A^t\}_{t \ge 1}$, like those established in [4], [5]. However, instead of max-plus powers of a single matrix we will consider max-plus inhomogeneous matrix products of the form $A_1 \otimes A_2 \ldots \otimes \ldots \otimes A_k$ where matrices $A_1, \ldots, A_k$ are taken from an infinite matrix set $\mathcal{X}$. We will make use of the assumptions made

DOI: 10.14736/kyb-2019-1-0012

in [6] and derive a bound for the *rank-one transient* of inhomogeneous products of matrices from $\mathcal{X}$ which is the minimal K such that $A_1 \otimes A_2 \otimes \ldots \otimes A_k$ for any $k \geq K$ can be represented as a max-plus outer product $\vec{x} \otimes \vec{y}^\top$, where column vectors $\vec{x}$ and $\vec{y}$ depend on the matrix product. In Theorem 4.1 we first obtain a sufficient condition for an inhomogenous product to be rank-one. The bound on the rank-one transient is then obtained in Corollaries 4.3 and 4.4.

A practical motivation of this study comes from the switching max-plus dynamical systems of the form $x(k+1) = A(k) \otimes x(k)$ where matrices $A(k)$ can vary. Such systems arise in some scheduling applications being related to the way that max-plus algebra is used in modeling discrete event dynamical systems [1]. Let us also note, in particular, a recent application of switching max-plus systems of above form in the legged locomotion of robots [3], where changing matrices $A(k)$ model the switch of gaits.

This paper is based on the ideas of [6] where the steady state properties of max-plus inhomogeneous matrix products were considered. The aim of [6] was to prove that, under certain assumptions, a sufficiently long max-plus matrix product is rank-one and it can be written as the outer product of two vectors. Components of these vectors are optimal weights of walks going to and from node 1 respectively. However, it seems to us that there is an oversight in [6, Corollary 3.1]. This oversight is that in order to prove that the initial and final parts of an optimal walk are bounded in length, paper [6] uses a method in which one removes part of a walk in order to create a more optimal walk. This would be fine if the matrices were the same however since they are different then removing matrices from the product changes the product and one ends up working with a different product. The result of [6] is also proved for a sufficient k that is large enough but no concrete bounds are established, so this invited us to look for a bound on the length of a max-plus inhomogeneous matrix product after which it becomes an outer product of two vectors. Such bound is the main result of this paper.

The structure of this paper will be as follows. Chapter 2 defines the key ideas and notation that will be used throughout the paper. In Chapter 3 we introduce and prove the lemmas required to prove the main theorem. Chapter 4 contains the proof of the main theorem as well as corollaries that follow from the theorem one being a coarser bound on k. Finally, Chapter 5 presents an example which demonstrates a long enough inhomogeneous matrix product which is an outer max-plus product of two vectors.

2. DEFINITIONS AND ASSUMPTIONS

2.1. Walks and digraphs

The aim of this subsection is to introduce some important definitions concerning 1) directed weighted graphs, associated with a matrix and 2) trellis digraphs associated with inhomogeneous matrix products. Note that Definitions 2.2 and 2.4 are standard [2], and Definition 2.3 follows [6].

Definition 2.1. A *directed graph (digraph)* is a pair (N, E) where N is a finite set of nodes and $E \subseteq N \times N = \{(i,j) \colon i,j \in N\}$ is the set of edges where (i,j) is a directed edge from node i to node j.

A *weighted digraph* is a digraph with associated weights $w_{i,j} \in \mathbb{R}_{\max}$ for each edge (i,j) in the digraph.

Definition 2.2. A *digraph associated with a square matrix* A is a digraph $\mathcal{D}_A = (N_A, E_A)$ where the set N_A has the same number of elements as the number of rows or columns in the matrix A. The set $E_A \subseteq N_A \times N_A$ is the set of edges in $\mathcal{D}_A$ where the weight of each edge (i, j) is associated with the respective entry in the matrix A, i. e. $w_{i,j} = a_{i,j} \in \mathbb{R}_{\max}$. If an entry in the matrix is negative infinity, this means that there is no edge connecting those nodes in that direction.

Definition 2.3. Matrices $A, B \in \mathbb{R}^{n\times n}_{\max}$ are called *geometrically equivalent* if $E_A = E_B$.

Definition 2.4. A sequence of nodes $W = (i_0, \ldots, i_l)$ is called a *walk on a weighted digraph* $D = (N, E)$ if $(i_{s-1}, i_s) \in E$ for each $s\colon 1 \le s \le l$. This walk is a *cycle* if the start node i_0 and the end node i_l are the same. It is a *path* if no two nodes in $i_0, \ldots, i_l$ are the same. The *length* of W is $l(W) = l$. The *weight* of W is defined as the max-plus product (i. e., the usual arithmetic sum) of the weights of each edge (i_{s-1}, i_s) traversed throughout the walk, and it is denoted by $p_D(W)$. Note that a sequence $W = (i_0)$ is also a walk (without edges), and we assume that it has weight and length 0.

A digraph is *strongly connected* if for any two nodes i and j there exists a walk connecting i to j. A matrix is *irreducible* if the graph associated with it in the sense of Definition 2.2 is strongly connected.

Definition 2.5. The *trellis digraph* $\mathcal{T}_{\Gamma(k)} = (\mathcal{N}, \mathcal{E})$ associated with the product $\Gamma(k) = A_1 \otimes A_2 \otimes \ldots \otimes A_k$ is the digraph with the set of nodes $\mathcal{N}$ and the set of edges $\mathcal{E}$, where:

(1) $\mathcal{N}$ consists of $k + 1$ copies of N which are denoted $N_0, \ldots, N_k$, and the nodes in N_l for each $0 \le l \le k$ are denoted by $1 : l, \ldots, n : l$;

(2) $\mathcal{E}$ is defined by the following rules:

  a) there are edges only between N_l and N_{l+1} for each l,

  b) we have $(i : (l-1), j : l) \in \mathcal{E}$ if and only if (i, j) is an edge of $\mathcal{D}_{A_l}$, and the weight of that edge is $(A_l)_{i,j}$.

The weight of a walk W on $\mathcal{T}_{\Gamma(k)}$ is denoted by $p_{\mathcal{T}}(W)$.

Definition 2.6. Consider a trellis digraph $\mathcal{T}_{\Gamma(k)}$.

By an *initial walk* connecting i to j on $\mathcal{T}_{\Gamma(k)}$ we mean a walk on $\mathcal{T}_{\Gamma(k)}$ connecting node $i : 0$ to $j : m$ where m is the first and the last time the walk arrives at node j and is such that $0 \le m \le k$.

By a *final walk* connecting i to j on $\mathcal{T}_{\Gamma(k)}$ we mean a walk on $\mathcal{T}_{\Gamma(k)}$ connecting node $i : l$ to $j : k$, where l is the first and the last time the walk leaves node i and is such that $0 \le l \le k$.

A *full walk* connecting i to j on $\mathcal{T}_{\Gamma(k)}$ is a walk on $\mathcal{T}_{\Gamma(k)}$ connecting node $i : 0$ to $j : k$.

2.2. Key notations

Here we will introduce the notation that will be used throughout the paper. We begin by introducing the following two matrices.

Notation 2.7. The "boundaries" of $\mathcal{X}$:

$A^{\sup}$: the entrywise *supremum over all matrices* in $\mathcal{X}$. More precisely, $A^{\sup}_{ij} = \sup_{X\in\mathcal{X}} (X)_{ij}$.

In max-plus matrix notation,

$$A^{\sup} = \bigoplus_{X\in\mathcal{X}} X.$$

The weight of a walk W on $\mathcal{D}_{A^{\sup}}$ will be denoted by $p_{\sup}(W)$.

$A^{\inf}$: the entrywise *infimum over all matrices* in $\mathcal{X}$. More precisely, $A^{\inf}_{ij} = \inf_{X\in\mathcal{X}} (X)_{ij}$.

We now introduce a number of useful parameters. The first group of parameters relates to $\mathcal{D}_{A^{\sup}}$ and $\mathcal{D}_{A^{\inf}}$, and the second to $\mathcal{T}_{\Gamma(k)}$.

Notation 2.8. λ^*: the largest cycle mean in the submatrix $(A^{\sup}_{i,j})_{i,j\neq 1}$:

$$\lambda^* = \max_{k\geq 1}\left(\max_{2\leq i_1,\ldots,i_k\leq n} \frac{A^{\sup}_{i_1 i_2} + \ldots + A^{\sup}_{i_k i_1}}{k}\right).$$

Notation 2.9. Weights of some paths and walks on $\mathcal{D}_{A^{\sup}}$ and $\mathcal{D}_{A^{\inf}}$:

- α_i: the maximal weight of paths on $\mathcal{D}_{A^{\sup}}$ connecting i to 1;
- β_j: the maximal weight of paths on $\mathcal{D}_{A^{\sup}}$ connecting 1 to j;
- γ_{ij}: the maximal weight of paths on $\mathcal{D}_{A^{\sup}}$ connecting i to j and not going through node 1;
- w_i: the maximal weight of walks of length not exceeding k on $\mathcal{D}_{A^{\inf}}$ connecting i to 1;
- v_j: the maximal weight of walks of length not exceeding k on $\mathcal{D}_{A^{\inf}}$ connecting 1 to j.

Notation 2.10. Weight of optimal walks on $\mathcal{T}_{\Gamma(k)}$:

- w_i^*: the maximal weight of initial walks on $\mathcal{T}_{\Gamma(k)}$ connecting i to 1;
- v_j^*: the maximal weight of final walks on $\mathcal{T}_{\Gamma(k)}$ connecting 1 to j.

Note that the length of any walk on $\mathcal{T}_{\Gamma(k)}$ does not exceed k.

2.3. Key assumptions

We will use the following main assumptions, which are very similar to those of [6].

Assumption 2.11. The matrices A_i, $i \in 1,\ldots,k$ are chosen from a set $\mathcal{X}$ of geometrically equivalent irreducible matrices, and the matrix $A^{\inf}$ is also geometrically equivalent to any of them.

Assumption 2.12. The digraph of each matrix in the set $\mathcal{X}$ has a unique critical cycle of length 1 at node 1 with weight 0.

Assumption 2.13. The digraph of the matrix A^{sup} has a unique critical cycle of length 1 at node 1 of weight 0.

Note that, if the unique critical cycle of length one is at any other node than 1, then none of our main results will change significantly. However, the given assumptions are still very limiting in terms of the type of matrix and the real world situation that this can apply to.

3. PRELIMINARY LEMMAS

The aim of this section is to prove some preliminary lemmas which will help us to construct the terms in the bound of Theorem 4.1 and its corollaries. The main ideas are that the lengths of optimal initial and final walks are bounded (Lemmas 3.1 and 3.2) and that after some transient on the length any optimal full walk should pass through node 1 (Lemma 3.3).

Lemma 3.1. Let W_1 be an optimal initial walk on trellis digraph $\mathcal{T}_{\Gamma(k)}$ connecting i to 1. Then we have the following upper bound on its length:

$$l(W_1) \leq \frac{w_i^* - \alpha_i}{\lambda^*} + (n-1). \tag{1}$$

Proof. Due to Assumptions 2.12 and 2.13 we have $\lambda^* < 0$. The weight of any optimal walk W_1 connecting i to 1 is less than or equal to that of a path P_1 connecting i to 1 on $\mathcal{D}_{A^{\text{sup}}}$ plus the remaining length multiplied by $\lambda^* < 0$. Thus

$$p_{\mathcal{T}}(W_1) \leq p_{\text{sup}}(P_1) + (l(W_1) - (n-1))\lambda^*.$$

Next we bound $p_{\text{sup}}(P_1) \leq \alpha_i$, hence

$$p_{\mathcal{T}}(W_1) \leq \alpha_i + (l(W_1) - (n-1))\lambda^*. \tag{2}$$

Now assume by contradiction that $l(W_1) > \frac{w_i^* - \alpha_i}{\lambda^*} + (n-1)$. However, this is equivalent to

$$\alpha_i + (l(W_1) - (n-1))\lambda^* < w_i^*. \tag{3}$$

Combining (2) with (3) we obtain $p_{\mathcal{T}}(W_1) < w_i^*$ meaning that W_1 is not optimal, a contradiction. The proof is complete. □

We now state an analogous lemma on the length of an optimal final walk. The proof is similar and will be omitted.

Lemma 3.2. Let W_2 be an optimal final walk on trellis digraph $\mathcal{T}_{\Gamma(k)}$ connecting 1 to j. Then we have the following upper bound on its length:

$$l(W_2) \leq \frac{v_j^* - \beta_j}{\lambda^*} + (n-1). \tag{4}$$

Lemma 3.3. Let
$$k > \frac{w_i^* - \alpha_i + v_j^* - \beta_j}{\lambda^*} + 2(n-1). \tag{5}$$
Then any optimal full walk W connecting i to j on $\mathcal{T}_{\Gamma(k)}$ and going through node 1 is decomposed as,

$$W = W_1 \circ C \circ W_2$$

where W_1 is an optimal initial walk and W_2 is an optimal final walk which satisfy

$$l(W_1) \leq \frac{w_i^* - \alpha_i}{\lambda^*} + (n-1),$$
$$l(W_2) \leq \frac{v_j^* - \beta_j}{\lambda^*} + (n-1),$$

C consists of several loops $1 \to 1$ and

$$p_{\mathcal{T}}(W) = w_i^* + v_j^*.$$

P r o o f. Let W be an optimal full walk connecting i to j that traverses node 1 at least once. Note first that all edges between the first and the last occurrence of 1 in W can be replaced with the copies of $(1,1)$, since these edges are present in every matrix X_α from $\mathcal{X}$. Assumption 2.13 implies that this leads to a strict increase of the weight, therefore we must have $W = \tilde{W}_1 \circ \tilde{C} \circ \tilde{W}_2$, where $\tilde{C}$ consists of several edges $(1,1)$, $\tilde{W}_1$ is an initial walk from i to 1 and $\tilde{W}_2$ is a final walk from 1 to j. We have $p_{\mathcal{T}}(\tilde{C}) = 0$, so $p_{\mathcal{T}}(W) = p_{\mathcal{T}}(\tilde{W}_1) + p_{\mathcal{T}}(\tilde{W}_2)$.

Now we note that by Lemmas 3.1 and 3.2 the length k is sufficient for constructing a walk $W' = V_1 \circ C' \circ V_2$ where V_1 is an optimal initial walk from i to 1, C' consists of several copies of $(1,1)$ and V_2 is an optimal final walk from 1 to j. The weight of this walk is $w_i^* + v_j^*$.

By the optimality of V_1 and V_2 we have $p_{\mathcal{T}}(\tilde{W}_1) \leq p_{\mathcal{T}}(V_1)$ and $p_{\mathcal{T}}(\tilde{W}_2) \leq p_{\mathcal{T}}(V_2)$. Since W is optimal, both inequalities should hold with equality.

That is, $\tilde{W}_1$ is an optimal initial walk connecting i to 1 and $\tilde{W}_2$ is an optimal final walk connecting 1 to j, so that $\tilde{W}_1$, $\tilde{W}_2$ and $\tilde{C}$ can be taken for W_1, W_2 and C respectively. The proof is complete. □

Lemma 3.4. Let
$$k > \frac{w_i^* + v_j^* - \gamma_{i,j}}{\lambda^*} + (n-1). \tag{6}$$
Then any full walk W connecting i to j on $\mathcal{T}_{\Gamma(k)}$ that does not go through node 1 has weight smaller than $w_i^* + v_j^*$.

P r o o f. Due to Assumption 3.1 and 3.2 the weight of any walk $p_{\mathcal{T}}(W)$ connecting $i \to j$ and not going through 1 will be less than or equal to the weight of a path P on $\mathcal{D}_{A^{\sup}}$ going from i to j plus the remaining length multiplied by λ^*:

$$p_{\mathcal{T}}(W) \leq p_{\sup}(P) + (k - (n-1))\lambda^*. \tag{7}$$

As P is a path from $i \to j$, its weight is bounded above by γ_{ij}. Therefore

$$p_{\sup}(P) + (k-(n-1))\lambda^* \leq \gamma_{ij} + (k-(n-1))\lambda^*. \tag{8}$$

We now see that (6) is equivalent to

$$\gamma_{ij} + (k-(n-1))\lambda^* < w_i^* + v_j^*. \tag{9}$$

Combining (7),(8) and (9) we see that $p_{\mathcal{T}}(W) < w_i^* + v_j^*$, thus the proof is complete.

□

4. MAIN RESULTS

Now we can move on to the main theorem of the paper, with its modifications and corollaries.

Theorem 4.1. Let $\Gamma(k)$ be an inhomogenous max-plus matrix product $\Gamma(k) = A_1 \otimes A_2 \otimes \ldots \otimes A_k$ with k satisfying

$$k > \max\Big(\frac{w_i^* + v_j^* - \gamma_{ij}}{\lambda^*} + (n-1), \frac{w_i^* - \alpha_i + v_j^* - \beta_j}{\lambda^*} + 2(n-1)\Big)\Big) \tag{10}$$

for some $i, j \in N$, then

$$\begin{aligned}\Gamma(k)_{i,j} &= \Gamma(k)_{i,1} \otimes \Gamma(k)_{1,j}\\ &= \Gamma(k)_{i,1} + \Gamma(k)_{1,j}.\end{aligned}$$

Proof. As seen by Lemma 3.4, if

$$k > \frac{w_i^* + v_j^* - \gamma_{ij}}{\lambda^*} + (n-1)$$

then any walk on $\mathcal{T}_{\Gamma(k)}$ not going through node 1 will have weight smaller than $w_i^* + v_j^*$. By Lemma 3.3, if

$$k > \frac{w_i^* - \alpha_i + v_j^* - \beta_j}{\lambda^*} + 2(n-1)$$

then any optimal full walk going through node 1 will consist of the three parts W_1, W_2 and C as defined in the Lemma and its weight will be $w_i^* + v_j^*$. Hence if k satisfies both inequalities then any optimal full walk goes through node1 and has weight

$$\Gamma(k)_{ij} = w_i^* + v_j^*$$

Observe that w_1^* and v_1^* are equal to 0, since the weight of any optimal initial or final walk on $\mathcal{T}_{\Gamma(k)}$ connecting 1 to 1 is 0. Therefore

$$\begin{aligned}\Gamma(k)_{i,1} &= w_i^* + v_1^* = w_i^*,\\ \Gamma(k)_{1,j} &= w_1^* + v_j^* = v_j^*,\end{aligned}$$

and

$$\Gamma(k)_{i,j} = \Gamma(k)_{i,1} + \Gamma(k)_{1,j}.$$

□

Let us extend Theorem 4.1 to a matrix form.

Corollary 4.2. If the matrix product $\Gamma(k)$ with length k satisfies

$$k > \max_{i,j\in N}\Big(\frac{w_i^* + v_j^* - \gamma_{ij}}{\lambda^*} + (n-1), \frac{w_i^* - \alpha_i + v_j^* - \beta_j}{\lambda^*} + 2(n-1)\Big)\Big)$$

for all $i, j \in N$, then $\Gamma(k)$ is rank one and

$$\Gamma(k) = \begin{bmatrix}\Gamma(k)_{1,1}\\ \Gamma(k)_{2,1}\\ \vdots \\ \Gamma(k)_{n,1}\end{bmatrix} \otimes \begin{bmatrix}\Gamma(k)_{1,1} & \Gamma(k)_{1,2} & \dots & \Gamma(k)_{1,n}\end{bmatrix}.$$

P r o o f. Using Theorem 4.1 for all $i, j \in N$, if k satisfies the condition (10) then

$$\Gamma(k)_{i,j} = \Gamma(k)_{i,1} + \Gamma(k)_{1,j}.$$

Since this applies for all $i, j \in N$, $\Gamma(k)_{i,1}$ and $\Gamma(k)_{1,j}$ can be written as vectors in $\mathbb{R}^n$. Using the max-plus outer product of these two vectors it becomes

$$\Gamma(k) = \begin{bmatrix}\Gamma(k)_{1,1}\\ \Gamma(k)_{2,1}\\ \vdots \\ \Gamma(k)_{n,1}\end{bmatrix} \otimes \begin{bmatrix}\Gamma(k)_{1,1}\\ \Gamma(k)_{1,2}\\ \vdots \\ \Gamma(k)_{1,n}\end{bmatrix}^\top$$

thus proving the corollary. □

The bounds of Theorem 4.1 and Corollary 4.2 are interesting to see but they are implicit. This is because in order to calculate w_i^* and v_j^* you need to calculate $\Gamma(k)$ in which the length of the product is dictated by the bound using w_i^* and v_j^*. However another bound can be derived from Theorem 4.1 using A^{inf}. From the definition of A^{inf}, w_i and v_j it is easy to see that for all $i, j \in N$

$$w_i \le w_i^* \quad \text{and} \quad v_j \le v_j^*$$

These inequalities, together with Theorem 4.1, imply the following results.

Corollary 4.3. Let $\Gamma(k)$ be an inhomogenous max-plus matrix product $\Gamma(k) = A_1 \otimes A_2 \otimes \ldots \otimes A_k$ with k satisfying

$$k > \max\Big(\frac{w_i + v_j - \gamma_{ij}}{\lambda^*} + (n-1), \frac{w_i - \alpha_i + v_j - \beta_j}{\lambda^*} + 2(n-1)\Big)\Big)$$

for some $i, j \in N$, then

$$\begin{aligned}\Gamma(k)_{i,j} &= \Gamma(k)_{i,1} \otimes \Gamma(k)_{1,j}\\ &= \Gamma(k)_{i,1} + \Gamma(k)_{1,j}.\end{aligned}$$

We now also extend Corollary 4.3 to a matrix form.

Corollary 4.4. Let $\Gamma(k)$ be an inhomogenous max-plus matrix product $\Gamma(k) = A_1 \otimes A_2 \otimes \ldots \otimes A_k$ with k satisfying

$$k > \max_{i,j\in N} \Big(\frac{w_i + v_j - \gamma_{ij}}{\lambda^*} + (n-1), \frac{w_i - \alpha_i + v_j - \beta_j}{\lambda^*} + 2(n-1)) \Big)$$

then $\Gamma(k)$ is rank one and

$$\Gamma(k) = \begin{bmatrix} \Gamma(k)_{1,1} \\ \Gamma(k)_{2,1} \\ \vdots \\ \Gamma(k)_{n,1} \end{bmatrix} \otimes \begin{bmatrix} \Gamma(k)_{1,1} & \Gamma(k)_{1,2} & \ldots & \Gamma(k)_{1,n} \end{bmatrix}.$$

Note that this bound is explicit, and in particular it can be found numerically without having to calculate $\Gamma(k)$ beforehand. This is a bound for the rank-one transient of inhomogeneous products.

5. AN EXAMPLE

To illustrate what has been achieved in the paper let us consider an example. Let D_A be a digraph consisting of five nodes with the generalised associated weight matrix (for convention let $\varepsilon = -\infty$),

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} & \varepsilon & \varepsilon \\ a_{2,1} & \varepsilon & \varepsilon & \varepsilon & a_{2,5} \\ \varepsilon & \varepsilon & \varepsilon & a_{3,4} & \varepsilon \\ \varepsilon & a_{4,2} & \varepsilon & \varepsilon & \varepsilon \\ a_{5,1} & \varepsilon & \varepsilon & a_{5,4} & \varepsilon \end{bmatrix},$$

where $a_{i,j} \in \mathbb{R}_{\max}$. Consider the set $\mathcal{X} = \{A_1, A_2, A_3\}$ where

$$A_1 = \begin{bmatrix} 0 & -1 & -2 & \varepsilon & \varepsilon \\ -3 & \varepsilon & \varepsilon & \varepsilon & -3 \\ \varepsilon & \varepsilon & \varepsilon & -4 & \varepsilon \\ \varepsilon & -5 & \varepsilon & \varepsilon & \varepsilon \\ -6 & \varepsilon & \varepsilon & -5 & \varepsilon \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 0 & -4 & -3 & \varepsilon & \varepsilon \\ -4 & \varepsilon & \varepsilon & \varepsilon & -3 \\ \varepsilon & \varepsilon & \varepsilon & -2 & \varepsilon \\ \varepsilon & -1 & \varepsilon & \varepsilon & \varepsilon \\ -1 & \varepsilon & \varepsilon & 1 & \varepsilon \end{bmatrix},$$

$$A_3 = \begin{bmatrix} 0 & 2 & -4 & \varepsilon & \varepsilon \\ -5 & \varepsilon & \varepsilon & \varepsilon & -6 \\ \varepsilon & \varepsilon & \varepsilon & -4 & \varepsilon \\ \varepsilon & -3 & \varepsilon & \varepsilon & \varepsilon \\ -2 & \varepsilon & \varepsilon & 2 & \varepsilon \end{bmatrix}.$$

It can be seen that these satisfy the assumptions with the top left entry of each matrix being zero. Using these we can calculate the coarser bounds of Corollaries 4.3 and 4.4. In order to do that we need $A^{\sup}$ and $A^{\inf}$, which are

$$A^{\sup} = \begin{bmatrix} 0 & 2 & -2 & \varepsilon & \varepsilon \\ -3 & \varepsilon & \varepsilon & \varepsilon & -3 \\ \varepsilon & \varepsilon & \varepsilon & -2 & \varepsilon \\ \varepsilon & -1 & \varepsilon & \varepsilon & \varepsilon \\ -1 & \varepsilon & \varepsilon & 2 & \varepsilon \end{bmatrix} \quad \text{and} \quad A^{\inf} = \begin{bmatrix} 0 & -4 & -4 & \varepsilon & \varepsilon \\ -5 & \varepsilon & \varepsilon & \varepsilon & -6 \\ \varepsilon & \varepsilon & \varepsilon & -4 & \varepsilon \\ \varepsilon & -5 & \varepsilon & \varepsilon & \varepsilon \\ -6 & \varepsilon & \varepsilon & -5 & \varepsilon \end{bmatrix}.$$

We now begin to calculate the bounds of Corollaries 4.3 and 4.4. The only cycle that does not got through node 1 is $(2 \to 5 \to 4 \to 2)$ which has average weight $\lambda^* = -\frac{2}{3}$. Using $A^{\sup}$ we get α_i, β_j and $\gamma_{i,j}$ as the entries of

$$\alpha = \begin{bmatrix} 0 \\ -3 \\ -6 \\ -4 \\ -1 \end{bmatrix}, \ \beta = \begin{bmatrix} 0 \\ 2 \\ -2 \\ 1 \\ -1 \end{bmatrix}, \ \gamma = \begin{bmatrix} \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & -2 & \varepsilon & -1 & -3 \\ \varepsilon & -3 & \varepsilon & -2 & -6 \\ \varepsilon & -1 & \varepsilon & -2 & -4 \\ \varepsilon & 1 & \varepsilon & 2 & -2 \end{bmatrix}.$$

Using $A^{\inf}$ we can also calculate w_i and v_j as the entries of

$$w = \begin{bmatrix} 0 \\ -4 \\ -13 \\ -9 \\ -6 \end{bmatrix}, \ v = \begin{bmatrix} 0 \\ -5 \\ -4 \\ -8 \\ -10 \end{bmatrix}.$$

With these pieces we can construct the bounds for k for each combination of i and j:

$$k > \max_{i,j \in N} \left(\begin{bmatrix} \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & 14.5 & \varepsilon & 20.5 & 20.5 \\ \varepsilon & 26.5 & \varepsilon & 32.5 & 29.5 \\ \varepsilon & 23.5 & \varepsilon & 26.5 & 26.5 \\ \varepsilon & 22 & \varepsilon & 28 & 25 \end{bmatrix}, \begin{bmatrix} 8 & 18.5 & 11 & 21.5 & 21 \\ 9 & 20 & 12.5 & 23 & 23 \\ 18 & 29 & 21.5 & 32 & 32 \\ 15 & 26 & 18.5 & 29 & 29 \\ 15.5 & 26 & 18.5 & 29 & 29 \end{bmatrix} \right) \Leftrightarrow k > 32.$$

This means that if a matrix product $\Gamma(k)$ has length greater then 32 then it will be rank-one. Let us now take a random product of length 44:

$$\begin{aligned} \Gamma(k) = {} & A_1 \otimes A_3 \otimes A_1 \otimes A_2 \otimes A_3 \otimes A_1 \otimes A_2 \otimes A_2 \otimes A_1 \otimes A_3 \otimes A_1 \\ & \otimes A_2 \otimes A_1 \otimes A_2 \otimes A_3 \otimes A_3 \otimes A_1 \otimes A_2 \otimes A_1 \otimes A_1 \otimes A_3 \otimes A_2 \\ & \otimes A_3 \otimes A_2 \otimes A_2 \otimes A_3 \otimes A_1 \otimes A_1 \otimes A_2 \otimes A_3 \otimes A_2 \otimes A_1 \otimes A_3 \\ & \otimes A_1 \otimes A_2 \otimes A_3 \otimes A_1 \otimes A_3 \otimes A_3 \otimes A_1 \otimes A_2 \otimes A_2 \otimes A_1 \otimes A_1. \end{aligned}$$

We obtain that

$$\Gamma(k) = \begin{bmatrix} 0 & -1 & -2 & -6 & -4 \\ -3 & -4 & -5 & -9 & -7 \\ -10 & -11 & -12 & -16 & -14 \\ -10 & -11 & -12 & -16 & -14 \\ -6 & -7 & -8 & -12 & -10 \end{bmatrix}.$$

We see that $\Gamma(k) = w_i^* \otimes (v_j^*)^\top = \Gamma(k)_{i,1} \otimes (\Gamma(k)_{1,j})^\top$ where

$$w^* = \begin{bmatrix} 0 \\ -3 \\ -10 \\ -10 \\ -6 \end{bmatrix}, \ v^* = \begin{bmatrix} 0 \\ -1 \\ -2 \\ -6 \\ -4 \end{bmatrix}.$$

Note that the bound appearing in Corollary 4.2 is equal to

$$\max_{i,j \in N} \left(\begin{bmatrix} \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & 7 & \varepsilon & 16 & 10 \\ \varepsilon & 16 & \varepsilon & 25 & 16 \\ \varepsilon & 19 & \varepsilon & 25 & 19 \\ \varepsilon & 16 & \varepsilon & 25 & 16 \end{bmatrix}, \begin{bmatrix} 8 & 12.5 & 8 & 18.5 & 12.5 \\ 8 & 12.5 & 8 & 18.5 & 12.5 \\ 14 & 18.5 & 14 & 24.5 & 18.5 \\ 17 & 21.5 & 17 & 27.5 & 21.5 \\ 15.5 & 20 & 15.5 & 26 & 20 \end{bmatrix} \right) = 27.5,$$

which is smaller than the coarser bound 33.

ACKNOWLEDGEMENT

This work was supported by EPSRC Grant EP/P019676/1. We would like to thank Dr. Oliver Mason for giving us an idea for this paper. We are also grateful to Dr. Glenn Merlet and our anonymous referees for their careful reading, useful suggestions and advice.

(Received February 28, 2018)

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