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AN ASSET – LIABILITY MANAGEMENT STOCHASTIC PROGRAM OF A LEASING COMPANY

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We build a multi–stage stochastic program of an asset–liability management problem of a leasing company, analyse model results and present a stress–testing methodology suited for financial applications. At the beginning, the business model of such a company is formulated. We introduce three various risk constraints, namely the chance constraint, the Value–at–Risk constraint and the conditional Value–at–Risk constraint along with the second–order stochastic dominance constraint, which are applied to the model to control risk of the optimal strategy. We also present the structure and the generation process of our scenarios. To capture the evolution of interest rates the Hull–White model is used. Thereafter, results of the model and the effect of the risk constraints on the optimal decisions are thoroughly investigated. In the final part, the performance of the optimal solutions of the problems for unconsidered and unfavourable crisis scenarios is inspected. The methodology of a stress test we used was proposed in such a way that it answers typical questions asked by asset–liability managers.

Keywords: asset–liability management, multi–stage stochastic programming, stress test

Classification: 90C15, 90B50, 90C31, 91G10

1. INTRODUCTION

Stochastic programming is a quickly developing area of mathematical optimization, which finds its application in many different fields. Especially the area of financial planning and control has seen plenty of implementations of stochastic programming models. One such an application arose in 1986, when [24] published their famous paper dealing with a bank asset–liability model. Two years later, [7] have introduced a different model, where they focused on immunisation of liabilities, and hence they for the first time took into account risks, which are associated with financial problems. Between other contributions, [4] presented a very first asset–liability model for an insurance company. Thereafter, with the arrival of new technologies aided by the availability of software, other, especially multi-stage models, have emerged. We should mention at least some focused on insurance companies [3, 19, 30], and pension funds [9, 15, 16], as these were the most popular types of financial companies, where stochastic programming was used.

These works however did not exhausted the potential of applications of stochastic programming in finance. [10, 11] have investigated the robustness of program's solution

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by scenario contamination. Other possibility was to analyse the effect of new, modern risk constraints on the optimal solution; for example [22] have applied a second–order stochastic dominance constraint. Important aspect of the modelling exercise is to focus on accuracy of the model formulation and capturing all the details associated with the investigated problem.[37] and [27] have paid a lot of attention to this part, as they aimed to cover all possibilities, how an insurance company could invest. They also focused on the credibility of cash-flow dynamic in the company and to the scenario generation procedure.

In our work, we focus on a stochastic programming formulation of an asset–liability model of a leasing company. The business of such a company stems from closing loans with clients, who need money to purchase some products, and with a bank, so the company has money to lend. The management has the right to decide how much and for how long period of time they borrow money. We aim to find the best borrowing strategy which the management could employ. Moreover, various risk constraints will be introduced to the model in order to control the portfolio value in the worst scenarios. Namely, we will discuss the effect of a chance constraint [35], a Value–at–Risk constraint [26], a conditional Value–at–Risk constraint [33, 34] and of a second–order stochastic dominance constraint [8, 17] on the portfolio value of the solution at the investment horizon. We generate scenarios from a calibrated Hull–White model [20], which is very popular between practitioners. Last, but not least, we introduce a stress test investigating the optimal value's sensibility to unconsidered (and unfavourable) scenarios. Such a test helps managers to determine how their portfolio of loans would perform in a crisis scenario. With respect to that information, they can adjust the scenario tree or reformulate the risk constraints if the results of the test are not satisfactory.

The paper is structured in the following way. In Section 2, we present a business model of a leasing company and construct a multi–stage stochastic programming model. We also discuss the scenario generating process and specify the risk and the second–order stochastic dominance constraints which will be employed in the asset–liability model. The third section is devoted to presentation of our model's results. We consider all reasonable values of the risk constraints' parameters and investigate how they affect the optimal value of the problem and the consequent decisions implied to the management. In Section 4, a stress test is performed to answer a question of what would be the optimal value of a predetermined strategy in the case of an unexpected development of interest rates. Finally, we summarise our results in final Section 5.

2. MODEL FORMULATION

This section will be devoted to a formulation of a multi–stage stochastic program of a leasing company. We will present the business objective of the company, the scenario generation process we adopted, as well as four concepts of how to control the interest rate risk which the company faces.

2.1. Business model of a leasing company

A leasing company is an enterprise, whose scope of business is to lend money to clients to purchase some products. On the other hand, the leasing company itself borrows money from a bank in order to have money to lend. For simplicity, we will not consider defaults of clients on their collateralized loans. Generalizing our model to consider such losses would not be difficult as we would just subtract the part of a mark–up on the interest rate corresponding to the risk profile of a client as an insurance.

We treat our problem as a discrete time, where all cash flows take place at times $t_0 < t_1 < \cdots < t_n$ where $n \in \mathbb{N}$. Time t_0 can be thought as present and time t_n as the investment horizon. All loans of the leasing company which were closed before time t_n and did not mature by the time t_n will be thought to be sold for their market value at the investment horizon. No contracts will be closed at time t_n or later.

2.1.1. Model variables

In the following, we formulate the stochastic programming model. We denote y_t^τ annualized, continuously compounded, risk free interest rate with time–to–maturity τ at time t. For $t > t_0$, this quantity is random. Next, we denote s_t^τ the rates, for which the leasing company borrows, while r_t^τ represent rates offered to clients. We define:

$$s_t^\tau = y_t^\tau + s(\tau), \qquad r_t^\tau = y_t^\tau + s(\tau) + m(\tau) = s_t^\tau + m(\tau).$$

Quantities $s(\tau)$ and $m(\tau)$ are the differences between the rates. Next, we denote the (random) demand for loans from time $t_i, i < n$ to time t_j as d_{t_i,t_j}. Such a loan is repaid at all times $t_k : t_i < t_k \le t_j$ with the same amount denoted as $d_{t_i,t_j}(t_k)$:

$$d_{t_i,t_j}(t_k) = \frac{(t_k - t_{k-1}) d_{t_i,t_j}}{\sum_{l=i+1}^{j} (t_l - t_{l-1}) \exp\{-r_{t_i}^{t_l - t_i}(t_l - t_i)\}}, \qquad k = i+1, \ldots, j. \tag{1}$$

From this formula, one can derive the amount of money repaid back by clients at time t_k, denoted by R_{t_k}, as well as the amount of money D_{t_k} which the leasing company has lent to clients at time t_k :

$$R_{t_k} = \sum_{\substack{t_i,t_j: \\ t_i < t_k \le t_j}} d_{t_i,t_j}(t_k), \qquad D_{t_k} = \sum_{t_j : t_k < t_j} d_{t_k,t_j}, \qquad k = 0, \ldots, n. \tag{2}$$

Similarly, we define the corresponding quantities for cash–flows with the bank. This time, the amount of money the leasing company borrows from the bank x_{t_i,t_j} from time $t_i, i < n$ to time $t_j, i < j$ is a decision variable. By a similar logic, we denote the amount of money repaid from such a loan at time $t_k, i < k \le j$ as $x_{t_i,t_j}(t_k)$. Moreover the total amount repaid to the bank is denoted by Q_{t_k} and borrowed from the bank by X_{t_k} both at time t_k. It holds that

$$x_{t_i,t_j}(t_k) = \frac{(t_k - t_{k-1}) x_{t_i,t_j}}{\sum_{l=i+1}^{j} (t_l - t_{l-1}) \exp\{-s_{t_i}^{t_l - t_i}(t_l - t_i)\}}, \qquad k = i+1, \ldots, j, \tag{3}$$

$$Q_{t_k} = \sum_{\substack{t_i,t_j: \\ t_i < t_k \le t_j}} x_{t_i,t_j}(t_k), \qquad X_{t_k} = \sum_{t_j : t_k < t_j} x_{t_k,t_j} \qquad k = 0, \ldots, n. \tag{4}$$

Now, let us denote $P(t_k, t_l)$ the discount factor from time t_k to time t_l. We also denote B_{t_k} the amount of money the leasing company has on its account immediately after time t_k and E_{t_k} the company's running costs from time t_k to t_{k+1}. We have that

$$B_{t_k} = B_{t_{k-1}}/P(t_{k-1}, t_k) - E_{t_{k-1}} + R_{t_k} - D_{t_k} + X_{t_k} - Q_{t_k}. \tag{5}$$

The relationship in (5) describes the movements on the current account of the company. Finally, we need to find a relationship for time t_k value of assets A_{t_k} and the value of liabilities L_{t_k}. These are only discounted future cash–flows agreed by time t_k.

$$A_{t_k} = \sum_{\substack{t_i,t_l,t_j:\\ t_i \le t_k < t_l \le t_j}} P(t_k, t_l) d_{t_i,t_j}(t_l), \qquad L_{t_k} = \sum_{\substack{t_i,t_l,t_j:\\ t_i \le t_k < t_l \le t_j}} P(t_k, t_l) x_{t_i,t_j}(t_l). \tag{6}$$

Finally, we will define V_{t_k} as the value of the portfolio at time t_k. It is defined as the sum of assets, liabilities and money on the company's current account. Formally,

$$V_{t_k} = A_{t_k} - L_{t_k} + B_{t_k}. \tag{7}$$

One can notice that given the definition of V_{t_n} as in (7) and the relationships (3), (4), (5) and (6), the value of a portfolio is a linear function of the decision variables x_{t_i,t_j}.

2.1.2. Benchmark strategy

We compare the optimal strategy suggested by our program to the current — benchmark strategy. Currently, the business is done in the following way. Consider a client who comes to the leasing company demanding a loan of d_{t_i,t_j}. The leasing company closes a deal with the client and it immediately closes a mirror deal with the bank too. By the term mirror deal, we mean a loan with the same principal and the same maturity — the company borrows d_{t_i,t_j}. Hence the benchmark strategy can be formulated as

$$x^0_{t_i,t_j} = d_{t_i,t_j}. \tag{8}$$

In the following, superscript ^{0} will denote corresponding quantities of the benchmark strategy. Under such a strategy, there is no way to manipulate with the leasing company's income. Yet, the company is still not assured to earn profit as it might not be able to gain enough to cover its costs. Hence we chose the costs of running the company such that the benchmark strategy just met the (survival) condition $B^0_{t_k} \ge 0, \forall k = 1, \dots, n$.

The main idea why it could be beneficial for the company to behave differently and not to follow the benchmark strategy is that the short term rates are usually lower than the long term rates. Hence the company instead of closing a mirror deal with maturity five years would borrow money multiple times only for shorter time periods. This could generate additional profit. However, that would require the company to open its interest rate position which could cause significant losses (or gains) in the case of an unexpected interest rate movement. For this reason, the introduction of certain measures of risk is required, as these could restrict the set of feasible strategies.

2.2. Scenario generation

Next, we denote $\omega = (\omega_0, \omega_1, \ldots, \omega_n)$ the random vector expressing the uncertainty of demands and interest rates in our problem. Its history up to time $t_k \le t_n$ will be denoted $\bar\omega_k = (\omega_0, \ldots, \omega_k)$. We consider ω_0 to be given — non–random, as it describes the information at current time. From now on, the components of the problem which depend on realizations of the random vector up to time t_k will be denoted by the established notation plus by a term $\bar\omega_k$ in brackets. We approximate the stochastic distribution of ω by scenarios, which are organized in the form of a scenario tree. At every time $t_k, k = 1, \ldots, n$ we will consider nodes $\bar\omega_k^s, s \in S_k$ with equal probability $1/|S_k|$. We require $|S_0| = 1$, as the only element of S_0 is the current state and that is given. Every node at time t_k will have for all times $t_0 \le t_i < t_k$ a unique ancestor. The non–anticipativity constraints are implemented implicitly as we have a single decision $x^s_{t_i,t_j}$ at every node $s \in S_i$. The stochastic program can be concisely written in the following way:

$$\begin{aligned} \max_{x^s_{t_i,t_j}} \quad & \frac{1}{|S_n|} \sum_{s \in S_n} V_{t_n}(\bar\omega_n^s) \\ \text{s.t.} \quad & B_{t_k}(\bar\omega_k^s) \ge 0, \qquad s \in S_k,\ \ 0 \le k \le n, \\ & x^s_{t_i,t_j} \ge 0, \qquad s \in S_i,\ \ 0 \le i \le n,\ \ i < j, \end{aligned} \tag{9}$$

where the variables $V_{t_n}(\bar\omega_n^s)$ and $B_{t_k}(\bar\omega_k^s)$ are calculated as described above. In our model, we set the time horizon to be $t_n = 6$ years, while the other decision points were one year apart. The branching was 8-4-2-2-2-2, so in the first year, we had 8 nodes, in the second 32, then 64 and in the last year 512. To generate scenarios of interest rates, we adopted the Hull–White model of [20] which belongs to a class of one factor short rate models introduced by [36]. It is defined by the following stochastic partial differential equation:

$$dr_t = (\theta(t) - \alpha r_t)dt + \sigma dW_t,$$

where r_t stands for short rate and W_t is the standard Brownian motion. Parameter α stands for the mean reversion factor and σ summarizes the volatility of the short rate. Finally, $\theta(t)$ is set such that the observed market prices are fitted perfectly, i. e.

$$\theta(t) = \frac{\partial f^M(0,u)}{\partial u}\Big|_{u=t} + \alpha f^M(0,t) + \frac{\sigma^2}{2\alpha}\left(1 - e^{-2\alpha t}\right),$$

where $f^M(0,t)$ is the market instantaneous forward rate at time 0 for time t. Thanks to the fact that the model uses exogenous information in the form of the observed market yield curve, predictions of yields based on this model are close to market expectations, which increases the model's credibility. The calibration of the model's parameters was inspired by [5] who estimated the Cox–Ingersoll–Ross model of [6] by the maximum likelihood method on observed yields. The estimated values of the parameters were $\hat\alpha = 0.03696, \hat\sigma = 0.0059585$. Scenarios of the short rate were chosen to be the quantiles of the model implied distribution, so they represent it as well as possible. We derived long rates $y^\tau_{t_i}(\bar\omega_i^s)$ and discount factors $P(t_i, t_j, \bar\omega_i^s)$ from the short rate based on well known formulas for zero-coupon bond prices implied by the Hull–White model, see for example [2] for more details. One can see the interest rate scenarios in Figure 1, where scenario tree with interest rate for one year maturity is depicted.

Demand in scenario $s_i \in S_i, i = 0, \dots, n-1$, was generated from gamma distribution with mean $\mu_i = \exp\{\beta_0 + \beta_1 y^1_{t_i}\}$ and the company was assumed to have a share of 1% the market. Parameters $\hat{\beta}_0 = 10.362, \hat{\beta}_1 = -0.020$ and a shape parameter $\hat{a} = 85.6$ were estimated given data of closed leasing loans in the Czech market, which were provided by Czech National Bank. We estimated slightly negative correlation between interest rates and demand for loans.

2.3. Risk management

Clearly, managers are not only interested in maximizing the value of their portfolio, but they also aim to control a risk (possible losses) associated with their strategy. To enable this, we introduce the following four risk constraints.

2.3.1. Chance constraint

The so called chance or probabilistic constraint was first formulated by [35] and used in asset–liability models for example by [9] or [21]. By employing such a constraint, we will aim to control the probability that the benchmark strategy over–performs the optimal

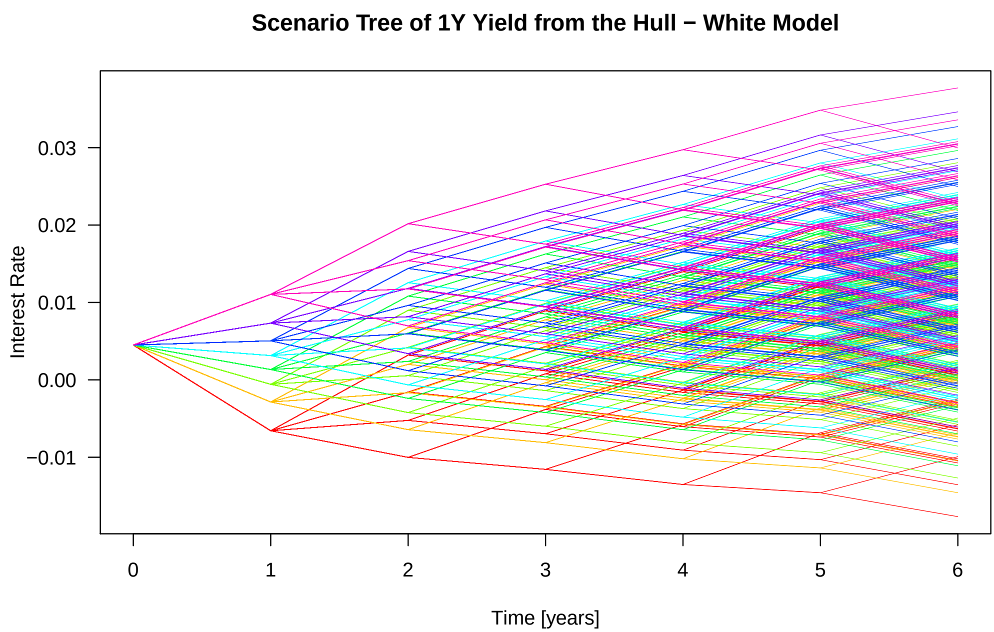

Fig. 1. Values of one year interest rate depicted in the form of a scenario tree of the program.

strategy. It has the following form:

$$\mathbb{P}\Big(V_{t_n}(\bar{\omega}_n) - V^0_{t_n}(\bar{\omega}_n) \geq 0\Big) \geq 1 - \alpha, \tag{10}$$

where $\alpha \in \langle 0, 1\rangle$ denotes a given probability and $\mathbb{P}$ is the probability measure of the random vector $\bar{\omega}_n$. The implementation is done with the help of binary variables $z^s_\circ \in \{0, 1\}, s \in S_n$ and a parameter M — some arbitrary large number, we have

$$\begin{gathered} V^0_{t_n}(\bar{\omega}^s_n) - V_{t_n}(\bar{\omega}^s_n) \leq M \cdot z^s_\circ, \quad z^s_\circ \in \{0, 1\}, \quad s \in S_n, \\ \sum_{s \in S_n} z^s_\circ \leq \alpha \cdot |S_n|. \end{gathered} \tag{11}$$

2.3.2. Value–at–risk constraint

The next constraint employs a well–known Value–at–Risk (VaR) of [26]. For a profit function V, and a given confidence level $\alpha \in (0, 1)$, $\mathrm{VaR}(V)$ measures what is the worst outcome which we experience among the best $100 \cdot \alpha\%$ cases. We shall use the following definition:

Definition 2.1. (Morgan [26]) Let Y be a random loss function with a cumulative distribution function F_Y and $\alpha \in (0, 1)$. Then $\alpha-$*Value–at–Risk of* Y $\mathrm{VaR}_\alpha(Y)$ is the $\alpha-$quantile of the random variable Y, i. e.

$$\mathrm{VaR}_\alpha(Y) = F_Y^{-1}(\alpha),$$

where F_Y^{-1} is the quantile function of Y.

In the model, we impose the following condition:

$$\mathrm{VaR}_\alpha(-V_{t_n}(\bar{\omega}_n)) \leq u_\alpha, \qquad \alpha \in (0, 1), \;\; u_\alpha \in \mathbb{R}. \tag{12}$$

This can be interpreted in such a way that we choose only from strategies whose Value–at–Risk at the confidence level α is less than or equal to some predetermined amount u_α. To implement it into the model formulation (9), we use again binary variables $z^s_\circ \in \{0, 1\}, s \in S_n$, each corresponding to one scenario and write:

$$\begin{gathered} -V_{t_n}(\bar{\omega}^s_n) - u_\alpha \leq M \cdot z^s_\circ, \quad z^s_\circ \in \{0, 1\}, \quad s \in S_n, \\ \sum_{s \in S_n} z^s_\circ \leq (1 - \alpha) \cdot |S_n|. \end{gathered} \tag{13}$$

2.3.3. Conditional value–at–risk constraint

The third constraint we will be dealing with controls the conditional Value–at–Risk (CVaR) of the final portfolio loss $(-V_{t_n}(\bar{\omega}_n))$. It measures the expected value from the worst $100 \cdot (1 - \alpha)\%$ cases. CVaR as a risk measure possesses number of desirable properties, as described for example by [28]. Most notably he mentions that CVaR is a coherent risk measure (see [1]), as it meets the four properties which are desirable for a measure of risk.

Definition 2.2. (Rockafellar and Uryasev [33]) Let Y be a random loss function and $\alpha \in (0,1)$. Then we define *α–conditional Value–at–Risk of Y* $\mathrm{CVaR}_\alpha(Y)$ as:

$$\mathrm{CVaR}_\alpha(Y) = \inf_{a \in \mathbb{R}} \Big\{ a \ + \frac{1}{1-\alpha} \mathbb{E}\big[(Y-a)^+\big] \Big\},$$

where $(\cdot)^+$ denotes a positive part of a real number.

Regarding our asset–liability problem, the loss function can be for CVaR defined in the same way as for VaR. The constraint has the following form:

$$\mathrm{CVaR}_\alpha(-V_{t_n}(\bar{\omega}_n)) \le v_\alpha, \tag{14}$$

where $v_\alpha \in \mathbb{R}$ is a given number and $\alpha \in (0,1)$. The conditional Value–at–Risk constraint can be formally implemented in the following way. Let $a, z^s, s \in S_n$ be variables, $a, z^s \in \mathbb{R}$. The scenario formulation of the constraint (14) takes the form:

$$\begin{aligned} z^s \ge -V_{t_n}(\bar{\omega}_n^s) - a, \quad z^s \ge 0, \quad s \in S_n, \\ a + \frac{1}{1-\alpha}\frac{1}{|S_n|}\sum_{s \in S_n} z^s \le v_\alpha, \quad a \ \in \mathbb{R}. \end{aligned} \tag{15}$$

The interpretation of this constraint is similar to one with VaR, as it enables us to find the best performing strategy which meets the predetermined limit value of CVaR.

2.3.4. Second–order stochastic dominance constraint

The final constraint we will employ in our asset–liability management problem is the *second–order stochastic dominance* (SSD) constraint, whose formal definition is given through the so called integrated cumulative distribution functions.

Definition 2.3. (Hadar and Russell [17]) Let X be a random variable and let F_X be its cumulative distribution function. Let us define the *integrated cumulative distribution function of X* as

$$F_X^{(2)}(x) = \int_{-\infty}^{x} F_X(u)\,\mathrm{d}u.$$

This allows us to state the following definition.

Definition 2.4. (Hadar and Russell [17]) Let V and B be random variables and let $F_V^{(2)}$ and $F_B^{(2)}$ be their integrated cumulative distribution functions. Then we say that V dominates B by a *second–order stochastic dominance* ($V \succeq_{SSD} B$) if

$$F_V^{(2)}(y) \le F_B^{(2)}(y), \qquad \forall y \in \mathbb{R}.$$

[17] have derived an important interpretation, as they showed that $V \succeq_{SSD} B$ if and only if every risk averse investor is either indifferent between V and B or prefers portfolio V to B. More on this topic can be found in [25]. The condition which we impose in our asset–liability problem is the following:

$$V_{t_n}(\bar{\omega}_n) \succeq_{SSD} V^0_{t_n}(\bar{\omega}_n) + b, \qquad b \in \mathbb{R} \tag{16}$$

and we will refer to it as to the *second–order stochastic b–dominance constraint.* Strategy which meets the constraint (16) for $b = 0$ has a very strong interpretation in the sense that every risk–averse manager would prefer (or at worst would be indifferent between) the new strategy to the benchmark strategy. The set of equations under which the condition (16) can be added into the model formulation (9), where all scenarios happen with equal probability, was derived by [23], who utilized results of [18]. The implementation is done by imposing the following (in)equalities.

$$V_{t_n}(\bar{\omega}_n^{s_i}) - b \geq \sum_{j=1}^{|S_n|} w_{ij} V^0_{t_n}(\bar{\omega}_n^{s_j}), \quad s_i \in S_n, \tag{17}$$

$$w_{ij} \geq 0, \ \sum_{i=1}^{|S_n|} w_{ij} = 1, \ \sum_{j=1}^{|S_n|} w_{ij} = 1.$$

3. RESULTS

In this section, we present results of our models of the leasing company. At the beginning, we focus on a model without risk constraints to find out which parameter values make sense to consider and also where the riskiness comes from. Thereafter, we give results of the programs for all meaningful values of the parameters and we analyse the objective values of the optimal portfolios. The program was written in GAMS and solved by CPLEX on standard laptop (Intel core i5 2.60 GHz, 8GB RAM). Scenario generation and results' analysis were performed in R. In the program, we set maturities $\tau = 1, \dots, 5$ as the ones for which it was allowed to borrow/lend money. We estimated $s(\tau) = (0.41, 0.49, 0.56, 0.58, 0.59)$ and $m(\tau) = (4.3, 5.9, 4.4, 4.2, 4.2)$ (both in percent) from real data. Costs of running of the company were set to 50 mil. CZK and 100 mil. CZK in first and second year and 125 mil. CZK for the other years.

The optimal solution of (9) states how a manager should behave if his only objective is to maximize expected profit. It suggests us to borrow basically only in one–year loans. This was not surprising as it generally holds for loans the shorter the cheaper. The expected value of such a strategy was 316.46 mil. CZK, while the benchmark achieved a mean return of 294.47 mil. CZK. We present Figure 2, where histograms and estimated densities of the benchmark and the optimal portfolio's value can be compared. On the right, we plot difference in final portfolio value of the two strategies against the final stage one-year interest rate. The second figure quantifies the effect of the final stage one-year interest rate on the performance of the optimal strategy.

We found out that the benchmark strategy betters the optimal strategy in about 19% of cases, hence we will consider $\alpha \in \langle 0, 0.19\rangle$ in the chance constraint. The values of VaR

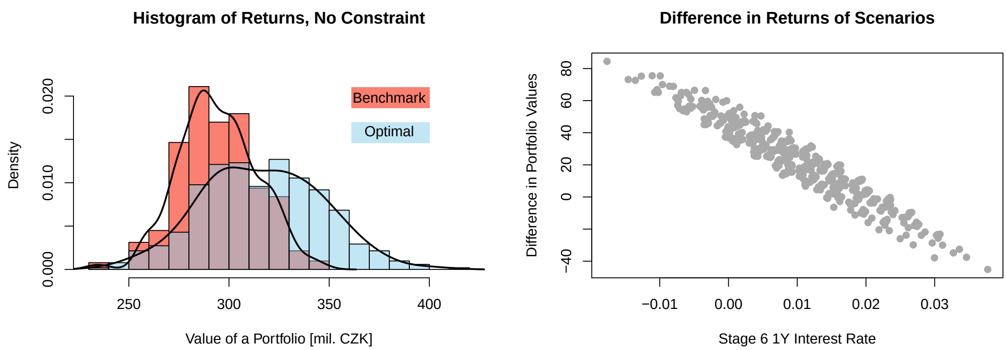

Fig. 2. Comparison of final values of portfolios of the benchmark and the no-risk–constraint optimal strategy. Histograms on the left and their differences against final stage one–year interest rate on the right.

and CVaR at the 0.95 level of significance of the benchmark portfolio are -262.32 and -254.88 mil. CZK respectively, while the optimal strategy gives -269.10 and -256.71 mil. CZK respectively. We can see, that both risk measures rank the optimal portfolio as less risky than the benchmark portfolio. Hence we will only consider VaR and CVaR limits even smaller than the ones of the currently optimal portfolio. Lower bounds will be determined by feasibility of the corresponding problems. Moreover, we found that the optimal strategy SSD dominates the benchmark strategy. Therefore, the only values of parameter b which come into considerations are the ones such that $b \geq 0$. The question will be to find b as large as possible so that (17) holds true.

Next, we will analyse the optimal solutions under risk constraints discussed in Section 2.3 and we will show model results for all reasonable parameter inputs. The most strict risk limits for which corresponding constraints were feasible were $u_\alpha = -288.25$ mil. CZK for VaR, $v_\alpha = -277.25$ mil. CZK for CVaR and $b = 15$ mil. CZK for the SSD constraint. As the benchmark strategy was feasible, also the chance constraint problem with $\alpha = 0$ was feasible. We illustrate the dependence of problem's optimal values for various risk limits in the risk constraints in Figure 3. We can see, that the most "expensive" is clearly the chance constraint, as for $\alpha = 0$, it forces to improve rapidly the worst scenarios. It is not a coincidence, that only in this constraint, the here–and–now decision (i. e. the decision for the initial period) was different from borrowing only one–year loans. There, it suggested us to invest into one and five year loans with the proportion depending on the parameter α.

However, when we employ risk constrains, we are not only interested in the expected final value of our portfolio, but also in other properties of the final portfolio value distribution. From this reason, we show Figure 4, where we present histograms with estimated densities of optimal final value of a portfolio of programs when the four risk constraints were applied with rather strict limits. This gives us the possibility to asses the effect of the constraints on the random variable of interest – its properties like variance,

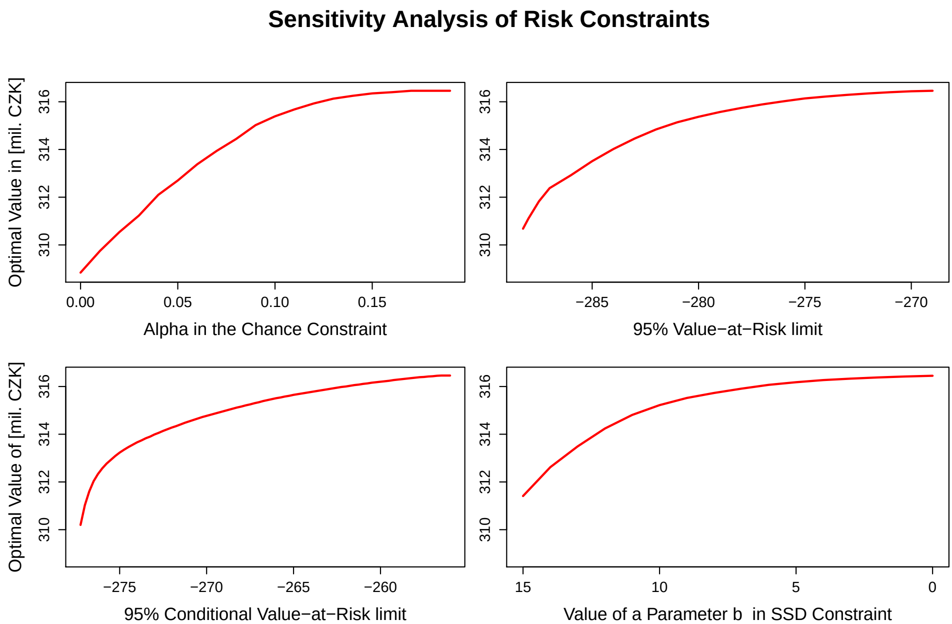

Fig. 3. The optimal values of the asset–liability problem for different risk constraints and all reasonable parameter values.

left tail, skewness etc... There, one can nicely see the effect of the second–order stochastic dominance on the optimal distribution or for example the 5% of scenarios which do not meet the VaR limit and were left without any improvement.

The ultimate conclusion is that the benchmark strategy is largely inefficient, and that we are able to propose different strategy, which gives much better performance in an arbitrary risk measure.

4. STRESS TESTING

The concept of stress testing is frequently used in the financial industry under various different meanings. The general idea behind it is however relatively clear, as we aim to find out what would happen to the object of interest in the case of an unexpected shock to the market environment. Some work on stress testing in the context of multi–stage stochastic programming has been done for example by [13] and [14]. Their idea was to contaminate (see [12] for meaning of contamination) the original distribution and investigate how the optimal decisions and optimal values change under such perturbations. However, our approach to the stress testing will be different, as we want to answer the manager utilising our stochastic program a question he is likely to ask. In the first part of this section, used methodology will bee described, while the second part will be devoted to the stress testing results.

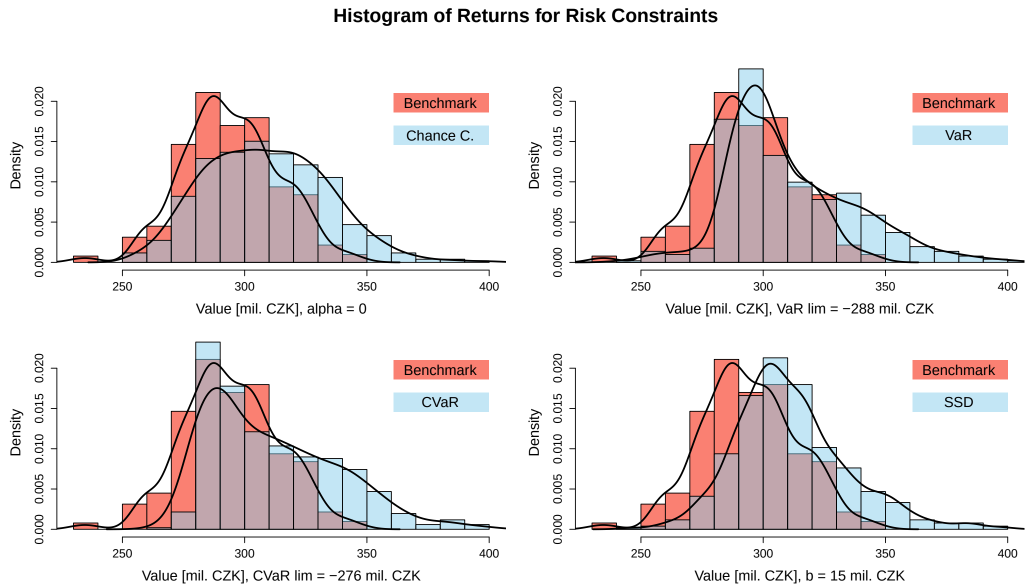

Fig. 4. Histograms of portfolio values at the time horizon t_n of the optimal strategies for different risk constraints with strict risk limits.

4.1. Methodology of stress testing

Let us recall the main reason why financial institutions do the stress testing. Their aim is to find out whether they are financially stable enough to overcome a deep financial crisis. Hence, we will focus on answering the below–stated question.

Assuming that we adopt some optimal strategy as suggested by the multi–stage program, what would happen if a deep financial crisis came?

Such a question can be answered by performing the following steps:

- Solve an asset–liability stochastic program, obtain the optimal here–and–now decision and act according to it.
- Then a crisis comes, we move to a crisis scenario at time one, where we can re-adjust our borrowing strategy.
- We solve a new stochastic program with a new crisis tree, while taking into account our initial decision. We learn our new optimal expected value.

We generate crisis scenarios by increasing interest rate, as we saw such movement hurts the most our optimal strategy. We will fix the value of the short rate in the first year, with the most severe scenario corresponding to an increase in the short rate of 2%. In the stress test, we will analyse strategies given by the following programs:

1. Program (9) without any risk constraint.

2. Program (9) with the chance constraint and $\alpha = 0$.
3. Program (9) with 0.95–VaR constraint and limit equal to the benchmark VaR.
4. Program (9) with 0.95–CVaR constraint and limit equal to the benchmark CVaR.
5. Program (9) with the SSD constraint and $b = 0$.

The programs were chosen in such a way that if they are feasible, the optimal solution is not considered worse than the benchmark by the corresponding risk constraint.

4.2. Stress testing analysis

The here-and-now decision for the chance constraint strategy was to borrow around 30% in five year loans and the rest in one year loans, while all other optimal strategies suggested to borrow everything in one year loans. Next, we move to the crisis scenario created by fixing the value of a short rate in the first year at some of the levels: $\{0\%, 0.1\%, 0.2\%, \dots, 2.4\%\}$. To put that into a context, we should mention that the expected value of the short rate in the first year is 0.35%, while its highest value at that stage in the original tree is 1.1%. From the crisis scenario, we model the interest rate to evolve in the way which is suggested by the Hull–White model. We calculate the optimal value as a solution of the program with the crisis tree and a strategy–specific initial wealth. These are plotted against the crisis short rate in Figure 5.

First, we comment on the feasibility of strategies. It can happen that the strategy which the management set, becomes infeasible (i. e. if the interest rate increases too much, one cannot set the decision so it still beats the benchmark). However, in such a situation, the management still have to make a decision. If this happens, we set the risk constraint as strict as possible. In Figure 5, we mark the point where we are forced to relax the initial assumption on the strategy of the management unit for every risk constraint by a small rhombus in the corresponding color. Expected values of the optimal solutions of such relaxed programs are then drawn by a dashed line.

Figure 5 summarizes how valuable are these five year loans in the case of unexpected interest rate movements. One can see that the optimal strategies which borrowed everything in one year loans suffer a lot once the value of the short rate exceeds 1.1%, which was the largest value of the short rate in the original scenario tree. On the other hand, the chance constraint strategy looks to cope well with the crisis. In Figure 5 we can see that the chance constraint strategy betters the return of the benchmark strategy even for extreme scenarios, as the five-year loans make it robust to such shocks. The large jump of the orange dashed line around the x-axis value of 1.5% in Figure 5 is a consequence of relaxing the parameter α in the chance constraint. Around that point, α rapidly increased from 6.3% to 22%, which allowed the optimal value to increase.

In this section, we have seen how one can approach the so called post–optimality analysis, which studies properties of optimal solutions. It can provide us with valuable information about the suitability of the model formulation. The stress test we adopted answers questions which one could expect to be asked by a user of the model. Given the results of this analysis, the manager would either get more belief in the optimal solution or he would identify properties where he wants the solution to improve. Consequently, we could reformulate the problem to achieve desired results.

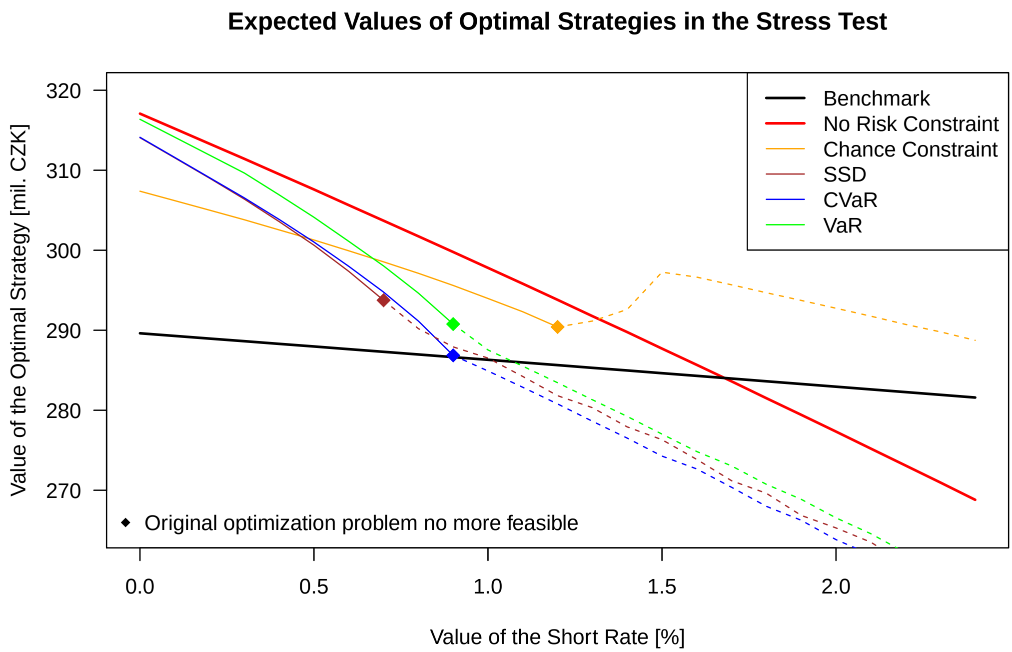

Fig. 5. The optimal value of the asset–liability model for different level of stress test and various strategies. Dashed lines show optimal values of the programs after relaxing the (infeasible) risk constraints.

5. CONCLUSION

We formulated a multi–stage stochastic program inspired by a business model of a leasing company, which as far as we know was not yet been subject to modelling by means of mathematical programming. We focused on showing how optimal decisions obtained as a solution of the optimization problem can improve the company's performance. The core of the model was formed by the dynamics of cash–flows of the company. Still, the risk constraints, which were employed to reflect the risk aversion, should be considered at least as equally important. They give desirable properties and a strong interpretation to the optimal solution. We also paid special attention to creating as most realistic model as possible, where all parameters were calibrated on real market data. The choice of the Hull – White model for the scenario generation was driven by the fact that the implied rates fit precisely the currently observed rates and hence we think the model gives realistic predictions of interest rates.

We found that the here–and–now solution to borrow money only for one year generates much higher return than the benchmark; and moreover under suitable behaviour in latter stages it is possible to achieve much better risk parameters of the strategy. The inefficiency of the benchmark is underlined by both the amount by how much it is beaten by the optimal strategy obtained from the stochastic program with the chance constraint with $\alpha = 0$ and by the maximum value of the parameter $b = 15$ for which we found an SSD b–dominating portfolio. If any institution ever considers implementing such a method in practice, we think they would also ask what performance they can expect from their strategy in crisis scenarios. The stress testing methodology we have introduced answers such a question. Most notably, the Figure 5 provides a clear graphical answer on the dependence of the optimal solution's expected value on the year–one interest rate. Such a figure can be of a great importance when choosing between optimal strategies or when assessing the suitability and the strictness of risk constraints. It could help us to reformulate the model so our strategy is more robust to such fluctuations. As far as we know, the methodology we have applied in the stress testing have not yet been used in any application of stochastic programming within asset–liability management.

There are multiple ways, how to extend our model. We see a natural one in applying multi-period risk measures [29] or other classes of stochastic dominance, such as third order [31] or DARA [32] as risk constraints to the model. Another interesting way how to improve the program might be to add a possibility to invest into market instruments, like interest rate caps and floors. These could be used to hedge against a possible increase in interest rate which could possibly lead to an improvement of the company's performance.

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