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CHANCE CONSTRAINED OPTIMAL BEAM DESIGN: CONVEX REFORMULATION AND PROBABILISTIC ROBUST DESIGN

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In this paper, we are concerned with a civil engineering application of optimization, namely the optimal design of a loaded beam. The developed optimization model includes ODE-type constraints and chance constraints. We use the finite element method (FEM) for the approximation of the ODE constraints. We derive a convex reformulation that transforms the problem into a linear one and find its analytic solution. Afterwards, we impose chance constraints on the stress and the deflection of the beam. These chance constraints are handled by a sampling method (Probabilistic Robust Design).

Keywords: optimal design, stochastic programming, chance constrained optimization, probabilistic robust design, geometric programming

Classification: 90C15, 90C30, 65C05, 49M25

1. INTRODUCTION

Optimal design problems in engineering frequently lead to optimization problems involving differential equations. One of the classes of these problems is shape optimization [11]. The particular shape optimization problem considered in this paper is the optimal design of a beam (be it a fixed beam, a cantilever beam, etc.) subjected to some kind of loading. Since shape optimization problems are inherently non-convex, most approaches use metaheuristics such as genetic algorithms [13] or cuckoo search [9]. A closely related field of topology optimization (where the size and shape of the structure can be manipulated) has developed a multitude of successful methods (level set, homogenization, topological derivative, etc.), see [19].

This problem was previously also examined in [21] and [24], where the authors used the finite element method (FEM) and the finite difference method to approximate the ordinary differential equations (ODE) and solve the problem by nonlinear programming techniques. Our paper shows that this beam design problem can be formulated as a geometric programming problem, which can be further transformed into a convex one, and thus can be efficiently solved (in comparison with the previous approaches). Geometric programming problems with random coefficients (although without chance constraints) were investigated in [8].

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An important issue regarding the design is its reliability (see [12]). In the context of this paper the reliability of the design will mean that the constraints in the resulting optimization program should hold with high probability. Depending on how reliable design is required, we can destinguish between the so-called chance constrained (or probabilistic constrained) optimization problems (see, e. g. [20]) and the robust optimization problems (see, e. g. [3]). Current approaches dealing with reliability constrained beam design, such as [2] and [25] use simple (point) loads and Gaussian distribution of the unknown parameters. In this paper we investigate the chance constrained beam design problem under more complicated random loads. We utilize the sampling approach (called Probabilistic Robust Design) developed in [5, 6] and [7] to obtain a manageable approximation of the chance constrained problem and use a scenario-deletion method to compute a trade-off between the reliability of the design and the objective value.

2. PROBLEM FORMULATION

The problem is best described by Figure 1. We consider a fixed beam of length l with rectangular cross-section that is subjected to a load $h(x)$ (with the opposite direction than the axis y), which is depicted in Figure 1a. The task is to find the optimal design, in terms of the cross-section dimensions a and b (Figure 1b), that minimizes the weight of the beam.

Naturally, given a load $h(x)$ the beam will deflect and will be subjected to a bending stress. The requirement for the design is that the maximum stress in the beam is less then a material-specific constant, that ensures that the design is safe (we use the value at which the material begins to deform plastically). The problem can be formulated as the following ODE-constrained optimization program:

$$\underset{a,b,v(x)}{\text{minimize}} \quad \rho a b l \tag{1}$$

$$\text{subject to} \quad E\frac{ab^3}{12}\frac{\mathrm{d}^4 v}{\mathrm{d}x^4}(x) = h(x), \qquad x \in [0,l], \tag{2}$$

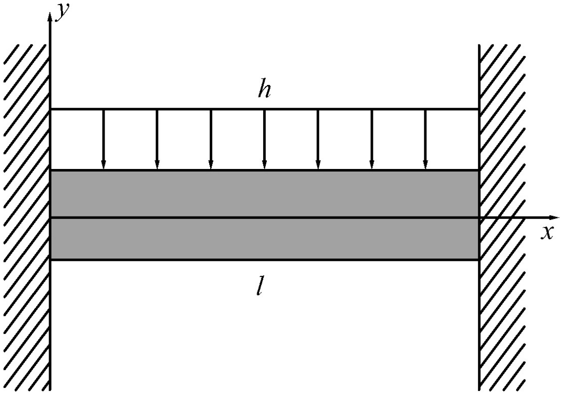
$$\left| E\frac{b}{2}\frac{\mathrm{d}^2 v}{\mathrm{d}x^2}(x)\right| \leqslant \sigma_M, \qquad x \in [0,l], \tag{3}$$

$$v(0) = 0, \ \frac{\mathrm{d}v}{\mathrm{d}x}(0) = 0, \ v(l) = 0, \ \frac{\mathrm{d}v}{\mathrm{d}x}(l) = 0, \tag{4}$$

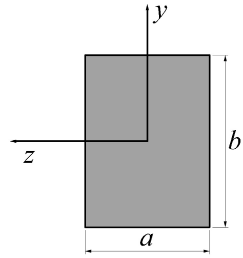
$$a_L \leqslant a \leqslant a_U, \quad b_L \leqslant b \leqslant b_U, \tag{5}$$

$$\tag{6}$$

where ρ is the density of the material, $v(x)$ is the deflection of the beam (with the opposite direction than the axis y) in a point $x \in [0,l]$, E is the Young modulus, σ_M is the maximum stress allowed, and a_L, a_U, b_L, b_U are the bounds on the cross-section dimensions. The constraint (2) is the ODE that governs the deflection of the beam $v(x)$ given a specific load $h(x)$. The constraint (3) is the maximum allowed stress in the beam. The constraint (4) defines the boundary conditions for the ODE (i. e. that we have a fixed beam).



a: The scheme of loaded beam.



b: Beam cross-section.

Fig. 1. The problem geometry.

2.1. FEM problem approximation and solution

To tackle the problem (1) – (5) we use the FEM to approximate the ODE in (2) and (3). Following [22] (p. 25 – 27), we divide the one-dimensional beam with the space dimension x into N finite elements. We will denote the nodal value of the deflection $v(x)$ in the node x_e as $V_e = v(x_e)$ and the nodal value of its derivative in the same node as $\theta_e = \frac{\mathrm{d}v}{\mathrm{d}x}(x_e)$. The continuous variable v is approximated by $\tilde{v}$ in terms of nodal values as follows:

$$\tilde{v}_e = [N_1\, N_2\, N_3\, N_4][V_{e-1}\, \theta_{e-1}\, V_e\, \theta_e]^T$$

where $N_1, \ldots, N_4$ are the following cubic shape functions:

$$N_1 = \frac{1}{d^3}(d^3 - 3\mathrm{d}x^2 + 2x^3), \qquad N_2 = \frac{1}{d^2}(\mathrm{d}^2 x - 2\mathrm{d}x^2 + x^3),$$
$$N_3 = \frac{1}{\mathrm{d}^3}(3\mathrm{d}x^2 - 2x^3), \qquad N_4 = \frac{1}{\mathrm{d}^2}(x^3 - \mathrm{d}x^2),$$

and $d = \frac{l}{N}$ is the length of one element. Substitution in (2) and application of Galerkin's method leads to four element equations:

$$\int_0^d \begin{bmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \end{bmatrix} E\frac{ab^3}{12}\frac{\mathrm{d}^4}{\mathrm{d}x^4}[N_1\, N_2\, N_3\, N_4]\,\mathrm{d}x \begin{bmatrix} V_{e-1} \\ \theta_{e-1} \\ V_e \\ \theta_e \end{bmatrix} = \int_0^d \begin{bmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \end{bmatrix} h(x)\,\mathrm{d}x.$$

To avoid differentiating four times, the following approximation is used:

$$\int N_i \frac{\mathrm{d}^4 N_j}{\mathrm{d}x^4}\mathrm{d}x \approx -\int \frac{\mathrm{d}N_i}{\mathrm{d}x}\frac{\mathrm{d}^3 N_j}{\mathrm{d}x^3}\mathrm{d}x \approx \int \frac{\mathrm{d}^2 N_i}{\mathrm{d}x^2}\frac{\mathrm{d}^2 N_j}{\mathrm{d}x^2}\mathrm{d}x.$$

The resulting system of linear equations has the form: $E\frac{ab^3}{12}\mathbb{K}\mathcal{V} = h$, where $\mathcal{V} = (V_0, \theta_0, \ldots, V_N, \theta_N)^T$. The dimensions of the stiffness matrix $\mathbb{K}$ are $(2N+2)\times(2N+2)$ and its precise description can be found in [21] or [22]. The order of accuracy of the finite element approximation is $\mathcal{O}(d^2)$.

Using this approximation of the deflection, the stress limit (3) on each element is given by

$$|E\frac{b}{2}[N_1^{''}\, N_2^{''}\, N_3^{''}\, N_4^{''}][V_{e-1}\,\theta_{e-1}\,V_e\,\theta_e]^T| \leqslant \sigma_M.$$

This equation describing the stress in one specific node holds only for the end nodes belonging to one element (the first one at x_0 and the last one at x_N). Since the rest of the nodes belongs to two adjacent elements, the stresses are not equal. Therefore, we consider the average stress from this discontinuity:

$$E\frac{b}{2}\frac{1}{2}\left|[N_1^{''}(0)\, N_2^{''}(0)\, N_3^{''}(0)\, N_4^{''}(0)]\begin{bmatrix} V_{e-1} \\ \theta_{e-1} \\ V_e \\ \theta_e \end{bmatrix} + [N_1^{''}(d)\, N_2^{''}(d)\, N_3^{''}(d)\, N_4^{''}(d)]\begin{bmatrix} V_e \\ \theta_e \\ V_{e+1} \\ \theta_{e+1} \end{bmatrix}\right|.$$

The system of inequalities that approximates (4) can be written as $|E\frac{b}{2}\mathbb{C}\mathcal{V}| \leqslant \sigma_M$, where the matrix $\mathbb{C}$ has dimensions $(N+1)\times(2N+2)$ and its complete description can be found in [21].

The FEM approximation of the problem (1)–(5) is then the following (using the notation described above):

$$\underset{a,b,\mathcal{V}}{\text{minimize}} \quad \rho abl \tag{7}$$

$$\text{subject to} \quad E\frac{ab^3}{12}\mathbb{K}\mathcal{V} = h, \tag{8}$$

$$|E\frac{b}{2}\mathbb{C}\mathcal{V}| \leqslant \sigma_M, \tag{9}$$

$$a_L \leqslant a \leqslant a_U, \quad b_L \leqslant b \leqslant b_U. \tag{10}$$

This problem has $2N$ variables ($2N+2$ in $\mathcal{V}$ of which 4 are fixed by boundary conditions, and 2 design variables a and b), $2N+2$ constraints and a box constraints on a and b, and is non-convex, meaning that the certification of global optimality is computationally very demanding.

The crucial realization (the one that is absent in [21] and [24]) is that the stiffness matrix $\mathbb{K}$ is, by design, always invertible. In other words – given a, b and h, the equation describing the deflection of the beam has a unique solution. Using this fact, we can rewrite (8) as:

$$\mathcal{V} = \frac{12}{Eab^3}\mathbb{K}^{-1}h, \tag{11}$$

and (9) becomes:

$$|\frac{6}{ab^2}\mathbb{C}\mathbb{K}^{-1}h| = \frac{1}{ab^2}|6\mathbb{C}\mathbb{K}^{-1}h| \leqslant \sigma_M. \tag{12}$$

Let us denote as v_M the maximum of $|6\mathbb{C}\mathbb{K}^{-1}h|$ over all the nodes of the FEM discretization. Since σ_M is the same for all $N+1$ nodes, the $N+1$ inequalities (9) are equivalent to a single inequality:

$$\frac{v_M}{ab^2} \leqslant \sigma_M. \tag{13}$$

Utilizing these results and neglecting the constants ρ and l in the objective (7), we can reformulated the problem (7) – (10) as the following equivalent problem:

$$\underset{a,b}{\text{minimize}} \quad ab \tag{14}$$

$$\text{subject to} \quad \frac{v_M}{ab^2} \leqslant \sigma_M, \quad a_L \leqslant a \leqslant a_U, \quad b_L \leqslant b \leqslant b_U, \tag{15}$$

which is a geometric program, that can be transformed into a convex program (this transformation is utilized in the following sections), with 2 variables, 1 constraint and box constraints on variables. This problem has the following analytic solution (that is derived in the Appendix A):

- if $\frac{v_M}{a_U b_U^2} > \sigma_M$, the problem is infeasible,
- if $\frac{v_M}{a_L b_L^2} \leqslant \sigma_M$, the solution is $a^* = a_L, b^* = b_L$,
- if $b = \sqrt{\frac{v_M}{a_L \sigma_M}}$ is within the bounds, $b^* = b, a^* = a_L$,
- else $a = \frac{v_M}{b_U^2 \sigma_M}$ and $a^* = a, b^* = b_U$.

This can be readily seen from the problem structure – a percentage increase in both a and b has the same result on the objective function value. However, percentage increase in b causes the left hand side of the inequality (13) to decrease faster than an equal percentage increase in a, making it preferable to increase b as much as needed (i. e. satisfying the inequality or the box constraint) before increasing a.

This result covers some of the numerical examinations done in [21] and [24] (which were more focused on the illustration of the combination of FEM and stochastic programming), without the need for using any optimization software (the only value one has to compute numerically is v_M). Another advantage is that for the same geometry (i. e. the same boundary conditions and number of elements) we can precompute the FEM matrices $\mathbb{C}$ and $\mathbb{K}$ (or its appropriate factorization, see [22], Chapter 3) and use them to quickly get optimal solution for different values of the load h.

2.2. Additional variable, constraints and convex reformulation

The structure of the problem allows us to consider the material constant E as a variable, without destroying the convexity of the upcoming reformulation. This means we can choose the quality of the material – higher E corresponding to better and more expensive one. To be able to perform the convex reformulation, the dependence of the cost on the material (per volume units) must be in the form cE^p, with $c > 0, p \in \mathbf{R}$. The objective function then becomes $cE^p abl$, where the constants c and l can be dropped during the optimization.

An additional restriction on the solution involves the maximum absolute deflection of the beam, which we denote as δ_M. In our FEM formulation, the vector $\mathcal{V}$ includes both the deflection of the beam and its first derivative in each node of the division. The condition on maximum deflection involves only the odd components in $\mathcal{V}$:

$$|\mathcal{V}_i| \leqslant \delta_M, i = 1, 3, 5, \ldots, 2N + 1, \tag{16}$$

which is equivalent to a single inequality

$$\max_{i=1,3,5,\ldots,2N+1} |\mathcal{V}_i| \leqslant \delta_M, \tag{17}$$

using (11) and denoting the maximum of the odd components of $|12\mathbb{K}^{-1}h|$ as w_M we get

$$\frac{w_M}{Eab^3} \leqslant \delta_M. \tag{18}$$

The final constraint restricts the ratio between b and a to be less then the maximum allowed r_M.

Adding these constraints to (14) – (15), treating E as a design variable (within the bounds $0 < E_L \leqslant E_U$) and changing the objective yields the following geometric program (presented here in its standard form):

$$\underset{a,b,E}{\text{minimize}} \quad E^p ab \tag{19}$$

$$\text{subject to} \quad \frac{v_M}{\sigma_M} a^{-1} b^{-2} \leqslant 1, \tag{20}$$

$$\frac{w_M}{\delta_M} E^{-1} a^{-1} b^{-3} \leqslant 1, \tag{21}$$

$$\frac{1}{r_M} b a^{-1} \leqslant 1, \tag{22}$$

$$a_L a^{-1} \leqslant 1, \frac{1}{a_U} a \leqslant 1, \;\; b_L b^{-1} \leqslant 1, \frac{1}{b_U} b \leqslant 1, \;\; E_L E^{-1} \leqslant 1, \frac{1}{E_U} E \leqslant 1, \tag{23}$$

where all the coefficients of the monomials in (19) – (23) are clearly positive, meaning we can use the following transformation to derive an equivalent convex program. First, we transform the variables: $y_a = \log a, y_b = \log b, y_E = \log E$. Then, we can write every monomial $f(a, b, E) = c a^{\alpha_1} b^{\alpha_2} E^{\alpha_3}$, where $c > 0, \alpha_1, \alpha_2, \alpha_3 \in \mathbf{R}$ in the form

$$f(a, b, E) = f(e^{y_a}, e^{y_b}, e^{y_E}) = c e^{\alpha_1 y_a} e^{\alpha_2 y_b} e^{\alpha_3 y_E} = e^{\alpha_1 y_a + \alpha_2 y_b + \alpha_3 y_E + \log c},$$

turning a monomial function into the exponential of an affine function. Next we transform the objective and the constraints, by taking the logarithm. Since every function both in the objective and the constraints is a monomial, the transformation results in a linear program:

$$\underset{y_a, y_b, y_E}{\text{minimize}} \quad y_a + y_b + p\, y_E \tag{24}$$

$$\text{subject to} \quad -y_a - 2y_b + \log v_M - \log \sigma_M \leqslant 0, \tag{25}$$

$$-y_a - 3y_b - y_E + \log w_M - \log \delta_M \leqslant 0, \tag{26}$$

$$-y_a + y_b - \log r_M \leqslant 0, \tag{27}$$

$$\log a_L \leqslant y_a \leqslant \log a_U, \;\; \log b_L \leqslant y_b \leqslant \log b_U, \;\; \log E_L \leqslant y_E \leqslant \log E_U. \tag{28}$$

Remark 2.1. If the dependence of the material cost was $\sum_{i \in I} c_i E^{p_i}$, $c_i > 0$, instead of the simple cE^p, there would still be a convex reformulation, but it would no longer result in a linear program – there would be a term involving a logarithm of a sum of exponentials in the objective (see [4], p. 160 – 162).

3. RANDOM LOADS AND ROBUST SOLUTION

Next we investigate how the problem changes, when we introduce uncertainty. The previous papers [21] and [24] dealt with the situation, when the Young modulus E was random. In this paper, we assume that the randomness is in the load h. Instead of specifying the distribution of h by its cumulative distribution function or moment generating function (that would allow us to use the Bernstein approximation, see [16]), we devised a mechanism that produces random samples/scenarios. The use of scenarios is typical for engineering applications because of the difficulty of identifying the probability distribution. In this way, we imitate the situation when one does not know the distribution of a certain random variable, but only has access to its realizations – in our experience a much more common case. The sampling procedure is the following ($\mathcal{U}(a, b)$ denotes a

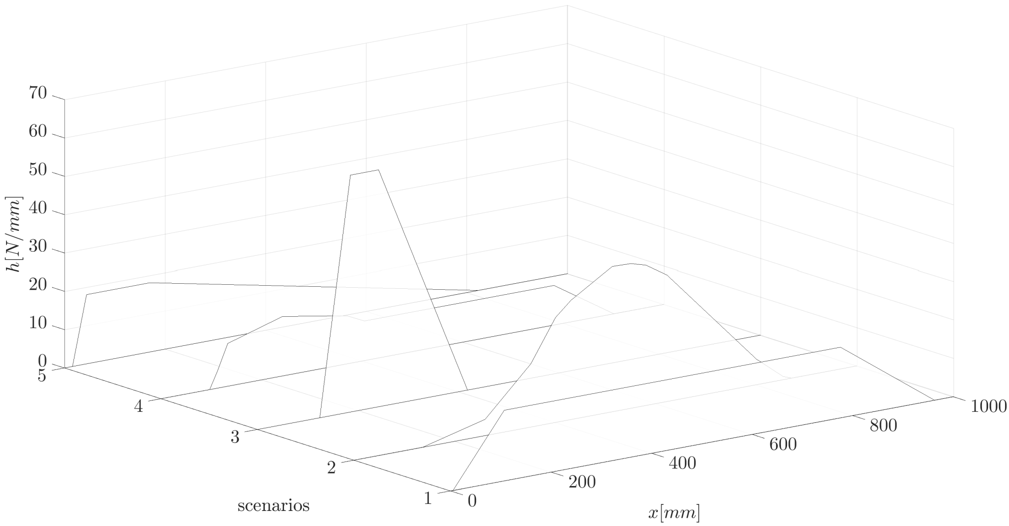


Fig. 2. A sample of 5 scenarios of the load $h(x)$.

uniform distribution):

0. Pick a random integer i between 1 and 4. Set $h(x) = 0$.

1. Repeat i times: Generate a Bernoulli trial.

   a) If 0, randomly pick 4 points $0 \leqslant x_a \leqslant x_b \leqslant x_c \leqslant x_d \leqslant l$ and add to $h(x)$ a trapezoidal load $h_a(x)$ between x_a and x_d. Height of the trapezoid is $h_M \sim \mathcal{U}(0, 1)$ (Figure 2, scenarios 1 and 3).

   b) If 1, sample $h_\mu \sim \mathcal{U}(0, l), h_\sigma \sim \mathcal{U}(0, l)$ and add to $h(x)$ the bell curve load:

$$h_b(x) = \frac{1}{h_\sigma \sqrt{2\pi}} e^{\frac{-(x-h_\mu)^2}{2h_\sigma^2}}.$$

2. Normalize the load $h(x)$: Pick $H \sim \mathcal{U}(8000\,\mathrm{N}, 15000\,\mathrm{N})$. Compute $h_i = \int_0^l h(x)\,\mathrm{d}x$, and set $h(x) = \frac{H}{h_i} h(x)$.

This sampling procedure generates very real-life like loads as can be seen in Figure 2 (see [15]). Because we transformed the original problem (1) – (5) into the problem (24) – (28), we are much more interested in the values v_M and w_M resulting from the the different load scenarios, and the actual loads $h(x)$ are of little importance. In Figure 3 we see the scatter plots and histograms of $\log v_M$ and $\log w_M$ using 2,000 scenarios of the load.

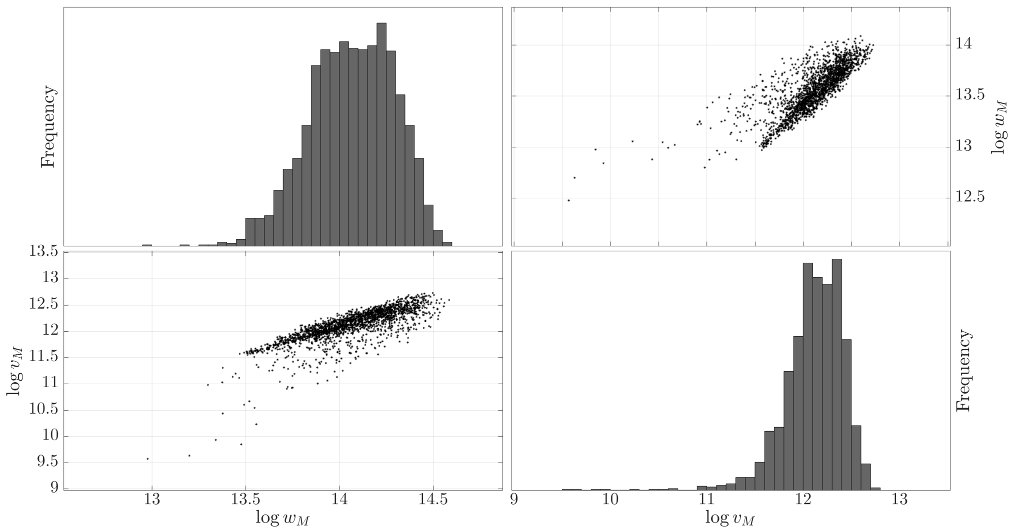


Fig. 3. Scatter plots and histograms of $\log v_M$ and $\log w_M$, 2,000 scenarios.

The important question is how to approach the optimization model (24) – (28) when some of its parameters, namely v_M and w_M, are random. One possibility is to use a so called robust formulation (see [3]), i. e. to enforce that the constraints will hold for any possible value of the random parameter. This results in the following formulation:

$$\underset{y_a,y_b,y_E}{\text{minimize}} \quad y_a + y_b + p\, y_E \tag{29}$$

$$\text{subject to} \quad -y_a - 2y_b + \log v_M(\xi) - \log \sigma_M \leqslant 0, \qquad \forall \xi \in \Xi, \tag{30}$$

$$-y_a - 3y_b - y_E + \log w_M(\xi) - \log \delta_M \leqslant 0, \qquad \forall \xi \in \Xi, \tag{31}$$

$$-y_a + y_b - \log r_M \leqslant 0, \tag{32}$$

$$\log a_L \leqslant y_a \leqslant \log a_U, \;\; \log b_L \leqslant y_b \leqslant \log b_U, \;\; \log E_L \leqslant y_E \leqslant \log E_U, \tag{33}$$

where ξ is a random outcome from a sample space Ξ. This formulation is best suited for situation, when the violation of the constraints would have disastrous consequences.

Given our scenario generation procedure, the robust formulation requires us to find the scenarios that result in the highest values of v_M and w_M, and then optimize the design with respect to these extreme values. The generation procedure allows for point loads (setting all 4 point of the trapezoid into a single point) and the magnitude of the point load is restricted to $15,000\,\text{N}$ by the normalization step. This allows us to find the worst-case scenarios simply by using the formulas for the deflection and stress of a fixed beam under point load (these can be found in [17] and [23]). The analysis of the worst case situations is carried out in the Appendix B.

4. CHANCE CONSTRAINTS AND PROBABILISTIC ROBUST DESIGN

The issue with the robust formulation is that it produces solutions that may be overly conservative. A different approach is to allow the possibility, that some of the constraints are violated, provided that the probability of violation is small. This corresponds to the following chance constrained (or probabilistic constrained, see [20]) formulation of the problem:

$$\underset{y_a,y_b,y_E}{\text{minimize}} \quad y_a + y_b + p\, y_E \tag{34}$$

$$\text{subject to} \quad P\begin{pmatrix} -y_a - 2y_b + \log v_M(\xi) - \log \sigma_M \leqslant 0, \\ -y_a - 3y_b - y_E + \log w_M(\xi) - \log \delta_M \leqslant 0 \end{pmatrix} \geqslant 1 - \epsilon, \tag{35}$$

$$-\, y_a + y_b - \log r_M \leqslant 0, \tag{36}$$

$$\log a_L \leqslant y_a \leqslant \log a_U,\ \ \log b_L \leqslant y_b \leqslant \log b_U,\ \ \log E_L \leqslant y_E \leqslant \log E_U, \tag{37}$$

where $1-\epsilon$ is the reliability level (or, alternatively, ϵ is the allowed violation probability). Except for some special cases, the formulation (34) – (37) is hard to solve exactly (see [20]).

One of the standard approaches (see [14]) to get an approximate solution is to fix the reliability level ϵ, draw a large number S of scenarios and construct a mixed-integer program, where for each scenario we have a binary decision variable, that corresponds to that scenario being neglected or not. One of the constraints then requires that we neglect less then ϵS scenarios. This method is clearly constrained by our ability to solve large mixed-integer programs. One of the most recent of the multiple approaches for solving the mixed-integer formulation was developed in [1].

In this paper we use a different approach based upon a method called Probabilistic Robust Design (see [5, 6] and [7]). This approach requires only that the objective is a convex functions and that the constraint functions are convex for any realization of ξ – there are no other restrictions on the position of the random variable (such as only right-hand side, linearly perturbed, etc.). The first part of the method is, again, to draw a large number S of scenarios (denoted by s) and solve the following problem:

$$\underset{y_a,y_b,y_E}{\text{minimize}} \quad y_a + y_b + p\, y_E \tag{38}$$

$$\text{subject to} \quad -\, y_a - 2y_b + \log v_M(s) - \log \sigma_M \leqslant 0, \qquad s = 1, \dots, S, \tag{39}$$

$$-\, y_a - 3y_b - y_E + \log w_M(s) - \log \delta_M \leqslant 0, \qquad s = 1, \dots, S, \tag{40}$$

$$-\, y_a + y_b - \log r_M \leqslant 0, \tag{41}$$

$$\log a_L \leqslant y_a \leqslant \log a_U,\ \ \log b_L \leqslant y_b \leqslant \log b_U,\ \ \log E_L \leqslant y_E \leqslant \log E_U, \tag{42}$$

where the $2N$ constraints (39) and (40) can be reduced to the following 2 constraints:

$$-\, y_a - 2y_b + \max_s(\log v_M(s)) - \log \sigma_M \leqslant 0, \tag{43}$$

$$-\, y_a - 3y_b - y_E + \max_s(\log w_M(s)) - \log \delta_M \leqslant 0. \tag{44}$$

For a high enough choice of S, the optimal solution to (38) – (44) yields a feasible solution for the chance constrained problem (29) – (33) with high probability (see [5]). As investigated in [18], the approach tends to be overly conservative (i. e., the feasibility result holds, but we get a solution that is far from optimal for the chance constrained problem).

The result regarding optimality for this approach was added in [6], where the idea of discarding scenarios was developed. The main idea is, in addition to drawing S scenarios, to determine a number $k < S$, such that if we remove any k scenarios, the optimal solution of this modified problem is, again, feasible for the chance constrained problem with high probability. Furthermore, if the k scenarios are removed in an optimal fashion (i. e. we select those whose removal decreases the optimal objective value the most), there is a direct link between the optimal solution of the modified problem and the optimal solution of the chance constrained problem (in the sense that we get closer the more scenarios S we draw). Although this basically recovers the standard mixed-integer approach discussed above, there is a crucial difference in how the scenario-removal is achieved.

As discussed in [6], we can remove the k scenarios at once (the mixed-integer variant) or we can use a greedy approach that removes just one scenario at a time. In our case, the greedy approach makes perfect sense – there are only two scenarios (called support scenarios in [7]) whose removal can decrease the optimal objective value of (38) – (44):

$$s_1 = \underset{s}{\operatorname{argmax}}(\log v_M(s)) \quad \text{and} \quad s_2 = \underset{s}{\operatorname{argmax}}(\log w_M(s)).$$

To determine, which one of the two scenarios should be removed, we must solve two additional linear problems (with s_1 or s_2 temporarily removed) and compare their optimal objective values – this is repeated k times. The individual optimization problems have three variables and differ only in the value of one coefficient in (40) or (41) and as such can be efficiently solved by warm-starting the optimization algorithm with the last solution.

There is one different approach we will discuss, and that is the approximation of the joint chance constraint (35) by individual chance constraints:

$$P(-y_a - 2y_b + \log v_M(\xi) - \log \sigma_M \leqslant 0) \geqslant 1 - \epsilon_1, \tag{45}$$

$$P(-y_a - 3y_b - y_E + \log w_M(\xi) - \log \delta_M \leqslant 0) \geqslant 1 - \epsilon_2, \tag{46}$$

which become

$$- y_a - 2y_b + \Phi_v^{-1}(1 - \epsilon_1) - \log \sigma_M \leqslant 0, \tag{47}$$

$$- y_a - 3y_b - y_E + \Phi_w^{-1}(1 - \epsilon_2) - \log \delta_M \leqslant 0, \tag{48}$$

where Φ_v^{-1} and Φ_w^{-1} are the (empirical) quantile functions of $\log v_M(\xi)$ and $\log w_M(\xi)$, and ϵ_1, $\epsilon_2 > 0$ are appropriately chosen. The problem then becomes:

$$\underset{y_a, y_b, y_E}{\text{minimize}} \quad y_a + y_b + p\, y_E \tag{49}$$

$$\text{subject to} \quad - y_a - 2y_b + \Phi_v^{-1}(1 - \epsilon_1) - \log \sigma_M \leqslant 0, \tag{50}$$

$$- y_a - 3y_b - y_E + \Phi_w^{-1}(1 - \epsilon_2) - \log \delta_M \leqslant 0, \tag{51}$$

$$- y_a + y_b - \log r_M \leqslant 0, \tag{52}$$

$$\log a_L \leqslant y_a \leqslant \log a_U, \ \ \log b_L \leqslant y_b \leqslant \log b_U, \ \ \log E_L \leqslant y_E \leqslant \log E_U. \tag{53}$$

The choice of ϵ_1 and ϵ_2 is crucial – simply setting $\epsilon_1 = \epsilon_2 = \epsilon$ does not guarantee that the reliability of the optimal solution of (49) – (53) is better than $1 - \epsilon$ (see Figure 4). To obtain a safe approximation of the joint chance constraint (35), ϵ_1 and ϵ_2 must satisfy (see [16]): $\epsilon_1 + \epsilon_2 \leqslant \epsilon$, the simplest values being $\epsilon_1 = \epsilon_2 = \frac{\epsilon}{2}$.

5. NUMERICAL RESULTS

Our goal is to obtain a trade-off curve between the optimal objective value (the weight of the beam) and the reliability of the design. To achieve this we used our scenarios generation technique to draw two large sets of scenarios, where the first one contained S_1 and the second S_2 scenarios. The first one was used for the optimization part (i. e. solving (38) – (44)), the second one was used for the estimate of the reliability level ϵ. The method proceeded as follows:

0. Generate the two sets of scenarios.
   Repeat k times:
1. Solve (38) – (44) using the first set of scenarios. Obtain an optimal design.
2. Estimate the reliability of the design using the second set of scenarios: given a design in the form of a, b and E, the constraints (39) – (40) either both hold, or at least one of them does not hold. This outcome describes a binomial random variable – compute its point estimate (a fraction of scenarios for which at least on of the constraints did not hold) and its 99.9% confidence interval (using the Clopper-Pearson interval).
3. Determine, which one of the two support scenarios to remove, and delete it from the first set of scenarios. Return to 1.

The problem setting under the numerical investigation was as follows: the length of the beam $l = 1\,\mathrm{m}$, number of elements for the FEM formulation $N = 1{,}000$, objective coefficient $p = \frac{1}{2}$, limits on the variables $a_L = b_L = 10^{-2}\,\mathrm{m}$, $a_U = b_U = 10^{-1}\,\mathrm{m}$, $E_L = 1.9{\cdot}10^5\,\mathrm{MPa}$ and $E_U = 2.2{\cdot}10^5\,\mathrm{MPa}$, maximum stress $\sigma_M = 120\,\mathrm{MPa}$, maximum deflection $\delta_M = 5{\cdot}10^{-4}\,\mathrm{m}$, maximum ratio between the variables $r_M = 5$, number of scenarios in the first set $S_1 = 50{,}000$, number of scenarios in the second set $S_2 = 100{,}000$, number of scenarios to discard $k = 2{,}500$.

The number of elements was chosen such that the length of one element $d = \frac{l}{N} = 10^{-3}\,\mathrm{m}$ results in accuracy $\mathcal{O}(10^{-6})$ of the FEM approximation, which is roughly of the same order as the accuracy of the optimization algorithm (termination criteria for optimality), that was set to 10^{-7}. The accuracy of the FEM was checked using the analytic results in the Appendix B and using ANSYS (commercial engineering simulation software). The FEM formulation was programmed and solved in MATLAB, the optimization parts were computed using the CVX modeling system (see [10]).

In Figure 4 is depicted the trade-off between the reliability level ϵ and the optimal objective value using the two approaches (38) – (44) and (49) – (53). In the first approach we gradually remove the scenarios (upto $k = 2{,}500$) – the computational time for each iteration (two optimization problems, scenario removal) was around 0.4 s. In the second approach (49) – (53) we vary the values of $\epsilon_1 = \epsilon_2$ between 0 and 0.05 – the computational time for each value was around 0.2 s. Furthermore, used a grid of 1,001 steps for ϵ_1 and ϵ_2 between 0 and 0.05 and computed the results for all of these grid values (they fill the

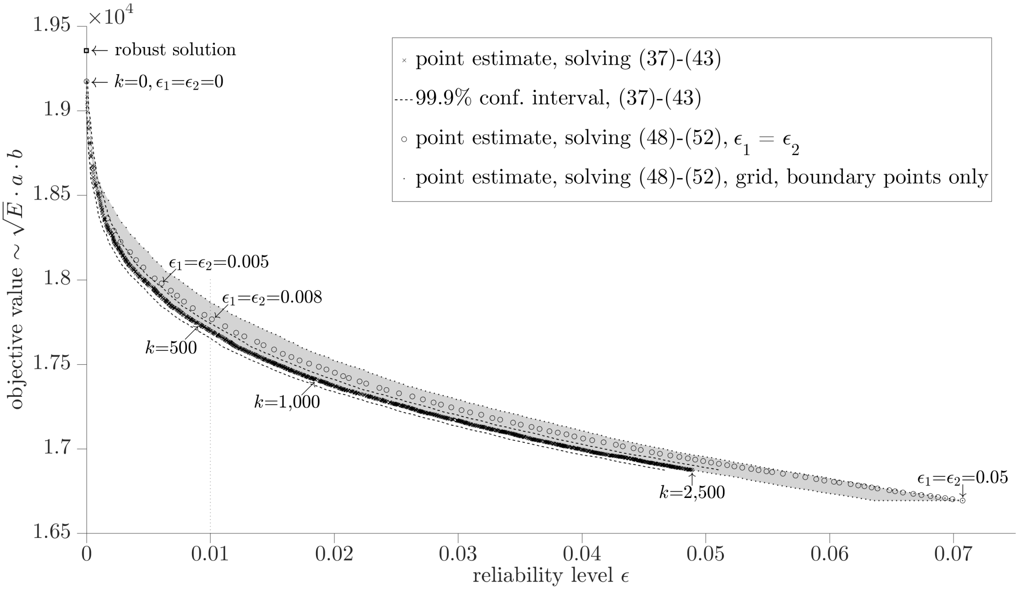


Fig. 4. The trade-off between reliability and optimal objective value.

grey area in Figure 4), this took 45 hours. The robust solution was computed using the results in the Appendix B (maximum point loads in $\frac{1}{2}l$ and $\frac{1}{3}l$).

The comparison between the two methods favours the scenario-removal one (38) – (44) over solving (49) – (53) with $\epsilon_1 = \epsilon_2$, as it produces designs with better objective value. For example, given the target (point estimate of) $\epsilon = 0.01$, the closest design produced by (49) – (53) is for $\epsilon_1 = \epsilon_2 = 0.008$, with the objective value $1.776{\cdot}10^4$, whereas the method using (38) – (44) with $k = 568$ deleted scenarios achieved the objective value $1.769{\cdot}10^4$. Moreover, the scenario-removal method (38) – (44) produced as good solutions as the best ones using the grid values for ϵ_1 and ϵ_2 and solving (49) – (53).

The shape of the trade-off heavily depends on the distribution of $h(x)$ (and, consequently, on the distribution of v_M and w_M). For the computation we used the scenario generation described earlier, which was constructed ad hoc to demonstrate the method. In a real situation (e. g., the one in [12]), the scenario generation will be swapped for the particular problem-specific outcomes.

6. CONCLUSION

In this paper, we have presented new reformulation for the optimal beam design problem, that serves as a test example for a larger set of problems solvable by similar techniques as presented. This reformulation leads to a geometric program and as such can be solved to global optimality. We then used this reformulation and extended the problem by considering randomness in the load and presented the robust and chance constrained problems. The chance constrained variant was handled by the Probabilistic Robust

Design approach. For the given scenario generation procedure we computed the trade-off between reliability and optimal objective value. Further research will be focused on situations, when the cross-section of the beam is not rectangular and the reformulation results in a possibly non-convex problem.

APPENDIX

A. The Analytic Solution

Here we derive the analytic solution for (14) – (15). We use the same convex reformulation as in (34) – (37) to derive an equivalent linear program:

$$\underset{y_a, y_b}{\text{minimize}} \quad y_a + y_b \tag{54}$$

$$\text{subject to} \quad - y_a - 2y_b + \log \frac{v_M}{\sigma_M} \leqslant 0, \tag{55}$$

$$\log a_L \leqslant y_a \leqslant \log a_U, \quad \log b_L \leqslant y_b \leqslant \log b_U. \tag{56}$$

0. a) If $\log \frac{v_M}{\sigma_M} \leqslant \log a_L + 2 \log b_L$, we are done, $a^* = a_L, b^* = b_L$.
   b) If $\log \frac{v_M}{\sigma_M} > \log a_U + 2 \log b_U$, the problem is infeasible.

1. Otherwise, we need $y_a, y_b : y_a + 2y_b = \log \frac{v_M}{\sigma_M}, \log a_L \leqslant y_a \leqslant \log a_U, \log b_L \leqslant y_b \leqslant \log b_U$.

2. The KKT conditions:

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \nu \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \lambda_1 \begin{pmatrix} -1 \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} 0 \\ -1 \end{pmatrix} + \lambda_3 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \lambda_4 \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{57}$$

$$\lambda_1(\log a_L - y_a) = 0, \lambda_2(\log b_L - y_b) = 0, \lambda_3(y_a - \log a_U) = 0, \lambda_4(y_b - \log b_U) = 0, \tag{58}$$

$$\log a_L \leqslant y_a \leqslant \log a_U, \log b_L \leqslant y_b \leqslant \log b_U, y_a + 2y_b = \log \frac{v_M}{\sigma_M}, \lambda_i \geqslant 0, i = 1, \ldots, 4. \tag{59}$$

3. From complementary slackness condition (58) we get 16 different possible situations - corresponding to a or b being at the specific bounds. From the outset it is clear that the variables cannot be at the lower and upper bound at the same time: λ_1 and λ_3 cannot be both nonzero, the same holds for λ_2 and λ_4. This rules out 7 possibilities.

4. If $\lambda_1 = 0, \lambda_2 = 0, \lambda_3 = 0, \lambda_4 = 0$, from (57) we have

$$\nu = -1, \quad \text{from the first row,} \qquad \nu = -\frac{1}{2}, \quad \text{from the second row,}$$

   which is not possible. This means that there cannot be an optimal solution such that $a_L < a^* < a_U$ and $b_L < b^* < b_U$ at the same.

5. If $\lambda_1 > 0, \lambda_2 = 0, \lambda_3 = 0, \lambda_4 = 0$, i. e. $a^* = a_L, y_a^* = \log a_L$. From (57) we have

$$\nu = -\frac{1}{2}, \quad \lambda_1 = \frac{1}{2} > 0,$$

meaning that $y_b^* = \frac{1}{2}\log\frac{v_M}{a_L\sigma_M}$ and $b^* = e^{y_b^*} = \sqrt{\frac{v_M}{a_L\sigma_M}}$ is a possible solution, provided $b_L < b^* < b_U$.

6. If $\lambda_1 > 0, \lambda_2 > 0, \lambda_3 = 0, \lambda_4 = 0$, i.e. $a^* = a_L, b^* = b_L$. This is the situation in 0. a).

7. If $\lambda_1 > 0, \lambda_2 = 0, \lambda_3 = 0, \lambda_4 > 0$, i.e. $a^* = a_L, b^* = b_U$. From (57) we have
$$\lambda_1 = 1 + \nu > 0 \Rightarrow \nu > -1, \quad \lambda_4 = -1 - 2\nu > 0 \Rightarrow \nu < -\frac{1}{2}, \quad \text{which is possible.}$$
This is the (arguably rare) situation when $\log a_L + 2\log b_U = \log\frac{v_M}{\sigma_M}$.

8. If $\lambda_1 = 0, \lambda_2 > 0, \lambda_3 = 0, \lambda_4 = 0$, i.e. $a^* = a_U$. From (57) we have
$$\nu = -1, \quad \lambda_2 = -1 > 0, \quad \text{which is not possible.}$$

9. If $\lambda_1 = 0, \lambda_2 > 0, \lambda_3 > 0, \lambda_4 = 0$, i.e. $a^* = a_U, b^* = b_L$ from (57) we have
$$\lambda_3 = -1 - \nu > 0 \Rightarrow \nu < -1, \quad \lambda_2 = 1 + 2\nu > 0 \Rightarrow \nu > -\frac{1}{2}, \quad \text{which is not possible.}$$

10. If $\lambda_1 = 0, \lambda_2 = 0, \lambda_3 > 0, \lambda_4 = 0$, i.e. $b^* = b_L$. From (57) we have
$$\nu = -\frac{1}{2}, \quad \lambda_3 = -\frac{1}{2} > 0, \quad \text{which is not possible.}$$

11. If $\lambda_1 = 0, \lambda_2 = 0, \lambda_3 > 0, \lambda_4 > 0$, i.e. $a^* = a_U, b^* = b_U$. From (57) we have
$$\lambda_3 = -1 - \nu > 0 \Rightarrow \nu < -1, \quad \lambda_4 = -1 - 2\nu > 0 \Rightarrow \nu < -\frac{1}{2}, \quad \text{which is possible.}$$
This is the situation when $\log a_U + 2\log b_U = \log\frac{v_M}{\sigma_M}$.

12. If $\lambda_1 = 0, \lambda_2 = 0, \lambda_3 = 0, \lambda_4 > 0$, i.e. $b^* = b_U$. From (57) we have
$$\nu = -1, \quad \lambda_4 = 1 > 0, \quad \text{which is possible, } y_a^* = \log\frac{v_M}{b_U^2\sigma_M}, a^* = \frac{v_M}{b_U^2\sigma_M}.$$

B. Worst case deflection and stress for point load

The following results are using the known formulas for deflection and bending moment for fixed beam under a point load that can be found in [17] and [23]. The Figure 5 depicts the situation and provides a graphical description of the used notation.

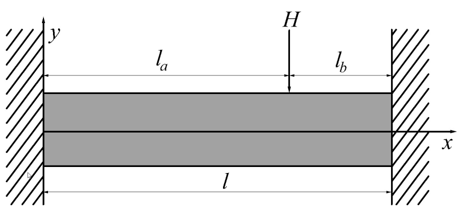


Fig. 5. Point load.

The maximum deflection of a fixed beam under point load is computed by the following formula (can be found in [23], p. 190):

$$\delta_M = \frac{2Hl_a^3 l_b^2}{3EI(3l_a + l_b)^2}, \tag{60}$$

where l_a and l_b correspond to the location of the point load ($l_a + l_b = l$), I is the moment of inertia of the cross-section and E is the Young modulus. In our case $I = \frac{ab^3}{12}$. If we look at (60) as a function of the location l_a of the point load, its maximum occurs when $l_a = l_b = \frac{l}{2}l$.

The maximum stress for each point $x \in [0, l]$ in the beam can be expressed in the following terms: $\sigma_M(x) = \frac{M(x)}{I} y_M$, where $M(x)$ is the bending moment and $y_M = \pm\frac{b}{2}$. This allows us to use the formulas for maximum bending moment of fixed beam under point load to find the critical points (the signs in the formulas are neglected, since the constraint (3) restricts the absolute value of the stress). The bending moment of a beam under point load changes linearly between the points $0, l_a$, and l, so it suffices to compute the bending moment in these three points. Given a point load at $x = l_a$ bending moment at the ends of the beam ($x = 0$ and $x = l$) is

$$\text{left end: } M(x=0) = \frac{Hl_a l_b^2}{l^2}, \quad \text{right end: } M(x=l) = \frac{Hl_a^2 l_b}{l^2},$$

the maximum occurs when $l_a = \frac{1}{3}l$ (or $l_a = \frac{2}{3}l$) resulting in $M(x = 0 \text{ or } x = l) = \frac{4}{27}lH$.

The moment at the location of the point is $M(x = l_a) = \frac{2Hl_a^2 l_b^2}{l^3}$, for which the maximum occurs when $l_a = l_b = \frac{1}{2}l$, resulting in $M(x = \frac{1}{2}l) = \frac{1}{8}lH$. This means that worst case occurs, when the point load is located in $l_a = \frac{1}{3}l$ or $l_a = \frac{2}{3}l$.

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