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ON THE ACCURACY OF APPROXIMATION OF THE DISTRIBUTION OF NEGATIVE-BINOMIAL RANDOM SUMS BY THE GAMMA DISTRIBUTION

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The main goal of this paper is to study the accuracy of approximation for the distributions of negative-binomial random sums of independent, identically distributed random variables by the gamma distribution.

Keywords: gamma distribution, negative-binomial random sums, Trotter's distance

Classification: 60F05, 60G50

1. INTRODUCTION

Let $X, X_1, X_2, \dots$ be a sequence of independent, identically distributed, positive random variables with mean $0 < \mathbb{E}(X) = m < +\infty$ and finite variance $0 < Var(X) = \sigma^2 < +\infty$. Let ν_p be a geometric random variable with parameter p (for short, $\nu_p \sim Geo(p), p \in (0,1)$), independent of all $X_i, i \geq 1$ and its probability mass distribution is given by

$$P(\nu_p = k) = p(1-p)^{k-1}, \quad k \geq 1, p \in (0,1).$$

Set $S_{\nu_p} = X_1 + X_2 + \cdots + X_{\nu_p}$. Then, the S_{ν_p} is said to be a geometric random sum or a geometric convolution of X (see [2]). Geometric random sums arise naturally in many applied probability models. The asymptotic behavior of the distribution of a geometric random sum is an object of interest in applied areas like finance, insurance, reliability, queue system and risk theory, etc. (see [2, 7, 12, 15, 16] and the references given there). It should be noted that $\mathbb{E}(\nu_p) \to +\infty$ as $p \to 0^+$. Therefore, the following Rényi's limit theorem (see [12] and [15] for more details) is one of well-known results for geometric random sums. Specifically,

$$\frac{S_{\nu_p}}{\mathbb{E}(\nu_p)} \xrightarrow{D} Z \quad \text{as} \quad p \to 0^+, \tag{1}$$

where Z is an exponential random variable with mean m $(m > 0)$, that is

$$P(Z \leq x) = 1 - e^{-m^{-1}x}, \quad x \in (0, +\infty),$$

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and $\xrightarrow{D}$ denotes the convergence in distribution.

The rate of convergence for limiting expression in (1) was estimated by Kalashnikov (see [12] for more details) as follows:

$$\sup_{0<x<+\infty}\left|P\left(\frac{S_{\nu_p}}{\mathbb{E}(\nu_p)}\leq x\right)-P\left(Z\leq x\right)\right|=o(1)\quad\text{as}\quad p\to 0^+. \tag{2}$$

In particular, when $\mathbb{E}(X_n)=1$, for $n=1,2,\dots$ and $\mathbb{E}(X_n^s)<\infty$ for $s\in(1,2)$, the rate of approximation in (1) is given by

$$\sup_{0<x<+\infty}\left|P\left(\frac{S_{\nu_p}}{\mathbb{E}(\nu_p)}\leq x\right)-1+e^{-x}\right|\leq C\left(\frac{1}{\mathbb{E}(\nu_p)}\right)^{s-1}\mathbb{E}(X_n^s), \tag{3}$$

where $C>0$ is an absolute constant (see [15] for more details).

It is worth pointing out that the upper bound for $d(S_{\nu_p},Z_{\mathbb{E}(S_{\nu_p})})$, the sup norm distance between geometric random sum S_{ν_p} and an exponential random variable with mean $\mathbb{E}(S_{\nu_p})$, is given by Brown (see [2] for more details) as follows:

$$d(S_{\nu_p},Z_{\mathbb{E}(S_{\nu_p})})\leq cp,\quad\text{for}\quad 0<p\leq 1/2,$$

where $c=\mathbb{E}X^2/(\mathbb{E}X)^2$, and $X,X_1,X_2,\dots$ are independent identically distributed positive random variables, and $\nu_p\sim Geo(p),p\in(0,1)$ is independent of all $X_i,i\geq 1$.

The problem to be considered in this paper is that of extending of known results in (1), (2) and (3). Specifically, the paper is concerned with the rates of weak convergence in following limiting expression

$$\frac{S_{N_{r,p}}}{\mathbb{E}(N_{r,p})}\xrightarrow{D}\mathcal{G},\quad\text{as}\quad p\to 0^+, \tag{4}$$

where $S_{N_{r,p}}:=X_1+\cdots+X_{N_{r,p}}$ is a negative-binomial random sum of independent and identically distributed random variables, $\mathcal{G}$ is a gamma distributed random variable, $N_{r,p}$ is a negative-binomial distributed random variable with two parameters $r\in\mathbb{N}\setminus\{0\}$, and $p\in(0,1)$, (written $N_{r,p}\sim\mathcal{NB}(r,p)$), with probability mass function (see for instance [1])

$$P(N_{r,p}=k)=\binom{k-1}{r-1}p^r(1-p)^{k-r},k\geq r;r\in\mathbb{N}\setminus\{0\},p\in(0,1). \tag{5}$$

In particular, when $r=1$ the relation (5) specifies the geometric distribution and the limit expression in (1) will be deduced by (4).

In recent years, the rate of convergence in limit expression (4) was investigated by Bevrani et al. in [1] and Gavrilenko et al. in [6]. It should be noticed that all estimates are established via uniform metrics (see [6] and [1] for more details).

Throughout this paper, the symbol $Gam(\alpha,\beta)$ denotes the gamma distribution of two parameters $\alpha>0$ and $\beta>0$. A random variable $\mathcal{G}$ is said to be a gamma distributed random variable with two parameters α and β, in short $\mathcal{G}\sim Gam(\alpha,\beta)$, if its probability density function is defined in following form:

$$f_{\mathcal{G}}(x)=\begin{cases}\frac{\beta^\alpha}{\Gamma(\alpha)}x^{\alpha-1}e^{-\beta x}, & \text{if}\quad x\in(0,+\infty),\\ 0, & \text{elsewhere,}\end{cases} \tag{6}$$

where $\Gamma(\alpha) = \int_0^\infty y^{\alpha-1}e^{-y}\mathrm{d}y$ is the gamma function ($\alpha > 0$). According to (6) the characteristic function of $Gam(\alpha, \beta)$ distribution is given by

$$\varphi_{\mathcal{G}}(t) := \mathbb{E}(e^{i\mathcal{G}t}) = \left(\frac{\beta}{\beta - it}\right)^{\alpha}, \quad t \in (-\infty, +\infty). \tag{7}$$

The gamma distribution is frequently a probability model for waiting times, for instance, in life testing, the waiting time until "death" is a random variable which frequently modeled with a gamma distribution (see [9] for more details). In particular, when $\alpha = 1$, and $\beta = \lambda$, then we have an exponential distributed random variable whose the probability density function is defined by

$$g(x) = \begin{cases} \lambda e^{-\lambda x}, & \text{if} \quad x \in (0, \infty), \\ 0, & \text{elsewhere}, \end{cases} \tag{8}$$

and its characteristic function is calculated as follows:

$$\varphi(t) = \frac{\lambda}{\lambda - it}, \quad t \in (-\infty, +\infty).$$

From on now, let $N_{r,p} \sim \mathcal{NB}(r, p)$ be a negative-binomial distributed random variable with two parameters $r \in \mathbb{N} \setminus \{0\}$ and $p \in (0, 1)$. The random sum $S_{N_{r,p}} := X_1 + X_2 + \cdots + X_{N_{r,p}}$ is said to be a negative-binomial random sum (see [21]). It can be verified that $\mathbb{E}(N_{r,p}) = rp^{-1}$. Through this paper we assume that parameter $p \to 0^+$ while the second parameter $r \in \mathbb{N} \setminus \{0\}$ is fixed. It is worth noticing that for a negative-binomial random variable $N_{r,p} \sim \mathcal{NB}(r, p)$, with two parameters $r \in \mathbb{N} \setminus \{0\}$ and $p \in (0, 1)$,

$$N_{r,p} \stackrel{D}{=} \nu_1 + \nu_2 + \cdots + \nu_r, \tag{9}$$

where $\nu_i := \nu_{p,i} \sim Geo(p), i \in \{1, 2, \ldots, r\}$, are independent geometric random variables with success probability $p \in (0, 1)$. The symbol $\stackrel{D}{=}$ denotes the equality in distribution.

The main purpose of this paper is to investigate the asymptotic behavior of the distributions of a desired negative-binomial random sum $\dfrac{S_{N_{r,p}}}{\mathbb{E}(N_{r,p})}$ when $p \to 0^+$, and establish the rate of convergence in limit theorems for negative-binomial random sums of independent and identically distributed random variables. Theorem 3.2 states that the gamma distribution is a weak limit of distribution of negative-binomial random sum $\dfrac{S_{N_{r,p}}}{\mathbb{E}(N_{r,p})}$, when $p \to 0^+$. It is clear that the Rényi's limit theorem in (1) (see [12]) is a direct consequent of Theorem 3.2 (see Remark 3.3). Moreover, the rates of convergence in weak limit theorems for negative-binomial random sums of independent identically distributed random variables are established via Theorem **??**, Theorem **??** and Theorem **??**. The estimates in limit theorems were established via Trotter distance, based on Trotter-operator method originated by Trotter (see [22] for more details). Up to now the Trotter-operator method had attracted much attention and it also had successfully been used and modified such as Rényi ([19]), Butzer et al. ([3, 4] and [5]), Rychlick ([20]), Kirschfink ([13]), and Hung ([10] and [11]).

Recently, using the Stein's method, some results of gamma approximation have been obtained by Peköz and Röllin ([17]), and Peköz et al. ([18]). Moreover, some estimates for the rate of convergence of the negative binomial distribution with parameters (r, p) to the gamma distribution with parameters (r, r) when $p \to 0$ are obtained by Gavrilenko et al. in [6].

It is worth noticing that some results related to central limit theorems for standardized negative-binomial random sums of independent identically distributed random variables with the upper bounds of the L_s norms $(1 \le s \le \infty)$ are investigated by Sunklodas (see [21] for more details).

This paper is organized as follows. Section 2 is devoted to a brief recall of the Trotter–operator method, Trotter distance and their properties. The definition of the modulus of continuity of a function and Lipschitz classes are recalled in this section, too. The Section 3 deals with the main results of this paper.

2. PRELIMINARIES

Before stating the main results we must review the definition of a probability metric (e. g. [12, 13] and [10]). From now on, we denote by Ξ the set of random variables in probability space $(\Omega, \mathcal{A}, P)$.

Definition 2.1. The mapping $d : \Xi \times \Xi \longrightarrow [0, +\infty)$ is said to be a probability metric for two random variables X and Y, denoted by $d(X, Y)$, if it possesses for the random variables $X, Y, Z \in \Xi$ the following properties

1. $P(X = Y) = 1 \Longrightarrow d(X, Y) = 0$;
2. $d(X, Y) = d(Y, X)$;
3. $d(X, Y) \le d(X, Z) + d(Z, Y)$.

Let $C_B(\mathbb{R})$ be a set of all real–valued, bounded, uniformly continuous functions f on the set of reals $\mathbb{R} = (-\infty, +\infty)$ with norm $\| f \| = \sup_{x \in \mathbb{R}} | f(x) |$, and write

$$C_B^k(\mathbb{R}) = \left\{ f \in C_B(\mathbb{R}) | f^{(i)} \in C_B(\mathbb{R}),\ i = 1, 2, \ldots, k; k \ge 1 \right\}.$$

Let us consider the Zolotarev metrics as an example of well known probability metrics which will be compared with desired Trotter distance in next part. The Zolotarev metric (see [13] for more details) for random variables X and Y is defined by

$$d_Z(X, Y) = \sup \left\{ \left| \mathbb{E}[f(X) - f(Y)] \right| ; f \in D_1(k; r+1; C_B(\mathbb{R})) \right\},$$

where

$$D_1(k; r+1; C_B(\mathbb{R})) = \left\{ f \in C_B^r(\mathbb{R}); \left| f^{(r)}(x) - f^{(r)}(y) \right| \le |x - y|^k \right\}.$$

It should be note that $D_1(k; r+1; C_B(\mathbb{R})) \subseteq C_B^r(\mathbb{R}) \subseteq C_B(\mathbb{R})$. For a deeper discussion of probability metrics we refer the reader to [10, 12, 13] and the references given there.

We need to recall the definition of Trotter operator which was mainly originated by Trotter in 1959 (see [22] for more details).

Definition 2.2. For every $f \in C_B(\mathbb{R})$, the Trotter operator of random variable X is defined by the mapping $T_X : C_B(\mathbb{R}) \longrightarrow C_B(\mathbb{R})$, such that

$$T_X f(y) := \mathbb{E}\Big[f(X+y)\Big] = \int_{-\infty}^{+\infty} f(x+y)\,\mathrm{d}F_X(x),$$

where $y \in \mathbb{R}$ and $F_X(x)$ is cumulative distribution function of a real-valued random variable X.

The definition, properties and applications of the Trotter operator T_X can be found in [3, 4, 5, 19, 20, 22] and [13]. It is necessary to recall the definition of Trotter distance recommended by Kirschfink (see [13] and [10] for more details).

Definition 2.3. The Trotter distance $d_T(X, Y; f)$ of two random variables X and Y associated to a function $f \in C_B^k(\mathbb{R}), k \geq 1$, is defined by the mapping $d_T : \Xi \times \Xi \longrightarrow [0, +\infty)$, such that

$$d_T(X, Y; f) = \sup_{y\in\mathbb{R}} \Big|\mathbb{E}[f(X+y)] - \mathbb{E}[f(Y+y)]\Big|.$$

We need in the sequel the following properties of the Trotter distance $d_T(X, Y; f)$. The proofs are easy to get from the properties of the Trotter operator (see [13] and [10] for more details).

1. Trotter distance $d_T(X, Y; f)$ is a probability metric.
2. Let X and Y be two random variables defined in space Ξ. For all $f \in C_B^k(\mathbb{R}), k \geq 1$, if
$$d_T(X, Y; f) = 0,$$
then $X \stackrel{D}{=} Y$.
3. Let $X_1, X_2, \ldots$ be a sequence of random variables and let X be a random variable defined on Ξ. If, for $f \in C_B^k(\mathbb{R}), k \geq 1$,
$$d_T(X_n, X; f) \longrightarrow 0 \quad \text{as} \quad n \to \infty,$$
then $X_n \stackrel{D}{\longrightarrow} X$ as $n \to \infty$.
4. Assume that $X_1, X_2, \ldots$ and $Y_1, Y_2, \ldots$ are two sequences of independent random variables (in each sequence). Then, for $f \in C_B(\mathbb{R})$,
$$d_T\left(\sum_{i=1}^{n} X_i, \sum_{i=1}^{n} Y_i; f\right) \leq \sum_{i=1}^{n} d_T(X_i, Y_i; f).$$
5. Suppose that $X_1, X_2, \ldots$ and $Y_1, Y_2, \ldots$ are two sequences of independent random variables (in each sequence). Let N be a positive integer-valued random variable, independent of all $X_1, X_2, \ldots$ and $Y_1, Y_2, \ldots$. Then, for $f \in C_B(\mathbb{R})$,
$$d_T\left(\sum_{i=1}^{N} X_i, \sum_{i=1}^{N} Y_i; f\right) \leq \sum_{n=1}^{\infty} P(N=n) \sum_{i=1}^{n} d_T(X_i, Y_i; f).$$

6. Assume that $X, X_1, X_2, \ldots$ and $Y, Y_1, Y_2, \ldots$ are two sequences of independent and identically distributed random variables (in each sequence). Let N be a positive integer-valued random variable, independent of all $X_1, X_2, \ldots$ and $Y_1, Y_2, \ldots$. Moreover, assume that $\mathbb{E}(N) < \infty$. Then, for $f \in C_B(\mathbb{R})$,

   (a) $d_T\left(\sum_{i=1}^{n} X_i, \sum_{i=1}^{n} Y_i; f\right) \leq n d_T(X, Y; f)$,

   (b) $d_T\left(\sum_{i=1}^{N} X_i, \sum_{i=1}^{N} Y_i; f\right) \leq \mathbb{E}(N) d_T(X, Y; f)$.

Based on discussion in [13] and [10], the following fact is connection between the Trotter distance and the Zolotarev metric.

Proposition 2.4.

$$\sup\{d_T(X, Y; f); f \in D_1(k; r+1; C_B(\mathbb{R}))\} = d_Z(X, Y).$$

It is to be noticed that the connections between the Trotter distance and other probability metrics as Kolmogorov metric, Lëvy metric, Prokhorov metric are discussed in [10, 13] and the references given there.

The concept of the modulus of continuity of a function $f \in C_B(\mathbb{R})$ plays a noticeable role in this paper. Firstly, we need to recall the definition of the modulus of continuity of a function $f \in C_B(\mathbb{R})$ (see [3] for more details)

Definition 2.5. For every function $f \in C_B(\mathbb{R})$ and $\forall \delta > 0$, the function

$$\omega(f; \delta) = \sup_{|h| \leq \delta} \sup_{x \in \mathbb{R}} \left| f(x+h) - f(x) \right|,$$

is called a modulus of continuity of a function f.

Some properties of modulus of continuity of a function f are defined as follows:

1. $\omega(f; \delta)$ is a monotonically increasing function of δ, i. e. if $0 < \delta_1 < \delta_2$, then
$$\omega(f; \delta_1) \leq \omega(f; \delta_2).$$
2. $\omega(f; \delta) \longrightarrow 0 \quad \text{as} \quad \delta \to 0^+$.
3. $\omega(f; \lambda\delta) \leq (1 + \lambda)\omega(f; \delta)$ for all $\lambda > 0$.

(See [3] for more details).

Definition 2.6. The function $f \in C_B(\mathbb{R})$ is said to satisfy the Lipschitz condition of order α, $(0 < \alpha \leq 1)$, if there exists a positive constant M, such that

$$\omega(f; \delta) \leq M\delta^\alpha.$$

The smallest constant M in above inequality is called the Lipschitz constant of f. We denote by $Lip(\alpha, M)$ the Lipschitz class of functions of order $\alpha, (0 < \alpha \leq 1)$, with Lipschitz constant M, that is

$$Lip(\alpha, M) = \{f \in C_B(\mathbb{R}) : \omega(f; \delta) \leq M\delta^\alpha\}.$$

Remark 2.7. Let X be a random variable with $\mathbb{E}\left(|X|^k\right) < +\infty$. Then $\mathbb{E}\left(|X|^j\right) < +\infty$ for any $1 \le j \le k$, and

$$\mathbb{E}\left(|X|^j\right) \le 1 + \mathbb{E}\left(|X|^k\right),$$

(see [3] for more details).

3. MAIN RESULTS

Before stating the main results we first need the following lemma.

Lemma 3.1. Let $Z, Z_1, Z_2, \ldots$ be a sequence of independent, exponential distributed random variables with mean $m > 0$. Let $N_{r,p}$ be a negative-binomial random variable with parameters $r \in \mathbb{N}$ and $p \in (0,1)$, independent of all $Z_n, n \ge 1$. Set $S_{N_{r,p}} = Z_1 + Z_2 + \cdots + Z_{N_{r,p}}$. Then, for $r \in \mathbb{N}, p \in (0,1)$,

$$S_{N_{r,p}} \sim Gam(r, m^{-1}p),$$

where $Gam(r, m^{-1}p)$ is a gamma distribution of parameters $r \in \mathbb{R}$ and $m^{-1}p$, with characteristic function is given in form $\left(\frac{p}{p-imt}\right)^r$.

P r o o f . It is easy to check that the probability generating function of geometric random variable ν_p is defined as follows:

$$\psi_{\nu_p}(t) := \mathbb{E}(t^{\nu_p}) = \frac{pt}{1-(1-p)t}, \quad | t |< \frac{1}{1-p}, \quad p \in (0,1).$$

We denote by $\psi_{N_{r,p}}(t)$ the probability generating function of a negative-binomial random variable $N_{r,p} \sim \mathcal{NB}(r,p)$. According to (8) and (9), the probability generating function of negative-binomial random variable $N_{r,p}$ with parameters $r \in \mathbb{N} \setminus \{0\}$ and $p \in (0,1)$ will be given by

$$\psi_{N_{r,p}}(t) = \left[\psi_{\nu_p}(t)\right]^r = \left[\frac{pt}{1-(1-p)t}\right]^r, \quad | t |< \frac{1}{1-p}, \quad p \in (0,1).$$

Let $\varphi_Z(t)$ be a characteristic function of the exponential random variable with mean $m > 0$. It is easily seen that

$$\varphi_Z(t) = \left(\frac{1}{1-imt}\right), \quad \text{for} \quad t \in (-\infty, +\infty).$$

On account of Theorem 9.3 ([8], page 193), the characteristic function of the negative-binomial random sum is given by

$$\varphi_{S_{N_{r,p}}}(t) = \psi_{N_{r,p}}\left[\varphi_Z(t)\right] = \left[\frac{p.\frac{1}{1-imt}}{1-(1-p).\frac{1}{1-imt}}\right]^r = \left(\frac{p}{p-imt}\right)^r.$$

Therefore, $S_{N_{r,p}} \sim Gam(r, m^{-1}p)$. The proof is complete. □

The following theorem confirms that the gamma distribution is a weak limiting distribution of a negative-binomial random sum of independent and identically distributed random variables.

Theorem 3.2. Let $X, X_1, X_2, \ldots$ be a sequence of independent and identically distributed positive–valued random variables with finite mean $0 < \mathbb{E}(X) = m < +\infty$. Let $N_{r,p}$ be a negative-binomial random variable with two parameters $r \in \mathbb{N} \setminus \{0\}$ and $p \in (0,1)$. Assume that $N_{r,p}$ is independent of all $X_n, n \geq 1$. Then,

$$\frac{S_{N_{r,p}}}{\mathbb{E}(N_{r,p})} \xrightarrow{D} \mathcal{G} \quad \text{as} \quad p \to 0^+, \tag{10}$$

where $S_{N_{r,p}} = \sum_{i=1}^{N_{r,p}} X_i$ and $\mathcal{G}$ is a gamma distributed random variable with parameters r and $m^{-1}r$ (for short, $\mathcal{G} \sim Gam\left(r, m^{-1}r\right)$).

P r o o f. Let us denote by $\psi_X(s) := \mathbb{E}[s^X]$ and $\varphi_X(t) := \mathbb{E}[e^{itX}]$ the generating function and the characteristic function of a random variable X, respectively. Then, direct computation shows that, for $r \in \mathbb{N} \setminus \{0\}$ and for $p \in (0,1)$,

$$\psi_{N_{r,p}}(s) = \left[\frac{ps}{1-(1-p)s}\right]^r, \quad \text{for} \quad |\, s \,| < \frac{1}{1-p} \quad \text{and} \quad p \in (0,1).$$

In view of [8] (Theorem 9.1 and 9.2, pages 193–194), the characteristic function of $\frac{S_{N_{r,p}}}{\mathbb{E}(N_{r,p})}$ is given by

$$\begin{aligned}
\varphi_{\frac{S_{N_{r,p}}}{\mathbb{E}(N_{r,p})}}(t) &= \varphi_{S_{N_{r,p}}}(r^{-1}pt) = \psi_{N_{r,p}}\left(\varphi_X(r^{-1}pt)\right) \\
&= \left[\frac{p\left(1 + r^{-1}pt\varphi'(\eta)\right)}{1-(1-p)\left(1+r^{-1}pt\varphi'(\eta)\right)}\right]^r = \left[\frac{1 + r^{-1}pt\varphi'(\eta)}{1-(1-p)r^{-1}t\varphi'(\eta)}\right]^r,
\end{aligned}$$

where the Maclaurin series of the differentiable function $\varphi_X(t)$ used as follows

$$\varphi_X(r^{-1}pt) = \varphi_X(0) + r^{-1}pt\varphi'_X(\eta) = 1 + r^{-1}pt\varphi'_X(\eta),$$

for $\theta \in (0,1)$ and $\eta = \theta r^{-1}pt \to 0$ as $p \to 0$. Letting $p \to 0^+$, with $\eta \to 0$ and $\mathbb{E}(X) = m$, using the fact that the function $\varphi'_X(t)$ is continuous at zero, we can assert that

$$\varphi'_X(\eta) \longrightarrow \varphi'_X(0) = im \quad \text{as} \quad p \to 0^+.$$

Consequently,

$$\lim_{p\to 0^+} \varphi_{\frac{N_{r,p}}{\mathbb{E}(N_{r,p})}}(t) = \left[\frac{1}{1 - ir^{-1}mt}\right]^r = \left[\frac{r}{r - imt}\right]^r = \varphi_{\mathcal{G}}(t) \quad \text{for} \quad t \in (-\infty, +\infty).$$

The proof is completed. □

Remark 3.3. It is worth pointing out that should be received as a trivial corollary of Theorem 18 due to Korolev (see [14] for more details).

The following theorems will be extends the results of [11].

Theorem 3.4. Let $X, X_1, X_2, \ldots$ be a sequence of independent, identically distributed positive-valued random variables with expectation $0 < \mathbb{E}(X) = m$ and variance $0 < Var(X) = \sigma^2 < +\infty$. Let $N_{r,p}$ be a negative-binomial distributed random variable with two parameters $r \in \mathbb{N} \backslash \{0\}$ and $p \in (0,1)$, independent of all random variables $X_n, n \geq 1$. Then, for all $f \in C_B^1(\mathbb{R})$,

$$d_T\left(\frac{S_{N_{r,p}}}{\mathbb{E}(N_{r,p})}, \mathcal{G}; f\right) \leq (2m + 3m^2 + \sigma^2)\omega(f'; pr^{-1}). \tag{11}$$

If, in addition $f' \in Lip(\alpha, M), (0 < \alpha \leq 1)$, then

$$d_T\left(\frac{S_{N_{r,p}}}{\mathbb{E}(N_{r,p})}, \mathcal{G}; f\right) \leq M(2m + 3m^2 + \sigma^2)(pr^{-1})^{\alpha}, \tag{12}$$

where $S_{N_{r,p}} = \sum_{i=1}^{N_{r,p}} X_i$ and $\mathcal{G}$ is a gamma distributed random variable with parameters $r \in \mathbb{N} \setminus \{0\}$ and rm^{-1}, M is a positive constant.

P r o o f. According to Lemma 3.1, we have

$$\mathcal{G} \stackrel{D}{=} \frac{S^*_{N_{r,p}}}{\mathbb{E}(N_{r,p})},$$

where $S^*_{N_{r,p}} = \sum_{i=1}^{N_{r,p}} Z_i$ and $Z_1, Z_2, \ldots$ is a sequence of independent, exponential random variables with mean m, i. e. $Z_1 \sim Exp(m^{-1})$.
For all $f \in C_B^1(\mathbb{R})$, it is easy to check that

$$d_T\left(\frac{S_{N_{r,p}}}{\mathbb{E}(N_{r,p})}, \mathcal{G}; f\right) \leq \left(\frac{r}{p}\right) d_T\left(\frac{p}{r}X_1, \frac{p}{r}Z_1; f\right).$$

Using the Taylor series expansion, we obtain

$$\mathbb{E}f\left(\frac{p}{r}X_1 + y\right) = f(y) + \frac{p}{r}mf'(y) + \frac{p}{r}\int_0^{+\infty} x[f'(\eta) - f'(y)]\,\mathrm{d}F_{X_1}(x),$$

where $|\eta - y| < \frac{p}{r}x$. By an analogous argument to the previous one, we get

$$\mathbb{E}f\left(\frac{p}{r}Z_1 + y\right) = f(y) + \frac{p}{r}mf'(y) + \frac{p}{r}\int_0^{+\infty} x[f'(\xi) - f'(y)]\,\mathrm{d}F_{Z_1}(x),$$

with $|\xi - y| < \frac{p}{r}x$.

Consider $\left|\mathbb{E}f\left(\frac{p}{r}X_1+y\right)-\mathbb{E}f\left(\frac{p}{r}Z_1+y\right)\right|$

$$\begin{aligned}
&\leq \frac{p}{r}\int_0^{+\infty} x|f'(\eta)-f'(y)|\,\mathrm{d}F_{X_1}(x)+\frac{p}{r}\int_0^{+\infty} x|f'(\xi)-f'(y)|\,\mathrm{d}F_{Z_1}(x)\\
&\leq \frac{p}{r}\int_0^{+\infty} x\omega\left(f';\frac{p}{r}x\right)\mathrm{d}F_{X_1}(x)+\frac{p}{r}\int_0^{+\infty} x\omega\left(f';\frac{p}{r}x\right)\mathrm{d}F_{Z_1}(x)\\
&\leq \frac{p}{r}\omega\left(f';\frac{p}{r}\right)\int_0^{+\infty} x(1+x)\,\mathrm{d}F_{X_1}(x)+\frac{p}{r}\omega\left(f';\frac{p}{r}\right)\int_0^{+\infty} x(1+x)\,\mathrm{d}F_{Z_1}(x)\\
&= \frac{p}{r}\omega\left(f';\frac{p}{r}\right)\left[\int_0^{+\infty}(x+x^2)\,\mathrm{d}F_{X_1}(x)+\int_0^{+\infty}(x+x^2)\,\mathrm{d}F_{Z_1}(x)\right]\\
&= \frac{p}{r}\omega\left(f';\frac{p}{r}\right)\left\{\mathbb{E}(X_1)+\mathbb{E}\left[(X_1)^2\right]+\mathbb{E}(Z_1)+\mathbb{E}\left[(Z_1)^2\right]\right\}\\
&= \frac{p}{r}\omega\left(f';\frac{p}{r}\right)\left(2m+3m^2+\sigma^2\right).
\end{aligned}$$

Hence

$$d_T\left(\frac{S_{N_{r,p}}}{\mathbb{E}(N_{r,p})},\mathcal{G};f\right)\leq\left(2m+3m^2+\sigma^2\right)\omega\left(f';\frac{p}{r}\right).$$

It is easy to prove that if $f'\in Lip(\alpha,M)$, then

$$d_T\left(\frac{S_{N_{r,p}}}{\mathbb{E}(N_{r,p})},\mathcal{G};f\right)\leq\left(2m+3m^2+\sigma^2\right)M\left(\frac{p}{r}\right)^{\alpha}.$$

The proof is complete. □

Theorem 3.5. Let $X_1, X_2, \dots$ be a sequence of independent, standard normal distributed random variables, in short $X_i\sim\mathcal{N}(0,1), i\geq 1$. Let $N_{r,p}$ be a negative-binomial random variable with two parameters $r\in\mathbb{N}\setminus\{0\}$ and $p\in(0,1)$. Additionally, assume that $N_{r,p}$ is independent of all $X_i, i=1,2,\dots$. Then, for all $f\in C_B^2(\mathbb{R})$,

$$d_T\left(\frac{S^2_{N_{r,p}}}{\mathbb{E}(N_{r,p})},\mathcal{G}^*;f\right)\leq\left(\frac{p}{2r}\right)\left[\|f''\|+26\omega(f'';\frac{p}{r})\right], \tag{13}$$

where, $S^2_{N_{r,p}}=\sum_{i=1}^{N_{r,p}}X_i^2$ and $\mathcal{G}^*$ is a gamma distributed random variables with equal parameters (written, $\mathcal{G}^*\sim Gam(r,r)$).

Proof. Since $X_i\sim\mathcal{N}(0,1),\ i\geq 1$, we have $X_i^2\sim\chi^2(1)$, where $\chi^2(1)$ is χ^2-distribution with 1 degrees of freedom. The density function function of $\chi^2(1)$ is given by (see [9] for more details)

$$f_{\chi^2(1)}(x)=\begin{cases}\dfrac{1}{\sqrt{2}\Gamma\left(\frac{1}{2}\right)}x^{-\frac{1}{2}}e^{-\frac{x}{2}} & \text{if}\quad x>0;\\ 0 & \text{if}\quad x\leq 0,\end{cases}$$

where $\Gamma\left(\frac{1}{2}\right) = \int_0^{+\infty} x^{-\frac{1}{2}} e^{-x}\, \mathrm{d}x$.

It can easily seen that

$$\mathbb{E}\left\{(X_i^2)^k\right\} = \int_0^{+\infty} x^k . \frac{1}{\sqrt{2}\Gamma\left(\frac{1}{2}\right)} x^{-\frac{1}{2}} e^{-\frac{x}{2}}\, \mathrm{d}x = \int_0^{+\infty} \frac{1}{\sqrt{2}\Gamma\left(\frac{1}{2}\right)} x^{k-\frac{1}{2}} e^{-\frac{x}{2}}\, \mathrm{d}x.$$

Putting $y = \dfrac{x}{2}$, we obtain

$$\mathbb{E}\left\{(X_i^2)^k\right\} = \frac{2^k}{\Gamma\left(\frac{1}{2}\right)} \int_0^{+\infty} y^{k-\frac{1}{2}} e^{-y} \mathrm{d}y = \frac{2^k}{\Gamma\left(\frac{1}{2}\right)} .\Gamma\left(k+\frac{1}{2}\right). \tag{14}$$

Using integration by parts, we have

$$\Gamma\left(k+\frac{1}{2}\right) = \left(k-\frac{1}{2}\right).\left(k-\frac{3}{2}\right)\ldots\frac{3}{2}.\frac{1}{2}.\Gamma\left(\frac{1}{2}\right). \tag{15}$$

Combining (14) and (15), it follows that

$$\mathbb{E}\left\{(X_i^2)^k\right\} = (2k-1).(2k-3)\ldots 3.1 = (2k-1)!!. \tag{16}$$

According to equation (16) it is obvious that

$$\mathbb{E}\left(X_i^2\right) = 1, \quad \mathbb{E}\left[\left(X_1^2\right)^2\right] = 3, \quad \mathbb{E}\left[\left(X_1^2\right)^3\right] = 15.$$

Let $Z_1, Z_2, \ldots$ be a sequence of independent, exponential distributed random variables with mean 1. Then, according to Lemma 3.1, it follows that

$$\mathcal{G}^* \stackrel{D}{=} \frac{p}{r} \sum_{i=1}^{N_{r,p}} Z_i.$$

Therefore, using the Trotter distance, we have

$$d_T\left(\frac{S_{N_{r,p}}^2}{\mathbb{E}(N_{r,p})}, \mathcal{G}^*; f\right) = d_T\left(\frac{S_{N_{r,p}}^2}{\mathbb{E}(N_{r,p})}, \frac{p}{r}\sum_{i=1}^{N_{r,p}} Z_i; f\right) \le \mathbb{E}(N_{r,p}) d_T\left(\frac{p}{r}X_1^2, \frac{p}{r}Z_1; f\right).$$

Hence,

$$d_T\left(\frac{S_{N_{r,p}}^2}{\mathbb{E}(N_{r,p})}, \mathcal{G}^*; f\right) \le \frac{r}{p} d_T\left(\frac{p}{r}X_1^2, \frac{p}{r}Z_1; f\right).$$

For every $f \in C_B^2(\mathbb{R})$, using Taylor expansion

$$f(x+y) = f(y) + xf'(y) + \frac{x^2}{2} f''(y) + \frac{x^2}{2}\left[f''(\eta) - f''(y)\right],$$

where $|\eta - y| < |x|$.

Consider

$$\mathbb{E}f\left(\frac{p}{r}X_1^2+y\right)=\int_0^{+\infty} f\left(\frac{p}{r}x+y\right)\mathrm{d}F_{X_1^2}(x)$$
$$=f(y)+\frac{p}{r}f'(y)+\frac{3\left(\frac{p}{r}\right)^2}{2}f''(y)+\frac{\left(\frac{p}{r}\right)^2}{2}\int_0^{+\infty} x^2\left[f''(\eta)-f''(y)\right]\mathrm{d}F_{X_1^2}(x),$$

with $|\eta-y|<\frac{p}{r}|x|$.

By an analogous argument to the previous one, we get

$$\mathbb{E}f\left(\frac{p}{r}Z_1+y\right)=\int_0^{+\infty} f\left(\frac{p}{r}x+y\right)\mathrm{d}F_{Z_1}(x)$$
$$=f(y)+\frac{p}{r}f'(y)+\left(\frac{p}{r}\right)^2 f''(y)+\frac{(\frac{p}{r})^2}{2}\int_0^{+\infty} x^2\left[f''(\xi)-f''(y)\right]\mathrm{d}F_{Z_1}(x),$$

where $|\xi-y|<\frac{p}{r}|x|$. Then,

$$\begin{aligned}
&\left|\mathbb{E}f\left(\tfrac{p}{r}X_1^2+y\right)-\mathbb{E}f\left(\tfrac{p}{r}Z_1+y\right)\right|\\
&\le \frac{\left(\frac{p}{r}\right)^2}{2}|f''(y)|+\frac{\left(\frac{p}{r}\right)^2}{2}\int_0^{+\infty} x^2\left|f''(\eta)-f''(y)\right|\mathrm{d}F_{X_1^2}(x)\\
&\quad+\frac{p^2}{2}\int_0^{+\infty} x^2\left|f''(\xi)-f''(y)\right|\mathrm{d}F_{Z_1}(x)\\
&\le \frac{\left(\frac{p}{r}\right)^2}{2}\|f''\|+\frac{\left(\frac{p}{r}\right)^2}{2}\int_0^{+\infty} x^2\omega\left(f'';\frac{p}{r}x\right)\mathrm{d}F_{X_1^2}(x)\\
&\quad+\frac{\left(\frac{p}{r}\right)^2}{2}\int_0^{+\infty} x^2\omega\left(f'';\frac{p}{r}x\right)\mathrm{d}F_{Z_1}(x)\\
&\le \frac{\left(\frac{p}{r}\right)^2}{2}\left[\|f''\|+\omega\left(f'';\frac{p}{r}\right)\int_0^{+\infty}(x^2+x^3)\mathrm{d}F_{X_1^2}(x)\right.\\
&\quad\left.+\,\omega\left(f'';\frac{p}{r}\right)\textstyle\int_0^{+\infty}(x^2+x^3)\,\mathrm{d}F_{Z_1}(x)\right].
\end{aligned}$$

Hence,

$$d_T\left(\frac{S^2_{N_{r,p}}}{\mathbb{E}\left(N_{r,p}\right)},\mathcal{G}^*;f\right)\le\left(\frac{p}{2r}\right)\left[||f''||+26\omega\left(f'';\frac{p}{r}\right)\right].$$

The proof is complete. □

Theorem 3.6. Let $X_1, X_2, \ldots$ be a sequence of independent, identically distributed, positive-valued random variables with $\mathbb{E}(\mid X_1 \mid^k) < +\infty, k \geq 2$. Let $Z_1, Z_2, \ldots$ be a sequence of independent, exponential distributed random variables with $\mathbb{E}(\mid Z_1 \mid^k) < +\infty, k \geq 2$. Moreover, assume that

$$\int\limits_{\mathbb{R}} x^j \mathrm{d}F_{X_1}(x) = \int\limits_{\mathbb{R}} x^j \mathrm{d}F_{Z_1}(x), \quad 0 \leq j < k; 2 \leq k, k \in \mathbb{N}. \tag{17}$$

Let $N_{r,p}$ be a negative-binomial distribution random variable with two parameters $r \in \mathbb{N}\backslash\{0\}$ and $p \in (0,1)$. Assume that $N_{r,p}$ is independent of all $X_i,\ i \geq 1$ and $Z_i, i \geq 1$. Then, for all $f \in C_B^{k-1}(\mathbb{R})$

$$d_T\left(\frac{S_{N_{r,p}}}{\mathbb{E}(N_{r,p})}, \mathcal{G}; f\right) \leq \frac{2\left(\frac{p}{r}\right)^{k-2}}{(k-1)!}\omega\left(f^{(k-1)}; \frac{p}{r}\right)\left[1 + \mathbb{E}\left(X_1^k\right) + \mathbb{E}\left(Z_1^k\right)\right].$$

If $f^{(k-1)} \in Lip(\alpha, M), (0 < \alpha \leq 1)$, then

$$d_T\left(\frac{S_{N_{r,p}}}{\mathbb{E}(N_{r,p})}, \mathcal{G}; f\right) \leq \frac{2M\left(\frac{p}{r}\right)^{k+\alpha-2}}{(k-1)!}\left[1 + \mathbb{E}\left(X_1^k\right) + \mathbb{E}\left(Z_1^k\right)\right],$$

where $S_{N_{r,p}} = \sum_{i=1}^{N_{r,p}} X_i$ and $\mathcal{G}$ is a gamma distributed random variable with parameters r and $\frac{r}{m}$.

P r o o f. On account of Lemma 3.1 we observe that

$$\mathcal{G} \stackrel{D}{=} \frac{p}{r} S^*_{N_{r,p}},$$

where $S^*_{N_{r,p}} = \sum_{i=1}^{N_{r,p}} Z_i$. For every $f \in C_B^{k-1}(\mathbb{R})$, using the Trotter distance, it follows that

$$d_T\left(\frac{S_{N_{r,p}}}{\mathbb{E}(N_{r,p})}, \mathcal{G}; f\right) = d_T\left(\frac{S_{N_{r,p}}}{\mathbb{E}(N_{r,p})}, \frac{S^*_{N_{r,p}}}{\mathbb{E}(N_{r,p})}; f\right) \leq \mathbb{E}(N_{r,p}) d_T\left(\frac{p}{r}X_1, \frac{p}{r}Z_1; f\right).$$

Let $f \in C_B^{k-1}(\mathbb{R})$, using the Taylor series expansion, we have

$$f(x+y) = f(y) + \sum_{i=1}^{k-1} \frac{x^i}{i!} f^{(i)}(y) + \frac{x^{k-1}}{(k-1)!}\left[f^{(k-1)}(\eta) - f^{(k-1)}(y)\right],$$

with $\eta \in (y, x+y)$.

Consider

$$\mathbb{E}f\left(\frac{p}{r}X_1 + y\right) = f(y) + \sum_{i=1}^{k-1} \frac{\left(\frac{p}{r}\right)^i f^{(i)}(y)}{i!} \int_0^{+\infty} x^i \mathrm{d}F_{X_1}(x)$$
$$+ \frac{\left(\frac{p}{r}\right)^{k-1}}{(k-1)!} \int_0^{+\infty} x^{k-1}\left[f^{(k-1)}(\eta) - f^{(k-1)}(y)\right] \mathrm{d}F_{X_1}(x),$$

where $|\eta - y| < \dfrac{p}{r}x$. In the same way, we have

$$\mathbb{E}f\left(\frac{p}{r}Z_1 + y\right) = f(y) + \sum_{i=1}^{k-1} \frac{\left(\frac{p}{r}\right)^i f^{(i)}(y)}{i!} \int_0^{+\infty} x^i \mathrm{d}F_{Z_1}(x)$$
$$+ \frac{\left(\frac{p}{r}\right)^{k-1}}{(k-1)!} \int_0^{+\infty} x^{k-1} \left[f^{(k-1)}(\xi) - f^{(k-1)}(y)\right] \mathrm{d}F_{Z_1}(x),$$

where $|\xi - y| < \dfrac{p}{r}x$. Therefore,

$$\begin{aligned}
&\left|\mathbb{E}f\left(\tfrac{p}{r}X_1 + y\right) - \mathbb{E}f\left(\tfrac{p}{r}Z_1 + y\right)\right| \\
&\le \frac{\left(\frac{p}{r}\right)^{k-1}}{(k-1)!} \int_0^{+\infty} x^{k-1} \left|f^{(k-1)}(\eta) - f^{(k-1)}(y)\right| \mathrm{d}F_{X_1}(x) \\
&\quad + \frac{\left(\frac{p}{r}\right)^{k-1}}{(k-1)!} \int_0^{+\infty} x^{k-1} \left|f^{(k-1)}(\xi) - f^{(k-1)}(y)\right| \mathrm{d}F_{Z_1}(x) \\
&\le \frac{\left(\frac{p}{r}\right)^{k-1}}{(k-1)!} \int_0^{+\infty} x^{k-1} \omega\left(f^{(k-1)}; \frac{p}{r}x\right) \mathrm{d}F_{X_1}(x) \\
&\quad + \frac{\left(\frac{p}{r}\right)^{k-1}}{(k-1)!} \int_0^{+\infty} x^{k-1} \omega\left(f^{(k-1)}; \frac{p}{r}x\right) \mathrm{d}F_{Z_1}(x) \\
&\le \frac{\left(\frac{p}{r}\right)^{k-1}}{(k-1)!} \omega\left(f^{(k-1)}; \frac{p}{r}\right) \left[\int_0^{+\infty} \left(x^{k-1} + x^k\right) \mathrm{d}F_{X_1}(x) + \int_0^{+\infty} \left(x^{k-1} + x^k\right) \mathrm{d}F_{Z_1}(x)\right] \\
&\le \frac{2\left(\frac{p}{r}\right)^{k-1}}{(k-1)!} w\left(f^{(k-1)}; \frac{p}{r}\right) \left[1 + \mathbb{E}\left(X_1^k\right) + \mathbb{E}\left(Z_1^k\right)\right].
\end{aligned}$$

Consequently,

$$d_T\left(\frac{S_{N_{r,p}}}{\mathbb{E}(N_{r,p})}, \mathcal{G}; f\right) \le \frac{2\left(\frac{p}{r}\right)^{k-2}}{(k-1)!} \omega\left(f^{(k-1)}; \frac{p}{r}\right) \left[1 + \mathbb{E}\left(X_1^k\right) + \mathbb{E}\left(Z_1^k\right)\right].$$

If $f^{(k-1)} \in Lip(\alpha, M)$, then we have

$$d_T\left(\frac{S_{N_{r,p}}}{\mathbb{E}(N_{r,p})}, \mathcal{G}; f\right) \le \frac{2M\left(\frac{p}{r}\right)^{k+\alpha-2}}{(k-1)!} \left[1 + \mathbb{E}\left(X_1^k\right) + \mathbb{E}\left(Z_1^k\right)\right].$$

The proof is complete. □

We conclude this paper with the following comment:

Remark 3.7. It is worth pointing out that all received results in theorems in this paper are valid in the Geometric distribution case by setting $r = 1$.

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REFERENCES

[1] H. Bevrani, V. E. Bening, and V. Yu. Korolev: On the accuracy of approximation of the negative-binomial distribution by the gamma distribution and convergence rate of the distributions of some statistics to the Student distribution. J. Math. Sci. *205* (2015), 1, 34–44. DOI:10.1007/s10958-015-2227-6

[2] M. Brown: Error bound for exponential approximations of geometric convolutions. Ann. Probab. *18* (1990), 3, 1388–1402. DOI:10.1214/aop/1176990750

[3] P. L. Butzer, L. Hahn, U. and Westphal: On the rate of approximation in the central limit theorem. J. Approx. Theory *13* (1975), 327–340. DOI:10.1016/0021-9045(75)90042-8

[4] P. L. Butzer and L. Hahn: General theorems on rates of convergence in distribution of random variables I. General limit theorems. J. Multivariate Anal. *8* (1978), 181–201. DOI:10.1016/0047-259x(78)90071-4

[5] P. L. Butzer, H. Kirschfink, and D. Schulz: An extension of the Lindeberg-Trotter operator-theoretic approach to limit theorems for dependent random variables. Acta Sci. Math. *51* (1987), 423–433.

[6] S. V. Gavrilenko, V. N. Zubov, and V. Yu. Korolev: The rate of convergence of the distributions of regular statistics constructed from samples with negatively binomially distributed random sizes to the student distribution. J. Math. Sci. *220* (2017), 6, 701–713. DOI:10.1007/s10958-016-3213-3

[7] J. Grandell: Risk Theory and geometric sums. Inform. Processes *2* (2002) 2, 180–181.

[8] A. Gut: Probability: A Graduate Course. Springer Texts in Statistics. Springer, New York 2005.

[9] R. V. Hogg, J. W. McKean, and A. T. Craig: Introduction to Mathematical Statistics. Seventh edition. Pearson Education, Inc. 2013.

[10] T. L. Hung: On a Probability metric based on Trotter operator. Vietnam J. Math. *35* (2007), 1, 22–33.

[11] T. L. Hung: On the rate of convergence in limit theorems for geometric sums. Southeast Asian J. Sci. *2* (2013) 2, 117–130.

[12] V. Kalashnikov: Geometric Sums: Bounds for Rare Events with Applications. Kluwer Academic Publisher 1997.

[13] H. Kirschfink: The generalized Trotter operator and weak convergence of dependent random variables in different probability metrics. Results Math. *15* (1989), 294–323.

[14] V. Yu. Korolev: Convergence of random sequences with independent random indices. I (in Russian). Teor. Veroyatnost. i Primenen. *39* (1994), 2, 313–333; translation in Theory Probab. Appl. *39* (1995), 2, 282–297.

[15] V. M. Kruglov and V. Yu. Korolev: Limit theorems for Random Sums (in Russian). Moskov. Gos. Univ., Moscow 1990.

[16] M. T. Malinowski: Geometrically strictly semi-stable law as the limit laws. Discuss. Math. Probab. Statist. *27* (2007), 79–97.

[17] E. Peköz and A. Röllin: New rates for exponential approximation and the theorems of Rényi and Yaglom. Ann. Probab. *39* (2011), 2, 587–608. DOI:10.1214/10-aop559

[18] E. Peköz, A. Röllin, and N. Ross: Generalized Gamma approximation with rates for urns, walks and trees. Ann. Probab. *44* (2016) 3, 1776–1816. DOI:10.1214/15-aop1010

[19] A. Rényi: Probability Theory. North-Holland Series in Applied Mathematics and Mechanics *10*, North-Holland Publishing Co., Amsterdam-London; American Elsevier Publishing Co., Inc., New York. 1970.

[20] Z. Rychlick: A central limit theorem for sums of a random number of independent random variables. Coll. Math. *35* (1976), 147–158.

[21] J. K. Sunklodas: On the normal approximation of a negative binomial random sum. Lith. Math. J. *55* (2015), 1, 150–158.

[22] H. F. Trotter: An elementary proof of the central limit theorem. Arch. Math. *10* (1959), 226–234.

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