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SOME LIMIT THEOREMS FOR M-PAIRWISE NEGATIVE QUADRANT DEPENDENT RANDOM VARIABLES

Yongfeng Wu and JiangYan Peng

The authors first establish the Marcinkiewicz–Zygmund inequalities with exponent p ($1 \leq p \leq 2$) for m-pairwise negatively quadrant dependent (m-PNQD) random variables. By means of the inequalities, the authors obtain some limit theorems for arrays of rowwise m-PNQD random variables, which extend and improve the corresponding results in [Y. Meng and Z. Lin (2009)] and [H. S. Sung (2013)]. It is worthy to point out that the open problem of [H. S. Sung, S. Lisawadi, and A. Volodin (2008)] can be solved easily by using the obtained inequality in this paper.

Keywords: m-pairwise negative quadrant dependent, Marcinkiewicz–Zygmund inequality, L^r convergence, complete convergence

Classification: 60F15, 60F25

1. INTRODUCTION

The concept of negative quadrant dependent (NQD) was introduced by [8].

Definition 1.1. Two random variables X and Y are said to be NQD if

$$P(X \leq x, Y \leq y) \leq P(X \leq x)P(Y \leq y) \quad \text{for all } x \text{ and } y.$$

A sequence of random variables $\{X_n, n \geq 1\}$ is said to be pairwise NQD if every pair of random variables in the sequence are NQD.

Remark 1.2. It is important to note that negatively orthant dependent (NOD, [4]), negatively associated (NA, [7]) or linearly negative quadrant dependent (LNQD, [14]) implies pairwise NQD.

It is well known that sequences of pairwise NQD random variables are a family of very wide scope and have been an attractive research topic in the recent papers. We refer reader to [1, 2, 3, 5, 6, 9, 10, 11, 12, 15, 16, 18, 19, 20, 21].

The literature [17] introduced a new concept of m-pairwise negative quadrant dependent (m-PNQD), which contains pairwise NQD.

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Definition 1.3. Let $m \geq 1$ be a fixed integer. A sequence of random variables $\{X_n, n \geq 1\}$ is said to be m-PNQD if for all $n \geq 2$ and all choices of $i_1, \ldots, i_n$ such that $|i_k - i_j| \geq m$ for all $1 \leq k \neq j \leq n$, $X_{i_1}, \ldots, X_{i_n}$ are pairwise NQD.

It is easily seen that this concept is a natural extension of the concept of pairwise NQD random variables (wherein $m = 1$). Indeed, if $\{X_n, n \geq 1\}$ is m-PNQD for some $m \geq 1$, then $\{X_n, n \geq 1\}$ is m'-PNQD for all $m' > m$.

Clearly the m-PNQD structure is substantially more comprehensive than the pairwise NQD structure. We can provide the following example to illustrate that this dependence indeed allows a wide range of dependence structures.

Example 1.4. Let $\{X_1, X_n, n \geq 3\}$ and $\{X_n, n \geq 2\}$ be sequences of pairwise NQD random variables respectively. Then $\{X_n, n \geq 1\}$ is a sequences of 2-PNQD random variables. In fact, there are no dependence restrictions between random variables X_1 and X_2. For instance, we can allow that X_1 and X_2 are positively quadrant dependent. Let X_1 and X_2 be dependent according to the Farlie-Gumbel-Morgenstern copula with the parameter $\theta \in [-1, 1]$ (see Example 3.12 in [13]),

$$C_\theta(u, v) = uv + \theta uv(1-u)(1-v), \ (u, v) \in [0, 1]^2,$$

which is absolutely continuous with density

$$c_\theta(u, v) = \frac{\partial^2 C_\theta(u, v)}{\partial u \partial v} = 1 + \theta(1-2u)(1-2v), \ (u, v) \in [0, 1]^2.$$

If we take $\theta \in (0, 1]$, X_1 and X_2 are positively quadrant dependent (see Section 5.2 in [13], p. 188).

For pairwise NQD random variables, the following Marcinkiewicz –Zygmund inequality with exponent $p = 2$

$$E\left|\sum_{k=1}^{n} X_k\right|^p \leq C \sum_{k=1}^{n} E|X_k|^p \tag{1.1}$$

has been proved by [18] (see Lemma 2.2). However, according to our knowledge, the above inequality with exponent p $(1 \leq p < 2)$ has not been discussed in previous literature. Because of the limitation of the exponent $p = 2$, many authors could not obtain desirable results of the convergence properties for pairwise NQD random variables. In this article, we will prove the above inequality with exponent p $(1 \leq p < 2)$ remains true for pairwise NQD random variables.

The literature [15] obtained the following L^r convergence result for weighted sums of arrays of rowwise pairwise NQD random variables.

Theorem 1.5. Let $\{X_{ni}, u_n \leq i \leq v_n, n \geq 1\}$ be an array of rowwise pairwise NQD random variables and $1 \leq r < 2$. Let $\{a_{ni}, u_n \leq i \leq v_n, n \geq 1\}$ be an array of constants. Suppose that

(i) $\sup_{n\geq 1} \sum_{i=u_n}^{v_n} |a_{ni}|^r E|X_{ni}|^r < \infty$,

(ii) $\sum_{i=u_n}^{v_n} |a_{ni}|^r E|X_{ni}|^r I(|a_{ni}|^r |X_{ni}|^r > \varepsilon) \to 0$ as $n \to \infty$ for any $\varepsilon > 0$. Then

$$\sum_{i=u_n}^{v_n} a_{ni}(X_{ni} - EX_{ni}) \to 0$$

in L^r and, hence, in probability as $n \to \infty$.

The literature [12] studied the weak laws of large numbers for the array of rowwise pairwise NQD random variables and obtained the following theorem.

Theorem 1.6. Let $\{X_{ni}, 1 \le i \le k_n \uparrow \infty, n \ge 1\}$ be a triangular array of random variables which is pairwise NQD in each row, and $EX_{ni} = 0$, $1 \le i \le k_n$ for each $n \ge 1$. Suppose that the uniform Cesàro-type condition

$$\lim_{x\to\infty} \sup_{n\ge1} k_n^{-1} \sum_{i=1}^{k_n} xP(|X_{ni}|^r > x) = 0 \tag{1.2}$$

for some $r \in (1, 2)$ holds. Then $k_n^{-1/r} \sum_{i=1}^{k_n} X_{ni} \xrightarrow{p} 0$ as $n \to \infty$.

In this work, we first establish the Marcinkiewicz–Zygmund inequality for m-PNQD random variables. Then we obtain two L^r convergence results for arrays of rowwise m-PNQD random variables, which extend and improve Theorem 1.5 and Theorem 1.6 respectively under the same conditions. In addition, we study the complete convergence for array of rowwise m-PNQD random variables, which was not considered by [15] and [12].

It is worthy to point out that we can easily solve the open problem of [16] by using the obtained inequality (See Remark 2.5). In addition, the method used in this article is much simpler than those in [15] and [12].

Throughout this paper, the symbol C represents positive constants whose values may change from one place to another. $I(A)$ will indicate the indicator function of A.

2. PRELIMINARIES

To prove our main results, we need some technical lemmas. By Definition 1.2 and Lemma 1 of [8], we can get the following lemma.

Lemma 2.1. Let $\{X_n, n \ge 1\}$ be a sequence of m-PNQD random variables. Let $\{f_n, n \ge 1\}$ be a sequence of increasing functions. Then $\{f_n(X_n), n \ge 1\}$ is a sequence of m-PNQD random variables.

Lemma 2.2. (Wu [18]) Let $\{X_n, n \ge 1\}$ be a sequence of pairwise NQD random variable with mean zero and $EX_n^2 < \infty$, and $T_j(k) = \sum_{i=j+1}^{j+k} X_i$, $j \ge 0$. Then

$$E(T_j(k))^2 \le C \sum_{i=j+1}^{j+k} EX_i^2, \quad E \max_{1\le k\le n} (T_j(k))^2 \le C \log^2 n \sum_{i=j+1}^{j+n} EX_i^2.$$

Lemma 2.3. Let $\{X_{ni}, 1 \le i \le k_n \uparrow \infty, n \ge 1\}$ be an array of any random variables satisfying (1.2) for some real number $r > 0$. Then the following statements hold:

(i) If $0 < \eta < r$, then

$$\lim_{n\to\infty} k_n^{-\eta/r} \sum_{i=1}^{k_n} E|X_{ni}|^\eta I(|X_{ni}|^r > k_n) = 0; \tag{2.1}$$

(ii) If $\delta > r$, then

$$\lim_{n\to\infty} k_n^{-\delta/r} \sum_{i=1}^{k_n} E|X_{ni}|^\delta I(|X_{ni}|^r \le k_n) = 0. \tag{2.2}$$

P r o o f. Firstly, we prove (2.1). Put $A = k_n^{-\eta/r} \sum_{i=1}^{k_n} E|X_{ni}|^\eta I(|X_{ni}|^r > k_n)$. Since

$$\begin{aligned} E|X_{ni}|^\eta I(|X_{ni}|^r > k_n) &= \Big(\int_0^{k_n^{\eta/r}} + \int_{k_n^{\eta/r}}^\infty\Big) P(|X_{ni}|^\eta I(|X_{ni}|^r > k_n) \ge t)\,\mathrm{d}t \\ &= \int_0^{k_n^{\eta/r}} P(|X_{ni}|^r > k_n)\,\mathrm{d}t + \int_{k_n^{\eta/r}}^\infty P(|X_{ni}|^\eta \ge t)\,\mathrm{d}t \\ &= k_n^{\eta/r} P(|X_{ni}|^r > k_n) + \int_{k_n^{\eta/r}}^\infty P(|X_{ni}|^\eta \ge t)\,\mathrm{d}t, \end{aligned}$$

we have

$$\begin{aligned} A &= \sum_{i=1}^{k_n} P(|X_{ni}|^r > k_n) + k_n^{-\eta/r} \sum_{i=1}^{k_n} \int_{k_n^{\eta/r}}^\infty P(|X_{ni}|^\eta \ge t)\,\mathrm{d}t \\ &=: A_1 + A_2. \end{aligned}$$

By letting $x = k_n$ in (1.2), we get $A_1 \to 0$ as $n \to \infty$. For A_2, let $t = u^{\eta/r}$, then

$$A_2 = C k_n^{-\eta/r} \sum_{i=1}^{k_n} \int_{k_n}^\infty u^{\eta/r-1} P(|X_{ni}|^r \ge u)\,\mathrm{d}u.$$

From (1.2), we know that, for any given $\varepsilon > 0$, there exists N such that

$$k_n^{-1} \sum_{i=1}^{k_n} P(|X_{ni}|^r > u) \le \varepsilon u^{-1} \tag{2.3}$$

if $u > N$. Since $k_n \uparrow \infty$, while n is sufficiently large, we can get $k_n > N$. Therefore, by $\eta < r$, we have

$$A_2 \le C\varepsilon k_n^{1-\eta/r} \int_{k_n}^\infty u^{\eta/r-2}\mathrm{d}u \le C\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, $A_2 \to 0$ as $n \to \infty$. The proof of (2.1) is complete.

Next we prove (2.2). Put $B = k_n^{-\delta/r} \sum_{i=1}^{k_n} E|X_{ni}|^\delta I(|X_{ni}|^r \le k_n)$, we have

$$\begin{aligned} B &= k_n^{-\delta/r} \sum_{i=1}^{k_n} \int_0^{k_n^{\delta/r}} P(|X_{ni}|^\delta I(|X_{ni}|^r \le k_n) \ge t)\,\mathrm{d}t \\ &\le k_n^{-\delta/r} \sum_{i=1}^{k_n} \int_0^{k_n^{\delta/r}} P(|X_{ni}|^\delta \ge t)\,\mathrm{d}t. \end{aligned}$$

Let $t = u^{\delta/r}$, we have

$$B \le Ck_n^{-\delta/r}\sum_{i=1}^{k_n}\int_0^{k_n} u^{\delta/r-1}P(|X_{ni}|^r \ge u)\,\mathrm{d}u.$$

By (2.3), we have

$$\begin{aligned} B &\le Ck_n^{-\delta/r}\sum_{i=1}^{k_n}\int_0^{N} u^{\delta/r-1}P(|X_{ni}|^r \ge u)\,\mathrm{d}u + C\varepsilon k_n^{1-\delta/r}\int_N^{k_n} u^{\delta/r-2}\mathrm{d}u \\ &=: B_1 + B_2. \end{aligned}$$

By $\delta > r$, we have

$$B_1 \le Ck_n^{-\delta/r}\sum_{i=1}^{k_n}\int_0^{N} u^{\delta/r-1}\mathrm{d}u \le Ck_n^{1-\delta/r} \to 0 \quad \text{as } n\to\infty.$$

Since $\varepsilon > 0$ is arbitrary, by $\delta > r$ we have

$$B_2 \le C\varepsilon k_n^{1-\delta/r}\Big[k_n^{\delta/r-1} - N^{\delta/r-1}\Big] \le C\varepsilon \to 0 \quad \text{as } n\to\infty.$$

The proof of (2.2) is completed. □

Now we present the Marcinkiewicz–Zygmund inequalities with exponent p $(1 \le p \le 2)$ for m-PNQD random variables, which is very important in the proofs of our main results.

Lemma 2.4. Let $\{X_n, n \ge 1\}$ be a sequence of m-PNQD random variables with mean zero and $E|X_n|^p < \infty$ for $1 \le p \le 2$. Then there exists a positive constant C depending only on p and m, such that

$$E\Big|\sum_{k=1}^n X_k\Big|^p \le C\sum_{k=1}^n E|X_k|^p, \quad E\max_{1\le j\le n}\Big|\sum_{k=1}^j X_k\Big|^p \le C\log^2 n\sum_{k=1}^n E|X_k|^p. \tag{2.4}$$

Proof. The proofs of the above inequalities are similar. Hence we need only to prove the former. We will consider the following cases.

(i) We first consider the case $p = 2$. If $n \le m$, obviously $\{X_n, n \ge 1\}$ is a sequence of pairwise NQD random variables. Therefore, we need only to consider the case $n > m$. Given any $1 \le k \le n$, take $\tau = [\frac{n}{m}]$. Let

$$V_k = \begin{cases} X_k, & \text{if } 1 \le k \le n \\ 0, & \text{if } k > n \end{cases} \quad \text{and} \quad T_{nj} = \sum_{i=0}^{\tau} V_{mi+j} \ \ (1 \le j \le m).$$

Clearly $\sum_{k=1}^n X_k = \sum_{j=1}^m T_{nj} = \sum_{j=1}^m\sum_{i=0}^{\tau} V_{mi+j}$. Therefore, by C_r-inequality and Lemma 2.2, we have

$$\begin{aligned} E\Big|\sum_{k=1}^n X_k\Big|^2 &= E\Big|\sum_{j=1}^m T_{nj}\Big|^2 \le m\sum_{j=1}^m E\Big|\sum_{i=0}^{\tau} V_{mi+j}\Big|^2 \\ &\le m\sum_{j=1}^m\sum_{i=0}^{\tau} E|V_{mi+j}|^2 = m\sum_{k=1}^n E|X_k|^2. \end{aligned}$$

(ii) Next we consider the case $1 \le p < 2$. Let $\varphi_n = \sum_{k=1}^n E|X_k|^p$. For all $t \ge \varphi_n$, let

$$\begin{aligned} Y_k &= -\varphi_n^{1/p} t^{1/p} I(X_k < -\varphi_n^{1/p} t^{1/p}) + X_k I(|X_k| \le \varphi_n^{1/p} t^{1/p}) + \varphi_n^{1/p} t^{1/p} I(X_k > \varphi_n^{1/p} t^{1/p}), \\ Z_k &= X_k - Y_k = (X_k + \varphi_n^{1/p} t^{1/p}) I(X_k < -\varphi_n^{1/p} t^{1/p}) + (X_k - \varphi_n^{1/p} t^{1/p}) I(X_k > \varphi_n^{1/p} t^{1/p}). \end{aligned}$$

By Lemma 2.1, it follows that $\{Y_k,\, k \ge 1\}$ and $\{Z_k,\, k \ge 1\}$ are sequences of m-PNQD random variables. Then

$$\begin{aligned} & E\Big|3^{-1}\varphi_n^{-1/p} \sum_{k=1}^n X_k\Big|^p \\ = \;& \int_0^\infty P\Big(\Big|\sum_{k=1}^n X_k\Big| \ge 3\,\varphi_n^{1/p}\, t^{1/p}\Big) \mathrm{d} \;\le\; 1 + \int_1^\infty P\Big(\Big|\sum_{k=1}^n X_k\Big| \ge 3\,\varphi_n^{1/p}\, t^{1/p}\Big) \mathrm{d}t \\ \le \;& 1 + \sum_{k=1}^n \int_1^\infty P\big(|X_k| > \varphi_n^{1/p} t^{1/p}\big) \mathrm{d}t + \int_1^\infty P\Big(\Big|\sum_{k=1}^n Y_k\Big| \ge 3\,\varphi_n^{1/p}\, t^{1/p}\Big) \mathrm{d}t \\ =: \;& 1 + I_1 + I_2. \end{aligned}$$

Noting that $\int_1^\infty P\big(|X_k| > \varphi_n^{1/p} t^{1/p}\big) \mathrm{d}t \le \varphi_n^{-1} E|X_k|^p I(|X_k| > \varphi_n^{1/p})$. Hence,

$$I_1 \;\le\; \varphi_n^{-1} \sum_{k=1}^n E|X_k|^p I(|X_k| > \varphi_n^{1/p}) \;\le\; 1.$$

By $EX_k = 0$ and $p \ge 1$, we have

$$\begin{aligned} & \sup_{t \ge 1} \varphi_n^{-1/p} t^{-1/p} \Big|\sum_{k=1}^n EY_k\Big| \\ = \;& \sup_{t \ge 1} \varphi_n^{-1/p} t^{-1/p} \Big|\sum_{k=1}^n \big\{-\varphi_n^{1/p} t^{1/p} P(X_k < -\varphi_n^{1/p} t^{1/p}) \\ & \qquad + EX_k I(|X_k| \le \varphi_n^{1/p} t^{1/p}) + \varphi_n^{1/p} t^{1/p} P(X_k > \varphi_n^{1/p} t^{1/p})\big\}\Big| \\ = \;& \sup_{t \ge 1} \varphi_n^{-1/p} t^{-1/p} \Big|\sum_{k=1}^n \big\{-\varphi_n^{1/p} t^{1/p} P(X_k < -\varphi_n^{1/p} t^{1/p}) \\ & \qquad - EX_k I(|X_k| > \varphi_n^{1/p} t^{1/p}) + \varphi_n^{1/p} t^{1/p} P(X_k > \varphi_n^{1/p} t^{1/p})\big\}\Big| \\ \le \;& \sup_{t \ge 1} \varphi_n^{-1/p} t^{-1/p} \sum_{k=1}^n \big\{\varphi_n^{1/p} t^{1/p} P(|X_k| > \varphi_n^{1/p} t^{1/p}) + E|X_k| I(|X_k| > \varphi_n^{1/p} t^{1/p})\big\} \\ \le \;& \sup_{t \ge 1} \sum_{k=1}^n P(|X_k| > \varphi_n^{1/p} t^{1/p}) + \sup_{t \ge 1} \varphi_n^{-1} t^{-1} \sum_{k=1}^n E|X_k|^p I(|X_k| > \varphi_n^{1/p} t^{1/p}) \\ \le \;& 2\varphi_n^{-1} \sum_{k=1}^n E|X_k|^p I(|X_k| > \varphi_n^{1/p}) \;\le\; 2. \end{aligned}$$

Hence, $\left|\sum_{k=1}^n EY_k\right| \le 2\,\varphi_n^{1/p}t^{1/p}$ holds uniformly for $t \ge 1$. Then

$$I_2 \;\le\; \int_1^\infty P\Big(\Big|\sum_{k=1}^n (Y_k - EY_k)\Big| \ge \varphi_n^{1/p}t^{1/p}\Big)\mathrm{d}t.$$

From the conclusion proved in the case (i), the Markov inequality and C_r-inequality, we have

$$\begin{aligned}
I_2 &\le \varphi_n^{-2/p}\int_1^\infty t^{-2/p}E\Big|\sum_{k=1}^n (Y_k - EY_k)\Big|^2\mathrm{d}t\\
&\le m\,\varphi_n^{-2/p}\sum_{k=1}^n\int_1^\infty t^{-2/p}E(Y_k - EY_k)^2\mathrm{d}t \;\le\; m\,\varphi_n^{-2/p}\sum_{k=1}^n\int_1^\infty t^{-2/p}EY_k^2\mathrm{d}t\\
&= m\,\varphi_n^{-2/p}\sum_{k=1}^n\int_1^\infty t^{-2/p}EX_k^2I(|X_k|\le\varphi_n^{1/p}t^{1/p})\mathrm{d}t + m\sum_{k=1}^n\int_1^\infty P(|X_k|>\varphi_n^{1/p}t^{1/p})\mathrm{d}t\\
&= m\,\varphi_n^{-2/p}\sum_{k=1}^n\int_1^\infty t^{-2/p}EX_k^2I(|X_k|\le\varphi_n^{1/p})\mathrm{d}t + m\sum_{k=1}^n\int_1^\infty P(|X_k|>\varphi_n^{1/p}t^{1/p})\mathrm{d}t\\
&\quad + m\,\varphi_n^{-2/p}\sum_{k=1}^n\int_1^\infty t^{-2/p}EX_k^2I(\varphi_n^{1/p}<|X_k|\le\varphi_n^{1/p}t^{1/p})\mathrm{d}t\\
&=: I_3 + I_4 + I_5.
\end{aligned}$$

By a similar argument as in the proof of $I_1 \le 1$, we can prove $I_4 \le m$. By $p < 2$, we get

$$\begin{aligned}
I_3 &= \frac{mp}{2-p}\varphi_n^{-2/p}\sum_{k=1}^n EX_k^2I(|X_k|\le\varphi_n^{1/p})\\
&\le \frac{mp}{2-p}\varphi_n^{-1}\sum_{k=1}^n E|X_k|^pI(|X_k|\le\varphi_n^{1/p}) \;\le\; \frac{m\,p}{2-p}.
\end{aligned}$$

Finally we consider I_5. Noting that $\sum_{m=s}^\infty m^{-2/p} \le 2/(2-p)s^{1-2/p}$ and $(s+1)/s \le 2$ for all $s \ge 1$. We can get

$$\begin{aligned}
I_5 &= m\,\varphi_n^{-2/p}\sum_{k=1}^n\sum_{m=1}^\infty\int_m^{m+1} t^{-2/p}EX_k^2I(\varphi_n^{1/p}<|X_k|\le\varphi_n^{1/p}t^{1/p})\mathrm{d}t\\
&\le m\,\varphi_n^{-2/p}\sum_{k=1}^n\sum_{m=1}^\infty m^{-2/p}EX_k^2I(\varphi_n<|X_k|^p\le\varphi_n(m+1))\\
&= m\,\varphi_n^{-2/p}\sum_{k=1}^n\sum_{m=1}^\infty m^{-2/p}\sum_{s=1}^m EX_k^2I(\varphi_n\,s<|X_k|^p\le\varphi_n(s+1))\\
&= m\,\varphi_n^{-2/p}\sum_{k=1}^n\sum_{s=1}^\infty EX_k^2I(\varphi_n\,s<|X_k|^p\le\varphi_n(s+1))\sum_{m=s}^\infty m^{-2/p}
\end{aligned}$$

$$\begin{aligned}
&\le \frac{2m}{2-p}\,\varphi_n^{-2/p}\sum_{k=1}^{n}\sum_{s=1}^{\infty} s^{1-2/p} E X_k^2 I(\varphi_n\, s < |X_k|^p \le \varphi_n(s+1)) \\
&\le 2^{2/p}\,\frac{m}{2-p}\,\varphi_n^{-1}\sum_{k=1}^{n}\sum_{s=1}^{\infty} E|X_k|^p I(\varphi_n\, s < |X_k|^p \le \varphi_n(s+1)) \\
&= 2^{2/p}\,\frac{m}{2-p}\,\varphi_n^{-1}\sum_{k=1}^{n} E|X_k|^p I(|X_k|^p > \varphi_n) \;\le\; 2^{2/p}\,\frac{m}{2-p}.
\end{aligned}$$

From $I_1 \le 1$, $I_3 \le m\,p/(2-p)$, $I_4 \le m$ and $I_5 \le 2^{2/p}\, m/(2-p)$, we have

$$E\left|3^{-1}\varphi_n^{-1/p}\sum_{k=1}^{n} X_k\right|^p \;\le\; 2 + \frac{m\,p}{2-p} + m + 2^{2/p}\,\frac{m}{2-p}.$$

Let $C = 3^p(2 + \frac{m\,p}{2-p} + m + 2^{2/p}\,\frac{m}{2-p})$. Clearly C depends only on p and m. Then we get

$$E\left|\sum_{k=1}^{n} X_k\right|^p \;\le\; C\sum_{k=1}^{n} E|X_k|^p.$$

The proof is completed. □

Remark 2.5. The above inequality is new even for the pairwise independent case. According to our knowledge, [3, 18] proved that the inequality (2.4) with $p = 2$ for sequence of pairwise NQD random variables. Because of the limitation of the exponent $p = 2$, many authors could not obtain desirable results of the convergence properties for pairwise NQD random variables.

We prove that the inequality (2.4) remains true for the case $1 < p < 2$, which will be very useful in establishing the convergence properties for pairwise NQD random variables. For example, we can easily solve the open problem in [16] (see Remark 3.1) by means of Lemma 2.4 and similar arguments as the proof of Theorem 3.3 in [16].

3. MAIN RESULTS AND THE PROOFS

In this section, we shall state some limit theorems for arrays of rowwise m-PNQD random variables. We first present the following theorem which extends Theorem 1.5.

Theorem 3.1. Let $\{X_{ni}, u_n \le i \le v_n, n \ge 1\}$ be an array of rowwise m-PNQD random variables and $1 \le r < 2$. Let $\{a_{ni}, u_n \le i \le v_n, n \ge 1\}$ be an array of constants. Suppose that

(i) $\sup_{n\ge 1}\sum_{i=u_n}^{v_n} |a_{ni}|^r E|X_{ni}|^r < \infty$,

(ii) $\sum_{i=u_n}^{v_n} |a_{ni}|^r E|X_{ni}|^r I(|a_{ni}|^r |X_{ni}|^r > \varepsilon) \to 0$ as $n \to \infty$ for any $\varepsilon > 0$.

Then

$$\sum_{i=u_n}^{v_n} a_{ni}(X_{ni} - EX_{ni}) \to 0 \tag{3.1}$$

in L^r and, hence, in probability as $n \to \infty$.

P r o o f. Without loss of generality, we may assume that $a_{ni} \ge 0$. For $u_n \le i \le v_n$, $n \ge 1$, let

$$\begin{aligned} Y_{ni} &= -\varepsilon^{1/r} I(a_{ni}X_{ni} < -\varepsilon^{1/r}) + a_{ni}X_{ni}I(a_{ni}|X_{ni}| \le \varepsilon^{1/r}) + \varepsilon^{1/r} I(a_{ni}X_{ni} > \varepsilon^{1/r}), \\ Z_{ni} &= a_{ni}X_{ni} - Y_{ni}. \end{aligned}$$

By Lemma 2.1, $\{Y_{ni}, u_n \le i \le v_n, n \ge 1\}$ and $\{Z_{ni}, u_n \le i \le v_n, n \ge 1\}$ are arrays of rowwise m-PNQD. Given $\varepsilon > 0$, by Lemma 2.4, we have

$$\begin{aligned} E\left|\sum_{i=u_n}^{v_n} a_{ni}(X_{ni} - EX_{ni})\right|^r &\le 2^{r-1}\left\{E\left|\sum_{i=u_n}^{v_n}(Z_{ni} - EZ_{ni})\right|^r + E\left|\sum_{i=u_n}^{v_n}(Y_{ni} - EY_{ni})\right|^r\right\} \\ &\le 2^{r-1}E\left|\sum_{i=u_n}^{v_n}(Z_{ni} - EZ_{ni})\right|^r + 2^{r-1}\left\{E\left|\sum_{i=u_n}^{v_n}(Y_{ni} - EY_{ni})\right|^2\right\}^{r/2} \\ &\le C\,2^{r-1}\sum_{i=u_n}^{v_n} E|Z_{ni}|^r + C\,2^{r-1}\left\{\sum_{i=u_n}^{v_n} EY_{ni}^2\right\}^{r/2} \\ &=: I_6 + I_7. \end{aligned}$$

We first prove $I_6 \to 0$ as $n \to \infty$. Noting that $|Z_{ni}| \le a_{ni}|X_{ni}|I(a_{ni}^r|X_{ni}|^r > \varepsilon)$. By the condition (ii), we have

$$I_6 \le C\sum_{i=u_n}^{v_n} a_{ni}^r E|X_{ni}|^r I(a_{ni}^r|X_{ni}|^r > \varepsilon) \to 0 \quad \text{as } n \to \infty.$$

Next we prove $I_7 \to 0$ as $n \to \infty$. Without loss of generality, we may assume $0 < \varepsilon < 1$. Then

$$\begin{aligned} I_7^{2/r} &\le C\sum_{i=u_n}^{v_n} a_{ni}^2 EX_{ni}^2 I(a_{ni}^r|X_{ni}|^r \le \varepsilon) + C\varepsilon^{2/r}\sum_{i=u_n}^{v_n} P(a_{ni}^r|X_{ni}|^r > \varepsilon) \\ &\le C\sum_{i=u_n}^{v_n} a_{ni}^2 EX_{ni}^2 I(a_{ni}^r|X_{ni}|^r \le \varepsilon^2) + C\sum_{i=u_n}^{v_n} a_{ni}^2 EX_{ni}^2 I(\varepsilon^2 < a_{ni}^r|X_{ni}|^r \le \varepsilon) \\ &\quad + C\varepsilon^{2/r-1}\sum_{i=u_n}^{v_n} a_{ni}^r E|X_{ni}|^r I(a_{ni}^r|X_{ni}|^r > \varepsilon) \\ &=: I_8 + I_9 + I_{10}. \end{aligned}$$

By $r < 2$ and (ii), we get $I_{10} \to 0$ as $n \to \infty$. For I_8, we have

$$\begin{aligned} I_8 &\le C\,\varepsilon^{4/r-2}\sum_{i=u_n}^{v_n} a_{ni}^r E|X_{ni}|^r I(a_{ni}^r|X_{ni}|^r \le \varepsilon^2) \\ &\le C\,\varepsilon^{4/r-2}\sup_{n\ge 1}\sum_{i=u_n}^{v_n} a_{ni}^r E|X_{ni}|^r. \end{aligned}$$

By $r < 2$ and (ii), we have

$$\begin{aligned} I_9 &\le C\,\varepsilon^{2/r-1}\sum_{i=u_n}^{v_n} a_{ni}^r E|X_{ni}|^r I(\varepsilon^2 < a_{ni}^r|X_{ni}|^r \le \varepsilon) \\ &\le C\,\varepsilon^{2/r-1}\sum_{i=u_n}^{v_n} a_{ni}^r E|X_{ni}|^r I(a_{ni}^r|X_{ni}|^r > \varepsilon^2) \to 0 \quad \text{as } n\to\infty. \end{aligned}$$

Therefore,

$$\lim_{n\to\infty}\sup E\left|\sum_{i=u_n}^{v_n} a_{ni}(X_{ni}-EX_{ni})\right|^r \le C\,\varepsilon^{2-r}\left(\sup_{n\ge 1}\sum_{i=u_n}^{v_n} a_{ni}^r E|X_{ni}|^r\right)^{r/2}.$$

Since $0 < \varepsilon < 1$ is arbitrary, by $r < 2$ and (i), the proof is completed. □

Remark 3.2. Since pairwise NQD implies m-PNQD, Theorem 3.1 extends Theorem 1.5. It is important to point out that, by using Lemma 2.4, the proof of Theorem 3.1 is much simple than that of Theorem 1.5 by [15].

Secondly, we state the following result which extends and improves Theorem 1.6 under the same conditions.

Theorem 3.3. Let $\{X_{ni}, 1 \le i \le k_n \uparrow \infty, n \ge 1\}$ be a triangular array of rowwise m-PNQD random variables, and $EX_{ni} = 0$, $1 \le i \le k_n$ for each $n \ge 1$. Suppose that the uniform Cesàro-type condition (1.2) for some $r \in (1, 2)$ holds. Then for $p \in (0, r)$,

$$k_n^{-1/r}\sum_{i=1}^{k_n} X_{ni} \to 0 \quad \text{in } L^p \quad \text{as } n\to\infty. \tag{3.2}$$

P r o o f. Let

$$\begin{aligned} Y_{ni} &= -k_n^{1/r}I(X_{ni} < -k_n^{1/r}) + X_{ni}I(|X_{ni}| \le k_n^{1/r}) + k_n^{1/r}I(X_{ni} > k_n^{1/r}), \\ Z_{ni} &= X_{ni} - Y_{ni} = (X_{ni} + k_n^{1/r})I(X_{ni} < -k_n^{1/r}) + (X_{ni} - k_n^{1/r})I(X_{ni} > k_n^{1/r}). \end{aligned}$$

By Lemma 2.4, we have

$$\begin{aligned} k_n^{-p/r}E\left|\sum_{i=1}^{k_n} X_{ni}\right|^p &\le C\,k_n^{-p/r}\left\{E\left|\sum_{i=1}^{k_n}(Z_{ni}-EZ_{ni})\right|^p + E\left|\sum_{i=1}^{k_n}(Y_{ni}-EY_{ni})\right|^p\right\} \\ &\le C\,k_n^{-p/r}E\left|\sum_{i=1}^{k_n}(Z_{ni}-EZ_{ni})\right|^p + C\,k_n^{-p/r}\left\{E\left|\sum_{i=1}^{k_n}(Y_{ni}-EY_{ni})\right|^2\right\}^{p/2} \\ &\le C\,k_n^{-p/r}\sum_{i=1}^{k_n} E|Z_{ni}|^p + C\left\{k_n^{-2/r}\sum_{i=1}^{k_n} EY_{ni}^2\right\}^{p/2} \\ &=: I_{11} + I_{12}. \end{aligned}$$

By $|Z_{ni}| \le |X_{ni}| I(|X_{ni}|^r > k_n)$ and Lemma 2.3(i), we have

$$I_{11} \le C\, k_n^{-p/r} \sum_{i=1}^{k_n} E|X_{ni}|^p I(|X_{ni}|^r > k_n) \to 0 \quad \text{as } n \to \infty.$$

By Lemma 2.3(ii) and (1.2) for $x = k_n$, we have

$$I_{12} \;=\; C \left\{ k_n^{-2/r} \sum_{i=1}^{k_n} EX_{ni}^2 I(|X_{ni}|^r \le k_n) + \sum_{i=1}^{k_n} P(|X_{ni}|^r > k_n) \right\}^{p/2} \to 0 \quad \text{as } n \to \infty.$$

The proof is completed. □

Remark 3.4. The above theorem shows that, we can improve Theorem 1.6 by considering L^p-convergence instead of convergence in probability under the same conditions. Since L^p-convergence implies convergence in probability, Theorem 3.3 improves Theorem 1.6.

The following theorem shows that, under some stronger conditions, we can obtain the complete convergence for the array of rowwise m-PNQD random variables.

Theorem 3.5. Let $\{X_{ni}, 1 \le i \le k_n \uparrow \infty, n \ge 1\}$ be an array of rowwise m-PNQD random variables with $EX_{ni} = 0$. $k_n = O(n)$. For $1 \le p < 2$ and $\delta > 2/p - 1$, suppose that

$$\lim_{x\to\infty} \sup_{n\ge 1} k_n^{-1} \sum_{i=1}^{k_n} x^{1+\delta} P\big(|X_{ni}|^p \ge x\big) = 0. \tag{3.3}$$

Then for $\alpha p \ge 1$,

$$\sum_{n=1}^{\infty} k_n^{\alpha p - 2} P\left(\max_{1\le j\le k_n} \left| \sum_{i=1}^{j} X_{ni} \right| > k_n^{\alpha} \varepsilon \right) < \infty, \quad \forall \varepsilon > 0. \tag{3.4}$$

P r o o f. For fixed $n \ge 1$, let $x = k_n^{\alpha(2-p)/4}$ and

$$\begin{aligned} Y_{ni} &= -xI(X_{ni} < -x) + X_{ni} I(|X_{ni}| \le x) + xI(X_{ni} > x), \\ Z_{ni} &= X_{ni} - Y_{ni} = (X_{ni} + x) I(X_{ni} < -x) + (X_{ni} - x) I(X_{ni} > x). \end{aligned}$$

Let $S_{nj} = \sum_{i=1}^{j} X_{ni}$, $S_{nj}^{*} = \sum_{i=1}^{j} Y_{ni}$ and $S_{nj}^{**} = \sum_{i=1}^{j} Z_{ni}$. For any $\varepsilon > 0$, we have

$$\begin{aligned} &\sum_{n=1}^{\infty} k_n^{\alpha p - 2} P\left(\max_{1\le j\le k_n} \left| \sum_{i=1}^{j} X_{ni} \right| > k_n^{\alpha} \varepsilon \right) \\ &\le \sum_{n=1}^{\infty} k_n^{\alpha p - 2} P\big(\max_{1\le j\le k_n} |S_{nj}^{*} - ES_{nj}^{*}| > k_n^{\alpha}\varepsilon/2 \big) + \sum_{n=1}^{\infty} k_n^{\alpha p - 2} P\big(\max_{1\le j\le k_n} |S_{nj}^{**} - ES_{nj}^{**}| > k_n^{\alpha}\varepsilon/2 \big) \\ &=: \; I_{13} + I_{14}. \end{aligned}$$

Noting that $|Y_{ni}| \le k_n^{\alpha(2-p)/4}$. Then by the Markov inequality and Lemma 2.4, we have

$$\begin{aligned} I_{13} &\le C\sum_{n=1}^{\infty} k_n^{\alpha p-2-2\alpha}\log^2 k_n \sum_{i=1}^{k_n} EY_{ni}^2 \\ &\le C\sum_{n=1}^{\infty} k_n^{-1-\alpha(2-p)/2}\log^2 k_n < \infty. \end{aligned}$$

By a similar argument as in the proof of Lemma 2.3, we have

$$EX_{ni}^2 I(|X_{ni}|>x) = x^2 P(|X_{ni}|>x) + \int_{x^2}^{\infty} P(|X_{ni}|^2 \ge t)\,\mathrm{d}t.$$

Hence by $|Z_{ni}| \le |X_{ni}| I(|X_{ni}|>x)$, the Markov inequality and Lemma 2.4, we get

$$\begin{aligned} I_{14} &\le C\sum_{n=1}^{\infty} k_n^{\alpha p-2-2\alpha}\log^2 k_n \sum_{i=1}^{k_n} EX_{ni}^2 I(|X_{ni}|>x) \\ &= C\sum_{n=1}^{\infty} k_n^{\alpha p-2-2\alpha}\log^2 k_n \sum_{i=1}^{k_n} x^2 P(|X_{ni}|>x) \\ &\quad + C\sum_{n=1}^{\infty} k_n^{\alpha p-2-2\alpha}\log^2 k_n \sum_{i=1}^{k_n}\int_{x^2}^{\infty} P(|X_{ni}|^2 \ge t)\,\mathrm{d}t \\ &=: I_{15}+I_{16}. \end{aligned}$$

From (3.3), $\exists M>0$, when $x>M$, we have

$$\sup_{n\ge1} k_n^{-1}\sum_{i=1}^{k_n} P\big(|X_{ni}|^p \ge x\big) \le x^{-(1+\delta)}. \tag{3.5}$$

By (3.5), $x=k_n^{\alpha(2-p)/4}$ and $\delta > 2/p-1$, we have

$$\begin{aligned} I_{15} &= C\sum_{n=1}^{\infty} k_n^{\alpha p-1-2\alpha}\log^2 k_n\, k_n^{-1}\sum_{i=1}^{k_n} x^2 P(|X_{ni}|>x) \\ &\le C\sum_{n=1}^{\infty} k_n^{\alpha p-1-2\alpha} x^{-p(1+\delta)+2}\log^2 k_n \\ &= C\sum_{n=1}^{\infty} k_n^{-1-\alpha(2-p)-\alpha p(2-p)(1+\delta-\frac{2}{p})/4}\log^2 k_n < \infty \end{aligned}$$

and

$$\begin{aligned} I_{16} &= C\sum_{n=1}^{\infty} k_n^{\alpha p-1-2\alpha}\log^2 k_n \int_{x^2}^{\infty} k_n^{-1}\sum_{i=1}^{k_n} P(|X_{ni}|^2 \geq t)\,\mathrm{d}t \\ &\leq C\sum_{n=1}^{\infty} k_n^{\alpha p-1-2\alpha}\log^2 k_n \int_{x^2}^{\infty} t^{-\frac{p}{2}(1+\delta)}\mathrm{d}t \\ &\leq C\sum_{n=1}^{\infty} k_n^{\alpha p-1-2\alpha} x^{-p(1+\delta)+2}\log^2 k_n \\ &\leq C\sum_{n=1}^{\infty} k_n^{-1-\alpha(2-p)-\alpha p(2-p)(1+\delta-\frac{2}{p})/4}\log^2 k_n < \infty. \end{aligned}$$

The proof is completed. □

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Yongfeng Wu, School of Mathematics and Finance, Chuzhou University, Chuzhou, 239000, PR China; College of Mathematics and Computer Science, Tongling University, Tongling 244000. P. R. China.
e-mail: wyfwyf@126.com

JiangYan Peng, School of Mathematics Science, University of Electronic Science and Technology of China, Chengdu 611731. P. R. China.
e-mail: cdpengjy@126.com