KYBERNETIKA — VOLUME 54 (2018), NUMBER 4, PAGES 736–747

REALIZATION OF NONLINEAR INPUT-OUTPUT EQUATIONS IN CONTROLLER CANONICAL FORM

Arvo Kaldmäe and Ülle Kotta

In this paper necessary and sufficient conditions are given which guarantee that there exists a realization of a set of nonlinear higher order differential input-output equations in the controller canonical form. Two cases are studied, corresponding respectively to linear and nonlinear output functions. The conditions are formulated in terms of certain sequence of vector spaces of differential 1-forms. The proofs suggest how to construct the transformations, necessary to obtain the specific state space realizations. Multiple examples are added, which describe different scenarios.

Keywords: realization, nonlinear systems, algebraic methods

Classification: 93B11, 93C10, 93B25

1. INTRODUCTION

The realization problem, i. e., transforming a set of higher order differential input-output (i/o) equations into a set of first order differential equations, called state equations, has been studied extensively for nonlinear control systems, both in continuous- and discrete-time [4, 5, 8, 9, 13, 18, 19, 20]. The reason is simple – most of the control theory is developed for systems described by the state equations, although system identification mostly yields the set of i/o equations. Often the obtained nonlinear state equations are further transformed into certain special forms in order to simplify analysis or control of the systems (observer form, feedforward form, linear equations, triangular form etc.). Therefore, the natural question appears: why not to transform the i/o equations directly into these special forms, whenever possible.

The contribution of the paper is to find a state-space realization of nonlinear multi-input multi-output (MIMO) equations in the controller canonical form, whenever possible. This possibility is an assumption of many control methods since systems in the controller canonical form can be easily linearized by a static state feedback. The problem has been studied before only for discrete-time single-input single-output (SISO) systems [14], where sufficient solvability conditions are found. We consider two cases – when the output function in the state-space realization is (i) linear (the case addressed partly in [14]) and (ii) any nonlinear function of states. For both cases we give necessary and

DOI: 10.14736/kyb-2018-4-0736

sufficient conditions under which such special realization exists. The obtained conditions are restrictions of general solution, given in [3, 5] and if the conditions are not satisfied, one can, without much additional computations, find a general state space realization. Additionally, we describe the structure of the i/o equations for which realization in the controller canonical form with linear output function is possible. Namely, the problem is solvable if and only if the i/o equations do not depend on the time-derivatives of input functions. This information can be used in the i/o model identification process, when realization in the controller canonical form is desired. Note that in empirical modeling the system structure is not given in advance, and a large collection of candidate models can be chosen. By choosing the model structure such that no time-derivatives of input variables are present, one can guarantee that realization in the controller canonical form is possible.

The problem of finding a realization in a specific state space form has been previously addressed in [9, 10, 11, 14]. The paper [9] investigates the problem of lowering the orders of input derivatives in the generalized state space equations (where the state equations depend also on time-derivatives of input variables) by a generalized state transformation. This problem is equivalent to the realization problem in the state-space form, since any set of i/o equations can be represented as a generalized state space system and removing all the input derivatives from the equations is equivalent to finding a classical state space realization. As a by-product, the authors of [9] derive conditions under which one can remove all the input derivatives and transform the generalized state space system into a linear one by a generalized state transformation. Note that the problem statement of this paper asks for less than [9], since controller canonical form may be nonlinear, though it is linearizable by a static state feedback. The paper [10] studies possibilities to find a realization of an autonomous system in the feedforward form and in [11] realization in the observer and controller canonical forms are studied using the transfer function approach. Finally, there are many papers, for instance [15, 16], where a nonlinear continuous-time system is transformed in the generalized observer form. Since the construction of transformation is based on the i/o representation of the system, the result is related to the realization problem, i. e., one can apply the results to systems, described by the i/o equations and get a state space representation which is in the generalized observer form.

The algebraic approach based on differential algebra and differential forms is used in the paper, exactly like in the monograph [7]. The interest of this approach is in generic properties that hold on some open and dense subsets of suitable domain if they hold at some point of this domain. In what follows, our theorems and transformations hold generically. By this is meant the following: for almost every point of the domain there is an open neighborhood on which some statement holds or some object is defined. If one assumes that some rank condition holds generically over the differential field $\mathcal{K}$, defined in Subsection 2.3, then around almost every point the solution exists though it is not necessarily global (that is, defined almost everywhere). By making the computations over the field $\mathcal{K}$, one can disregard the cases when the functions of system variables have zero values since these functions are considered to be the elements of $\mathcal{K}$ and not the functions of time. The study of generic properties allows to express the solutions in a more compact way, since there is no need to specify the working point and its neighborhood. Of course, when addressing concrete examples, the singularities and the points where some functions are not defined have to be taken into account.

The paper is organized as follows. In Section 2 we give an overview of the algebraic approach and the problem statement. The main results are presented in Section 3. Section 4 consists of examples illustrating the results and finally, the paper ends with the Conclusion section.

2. PRELIMINARIES

2.1. I/o equations

A nonlinear MIMO control system can be described by the set of higher order differential equations, called input-output (i/o) equations, that relate the system inputs $u = (u_1, \ldots, u_m)^{\mathrm{T}}$, outputs $y = (y_1, \ldots, y_m)^{\mathrm{T}}$ and their time-derivatives:

$$y_i^{(n_i)}(t) = \Phi_i(y_j(t), \ldots, y_j^{(n_{ij})}(t), u_j(t), \ldots, u_j^{(s_i)}(t)) \tag{1}$$

for $i, j = 1, \ldots, m$. The functions Φ_i are assumed to be real analytic in variables from the set $\{y_j, \ldots, y_j^{(n_{ij})}, u_j, \ldots, u_j^{(s_i)}; j = 1, \ldots, m\}$. The indices in (1) are typically supposed to satisfy the conditions

$$\begin{array}{lll} n_1 \leq n_2 \leq \cdots \leq n_m, & n_{ij} < n_j & s_i < n_i \\ n_{ij} < n_i, & j \leq i & \\ n_{ij} \leq n_i, & j > i. & \end{array} \tag{2}$$

The restrictions (2) mean that the equations (1) are assumed to be in the strong Popov form, which guarantees the uniqueness of the indices n_i up to output permutation. Note that under mild conditions, one can always transform an arbitrary set of i/o equations, at least locally, into the form (1),(2), see [19]. Define the order n of the system (1) as $n := \sum_i n_i$ and let $s := \max_i\{s_i\}$.

2.2. State equations

Another way to describe a MIMO nonlinear continuous-time control system is by the state equations

$$\begin{array}{rcl} x^{(1)}(t) & = & f(x(t), u(t)) \\ y(t) & = & h(x(t)), \end{array} \tag{3}$$

where $x(t) \in \mathbb{R}^n$ is the state at time t, $u(t) \in \mathbb{R}^m$ is the input and $y(t) \in \mathbb{R}^m$ the output of the system. In this paper we assume that the system (3) is observable in the sense that

$$\operatorname{rank} \frac{\partial(h, h^{(1)}, \ldots, h^{(n-1)})^{\mathrm{T}}}{\partial x} = n,$$

where $h^{(k)}$, $k \geq 1$, is kth time-derivative of the output function h of (3), is true generically [7]. In this paper we are interested in a controller canonical form of (3):

$$\begin{array}{rcl} x_{i,1}^{(1)}(t) & = & x_{i,2}(t) \\ x_{i,2}^{(1)}(t) & = & x_{i,3}(t) \\ & \vdots & \\ x_{i,n_i}^{(1)}(t) & = & F_i(x(t), u(t)), \end{array} \tag{4}$$

where either

$$y_i(t) = x_{i,1}(t) \tag{5}$$

or

$$y_i(t) = h_i(x(t)) \tag{6}$$

for $i = 1, \ldots, m$. Note that the system (4),(5) is observable for any functions F_i, $i = 1, \ldots, m$.

2.3. Algebraic framework

Next a brief overview of the algebraic approach used in the paper is given, see [7] for more details. Denote by $\mathcal{K}$ the field of meromorphic functions, depending on finite number of variables from the set $\mathcal{C} = \{y_i, \ldots, y_i^{(n_i-1)}, u_i^{(k)}; i = 1, \ldots, m; k \in \mathbb{N}\}$. The elements of the set $\mathcal{C}$ are viewed as independent variables of the field $\mathcal{K}$, and not as functions of time. In the following all the functions and transformations considered in the paper are assumed to belong to the field $\mathcal{K}$. On the field $\mathcal{K}$ the standard time-derivative operator $\mathrm{d}/\mathrm{d}t$ is defined and the pair $(\mathcal{K}, \mathrm{d}/\mathrm{d}t)$ forms a differential field [12], which we denote simply by $\mathcal{K}$. It is important to distinguish that due to (1), $\frac{\mathrm{d}}{\mathrm{d}t} y_i^{(n_i-1)} = \Phi_i(\cdot)$, since $y_i^{(n_i)}$ is not independent system variable, whereas $\frac{\mathrm{d}}{\mathrm{d}t} y_i^{(k)} = y_i^{(k+1)}$ for $k = 0, \ldots, n_i - 2$.

Define the vector space of 1-forms over $\mathcal{K}$ as $\mathcal{E} = \mathrm{span}_{\mathcal{K}}\{\mathrm{d}\varphi | \varphi \in \mathcal{K}\}$. The time-derivative operator $\mathrm{d}/\mathrm{d}t$ is naturally extended to $\mathcal{E}$ as $\mu : \mathcal{E} \to \mathcal{E}$, where

$$\mu\Big(\sum_j a_j \mathrm{d}\xi_j\Big) = \sum_j \big(a_j^{(1)} \mathrm{d}\xi_j + a_j \mathrm{d}\xi_j^{(1)}\big)$$

for $a_j \in \mathcal{K}$ and $\xi_j \in \mathcal{C}$.

A 1-form $\omega \in \mathcal{E}$ is said to be exact if it is a total differential of some function $\varphi \in \mathcal{K}$, i. e., $\omega = \mathrm{d}\varphi$. A subspace $\mathrm{span}_{\mathcal{K}}\{\omega_1, \ldots, \omega_k\}$ of $\mathcal{E}$ is said to be integrable if it has locally a basis, consisting of exact 1-forms.

Similar algebraic framework can be constructed for systems of the form (3), with the difference that instead of $\mathcal{C}$ one has $\bar{\mathcal{C}} = \{x, u^{(k)}; k \in \mathbb{N}\}$, see more in detail in [7].

2.4. Problem statement

The general realization problem deals with possibilities to transform the equations (1) into the form (3) such that starting from the corresponding initial conditions, equal inputs yield equal output behaviors for systems (1) and (3). Precise definition of realization is given as follows.

Definition 2.1. The set of the nth order i/o equations (1) is said to be realizable in the state space form (3) if there exist state coordinates $x = \varphi(y, y^{(k)}, u, u^{(k)}; k \geq 0)$ such that in these coordinates the state equations take the form (3) and the solutions $\{y(t), u(t); t \geq 0\}$, generated by (1) and (3), coincide. The system (3) is called the nth order state space realization of the set of the nth order i/o equations (1).

In this paper our goal is to study the realization in the special form, namely the possibilities to transform the equations (1) into the form (4),(5) or (4),(6). Note that the state equations of the form (4) are static state feedback linearizable.

3. RESULTS

The solution to the problem statement above is given in terms of the sequence of non-increasing vector spaces $\mathcal{H}_i$, $i \geq 1$, which are the subspaces of $\mathcal{E}$ and defined [3] recursively by

$$\begin{array}{rcl} \mathcal{H}_1 & = & \mathrm{span}_{\mathcal{K}}\{\mathrm{d}y_j^{(n_j-1)}, \ldots, \mathrm{d}y_j, \mathrm{d}u_j^{(s)}, \ldots, \mathrm{d}u_j; j = 1, \ldots, m\} \\ \mathcal{H}_{i+1} & = & \{\omega \in \mathcal{H}_i | \mu(\omega) \in \mathcal{H}_i\}. \end{array} \tag{7}$$

The sequence $\{\mathcal{H}_i\}$ converges, by which one means that there exists $k_* \in \mathbb{N}$ such that $\mathcal{H}_{k_*} = \mathcal{H}_{k_*+i}$, $i \geq 1$, but $\mathcal{H}_{k_*-1} \neq \mathcal{H}_{k_*}$. The latter results from each $\mathcal{H}_k$ being a finite-dimensional vector space so that, at each step either the dimension decreases by at least one or $\mathcal{H}_{k+1} = \mathcal{H}_k$. Define $\mathcal{H}_\infty := \mathcal{H}_{k_*}$. From now on we make the assumption that $\mathcal{H}_\infty = \{0\}$, which means that the system (1) is generically accessible in the sense that (1) does not admit autonomous elements in $\mathcal{K}$, see [2]. Note that the latter is necessary for transforming a system (1) or (3) into the form (4),(5). Also, note that directly from the definition (7) of the sequence $\mathcal{H}_i$, one may conclude the following.

Lemma 3.1. A 1-form $\omega \in \mathcal{E}$ is an element of $\mathcal{H}_i$ if and only if $\mu^{i-1}(\omega) \in \mathcal{H}_1$.

Another observation, which will be useful later, is the following.

Lemma 3.2. For $i = 1, \ldots, s+2$ one has $\dim \mathcal{H}_i = n + (s+2-i)m$.

Proof. In Theorem 4 of [5] the basis of $\mathcal{H}_i$, $i = 1, \ldots, s+2$, is computed for systems on homogeneous time scales. Note that the continuous-time case is a special case of that. More precisely, the theorem states that

$$\mathcal{H}_i = \mathrm{span}_{\mathcal{K}}\{\omega_1, \ldots, \omega_n, \mathrm{d}u_j, \ldots, \mathrm{d}u_j^{(s+1-i)}; j = 1, \ldots, m\}$$

for $i = 1, \ldots, s+1$ and

$$\mathcal{H}_{s+2} = \mathrm{span}_{\mathcal{K}}\{\omega_1, \ldots, \omega_n\}$$

for certain 1-forms $\omega_1, \ldots, \omega_n$, computed in [5]. Thus, $\dim \mathcal{H}_i = n + (s+2-i)m$ for $i = 1, \ldots, s+2$. □

First, we give the result for general realization problem in the form (3)[1].

Theorem 3.3. The set of i/o equations (1) has an observable state-space realization in the form (3) if and only if the subspace $\mathcal{H}_{s+2}$, defined by (7), is completely integrable.

Proof. *Sufficiency.* Because of Lemma 3.2 and integrability of $\mathcal{H}_{s+2}$ one has $\mathcal{H}_{s+2} = \mathrm{span}_{\mathcal{K}}\{\mathrm{d}\varphi_1, \ldots, \mathrm{d}\varphi_n\}$. Also, by Lemma 3.1, $\mathrm{d}u_j \in \mathcal{H}_{s+1}$ for $j = 1, \ldots, m$. Thus, by definition of $\mathcal{H}_{s+1}$ one has $\mathcal{H}_{s+1} = \mathcal{H}_{s+2} \oplus \mathrm{span}_{\mathcal{K}}\{\mathrm{d}u\}$. Now, define the state variables as $x_i = \varphi_i$, $i = 1, \ldots, n$. Since, $\mathrm{d}\varphi_i \in \mathcal{H}_{s+2}$, then $\mu(\mathrm{d}\varphi_i) = \mathrm{d}\varphi_i^{(1)} \in \mathcal{H}_{s+1}$. Thus,

[1] The analogue of this theorem has been stated and proved in a conference paper [3] for nonlinear continuous- and discrete-time systems using the formalism of pseudo-linear algebra [6]. Compared to the theorem in [3], the proof of the result has been improved

$d\varphi_i^{(1)} = df_i(\varphi_1,\dots,\varphi_n,u)$ and $x_i^{(1)} = f_i(x,u)$. Finally, by Lemma 3.1, $dy_i \in \mathcal{H}_{s+2}$, which means that there exists a function $h_i \in \mathcal{K}$, such that $y_i = h_i(x)$ for $i = 1,\dots,m$.

Necessity. Since the realization (3) is observable, then there exist functions φ_i, $i = 1,\dots,n$, such that $x_i = \varphi_i(\cdot)$ and $d\varphi_i \in \mathcal{H}_1$. By Lemma 3.1, $du_j^{(i)} \in \mathcal{H}_{s+1-i}$, but $du_j^{(i)} \notin \mathcal{H}_{s+2-i}$. Also, by Lemma 3.2 one knows the dimensions of $\mathcal{H}_i$, $i = 1,\dots,s+2$. By combining these observations, one has

$$\mathcal{H}_\ell = \mathcal{H}_{s+2} \oplus \operatorname{span}_{\mathcal{K}}\{du^{(k)}; k = 0,\dots,s-\ell+1\}$$

for $\ell = 1,\dots,s+1$. Since $d\varphi_i \in \mathcal{H}_1$, but $d\varphi_i \notin \operatorname{span}_{\mathcal{K}}\{du,\dots,du^{(s)}\}$, then necessarily, $d\varphi_i \in \mathcal{H}_{s+2}$ for $i = 1,\dots,n$. Because the 1-forms $d\varphi_i$ are all linearly independent and the dimension of $\mathcal{H}_{s+2}$ is n, then $\mathcal{H}_{s+2} = \operatorname{span}_{\mathcal{K}}\{d\varphi_1,\dots,d\varphi_n\}$, i. e., the subspace $\mathcal{H}_{s+2}$ is integrable. □

The proof of Theorem (3.3) also suggest that the state variables can be found by integrating the subspace $\mathcal{H}_{s+2}$.

Now, we are looking for a realization of the form (4) with the output function (5).

Theorem 3.4. The set of the nth order i/o equations (1) has the nth order realization in the form (4),(5) if and only if

$$\mathcal{H}_{s+2} = \operatorname{span}_{\mathcal{K}}\{dy_i,\dots,dy_i^{(n_i-1)}; i = 1,\dots,m\}. \tag{8}$$

Proof. *Sufficiency.* By Theorem 3.3 the equations (1) have observable state-space realization if and only if $\mathcal{H}_{s+2}$ is integrable. Moreover, the differentials of the state variables can be chosen as any set of exact basis of $\mathcal{H}_{s+2}$. Therefore, one may define $x_{i,j+1} = y_i^{(j)}$, $i = 1,\dots,m$, $j = 0,\dots,n_i-1$. This yields $y_i = x_{i,1}$, i. e., the output in the form (5), as well as state equations in the form (4).

Necessity. From the special form (4),(5) one can see that $x_{i,j+1} = y_i^{(j)}$ for $i = 1,\dots,m$, $j = 0,\dots,n_i-1$. Since the realization problem has a solution, the subspace $\mathcal{H}_{s+2}$ has to be integrable and one nth order state-space realization of (1) is (4),(5). Thus, by Theorem 3.3, the differentials of the states of system (4) form a basis for $\mathcal{H}_{s+2}$. Therefore, (8) must be satisfied. □

Proposition 3.5. The subspace $\mathcal{H}_{s+2}$ can be written as (8) if and only if in (1) $s := \max_i\{s_i\} = 0$.

Proof. *Sufficiency.* If $s = 0$, then the equations (1) can be rewritten as

$$y_i^{(n_i)} = \Phi_i(y_j,\dots,y_j^{(n_{ij})},u_j)$$

for $i,j = 1,\dots,m$. Thus, directly from the definition of $\mathcal{H}_{s+2} = \mathcal{H}_2$ one gets

$$\mathcal{H}_2 = \operatorname{span}_{\mathcal{K}}\{dy_i,\dots,dy_i^{(n_i-1)}; i = 1,\dots,m\}.$$

Necessity. It has been shown in the proof of Theorem 3.3 that $\mathcal{H}_{s+1} = \mathcal{H}_{s+2} \oplus \text{span}_{\mathcal{K}}\{\mathrm{d}u\}$. On the other hand, by the definition of subspace $\mathcal{H}_{s+2}$, $\mu(\mathrm{d}y_i^{(n_i-1)}) = \mathrm{d}y_i^{(n_i)} = \mathrm{d}\Phi_i(\cdot)$ has to belong to $\mathcal{H}_{s+1}$ for every $i = 1,\dots,m$. Thus,

$$\mathrm{d}\Phi_i(\cdot) \in \text{span}_{\mathcal{K}}\{\mathrm{d}y_j,\dots,\mathrm{d}y_j^{(n_i-1)},\mathrm{d}u_j\}$$

for $i,j = 1,\dots,m$. This yields that $s = 0$. □

Remark 3.6. For systems of the form (3), by using the subspaces $\mathcal{H}_i$, one can in a similar manner characterize the existence of an invertible state transformation $z = \varphi(x)$ $(x = \varphi^{-1}(z))$, where $\varphi, \varphi^{-1} \in \mathcal{K}^n$, which transforms the equations (3) into the form (4),(5). In this case the sequence $\{\mathcal{H}_i\}$ is initialized by

$$\mathcal{H}_0 = \text{span}_{\mathcal{K}}\{\mathrm{d}x, \mathrm{d}u\}.$$

Then the system (3) can be transformed into the form (4),(5) by an invertible state transformation if and only if

$$\mathcal{H}_1 = \text{span}_{\mathcal{K}}\{\mathrm{d}y_i,\dots,\mathrm{d}y_i^{(n_i-1)}; i = 1,\dots,m\}$$

for some $n_i \in \mathbb{N}$. The latter is possible if and only if $\sum_i r_i = n$, where r_i is the relative degree of y_i with respect to the input u, i.e., $r_i \in \mathbb{N}$ is the minimal number such that $y_i^{(r_i)}$ depends on u.

Second, we are looking for realizations in the form (4) with an arbitrary output function (6).

Theorem 3.7. The set of the nth order i/o equations (1) has the nth order realization in the form (4),(6) if and only if all the vector spaces $\mathcal{H}_j$, $j = s+2,\dots,k_*-1$ are integrable.

Proof. *Sufficiency.* First, note that since $\mathcal{H}_{s+2}$ is integrable, then the general realization problem has a solution. When all the vector spaces $\mathcal{H}_j$, $j = s+2,\dots,k_*-1$ are integrable, then one can write, by the definition of the subspaces $\mathcal{H}_i$,

$$\begin{array}{rcl} \mathcal{H}_{k_*-1} & = & \text{span}_{\mathcal{K}}\{\mathrm{d}\varphi_1\} \\ \mathcal{H}_{k_*-2} & = & \text{span}_{\mathcal{K}}\{\mathrm{d}\varphi_1, \mathrm{d}\varphi_1^{(1)}, \mathrm{d}\varphi_2\} \\ & \vdots & \\ \mathcal{H}_{s+2} & = & \text{span}_{\mathcal{K}}\{\mathrm{d}\varphi_1,\dots,\mathrm{d}\varphi_1^{(k_*-s-3)},\dots,\mathrm{d}\varphi_{k_*-s-2}\} \end{array} \tag{9}$$

for some vectors $\mathrm{d}\varphi_i = (\mathrm{d}\varphi_{i,1},\dots,\mathrm{d}\varphi_{i,\rho_i})$ of 1-forms, $i = 1,\dots,k_*-s-2$, which can for some i be also zero vectors. Now, in terms of $\mathrm{d}\varphi_{i,j}$, $i = 1,\dots,k_*-s-2$, $j = 1,\dots,\rho_i$, the subspace $\mathcal{H}_{s+2}$ is in a form (8). Based on the structure of $\mathcal{H}_{s+2}$, as in the sufficiency proof of Theorem 3.4, the state equations can be written in the form (4). It remains to show that $\mathrm{d}y_i \in \mathcal{H}_{s+2}$ for $i = 1,\dots,m$. Really, since $\mathrm{d}y_i^{(n_i)} = \mathrm{d}\Phi_i \in \mathcal{H}_1$, then always $\mathrm{d}y_i \in \mathcal{H}_{s+2}$ by Lemma 3.1. The latter means that the outputs y_i can be written as functions h_i of the state variables, which are defined by the basis of $\mathcal{H}_{s+2}$.

Necessity. By Theorem 3.3 the subspace $\mathcal{H}_{s+2}$ is integrable and the exact basis can be chosen as differentials of the state variables

$$\mathcal{H}_{s+2} = \mathrm{span}_{\mathcal{K}}\{\mathrm{d}x_{i,1}, \ldots, \mathrm{d}x_{i,n_i}; i = 1, \ldots, m\}.$$

Note that because of the observability assumption all the states can be expressed as functions on variables from the set $\{y_i, \ldots, y_i^{(n_i-1)}, u_i, \ldots, u_i^{(s)}\}$. From the special structure of the state equations (4) and definition of the subspaces $\mathcal{H}_i$, one has

$$\begin{array}{rcl} \mathcal{H}_{s+3} & = & \mathrm{span}_{\mathcal{K}}\{\mathrm{d}x_{i,1}, \ldots, \mathrm{d}x_{i,n_i-1}; i = 1, \ldots, m\} \\ \mathcal{H}_{s+4} & = & \mathrm{span}_{\mathcal{K}}\{\mathrm{d}x_{i,1}, \ldots, \mathrm{d}x_{i,n_i-2}; i = 1, \ldots, m\} \\ & \vdots & \end{array}$$

Therefore, all the subspaces $\mathcal{H}_i$, $i = s+2, \ldots, k_* - 1$, are integrable. □

4. EXAMPLES

Example 4.1. Consider the classical example of inverted pendulum of unit length with a point mass m attached at the end of the beam and which is actuated by the torque u applied at the base of the beam [7, 17]. This system can be described by the equation

$$y^{(2)} = g \sin y + \frac{u}{m}, \tag{10}$$

where g is the gravitational constant and y is the angular position of the pendulum with respect to the vertical position. By Proposition 3.5 one can say that the equation (10) can be transformed in the form (4) with the output function (5). We also know from the proof of Theorem 3.4 that the state coordinates can be defined as $x_1 = y$ and $x_2 = y^{(1)}$, which yields

$$\begin{array}{rcl} x_1^{(1)} & = & x_2 \\ x_2^{(1)} & = & g \sin x_1 + \frac{u}{m}. \end{array} \tag{11}$$

This realization is valid for all values of $y(t)$ and $u(t)$.

Example 4.2. Consider a MIMO nonlinear system described by the i/o equations

$$\begin{array}{rcl} y_1^{(2)} & = & y_1^{(1)} u_1^{(1)} - y_2 \\ y_2^{(1)} & = & y_1 u_2. \end{array} \tag{12}$$

Since there exists the time-derivative of u_1 in the equations (12), we can conclude, by Proposition 3.5, that the equations (12) are not transformable into the controller canonical form (4),(5). Compute

$$\begin{array}{rcl} \mathcal{H}_{s+2} & = & \mathcal{H}_3 = \mathrm{span}_{\mathcal{K}}\{\mathrm{d}y_1, \mathrm{d}y_2, \mathrm{d}(u_1 - \ln y_1^{(1)})\} \\ \mathcal{H}_{s+3} & = & \mathcal{H}_4 = \mathrm{span}_{\mathcal{K}}\{y_1^{(1)} \mathrm{d}(u_1 - \ln y_1^{(1)}) - y_2 \mathrm{d}y_1\} \\ \mathcal{H}_{s+4} & = & \mathcal{H}_5 = \{0\}. \end{array}$$

Since the subspace $\mathcal{H}_4$ is not integrable, the realization in the state-space form (4),(6) is also not possible. However, integrability of $\mathcal{H}_3$ means that there exists a general realization

$$\begin{array}{rcl} x_1^{(1)} & = & \mathrm{e}^{u_1 - x_2} \\ x_2^{(1)} & = & x_3 \mathrm{e}^{x_2 - u_1} \\ x_3^{(1)} & = & x_1 u_2 \\ y_1 & = & x_1 \qquad y_2 = x_3 \end{array} \tag{13}$$

in the form (3), by defining $x_1 = y_1$, $x_2 = u_1 - \ln y_1^{(1)}$ and $x_3 = y_2$. Note that the state variables of system (13) cannot be defined for $\dot{y}_1(t) \leq 0$. Thus, the realization (13) is valid, when $\dot{y}_1(t) > 0$.

Example 4.3. Under some simplified assumptions the model of the tilt of a bicycle can be described by the i/o equation [1]

$$y^{(2)} = \alpha_1 u^{(1)} + \alpha_2 \sin y + \alpha_3 u, \tag{14}$$

where $\alpha_1 = \frac{Dv}{bJ}$, $\alpha_2 = \frac{mgh}{J}$, $\alpha_3 = \frac{mhv^2}{bJ}$, y is the tilt angle, u is the steering angle, D is the product of inertia, v is the velocity of the bicycle, b is the wheel base, J is the moment of inertia, m is the mass of the system, g is the gravitational constant and h is the height of the center of mass.

Since the time-derivative of u is present in the equation (14), by Proposition 3.5, the system (14) is not transformable into the form (4),(5). This can be also verified by Theorem 3.4. Compute the $\mathcal{H}_i$ subspaces for the system:

$$\begin{array}{rcl} \mathcal{H}_1 & = & \mathrm{span}_{\mathcal{K}}\{\mathrm{d}y, \mathrm{d}y^{(1)}, \mathrm{d}u, \mathrm{d}u^{(1)}\} \\ \mathcal{H}_2 & = & \mathrm{span}_{\mathcal{K}}\{\mathrm{d}y, \mathrm{d}y^{(1)}, \mathrm{d}u\} \\ \mathcal{H}_3 & = & \mathrm{span}_{\mathcal{K}}\{\mathrm{d}y, \mathrm{d}(y^{(1)} - \alpha_1 u)\} \\ \mathcal{H}_4 & = & \mathrm{span}_{\mathcal{K}}\{\mathrm{d}(\alpha_1 y^{(1)} - \alpha_1^2 u - \alpha_3 y)\} \\ \mathcal{H}_5 & = & \{0\}. \end{array}$$

Obviously, the subspace $\mathcal{H}_{s+2} = \mathcal{H}_3$ cannot be written as (8), which also shows that the realization in the form (4),(5) does not exist for system (14).

On the other hand, the subspaces $\mathcal{H}_3$ and $\mathcal{H}_4$ are integrable and thus, by Theorem 3.7, the system (14) can be transformed in the state-space form (4),(6). Note that eventhough the existence of nonlinear inverse transformation can be proven, one cannot always express the solution in terms of elementary functions. This happens also in the current example. One has to choose the state variables as

$$\begin{array}{rcl} x_1 & = & \alpha_1 y^{(1)} - \alpha_1^2 u - \alpha_3 y \\ x_2 & = & \alpha_1 \alpha_2 \sin y + \alpha_1 \alpha_3 u - \alpha_3 y^{(1)}, \end{array} \tag{15}$$

but cannot express $x_1^{(1)}$ and $x_2^{(1)}$ in terms of x_1, x_2 and u, since equations (15) cannot be solved for y and $y^{(1)}$ in terms of elementary functions. This is not the problem of the specific solution of this paper as the same problem appears also when one tries to transform any other realization of (14) into the controller canonical form by state transformation.

Example 4.4. Consider a nonlinear system described by the equations

$$\begin{aligned} y_1^{(2)} &= y_1^{(1)}y_2 - u_1^{(1)} \\ y_2^{(2)} &= \mathrm{e}^{y_1} - u_2. \end{aligned} \tag{16}$$

Again, existence of time-derivatives of inputs in the system equations (16) indicates that a realization of the form (4),(5) does not exist. To check whether the equations can be transformed into the form (4) with the output (6), one has to compute the following subspaces:

$$\begin{aligned} \mathcal{H}_1 &= \mathrm{span}_{\mathcal{K}}\{\mathrm{d}y_1, \mathrm{d}y_1^{(1)}, \mathrm{d}y_2, \mathrm{d}y_2^{(1)}, \mathrm{d}u_1, \mathrm{d}u_1^{(1)}, \mathrm{d}u_2, \mathrm{d}u_2^{(1)}\} \\ \mathcal{H}_2 &= \mathrm{span}_{\mathcal{K}}\{\mathrm{d}y_1, \mathrm{d}y_1^{(1)}, \mathrm{d}y_2, \mathrm{d}y_2^{(1)}, \mathrm{d}u_1, \mathrm{d}u_2\} \\ \mathcal{H}_3 &= \mathrm{span}_{\mathcal{K}}\{\mathrm{d}y_1, \mathrm{d}y_2, \mathrm{d}(y_1^{(1)} + u_1), \mathrm{d}y_2^{(1)}\} \\ \mathcal{H}_4 &= \mathrm{span}_{\mathcal{K}}\{\mathrm{d}y_2, \mathrm{d}(y_1^{(1)} + u_1 - y_1y_2)\} \\ \mathcal{H}_5 &= \{0\}. \end{aligned}$$

Since all the subspaces are integrable, then one can conclude by Theorem 3.7 that it is possible to transform the equations (16) in the form (4),(6). To find the state coordinates, the subspace $\mathcal{H}_3 = \mathcal{H}_{s+2}$ should be given in the form (9). For that, take $\varphi_1 = (y_2, y_1^{(1)} + u_1 - y_1y_2)^{\mathrm{T}}$. Then $\varphi_1^{(1)} = (y_2^{(1)}, -y_1y_2^{(1)})^{\mathrm{T}}$ and φ_2 is zero vector. The basis for $\mathcal{H}_3$ can be chosen as $\{\mathrm{d}\varphi_1, \mathrm{d}\varphi_1^{(1)}\}$. This means that we may define the state variables as $(x_1, x_3, x_2, x_4)^{\mathrm{T}} = (\varphi_1, \varphi_1^{(1)})^{\mathrm{T}}$, resulting in the state equations

$$\begin{aligned} x_1^{(1)} &= x_2 \\ x_2^{(1)} &= \mathrm{e}^{-\frac{x_4}{x_2}} - u_2 \\ x_3^{(1)} &= x_4 \\ x_4^{(1)} &= \frac{x_4}{x_2}(\mathrm{e}^{-\frac{x_4}{x_2}} - u_2) - x_2(x_3 - u_1) + x_1x_4 \\ y_1 &= -\frac{x_4}{x_2} \\ y_2 &= x_1. \end{aligned}$$

The transformation to the latter state-space form is valid everywhere except when $\dot{y}_2(t) = 0$.

5. CONCLUSION

In this paper the problem of transforming the set of nonlinear i/o equations into the controller canonical form was studied. Two cases were addressed – when the output function is linear or nonlinear. For both cases necessary and sufficient conditions were given for existence of realization in the controller canonical form. Since realization theory is very similar (except for computations) for nonlinear continuous and discrete-time systems, the results of the paper are easily extendable for discrete-time case. In fact, the sufficiency of analogue of Proposition 3.5 was already proved in [14] for SISO discrete-time systems.

ACKNOWLEDGEMENT

The work of A. Kaldmäe was supported by the Estonian Centre of Excellence in IT (EXCITE), funded by the European Regional Development Fund.

(Received May 18, 2017)

REFERENCES

[1] K. J. Aström and R. M. Murray: Feedback Systems: An Introduction for Scientists and Engineers. Princeton University Press, 2008. DOI:10.1086/596297

[2] Z. Bartosiewicz, Ü. Kotta, M. Tõnso, and M. Wyrwas: Accessibility conditions of MIMO nonlinear control systems on homogeneous time scales. Math. Control Relat. Fields *6* (2016), 217–250. DOI:10.3934/mcrf.2016002

[3] J. Belikov, P. Kotta, Ü. Kotta, and M. Tõnso: Practical polynomial formulas in MIMO nonlinear realization problem. In: 51st IEEE Conference on Decision and Control, Hawaii 2012, pp. 1253–1258. DOI:10.1109/cdc.2012.6427109

[4] J. Belikov, Ü. Kotta, and M. Tõnso: Adjoint polynomial formulas for nonlinear state-space realization. IEEE Trans. Automat. Control *59* (2014), 256–261. DOI:10.1109/cdc.2012.6427109

[5] J. Belikov, Ü. Kotta, and M. Tõnso: Realization of nonlinear MIMO system on homogeneous time scales. Eur. J. Control *23* (2015), 48–54. DOI:10.1016/j.ejcon.2015.01.006

[6] M. Bronstein and M. Petkovsek: An introduction to pseudo-linear algebra. Theoret. Comput. Sci. *157* (1996), 3–33. DOI:10.1016/0304-3975(95)00173-5

[7] G. Conte, C. H. Moog, and A. M. Perdon: Algebraic Methods for Nonlinear Control Systems. Theory and Applications. Springer, London 2007. DOI:10.1007/978-1-84628-595-0

[8] P. E. Crouch and F. Lamnabhi-Lagarrigue: State space realizations of nonlinear systems defined by input-output differential equations. In: Analysis and Optimization of Systems, Springer, Berlin, Heidelberg 1988, pp 138–149. DOI:10.1007/bfb0042209

[9] E. Delaleau and W. Respondek: Lowering the orders of derivatives of controls in generalized state space systems. J. Math. Systems, Estimation, Control *5* (1995), 1–27.

[10] M. Halas, Y. Kawano, C. H. Moog, and T. Ohtsuka: Realization of a nonlinear system in the feedforward form: a polynomial approach. In: 19th IFAC World Congress, Cape Town 2014, pp. 9480–9485. DOI:10.3182/20140824-6-za-1003.00990

[11] M. Halas and Ü. Kotta: A transfer function approach to the realisation problem of nonlinear systems. Int. J. Control *85* (2012), 320–331. DOI:10.1080/00207179.2011.651748

[12] E. R. Kolchin: Differential Algebra and Algebraic Groups. Academic Press, New York 1973.

[13] Ü. Kotta and T. Mullari: Equivalence of realizability conditions for nonlinear control systems. Proc. Est. Acad. Sci. Physics. Math. *55* (2006), 24–42.

[14] Ü. Kotta and N. Sadegh: Two approaches for state space realization of NARMA models: bridging the gap. Math. Comput. Model. Dyn. Syst. *8* (2002), 21–32. DOI:10.1076/mcmd.8.1.21.8340

[15] V. L. Morales, F. Plestan,and A. Glumineau: Linearization by completely generalized input-output injection. Kybernetika *35* (1999), 793–802.

[16] F. Plestan and A. Glumineau: Linearization by generalized input-output injection. Systems Control Lett. *31* (1997), 115–128. DOI:10.1016/s0167-6911(97)00025-x

[17] E. D. Sontag: Mathematical Control Theory. Springer–Verlag, New York 1998. DOI:10.1007/978-1-4612-0577-7

[18] M. Tõnso and Ü. Kotta: Realization of continuous-time nonlinear input-output equations: polynomial approach. In: 12th International Conference on Computer Aided Systems Theory, Gran Canaria 2009, pp. 633–640. DOI:10.1007/978-3-642-04772-5_82

[19] A. J. van der Schaft: On realization of nonlinear systems described by higher-order differential equations. Math. Systems Theory *19* (1987), 239–275. DOI:10.1007/bf01704916

[20] J. Zhang, C. H. Moog, and X. Xia: Realization of multivariable nonlinear systems via the approaches of differential forms and differential algebra. Kybernetika *46* (2010), 799–830.

Arvo Kaldmäe, Tallinn University of Technology, Akadeemia tee 21, 12618 Tallinn. Estonia.
e-mail: arvo@cc.ioc.ee

Ülle Kotta, Tallinn University of Technology, Akadeemia tee 21, 12618 Tallinn. Estonia.
e-mail: kotta@cc.ioc.ee