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# DESIGN OF ROBUST GAIN SCHEDULED CONTROLLER USING $L_2$ GAIN PERFORMANCE

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This paper is devoted to robust gain scheduled PID controller design with $L_2$ performance for the linear time varying (LPV) uncertain system with polytopic uncertainties. The novel approach of robust controller design ensures that the obtained design procedure is convex with respect to both plant uncertainties (polytopic system) and gain scheduling parameters and gives less conservative results. Modified design procedure should be used to obtain a robust controller or robust switched controller (ideal, non-ideal switching) with arbitrarily switching algorithm. The effectiveness of the proposed approach is illustrated on the simulation examples.

*Keywords:* gain scheduled controller, Linear parameter varying systems (LPV), robust controller, switched controller, $L_2$ gain performance

*Classification:* 70E60, 93B36

## 1. INTRODUCTION

Linear parameter varying systems (LPV) are a class of linear systems where the plant state matrices are affinelly dependent on measurable vector of time-varying parameters. The LPV systems should be viewed as a linear time invariant (LTI) plants derived from time-varying parameters or they should be also obtained as a result from the nonlinear systems linearization along the trajectories of the parameter $\theta(t)$. The concept of LPV systems has been originally introduced in [9] and their modeling and identification have been published in the book [10]. The nonlinear and time-varying behavior of the system should be embedded in the solution of the linear dynamic input-output relationship which depends on the scheduling variables. The existing analysis and synthesis results for LPV systems provide a rigorous framework to obtain the gain scheduled controller design procedure [5, 7, 8, 11, 12, 16, 17].

The several classes of LPV systems are categorized based on the state matrices which depend on the scheduling parameters [5]. The LPV systems should be divided to two different classes, where the first class assumes that state matrices have a rational dependence on the parameters and the second class assumes that they have an arbitrary dependence. This paper deals with the first class of the LPV systems, so the state matrices used in this paper have a rational (affine) dependence on the gain scheduling parameters. General approach of the worst case analysis has been introduced in the

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paper [5], where input/output interconnection gain of LPV system and an uncertain system have been described by an integral quadratic constraints (IQC). The dissipation inequality has been used here to compute an upper bound for $L_2$ norm gain. The design problem of gain scheduled output feedback controllers for LPV systems has been addressed in the paper [8] where the scheduling parameters have been uncertain and the sufficient condition for the $H_\infty$ design problem has been in the terms of parameter dependent matrix inequalities. Solved problem has been given in the terms of LMI. Wang and Seiler have used the $H_\infty$ wind turbine controller designed at several operating points to coordinate the blade pitch angle and the generator torque [16]. The derived gain scheduled controller compensated the nonlinear turbine dynamics and has been dependent on the wind speed and the power output. Another approach has been used in the paper [12], where the authors addressed the problem of robust PI gain scheduled controller design with $H_2$ performance using the Lyapunov function and novel equivalent systems approach. The obtained controller design procedures have been in the LMI form.

The contribution of this paper is to provide gain scheduled controller with robust performance conditions derived by minimization of $L_2$ gain with respect to the measured output and disturbance input for the feedback interconnection of the LPV system and the controller. The modified design procedure should be also used to obtain other $L_2$ gain performance criteria controllers e. g. robust controller or robust switched controller.

This paper has been organized to the following sections. Section 2 presents preliminaries and remark the $L_2$ gain design procedure. Section 3 addresses the new obtained output feedback PID gain scheduled controller design procedure for $L_2$ gain performance and convex robust stability conditions and the final Section 4 verifies the proposed design procedure with the examples.

Notation used in this paper is standard, $P \in R^{m\times n}$ denotes the set of real $m \times n$ matrices, $P > 0 (P \geq 0) \in R^{n\times n}$ is a real symmetric, positive definite (semidefinite) matrix. "*" in matrices denotes the respective transposed (conjugate) term to make matrix symmetric. $I_m$ is an $m \times m$ identity matrix, $0_m$ denotes the zero matrix.

## 2. PRELIMINARIES AND PROBLEM FORMULATION

Consider a linear parameter varying (LPV) uncertain systems with state space matrices which are fixed function of known gain scheduled time varying parameter $\theta$, [10] in the form:

$$\begin{aligned} G : \dot{x} &= A(\xi,\theta)x + B(\xi,\theta)u + B_w(\xi,\theta)w \\ y &= Cx \quad z = C_z x \end{aligned} \tag{1}$$

where

$$A(\xi,\theta) = A_0(\xi) + \sum_{j=1}^{p} A_j(\xi)\theta_j \in R^{n\times n}$$

$$B(\xi,\theta) = B_0(\xi) + \sum_{j=1}^{p} B_j(\xi)\theta_j \in R^{n\times m}$$

$$B_w(\xi,\theta) = B_{w0}(\xi) + \sum_{j=1}^{p} B_{wj}(\xi)\theta_j \in R^{n\times k}$$

$$x \in R^n, u \in R^m, y \in R^l, w \in R^k, z \in R^l \tag{2}$$

denotes the state, control input, controlled output, disturbance input and system output respectively. Disturbance input $w(t)$ and output $z(t)$ are assumed to be square integrable, so the $w(t), z(t) \in L_2[0,\infty)$. The system in equation (1) with $\theta = 0, \xi = const$ is supposed to be stabilizable via static output feedback controller. Matrices $A_j(\xi), B_j(\xi), B_{wj}(\xi), j = 0,1,2,\ldots,p$ belong to the convex set of polytope with N vertices that can be formally defined as:

$$\Psi := \{A_j(\xi), B_j(\xi), B_{wj}(\xi) := \sum_{i=1}^{N}(A_{ji}, B_{ji}, B_{wji})\xi_i\}$$

$$\sum_{i=1}^{N}\xi_i = 1,\ \sum_{i=1}^{N}\dot{\xi}_i = 0,\ \xi_i \geq 0,\ j = 0,1,2,\ldots,p. \tag{3}$$

Where $\xi_i, i = 1,2,\ldots,N$ are constant or time varying but unknown parameters respective uncertainties of system matrices in equation (1) satisfying (3). $A_{ji}, B_{ji}, B_{wji}, C, C_z$ are constant matrices of corresponding dimensions. $\theta \in R^p$ is a vector of known constant or time varying real gain scheduled parameters. Assume that both lower and upper bounds for these parameters and variation rates are available. Specifically:

- Each parameter $\theta_i, i = 1,2,\ldots,p$ belongs between known extremal values:

$$\theta_i \in \Omega_p = \{\theta_i \in \langle \underline{\theta_i} \quad \overline{\theta_i}\rangle, i = 1,2,\ldots,p\}. \tag{4}$$

- The rate of changes of $\theta$ is well defined at all times and satisfies:

$$\dot{\theta}_i \in \Omega_t = \{\dot{\theta}_i \in \langle \underline{\dot{\theta}_i} \quad \overline{\dot{\theta}_i}\rangle, i = 1,2,\ldots,p\}. \tag{5}$$

- The rate of changes of $\dot{\xi}$ is well defined at all times and satisfies:

$$\dot{\xi}_i \in \Omega_\rho = \{\dot{\xi}_i \in \langle \underline{\rho_i} \quad \overline{\rho_i}\rangle, i = 1,2,\ldots,N\}. \tag{6}$$

The plant states in equation (1) have to be extended to design the integration part of gain scheduled controller, where the static output feedback control algorithm should provide proportional (P) and integral (I) parts of the designed PID controller. The more detailed explanation of the plant states extension has been published in [15]. It should be assumed that, the PID controller with static output feedback should be designed without any denotation changes in the system output equation (1). The following definition and lemma is crucial for the PID controller design process.

**Definition 2.1.** (Boyd et al. [1]) $L_2$ gain of the uncertain system in equation (1) is the quantity:

$$sup_{||w||_2\neq 0}\frac{||z||_2}{||w||_2} \leq \gamma \tag{7}$$

where $L_2$ norm of variable $w$ is $||w||_2 = \int_0^\infty w^T w \, \mathrm{d}t$ and supreme is taken over all non zero trajectories of the system, starting from $x(0) = 0$.

**Lemma 2.2.** (Boyd et al. [1]) Suppose there exists a quadratic function $V(x) = x^T P(\xi, \theta) x$, where $P(\xi, \theta) > 0$ and scalar $\gamma \geq 0$ such that for all $t, \theta \in \Omega_\theta, \xi \in \Omega_\xi$, and $x \in R^n$ hold:

$$L = \frac{\mathrm{d}V(x)}{\mathrm{d}t} + z^T z - \gamma^2 w^T w \leq 0 \tag{8}$$

then the $L_2$ gain of system (1) is less than $\gamma$.

Proof of Lemma 1 is in the integration of equation (8) from 0 to $T_0 > 0$, with the initial condition $x(0) = 0$.

$$V(x(T_0)) + \int_0^{T_0} (z^T z - \gamma^2 w^T w) \, \mathrm{d}t \leq 0.$$

Since $V(x(T_0)) \geq 0$, implies:

$$\gamma^2 w^T w - z^T z \geq 0 \rightarrow \gamma \geq \frac{||z||_2}{||w||_2}. \tag{9}$$

If equations (8) and (9) hold, then the trajectory $x(t)$ of the system (1) driven by $w(t)$ lies within the limits and then the system is stable, [2] – [6].

The following lemma of Isidori plays an important role in the next development [3].

**Lemma 2.3.** Let $\gamma > 0$ be a fixed number. The positive definite matrix $P(\xi, \theta)$ satisfying equation (8) exists if and only if there exists a positive definite matrix $X(\xi, \theta)$ satisfying

$$\begin{bmatrix} A_c(\xi,\theta)^T X(\xi,\theta) + X(.)A(.) & X(.)B_w(.) & C_z \\ B_w(.)^T X(.) & -\gamma I & 0 \\ C_z & 0 & -\gamma I \end{bmatrix} < 0$$

where $A_c(\cdot)$ is the closed loop system matrix for the uncertain system (1) and gain scheduled controller (11).

The Lemma 2.3 is fulfilled if exists $\gamma$ and positive definite matrix $X(.)$ and the system is asymptotically stable while the $H_\infty$ norm of its closed loop transfer function is strictly less than $\gamma$. Note that inequality in Lemma 2 without modification is known as the bounded real lemma [3], [4]. If the inequality condition in equation (9) is satisfied then inequality in equation (8) should be interpreted as well known Bellman-Lyapunov equation, which holds for the time invariant and variant systems. However it is complicated to apply the Lemma 2 for the robust gain scheduled controller design, because the above inequality is non convex with respect to uncertainty $\xi$ and gain scheduled variables $\theta$. This paper offers a new robust PID gain scheduled controller design procedure where uncertainty and gain scheduled parameter are affine (convex) with respect to the obtained robust stability conditions. Convex properties may not hold for other variables of the controller design procedure. The plant states have to be extended that the static output feedback control algorithm should provide proportional (P) and integral (I) parts of the designed PID controller. The more detailed plant states extension is described in [15]. If we assume that the plant states are already extended with the integration part without any changes in denotation, then the plant outputs of system (1) allows us to design PID controller with static output feedback. The following problem has been studied in this paper.

**Problem 2.4.** For uncertain gain scheduled plant design a robust static output feedback gain scheduled PID controller with control algorithm

$$u = K(\theta)y + K_d(\theta)\dot{y} = K(\theta)Cx + K_d(\theta)C_d\dot{x} \tag{10}$$

where

$$K(\theta) = [K_p(\theta) \quad K_i(\theta)]$$

is the PI part of the controller [15] and the controller in equation (10) ensures the closed loop system robust parameter dependent quadratic stability and optimal value of $L_2$ gain $\gamma$ (7) for all $\theta \in \Omega_\theta, \dot{\theta} \in \Omega_t, \dot{\xi} \in \Omega_\rho$. Matrices $K(\theta), K_d(\theta)$ represent gains of the PID controller in the equation (10).

$$K(\theta) = K_0 + \sum_{j=1}^{p} K_j\theta_j, \quad K_d(\theta) = K_{d0} + \sum_{j=1}^{p} K_{dj}\theta_j.$$

Note that $C_d$ is the output matrix for the D-part of the controller.

## 3. GAIN SCHEDULED CONTROLLER DESIGN

This section formulates the theoretical approach to the robust gain scheduled controller design with less conservative results and convex robust stability conditions of gain scheduled and uncertain parameters. The uncertain gain scheduled polytopic system is given by the equation (1) and designed gain scheduled controller in equation (10) ensures the closed loop system parameter dependent quadratic stability and robust performance conditions by minimizing of $L_2$ gain with respect to measurable output vector and disturbance input for all uncertain plant parameters $\Pi \in \Psi$ and gain scheduled parameters $\theta \in \Omega_\theta, \dot{\theta} \in \Omega_t, \dot{\xi} \in \Omega_\rho$. The main paper results are interpreted in the next theorem.

**Theorem 3.1.** The closed-loop system, that consists of the uncertain plant (1) and gain scheduled controller (10), is robust parameter dependent quadratically stable with minimal value of $L_2$ gain $\gamma$ (7) for all uncertain plant parameters $\Pi \in \Psi$, gain scheduling parameters $\theta \in \Omega_\theta, \dot{\theta} \in \Omega_t$ and rate of uncertain parameter changes $\dot{\xi} \in \Omega_\rho$ if for all $i$ the following BMI conditions hold:

$$W_i = W_{0i} + \sum_{j=1}^{p} W_{ji}\theta_j < 0 \quad i = 1, 2, \ldots, N \tag{11}$$

where

$$\begin{aligned}
W_{0i} =& \{w_{0ikl}\}_{4\times 4} \quad W_{ji} = \{w_{jikl}\}_{4\times 4} \\
w_{0i11} =& N_1^T + N_1 - N_5^T K_{d0} C_d - C_d^T K_{d0}^T N_5 \\
w_{ji11} =& - N_5^T K_{dj} C_d - C_d^T K_{dj}^T N_5 \\
w_{0i12} =& P_{0i} + N_2 - N_1^T A_{0i} - N_5^T K_0 C - C_d^T K_{d0}^T N_6 \\
w_{ji12} =& P_{ji} - N_1^T A_{ji} - N_5^T K_j C - C_d^T K_{dj}^T N_6 \\
w_{0i13} =& N_3 - N_1^T B_{0i} + N_5^T - C_d^T K_{d0}^T N_7 \\
w_{ji13} =& - N_1^T B_{ji} - C_d^T K_{dj}^T N_7 \\
w_{0i14} =& N_4 - N_1^T B_{w0i} - C_d^T K_{d0}^T N_8 \\
w_{ji14} =& - N_1^T B_{wji} - C_d^T K_{dj}^T N_8 \\
w_{0i22} =& - N_2^T A_{0i} - A_{0i}^T N_2 - N_6^T K_0 C \\
& - C^T K_0^T N_6 + C_z^T C_z + \sum_{i=1}^{N} P_{0i}\dot{\xi} + \sum_{j=1}^{p} P_{ji}\dot{\theta}_j \\
w_{ji22} =& - N_2^T A_{ji} - A_{ji}^T N_2 - N_6^T K_j C \\
& - C^T K_j^T N_6 + \sum_{i=1}^{N} P_{ji}\dot{\xi}_i \\
w_{0i23} =& - A_{0i}^T N_3 - N_2^T B_{0i} + N_6^T - (K_0 C)^T N_7 \\
w_{ji23} =& - A_{ji}^T N_3 - N_2^T B_{ji} - (K_j C)^T N_7 \\
w_{0i24} =& - A_{0i}^T N_4 - N_2^T B_{w0} - (K_0 C)^T N_8 \\
w_{ji24} =& - A_{ji}^T N_4 - N_2^T B_{wj} - (K_j C)^T N_8 \\
w_{0i33} =& N_7^T + N_7 - N_3^T B_{0i} - B_{0i}^T N_3 \\
w_{ji33} =& - N_3^T B_{ji} - B_{ji}^T N_3 \\
w_{0i34} =& - B_{0i}^T N_4 - N_3^T B_{woi} + N_8 \\
w_{ji34} =& - B_{ji}^T N_4 - N_3^T B_{wji} \\
w_{0i44} =& - N_4^T B_{w0i} - B_{w0i}^T N_4 - \gamma^2 I \\
w_{ji44} =& - N_4^T B_{wji} - B_{wji}^T N_4
\end{aligned}$$

where dimensions of auxiliary matrices are: $N_i \in R^{n\times n}, i = 1, 2, N_3 \in R^{m\times n}, N_4 \in R^{k\times n}, N_i \in R^{n\times m}, i = 5, 6, N_7 \in R^{m\times m}, N_8 \in R^{k\times m}$.

**Remark 3.2.**
1. Time derivative of gain scheduling parameters $\theta$ and uncertainties $\xi$ are set in the diagonal entries of matrices $W_{0i}, W_{ji}$. The corresponding terms of $w_{22}$ should be changed to decrease the computational load as

$$\sum_{i=1}^{N} P_{ji}\dot{\xi} \le \sum_{i=1}^{N} P_{ji}\rho_i \quad \sum_{i=1}^{N} P_{ji}\dot{\theta}_j \le \sum_{i=1}^{N} P_{ji}\overline{\dot{\theta}_j} \tag{12}$$

assuming that $P_{ji} > 0$ for all $i, j$. Because the inequality of (11) with respect to $\theta$ and $\xi$ is convex, it is sufficient to ensure the negative definiteness at all corners of $\theta$ and uncertainties $\xi$ to guarantee the negative definiteness of (11). The inequality of (11) is negative if and only if it is negative in $2^p$ vertices ($\theta$-vertices) and i = 1, 2, ...,N $\xi$-vertices.

2. Assume that instead of equation (4) holds:

$$\sum_{j=1}^{p} \theta_j = 1 \quad \sum_{j=1}^{p} \dot{\theta}_j = 0 \quad \theta_j \in \langle 0, 1 \rangle. \tag{13}$$

The gain scheduled parameter $\theta$ is transformed to switching variable with arbitrarily switching algorithm. The rate of changes of $\theta$ is infinite for ideal switching and in this case $P_{ji} = 0, j = 1, 2, \ldots, p, i = 1, 2, \ldots, N$. In the case of non ideal switching the rate of changes of $\theta$ is finite. Using the switching algorithm the number of $\theta$ vertices decreases to $p$. Inequality of the equation (11) for the case of equation (13) serves to design the robust PID switching controller with $p$ plant mode, arbitrarily switching algorithm and $L_2$ gain performance.

3. Assume that $\theta_j = 0, \overline{\dot{\theta}_j} = 0, j = 1, 2, \ldots, p$. Inequality in the equation (11) serves to design of robust PID controller with polytopic uncertainties, parameter dependent quadratic stability and $L_2$ gain performance.

4. The robust PI controller design procedure should be obtained by setting the $C_d = 0$.

5. The robust PD gain scheduled controller design procedure could by obtained by setting the $K_i(\theta) = 0$ in (11).

P r o o f. To proof the Theorem 3.1 assume that the Lyapunov function in the equation (8) is in the form:

$$V(x, \theta, \xi) = x^T P(\xi, \theta) x \tag{14}$$

where:

$$P(\xi, \theta) = P_0(\xi) + \sum_{j=1}^{p} P_j(\xi)\theta_j \quad P_j(\xi) = \sum_{i=1}^{N} P_{ji}\xi_i.$$

Note that for quadratic stability $P_j(\xi) = 0, j = 1, 2, \ldots, p$.
Time derivative of the equation (14) is given as:

$$\frac{\mathrm{d}V}{\mathrm{d}t} = \dot{x}^T P(\xi, \theta) x + x^T P(\dot{\xi}, \dot{\theta}) x + x^T P(\xi, \theta) \dot{x}$$

and by equation (8) should be obtained:

$$L = v^T \begin{bmatrix} 0 & P(\xi, \theta) & 0 & 0 \\ P(.) & P(\dot{\xi}, \dot{\theta}) + C_z^T C_z & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\gamma^2 I \end{bmatrix} v \tag{15}$$

where $v^T = [\dot{x}^T x^T u^T w^T]$

$$P(\dot{\xi},\dot{\theta}) = \sum_{i=1}^{N} P_{0i}\dot{\xi}_i + \sum_{j=1}^{p}\sum_{i=1}^{N} P_{ji}\dot{\xi}_i\theta_j + \sum_{i=1}^{N}\sum_{j=1}^{p} P_{ji}\dot{\theta}_j\xi_i.$$

To join system matrices to time derivative of Lyapunov function and separate matrix $P(.)$ from matrices $A(.), B(.), B_w(.)$ we have introduced auxiliary matrices $N_i, i = 1, 2, \dots, 8$ of corresponding dimensions in the following form:

$$2(N_1\dot{x} + N_2x + N_3u + N_4w)^T(\dot{x} - A(\xi,\theta)x - B(\xi,\theta)u - B_w(\xi,\theta)w) = 0 \tag{16}$$

$$2(N_5\dot{x} + N_6x + N_7u + N_8w)^T(u - K(\theta)Cx - K_d(\theta)C_d\dot{x}) = 0. \tag{17}$$

The stability condition in the equation (18) should be obtained by summarizing equations (15),(16) and (17).

$$L = v^TWv < 0 \tag{18}$$

where $W = \{w_{kl}\}_{4\times4}$ and for each element of $W$ holds:

$$\begin{aligned}
w_{11} =& N_1^T + N_1 - N_5^TK_d(\theta)C_d - (K_dC_d)^TN_5\\
w_{12} =& P(\xi,\theta) + N_2 - N_1^TA(\xi,\theta) - N_5^TKC\\
& - (K_dC_d)^TN_6\\
w_{13} =& N_3 - N_1^TB(\xi,\theta) + N_5^T - (K_dC_d)^TN_7\\
w_{14} =& N_4 - N_1^TB_w(\xi,\theta) - (K_dC_d)^TN_8\\
w_{22} =& - N_2^TA(\xi,\theta) - A(\xi,\theta)^TN_2 - N_6^TKC\\
& - (KC)^TN_6 + P(\dot{\xi},\dot{\theta}) + C_z^TC_z\\
w_{23} =& - A(\xi,\theta)^TN_3 - N_2^TB(\xi,\theta) + N_6^T - (KC)^TN_7\\
w_{24} =& - A(\xi,\theta)^TN_4 - N_2^TB_w(\xi,\theta) - (KC)^TN_8\\
w_{33} =& N_7^T + N_7 - N_3^TB(\xi,\theta) - B(\xi,\theta)^TN_3\\
w_{34} =& - B(\xi,\theta)N_4 - N_3^TB_w(\xi,\theta) + N_8\\
w_{44} =& - N_4^TB_w(\xi,\theta) - B_w(\xi,\theta)^TN_4 - \gamma^2I.
\end{aligned}$$

The inequality in the equation (18) is convex with respect to uncertainty $\xi$ and gain scheduled parameter $\theta$ and therefore the equation (18) should be divided to $N$ inequalities of the equation (11) which prove the sufficient robust stability conditions of Theorem 3.1. □

## 4. EXAMPLES

**Example 4.1.** Consider the following uncertain SISO linear parameter varying continuous time system (extended model for PI controller design) in the form of the equation (1) :

$$A(\theta,\xi) = A_0(\xi) + A_1(\xi)\theta_1$$

$$B(\theta,\xi) = B_0(\xi) + B_1(\xi)\theta_1$$
$$B_w(\theta,\xi) = B_{w0}(\xi) + B_{w1}(\xi)\theta_1$$

where $p = 1, N = 2$

$$A_{01} = \begin{bmatrix} -0.5 & 0.7 & 0 \\ -0.1 & -0.8 & 0 \\ 0 & 1 & 0 \end{bmatrix}, B_{01} = B_{w01} = \begin{bmatrix} 0.1 \\ 1 \\ 0 \end{bmatrix}$$
$$A_{02} = \begin{bmatrix} -0.49 & 0.6 & 0 \\ -.2 & -0.7 & 0 \\ 0 & 1 & 0 \end{bmatrix}, B_{02} = B_{w02} = \begin{bmatrix} 0.15 \\ 0.95 \\ 0 \end{bmatrix}$$
$$A_{11} = \begin{bmatrix} -0.1 & 0.20 & 0 \\ 0.25 & 0.05 & 0 \\ 0 & 0 & 0 \end{bmatrix}, B_{11} = B_{w11} = \begin{bmatrix} -0.05 \\ 0.1 \\ 0 \end{bmatrix}$$
$$A_{12} = \begin{bmatrix} -0.15 & 0.19 & 0 \\ 0.2 & 0.04 & 0 \\ 0 & 0 & 0 \end{bmatrix}, B_{12} = B_{w12} = \begin{bmatrix} 0.0 \\ 0.15 \\ 0 \end{bmatrix}.$$

The following output matrices were concerned to obtain the PI and derivative controller parts:

$$C = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, C_d = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}.$$

Consider the gain scheduled parameter $\theta_1$ variation: if output $y = 0$ then $\theta_1 = -1$ and if $y = 1$ then $\theta_1 = 1$.

The problem is to design the robust PID controller which ensures the closed-loop robustness properties, $L_2$ gain optimization, quadratic stability or parameter dependent quadratic stability for the cases of:

- gain scheduled controller design with quadratic stability (QS) and parameter dependent quadratic stability (PDQS) under the conditions $\theta \in \langle -1, 1\rangle, \gamma \in \langle 0.3, 1.2\rangle$, maximal value of rate of gain scheduled parameter and uncertainties changes $\dot{\theta} = 2/sec, \dot{\xi} = 0.2/s$ and $\dot{\theta} = 10/s, \dot{\xi} = 1/s$;
- robust controller design with QS and PDQS when maximal value of rate of uncertain parameter changes are $\dot{\xi} = 0.025/s$ and $\dot{\xi} = 1/s$.

Numerical solution has been carried out by MATLAB 7.5 using YALMIP with solver PENBMI21. The obtained controllers for the desired cases have been summarized as:

**Case 1.** Gain scheduled controller designed for $\dot{\theta} = 2/s, \dot{\xi} = 0.2/s$ QS:

$$\begin{aligned} R(s) = &-33.1829 - \frac{20.9522}{s} - 3.2728s \\ &+ \theta(-0.4192 + \frac{0.1270}{s} + 0.0033s). \end{aligned}$$

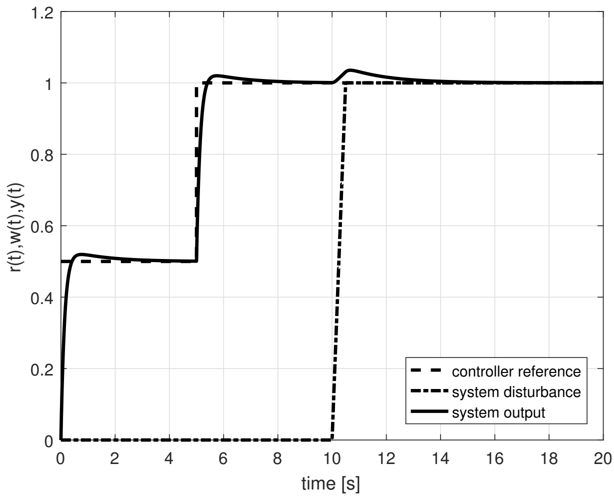


**Fig. 1.** Simulation results of output and disturbance input for GSC designed for case 1 when $\dot{\theta}(t) \in \langle -2, 2\rangle$, $\dot{\xi} \in \langle -0.2, 0.2\rangle$.

Computed $L_2$ gain for QS $\gamma = 0.6471$ and for the case of $\theta = 0$ maximal closed loop eigenvalue is $-0.4492$. The obtained controller has been verified by simulation, where the controller reference input has been step changed from 0.5 p.u. to 1 p.u. in 5s. The system disturbance has changed the parameter $\theta$ in 10s with rate of $\dot{\theta} = 2/s$. The designed controller has suppressed the system disturbance and provided stable control for the system as is shown in Figure 1. The influence of a sine wave parameter disturbance has been studied in the experiment which results are shown in Figure 2. The controller has successfully suppressed the disturbance influence and the calculated $L_2$ gain value has been $\gamma = 0.034$. The effectiveness of the disturbance suppression by the designed controller is shown by amplitude frequency response in Figure 3.

**Case 2.** Gain scheduled controller designed for $\dot{\theta} = 2/s, \dot{\xi} = 0.2/s$ PDQS:

$$\begin{aligned} R(s) = &- 25.0871 - \frac{-5.5362}{s} - 9.3848s \\ &+ \theta(-1.4561 - \frac{0.6251}{s} + 0.0789s) \end{aligned}$$

$L_2$ gain for PDQS $\gamma = 0.75$ and for the case of $\theta = 0$ maximal closed loop eigenvalue is $-0.2303$.

**Case 3.** Gain scheduled controller designed for $\dot{\theta} = 10/s, \dot{\xi} = 1/s$ QS:

$$\begin{aligned} R(s) = &- 33.1829 - \frac{20.9522}{s} - 3.2728s \\ &+ \theta(-0.4192 + \frac{0.1270}{s} + 0.0033s) \end{aligned}$$

$L_2$ gain $\gamma = 0.6471$ and for the case of $\theta = 0$ maximal closed loop eigenvalue is $-0.4492$.

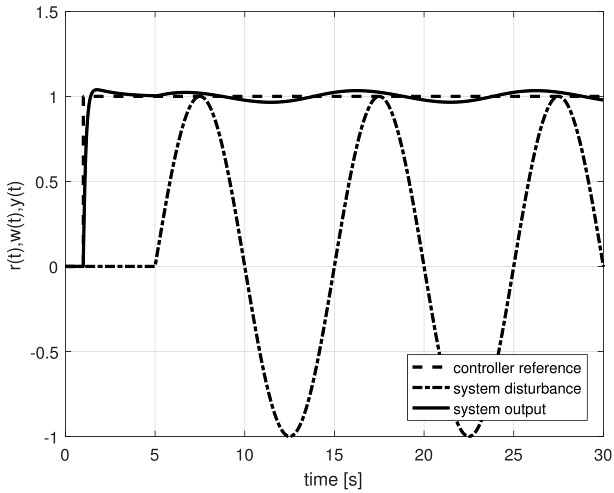


**Fig. 2.** Response of gain scheduled controller designed for case 1 to sine wave disturbance input.

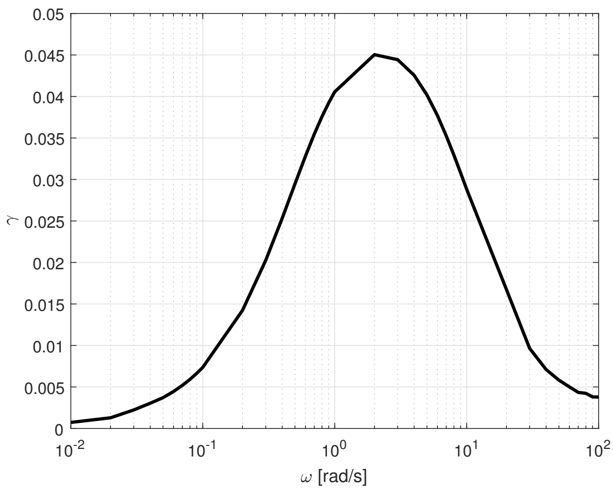


**Fig. 3.** Amplitude frequency response of closed loop system $\gamma$ for gain scheduled controller designed for case 1.

**Case 4.** Gain scheduled controller designed for $\dot{\theta} = 10/s, \dot{\xi} = 1/s$ PDQS:

$$R(s) = -17.5965 - \frac{11.4286}{s} - 7.4858s + \theta(-1.0835 - \frac{1.8462}{s} - 0.0745s)$$

$L_2$ gain $\gamma = 0.75$ and for the case of $\theta = 0$ maximal closed loop eigenvalue is $-0.4463$.

**Case 5.** Robust controller design. Maximal value of rate of uncertain parameter change is $\dot{\xi} = 0.025/s$ QS:

$$R(s) = -13.7207 - \frac{9.8584}{s} - 1.482s$$

$L_2$ gain $\gamma = 0.7489$ and for the case of $\theta = 0$ maximal closed loop eigenvalue is $-0.4434$.

**Case 6.** Robust controller design. Maximal value of rate of uncertain parameter change is $\dot{\xi} = 0.025/s$ PDQS:

$$R(s) = -15.894 - \frac{10.4877}{s} - 1.6189s$$

$L_2$ gain $\gamma = 0.7487$ and for the case of $\theta = 0$ maximal closed loop eigenvalue is $-0.4421$.

**Case 7.** Robust controller design. Maximal value of rate of uncertain parameter change is $\dot{\xi} = 1/s$ QS:

$$R(s) = -13.7207 - \frac{9.8584}{s} - 1.482s$$

$L_2$ gain $\gamma = 0.7489$ and maximal closed loop eigenvalue is $-0.4432$.

**Case 8.** Robust controller design. Maximal value of rate of uncertain parameter change is $\dot{\xi} = 1/s$ PDQS:

$$R(s) = -4.1712 - \frac{7.0321}{s} - 1.7332s$$

$L_2$ gain $\gamma = 0.75$ and maximal closed loop eigenvalue is $-0.4473$. This case should be interpreted as a design of robust switched controller with arbitrarily switching algorithm for non-ideal switching due to the large value of $\dot{\theta} = 10/s$. The controller has been verified by simulation experiment, which results are shown in Figure 4.

**Example 4.2.** Plant model of the second example has been introduced in [13]. The following nonlinear model should be obtained for $L_2$ gain case after the small modification:

$$\dot{x} = -asinx + bu + b_w w \quad y = x$$

where $a \in \langle 0.8, 1\rangle$, when a=0.8 then b=1 and a=1, b=0.5, $b_w = b$. The above model should be linearized in three working points $x_0 = \{0, \pi/4, \pi/2\}$. The system state space has to be increased for PI $L_2$ gain scheduled controller design. The matrices $A(\xi,\theta), B(\xi,\theta), B_w(\xi,\theta)$ which have to be obtained for PI controller design for the case

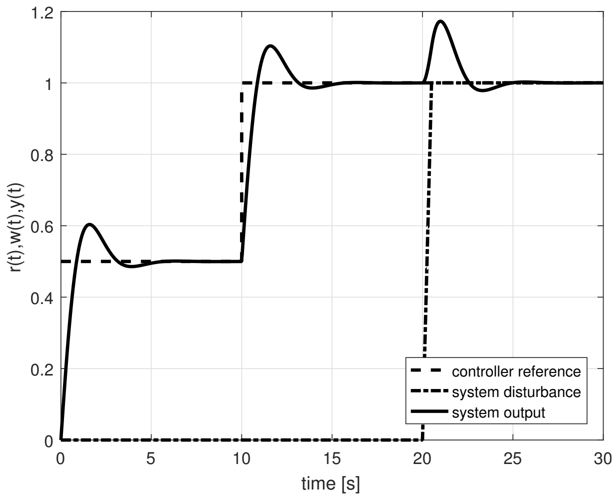


**Fig. 4.** Simulation results of output and disturbance input for robust controller designed in case 8 when $\dot{\xi} \in \langle -1, 1 \rangle$.

of $\theta_i \in \langle -1, 1 \rangle$ are:

$$
\begin{aligned}
A(\xi,\theta) = & \{\begin{bmatrix} -0.4 & 0 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 0.117 & 0 \\ 0 & 0 \end{bmatrix}\theta_1 \\
& + \begin{bmatrix} -0.28 & 0 \\ 0 & 0 \end{bmatrix}\theta_2\}\xi_1 + \\
& \{\begin{bmatrix} -0.5 & 0 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 0.14645 & 0 \\ 0 & 0 \end{bmatrix}\theta_1 \\
& + \begin{bmatrix} -0.35355 & 0 \\ 0 & 0 \end{bmatrix}\theta_2\}\xi_2
\end{aligned}
$$

$$
B(\xi,\theta) = B_w(\xi,\theta) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}\xi_1 + \begin{bmatrix} 0.5 \\ 0 \end{bmatrix}\xi_2.
$$

Gain scheduled parameters $\theta_i, i = 1, 2$ depend on the following relation: when output $y = 0, \theta_1 = -1, \theta_2 = -1$, $y = \pi/4, \theta_1 = 1, \theta_2 = -1$ and $y = \pi/2, \theta_1 = 1, \theta_2 = 1$.
The problem is to design the robust PID gain scheduled controller which ensures the closed-loop robustness properties, $L_2$ gain optimization, quadratic stability or parameter dependent quadratic stability for the case that maximal value of rate of gain scheduled parameter and uncertainties variations are $\dot{\theta} = 5/sec, \dot{\xi} = 0.5/s$.

The obtained GSC for QS:

$$\begin{aligned} R(s) =& -97.7091 - \frac{68.2203}{s} - 5.3048s \\ &+ \theta_1(-5.1508 - \frac{1.0318}{s} - 0.4874s) \\ &+ \theta_2(9.1057 + \frac{1.7470}{s} + 0.1324s) \end{aligned}$$

$L_2$ gain $\gamma = 0.1882$ and for the case of $\theta = 0$ maximal closed loop eigenvalue is $-0.7296$. The obtained GSC for PDQS:

$$\begin{aligned} R(s) =& 151.5784 + \frac{141.5023}{s} + 30.4907s \\ &+ \theta_1(32.3374 + \frac{14.9872}{s} + 8.6029s) \\ &+ \theta_2(-5.3939 + \frac{-20.5362}{s} - 6.9868s) \end{aligned}$$

$L_2$ gain for PDQS $\gamma = 0.198$ and for the case of $\theta = 0$ maximal closed loop eigenvalue is $-1.2322$. Simulation results for the second example are given in the Figure 5.

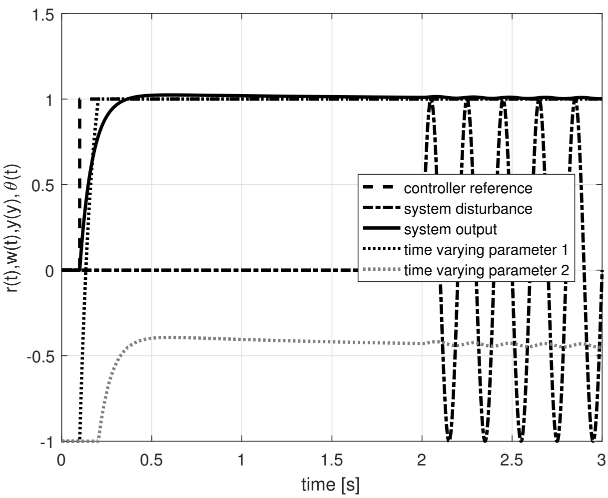


**Fig. 5.** Response of robust gain scheduled controller obtained for QS in example 2 to step change of reference value and variation of parameters $\theta_1$ and $\theta_2$.

**Example 4.3.** The following extended matrices of the system described by the equation (1) for the PI controller design have been used from the paper of Sato [7]:

$$\begin{aligned} A(\xi, \theta) &= A_0(\xi) + A_1(\xi)\theta_1 \\ B(\xi, \theta) &= B_w(\xi, \theta) = \dots \end{aligned}$$

$$A_{01}=\begin{bmatrix} -4 & 3 & 5 & 0\\ 0 & 7 & -5 & 0\\ 0.1 & -2 & -3 & 0\\ 1 & 1 & 0 & 0 \end{bmatrix} \qquad A_{02}=\begin{bmatrix} -4.4 & 3.3 & 5.5 & 0\\ 0 & 7.7 & -5.5 & 0\\ 0.1 & -1.8 & -3.3 & 0\\ 1 & 1 & 0 & 0 \end{bmatrix}$$

$$A_{11}=\begin{bmatrix} 1 & 0 & 1 & 0\\ 2 & 0 & -5 & 0\\ 2 & 5 & 1.5 & 0\\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad A_{12}=\begin{bmatrix} .8 & 0 & 0.8 & 0\\ 1.8 & 0 & -4.6 & 0\\ 1.8 & 4.5 & 1.75 & 0\\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$B_{01}=\begin{bmatrix} 0\\ 16\\ -10\\ 0 \end{bmatrix} \quad B_{02}=\begin{bmatrix} 0\\ 13.4\\ -12\\ 0 \end{bmatrix} \quad B_{11}=\begin{bmatrix} 1\\ -5\\ 3.5\\ 0 \end{bmatrix}$$

$$B_{12}^T=\begin{bmatrix} 0.8 & -4.5 & 3.15 & 0 \end{bmatrix} \quad C=\begin{bmatrix} 1 & 1 & 0 & 0\\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Gain scheduled parameter $\theta_1$ depends on the following relation: when $y=0, \theta_1=-1$ and $y=1, \theta_1=1$.
For the case of $\max(\dot{\theta})=50p.u./s, \max(\dot{\xi})=0.5p.u./s$ and $\rho * I \geq P(\xi,\theta)$ the problem is to design robust PID gain scheduled controller which will guarantee the $L_2$ gain performance. The following obtained results for two cases as QS and PDQS have been summarized as:

1. The gain scheduled controller transfer function obtained for QS:

$$\begin{aligned} R(s)= & -193.1135-\frac{70.992}{s}-99.8125s \\ & +(56.1446-\frac{11.73}{s}-7.8123s)\theta_1 \end{aligned}$$

$L_2$ gain $\gamma=0.5$ and for the case of $\theta=0$ maximal closed loop eigenvalue is $-0.4796$.

2. PENBMI solver has failed to solve the problem for PDQS.

The above example should be used as effective solution for the non ideal switching systems with arbitrarily switching algorithm, [14] due to large value of gain scheduled parameter rate $\max(\dot{\theta})=50p.u./s$ Simulation results for the third example are given in the Figure 6.

## 5. CONCLUSION

This paper analysis the linear uncertain parameter varying system with polytopic uncertainties and gives the new robust gain scheduled PID controller design procedure with $L_2$ gain performance. The main paper result, the novel controller design approach in Theorem 3.1, ensures that the plant uncertainties and gain scheduled parameters are convex in the obtained robust stability conditions. The obtained design procedure should be used under a mild modifications for robust controller design or robust switched controller design with arbitrarily switching algorithm and guarantees the $L_2$ gain performance for the designed robust controllers. Simulation results imply that the robust gain

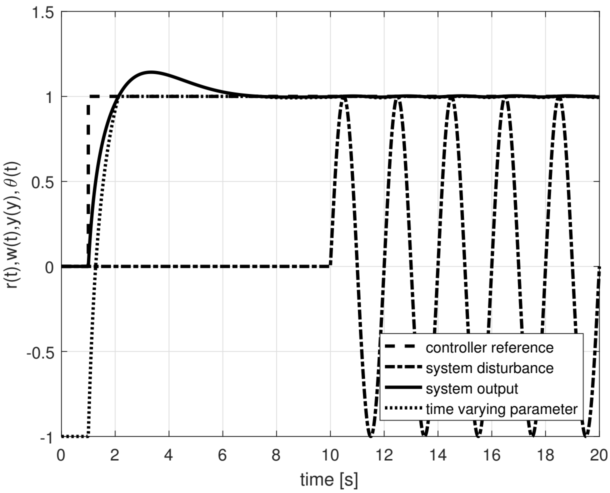


**Fig. 6.** Response of robust gain scheduled controller obtained by QS for example 3.

scheduled controller achieves better dynamic properties than the robust controller. The gain scheduled parameters uncertainty problem is very important from the viewpoint of the future research.

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