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UPPER BOUND ESTIMATION OF THE SPECTRAL ABSCISSA FOR SWITCHED LINEAR SYSTEMS VIA COORDINATE TRANSFORMATIONS

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In this paper, we develop computational procedures to approximate the spectral abscissa of the switched linear system via square coordinate transformations. First, we design iterative algorithms to obtain a sequence of the least μ_1 measure. Second, it is shown that this sequence is convergent and its limit can be used to estimate the spectral abscissa. Moreover, the stopping condition of Algorithm 1 is also presented. Finally, an example is carried out to illustrate the effectiveness of the proposed method.

Keywords: switched linear systems, matrix set measure, spectral abscissa, coordinate transformations

Classification: 93D20

Notation: Let $\mathbf{R}$ be the set of real numbers, and $\mathbf{N}$ be the set of nonnegative integers. For a positive integer k, denote $\bar{\mathbf{k}} = \{1, \ldots, k\}$. For two positive integers i and j, denote $\delta(i,j)$ the minimal positive integer such that $i = lj + \delta(i,j)$ for some nonnegative integer l. Let $\lambda(\mathbf{A})$ be the largest real part of the eigenvalues of matrices $A_i,\ i = 1, \ldots, m$. For given matrices $A, T \in \mathbf{R}^{n\times n}$ with $\operatorname{rank}(T) = n$, denote

$$\zeta_T A = T^{-1}AT.$$

An elementary matrix $T \in \mathbf{R}^{n\times n}$ is called

- of type 1 if $T = T(i,j)$, $\forall i, j \in \bar{\mathbf{n}}, j \neq i$, where $T(i,j)$ is obtained from the identity matrix I_n by interchanging rows i and j, i. e.,

$$T(i,j) = \begin{bmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 0 & & 1 & & \\ & & & \ddots & & & \\ & & 1 & & 0 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{bmatrix}.$$

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- of type 2 if $T = T_i(q)$, $\forall i \in \bar{\mathbf{n}}$, where $T_i(q)$ is obtained by multiplying the ith row of I_n by a nonzero constant q, i. e.,

$$T_i(q) = \begin{bmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & & & \\ & & & q & & & \\ & & & & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{bmatrix}.$$

- of type 3 if $T = T_i^j(q)$, $\forall\, i, j \in \bar{\mathbf{n}}, j \neq i$, where $T_i^j(q)$ is obtained from I_n by adding q times the ith row of I_n to the jth row, i. e.,

$$T_i^j(q) = \begin{bmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & & & \\ & & & \ddots & & & \\ & & q & & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{bmatrix}.$$

1. INTRODUCTION

In this paper, we address the performance analysis issue for continuous-time switched linear systems

$$\dot{x}(t) = A_{\sigma(t)}x(t), \quad x(t) \in \mathbf{R}^n, \tag{1}$$

where $\sigma(t)$ is a switching signal taking values from the index set $\{1, \ldots, m\}$ and $A_i \in \mathbf{R}^{n\times n}$ are real constant matrices for $i \in \{1, \ldots, m\}$.

In recent decades, much effort has been devoted to investigating the stability of the switched linear systems [3, 5, 6, 7, 11, 19, 20]. In particular, asymptotic stability under arbitrary switching, named guaranteed stability, has attracted enormous interest. So far, most available results on this problem were based on the existence of a common Lyapunov function [10]. Current studies in this field mainly concentrate on the construction of the quadratic Lyapunov functions [8, 12, 19]. Due to the fact that quadratic functions are not universal for characterizing guaranteed stability of switched linear systems, various non-quadratic functions have to be sought, which makes effective computations intractable.

For system (1), it was established that, there exist a positive real number c and a non-negative integer $k \leq n-1$ such that

$$\|x(t)\| \leq c(1 + t^k)e^{\varrho t}\|x(0)\|$$

for any system solution $x(\cdot)$, where ϱ is the spectral abscissa of matrix set $\{A_1, \cdots, A_m\}$ [4, 17]. Therefore, spectral abscissa gives a quantitative measure on the asymptotic

performance of switched systems. It was established in [1, 18] that the spectral abscissa is equal to the least possible common matrix set measure of the subsystems, which provides an algebraic approach to characterizing the spectral abscissa. Several least-measure-based computational schemes have recently been proposed along this approach. In [21], an algorithm based on the sum-of-squares (SOS) technique was used to approximate the system's least matrix set measure. However, it fails when the eigenvalues of subsystems are not concentrate or close to the imaginary axis. Another possible way is to compute the least common μ_1 measure of the transformed matrix set via extended algebraic transforms. In our previous work [9], square coordinate transformation has been proposed to get the least μ_1 measure via the fact that any invertible square matrix can be expressed as a product of elementary matrices.

In this paper, by examining the property of the least μ_1 measure obtained by coordinate transformations of type 2, a revised algorithm is presented to calculate the least μ_1 measure via iteratively performing the transformations on some particular columns only, which reduce the iterative steps and computational cost compared with the row-by-row iteration in [9]. Furthermore, by constructing a convex function and applying the subgradient inequality, a stopping condition of this algorithm is proposed. On the other hand, the least μ_1 measure obtained by coordinate transformations of type 2 can be solved by linear programming and bisection method. For the coordinate transformations of type 3, a new algorithm is designed to search for the least μ_1 measure which can be seen as the upper bound estimation of the spectral abscissa.

2. PRELIMINARIES

Definition 2.1. For switched linear system (1), the spectral abscissa is

$$\varrho(\mathbf{A}) = \limsup_{t\to\infty, x(\cdot)\in\Lambda} \frac{\ln \|x(t)\|}{t},$$

where $\mathbf{A} = \{A_1, \ldots, A_m\}$, and Λ is the set of non-trivial solutions of system (1).

Note that the spectral abscissa in Definition 2.1 is also known as the maximum Lyapunov exponent or largest divergence rate in the literature [4, 5, 14, 15, 16].

Definition 2.2. (Sun and Ge [20]) For any vector norm $\|\cdot\|$ in $\mathbf{R}^n$ and a given set of matrices $\mathbf{A} = \{A_1, A_2, \ldots, A_m\}$, the induced matrix set measure is defined as

$$\mu_{\|\cdot\|}(\mathbf{A}) = \max\left\{\mu_{\|\cdot\|}(A_1), \ldots, \mu_{\|\cdot\|}(A_m)\right\},$$

where $\mu_{\|\cdot\|}(A_i)$ is a matrix measure formulated as

$$\mu_{\|\cdot\|}(A_i) = \limsup_{\tau\to 0^+, 0\neq \boldsymbol{z}\in\mathbf{R}^n} \frac{\|\boldsymbol{z} + \tau A_i \boldsymbol{z}\| - \|\boldsymbol{z}\|}{\tau\|\boldsymbol{z}\|}, i = 1, \ldots, m.$$

For example, for a matrix $A = (a_{ij})_{n\times n}$, the induced matrix measure of norm ℓ_1, denoted by $\mu_1(A)$ measure for simplicity, is the maximum of the column sum

$$\mu_1(A) = \max_{j\in\bar{\mathbf{n}}} \psi_j(A), \tag{2}$$

where the jth column sum is defined as

$$\psi_j(A) = a_{jj} + \sum_{i=1, i\neq j}^{n} |a_{ij}|,\ j \in \bar{\mathbf{n}}. \tag{3}$$

Definition 2.3. (Sun and Ge [20]) For a given matrix set $\mathbf{A} = \{A_1, \ldots, A_m\}$, the least measure is defined as

$$\mu_*(\mathbf{A}) = \inf_{\|\cdot\| \in \mathbf{V}} \mu_{\|\cdot\|}(\mathbf{A}),$$

where $\mathbf{V}$ is the set of vector norms.

For any norm $\|\cdot\|$, the following inequality holds true,

$$\|\boldsymbol{x}(t)\| \leq e^{\mu_{\|\cdot\|}(\mathbf{A})t}\|\boldsymbol{x}(0)\|,\ \forall\ \boldsymbol{x}(\cdot) \in \Lambda,\ t \geq 0,$$

which indicates that $\mu_{\|\cdot\|}(\mathbf{A})$ is an upper bound of the spectral abscissa, *i. e.*, $\mu_{\|\cdot\|}(\mathbf{A}) \geq \varrho(\mathbf{A})$. Furthermore, the following lemma establishes that the least matrix set measure is exactly the spectral abscissa.

Lemma 2.4. (Barabanov [1]) For any matrix set $\mathbf{A}$, we have

$$\mu_*(\mathbf{A}) = \varrho(\mathbf{A}).$$

The significance of the next lemma, an extension of Theorem 4.1 in [2], is that the estimation of the least measure is made at an arbitrary precision by the μ_1 measure of the transformed matrix set.

Lemma 2.5. For any matrix set $\mathbf{A}$ and any $\epsilon > 0$, there exist a natural number $r \geq n$, a matrix $T_{n\times r}$ of rank n, and $r \times r$ matrices H_i, such that

$$A_i T = T H_i, \qquad i = 1, 2, \ldots, m,$$

and

$$\mu_1(H_1, \ldots, H_m) < \mu_*(\mathbf{A}) + \epsilon.$$

Remark 2.6. Note that the inequality $\mu_*(\mathbf{A}) \leq \mu_1(H_1, \ldots, H_m)$ is obvious, because the size of matrices H_i is greater than or equal to that of A_i.

3. COMPUTATION OF THE LEAST μ_1 MEASURE

From the characteristic property and Definition 2.2, it is easy to verify that the matrix set measure obtained after each transformation of type 1 is invariant. Therefore, in this section, based on the result obtained in [9], we concentrate on searching for the proper coordinate transformations of type 2 and 3, such that the spectral abscissa can be approximated by the least μ_1 measure of the transformed matrix set. For clarity, we focus on two-form switched systems, *i. e.*, $\mathbf{A} = \{A_1, A_2\}$ with $A_1 = (a_{ij})_{n\times n}, A_2 = (b_{ij})_{n\times n}$.

3.1. Coordinate transformations of type 2

In this subsection, we aim to get the least μ_1 measure obtained by a series of transformations of type 2. That is, to get the least μ_1 measure of matrices $\zeta_T A_1$ and $\zeta_T A_2$,

$$\begin{aligned}\inf_T \mu_1(\zeta_T A_1, \zeta_T A_2) &= \inf_T \max_{j\in\bar{\mathbf{n}}}\{\psi_j(\zeta_T A_1), \psi_j(\zeta_T A_2)\} \\ &= \inf_T \max_{j\in\bar{\mathbf{n}}} \left\{ \sum_{s=1,s\neq j}^{n} |\frac{z_j}{z_s} a_{sj}| + a_{jj}, \sum_{s=1,s\neq j}^{n} |\frac{z_j}{z_s} b_{sj}| + b_{jj} \right\}, \quad (4)\end{aligned}$$

where $T = \mathrm{diag}(z_1, \ldots, z_n)$ with $z_i \neq 0, i = 1, \ldots, n$ being free variables to be optimized. For more details of Problem (4), the reader is referred to [9]. To solve Problem (4), we start with the following definition.

Definition 3.1. The index of the minimum column sum, $k \in \bar{\mathbf{n}}$, of matrix set $\{A_1, A_2\}$ is the index of a column sum of $A_i, i \in \{1, 2\}$ with

$$\max_{i=1,2} \psi_k(A_i) = \min_{j\in\bar{\mathbf{n}}} \max_{i=1,2} \psi_j(A_i).$$

Next, we propose the following algorithm for the computation of the optimal value h^* to Problem (4).

Algorithm 1: Iterative procedure for solving Problem (4)

Step 1. Set $i := 1$, $P_1^1 := A_1$ and $P_2^1 := A_2$.

Step 2. Find the index of the minimum column sum, k, of matrix set $\{P_1^i, P_2^i\}$. Set $T_i(z_i) := \mathrm{diag}(1, \ldots, z_i, \ldots, 1)$ with the kth diagonal element being z_i.

Step 3. Solve the problem

$$\min_{z_i\neq 0} \max_{j\in\bar{\mathbf{n}}} \{\psi_j(\zeta_{T_i(z_i)} P_1^i), \psi_j(\zeta_{T_i(z_i)} P_2^i)\}, \quad (5)$$

to get the optimal value h_i and obtain the corresponding minimizer $z_i = d_i$.

Step 4. Update the matrices

$$P_1^{i+1} := \zeta_{T_i(d_i)} P_1^i, \qquad P_2^{i+1} := \zeta_{T_i(d_i)} P_2^i. \quad (6)$$

Set $i := i + 1$ and then go to Step 2.

Remark 3.2. In Step 2, the index of the minimum column sum, k, of matrix set is introduced to guarantee that the ith transformation is performed on the minimum column sum of matrices P_1^i and P_2^i for each $i \in \mathbf{N}$. Therefore, for $i = 1$, by solving Problem (5), we obtained the optimal value h_1 and the corresponding d_1. And then, we can obtain the transformed matrix set $\{P_1^2, P_2^2\}$ and get the index of the minimum column sum.

After that, by Step 2, we derive the transformation matrix T_2 and resolve Problem (5) to obtain h_2 and d_2. Using a similar idea, it follows that a sequence of the minimum of matrix set measure $\{h_i\}$ is obtained.

We now present a slightly different statement of Theorem 3.3 in [9]. The proof of the following theorem is similar as Theorem 3.3 in [9] and then omitted.

Theorem 3.3. Suppose that each column and row of P_1^i or P_2^i has at least one nonzero off-diagonal element for $i = 1, 2, \ldots$. The optimal value, h_i, to Problem (5) is

$$h_i = \max\{\psi_k(\zeta_{T_i(d_i)}P_1^i), \psi_k(\zeta_{T_k(d_i)}P_2^i)\} \tag{7}$$

with k defined in Step 2, $d_i = \max_{y\in\mathcal{W}_i} |y|$, where

$$\mathcal{W}_i = \{y : \exists\ (j, r) \in (\bar{\mathbf{n}}, \bar{\mathbf{2}}),\ j \neq k,\ \text{s.t. } \psi_j(\zeta_{T_i(y)}P_r^i) = \psi_k(\zeta_{T_i(y)}P_s^i)\},$$

and s is determined by the following Rules. For $z_i \in (0, +\infty)$,

1. if
$$\frac{d\psi_k(\zeta_{T_i(z_i)}P_1^i)}{dz_i} = \frac{d\psi_k(\zeta_{T_i(z_i)}P_2^i)}{dz_i},$$
then s is the index of the matrix whose column sum is $\max\{\psi_k(\zeta_{T_i(z_i)}P_1^i), \psi_k(\zeta_{T_i(z_i)}P_2^i)\}$.

2. get the value $\tilde{y}$ such that $\psi_k(\zeta_{T_i(\tilde{y})}P_1^i) = \psi_k(\zeta_{T_i(\tilde{y})}P_2^i)$. If
$$\psi_k(\zeta_{T_i(|\tilde{y}|)}P_1^i) = \max_{j\in\bar{\mathbf{n}}}\{\psi_j(\zeta_{T_i(|\tilde{y}|)}P_1^i), \psi_j(\zeta_{T_i(|\tilde{y}|)}P_2^i)\},$$
then s is the index of the matrix whose column sum's derivative is $\min\{\psi_k'(\zeta_{T_i(z_i)}P_1^i), \psi_k'(\zeta_{T_i(z_i)}\ P_2^i)\}$; otherwise, s is the index of the matrix whose column sum's derivative is $\max\{\psi_k'(\zeta_{T_i(z_i)}P_1^i), \psi_k'(\zeta_{T_i(z_i)}P_2^i)\}$.

Remark 3.4. The hypothesis of Theorem 3.3 is reasonable. In fact, since $d_i \neq 0$, the hypothesis holds true if each column and row of given matrices A_1 or A_2 have at least one nonzero off-diagonal element. Theorem 3.3 describes a convenient computation method for the minimum of matrix set measure after each transformation. It doesn't need to solve the equation $\psi_j(\zeta_{T_i(z_i)}P_r^i) - \max_{l=1,2}\psi_k(\zeta_{T_i(z_i)}P_l^i) = 0, \forall j \in \bar{\mathbf{n}}\backslash\{k\}, r \in \{1, 2\}$, where the maximum function here is piecewise. Figure 1 shows a graphical explanation of Theorem 3.3. Since $\psi_1(\zeta_{T_1(|\tilde{y}|)}P_1^1) < \max_{j\in\bar{\mathbf{n}}}\{\psi_j(\zeta_{T_1(|\tilde{y}|)}P_1^1), \psi_j(\zeta_{T_1(|\tilde{y}|)}P_2^1)\}$, according to Rule 2, get the zeros, y_s, of equations $\psi_j(\zeta_{T_1(z_1)}P_r^1) - \psi_1(\zeta_{T_1(z_1)}P_1^1) = 0, r \in \{1, 2\},\ j = 2, 3$. Then we have $d_1 = \max_{s=1,2,3,4}|y_s|$.

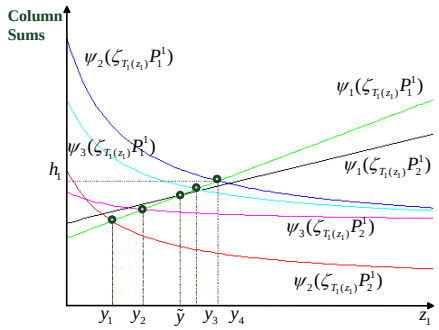

Fig. 1. A graphical explanation of Theorem 3.3.

In the next proposition, we prove that the sequence of minimum of matrix set measure, $\{h_1, h_2, \cdots\}$, generated by Algorithm 1 is convergent.

Proposition 3.5. Algorithm 1 produces a sequence of the minima of matrix set measure $\{h_i\}$ and it is decreasing and convergent. The sequence admits either of the following properties:

(i) There is a natural number $i^\sharp$, such that $h_{i^\sharp} = h_{i^\sharp+1}$, and $d_{i^\sharp} \neq 1$.

(ii) The sequence $\{h_i\}$ is convergent to h^*. Moreover, the sequence reaches h^* if and only if there is a natural number i^*, such that $d_{i^*} = 1$.

P r o o f. The proof of decreasing and convergence is exactly the same as that of Proposition 3.6 in [9]. From the proof of Proposition 3.6 in [9], it can be seen that the sequence $\{h_i\}$ reaches h^* if and only if the following equalities hold true,

$$\max_{r=1,2} \psi_s(P_r^{i^*}) = \max_{r=1,2} \psi_j(P_r^{i^*}), \ \forall s, j \in \bar{\mathbf{n}}.$$

Obviously, this is equivalent to $d_{i^*} = 1$.

The proof of the statement (i) and the former part of statement (ii) is similar as that in Proposition 3.6 in [9], and is omitted. □

Remark 3.6. Proposition 3.5 derives that Algorithm 1 can get the global optimal value to Problem (4) in case (ii). Compared with that in [9], the transformations are performed on the minimum of the column sums and then the computation load is reduced.

From the above result, Algorithm 1 is obtained to approximate the spectrum abscissa of system **A** via coordinate transformations of type 2. However, it cannot tell the number of steps we needed to attend a sufficiently small $|h_i - h^*|$ a prior. Generally, a natural criterion is to stop this algorithm when $h_i - h^* \leq \epsilon$, where ϵ is an absolute error tolerance. The following is devoted to getting the stopping rule which is inspired by [13].

Let $\boldsymbol{d} = [d_1^*, \ldots, d_n^*]^T$ is a solution to Problem (4) obtained by Algorithm 1 and $T^* = \mathrm{diag}(d_1^*, \ldots, d_n^*)$ with $d_i^*, i = 1, \cdots, n$ being the element of $\boldsymbol{d}$. Denote $P_1^* = \zeta_{T^*} A_1 = (p_{ij})_{n\times n}$ and $P_2^* = \zeta_{T^*} A_2 = (q_{ij})_{n\times n}$. Let

$$\varpi(\boldsymbol{z}) = \mu_1(A_1 T - T P_1^*, A_2 T - T P_2^*),$$

where $T = \mathrm{diag}(z_1, \cdots, z_n)$ and $\boldsymbol{z} = (z_1, \ldots, z_n)^T$ are defined in Problem (4). Simple calculation yields that the function $\varpi(\boldsymbol{z})$ is convex, continuous and differentiable almost everywhere.

Let $\prod_{i=1}^{l} d_i(k_j)$ represent the product of all the d_i which satisfies that $k = j$ for all $i \leq l$. Let $T^l = \prod_{i=1}^{l} T_i(d_i)$ and $\boldsymbol{d}^l = (\prod_{i=1}^{l} d_i(k_1), \cdots, \prod_{i=1}^{l} d_i(k_n))^T$.

Lemma 3.7. The sequence $\{\boldsymbol{d}^l\}$ generated by Algorithm 1 is bounded, that is, there exists a positive constant c_0, such that $\|\boldsymbol{d}^l\|_2 \leq c_0$.

P r o o f. Without loss of generality, suppose by contradiction that $\prod_{i=1}^{l} d_i(k_j) \to +\infty$. Therefore, there exists at least one $d_i(k_j)$, such that $|d_i(k_j)| \to +\infty$. It follows that $h_i \to +\infty$, which contradicts the decrease of sequence $\{h_i\}$. Therefore, $\{\boldsymbol{d}^l\}$ is bounded. □

Proposition 3.8. For any given $\epsilon > 0$, we have

$$\varpi(\boldsymbol{d}^{l_\sharp}) \leq \epsilon,$$

where $l_\sharp = \frac{2cc_0^2}{\epsilon^2}$, $c = n(\mu_1(A_1, A_2) + \max_{1\leq j\leq n}\{|a_{jj}|, |b_{jj}|\})^2$.

P r o o f. From (10) in [9], it is clear that, for every $i \in \bar{\mathbf{n}}$,

$$\begin{aligned}
\frac{\partial \varpi(\boldsymbol{z})}{\partial z_i^+}|_{\boldsymbol{z}=\boldsymbol{d}^l} &\leq \max_{1\leq s\leq n}\left\{\frac{\partial \sum_{j=1,j\neq s}^{n} |a_{sj}z_j - p_{js}z_s|}{\partial z_i^+}|_{\boldsymbol{z}=\boldsymbol{d}^l}, \right. \\
&\qquad \left. \frac{\partial \sum_{j=1,j\neq s}^{n} |b_{sj}z_j - q_{js}z_s|}{\partial z_i^+}|_{\boldsymbol{z}=\boldsymbol{d}^l}\right\}, \\
&\leq \max\left\{\sum_{j=1,j\neq i}^{n} |a_{ij}|, \max_{1\leq s\leq n, s\neq i}\{|p_{is}|\}, \sum_{j=1,j\neq i}^{n} |b_{ij}|, \max_{1\leq s\leq n, s\neq i}\{|q_{is}|\}\right\}, \\
&\leq \mu_1(A_1, A_2) + \max_{1\leq j\leq n}\{|a_{jj}|, |b_{jj}|\}.
\end{aligned}$$

Similarly, we could prove that

$$\frac{\partial \varpi(\boldsymbol{z})}{\partial z_i^-}|_{\boldsymbol{z}=\boldsymbol{d}^l} \geq -\mu_1(A_1, A_2) - \max_{1\leq j\leq n}\{|a_{jj}|, |b_{jj}|\}, \forall i \in \bar{\mathbf{n}}.$$

Thus, for any $l \in \mathbf{N}$, we have

$$||\boldsymbol{g}_l||_2^2 \leq n(\mu_1(A_1, A_2) + \max_{1\leq j\leq n}\{|a_{jj}|, |b_{jj}|\})^2 \triangleq c,$$

where $\boldsymbol{g}_l$ is any subgradient of $\varpi(\boldsymbol{z})$ at $\boldsymbol{d}^l$.

Secondly, Lemma 3.7, together with the fact $d_1 > 0, 1 \leq d_i < +\infty, i = 2, 3, \ldots,$ implies that $\{\boldsymbol{d}^l\}$ is convergent to $\boldsymbol{d}^*$.

It is clear that

$$\begin{aligned} & ||\boldsymbol{d}^{l+1} - \boldsymbol{d}^*||_2^2 \leq ||\boldsymbol{d}^l - \boldsymbol{d}^*||_2^2 \\ \leq \; & ||\boldsymbol{d}^l - \boldsymbol{d}^* - \boldsymbol{g}_l w||_2^2 + ||\boldsymbol{g}_l w||_2^2 \\ = \; & ||\boldsymbol{d}^l - \boldsymbol{d}^*||_2^2 - 2(\boldsymbol{d}^l - \boldsymbol{d}^*)^T \boldsymbol{g}_l w + 2||\boldsymbol{g}_l w||_2^2, \end{aligned} \tag{8}$$

where w is an arbitrary positive constant. The subgradient inequality implies that

$$0 = \varpi(\boldsymbol{d}^*) \geq \varpi(\boldsymbol{d}^l) + \boldsymbol{g}_l^T(\boldsymbol{d}^* - \boldsymbol{d}^l). \tag{9}$$

It follows from (9) and (8) that

$$||\boldsymbol{d}^{l+1} - \boldsymbol{d}^*||_2^2 \leq ||\boldsymbol{d}^l - \boldsymbol{d}^*||_2^2 - 2\varpi(\boldsymbol{d}^l)w + 2||\boldsymbol{g}_l w||_2^2. \tag{10}$$

Thirdly, we are ready to prove $\varpi(\boldsymbol{d}^{l_\sharp}) \leq ||\boldsymbol{g}_{l_\sharp}||_2^2 w + \frac{\epsilon}{2}$ when $l_\sharp = \frac{c_0^2}{w\epsilon}$. Assume by contradiction that $\varpi(\boldsymbol{d}^{l_\sharp}) > ||\boldsymbol{g}_{l_\sharp}||_2^2 w + \frac{\epsilon}{2}$. According to inequality (10), recursively applying inequality (9), we have

$$||\boldsymbol{d}^{l_\sharp+1} - \boldsymbol{d}^*||_2^2 \leq ||\boldsymbol{d}^1 - \boldsymbol{d}^*||_2^2 - l_\sharp w\epsilon.$$

It implies that $l_\sharp w\epsilon \leq ||\boldsymbol{d}^1 - \boldsymbol{d}^*||_2^2$ which is not true since $l_\sharp = \frac{c_0^2}{w\epsilon}$.

Finally, taking $w = \frac{\epsilon}{2c}$ derives the result. □

Base on Proposition 3.8, we can establish the following result which addresses the question: by how many iteration steps one should take to guarantee that the minimum of matrix set measure h_i is good enough to approximate the value h^*?

Theorem 3.9. For any given $\epsilon > 0$, we have

$$h_{l_\sharp} \leq \mu_1(P_1^*, P_2^*) + \epsilon, \tag{11}$$

where $l_\sharp = \frac{2cc_0^2}{\epsilon^2}$.

P r o o f . It is easy to verify that

$$\begin{aligned} & \mu_1(\zeta_{T^l} A_1) - \mu_1(P_1^*) \\ \leq \; & \max_{1\leq j\leq n} \left\{ (a_{jj} - p_{jj}) + \sum_{i=1, i\neq j}^{n} \left| \frac{a_{ij} \prod_{i=1}^{l} d_i(k_j)}{\prod_{i=1}^{l} d_i(k_i)} - p_{ij} \right| \right\} \\ = \; & \mu_1((T^l)^{-1}(A_1 T^l - T^l P_1^*)) \\ \leq \; & \mu_1((T^l)^{-1}) \mu_1(A_1 T^l - T^l P_1^*). \end{aligned}$$

The last inequality holds true since $(T^l)^{-1}$ is a diagonal matrix. From Proposition 3.8, for $l = l_\sharp = \frac{2cc_0^2}{\epsilon^2}$, we have

$$\mu_1(\zeta_{T^{l_\sharp}} A_1) - \mu_1(P_1^*) \leq \frac{\epsilon}{\mu_1(T^{l_\sharp})}. \tag{12}$$

By exploiting a similar idea, we have

$$\mu_1(\zeta_{T^{l_\sharp}} A_2) - \mu_1(P_2^*) \leq \frac{\epsilon}{\mu_1(T^{l_\sharp})}. \tag{13}$$

Since $\mu_1(T^{l_\sharp}) \geq 1$, we have

$$\mu_1(\zeta_{T^{l_\sharp}} A_i) - \mu_1(P_i^*) \leq \epsilon, i = 1, 2. \tag{14}$$

This implies that, $h_{l_\sharp} - \mu_1(P_1^*, P_2^*) \leq \epsilon$. □

Remark 3.10. The above theorem indicates that we do have an upper bound of the steps needed to get the least μ_1 matrix set measure with the accuracy we asked for. Obviously, the method to get this stopping condition, *i. e.*, Proposition 3.8 and Theorem 3.9, can also be used to obtain the stop condition of the corresponding algorithm in [9].

Remark 3.11. On the other hand, Problem (4) is equivalent to $\inf_{T^{-1}} \mu_1(\zeta_{T^{-1}} A_1, \zeta_{T^{-1}} A_2)$ for the special structure of the transformation matrix T. Therefore, Problem (4) can be efficiently solved by linear programming tools. In fact, we consider the following problem

$$\begin{aligned} &\min \ \varsigma \\ &\text{s.t.} : a_{jj} z_j + \sum_{s=1, s\neq j}^{n} |a_{sj}| z_s \leq \varsigma z_j, j = 1, \ldots, n, \\ &\quad b_{jj} z_j + \sum_{s=1, s\neq j}^{n} |b_{sj}| z_s \leq \varsigma z_j, j = 1, \ldots, n, \\ &\quad z_s > 0, s = 1, \ldots, n. \end{aligned} \tag{15}$$

For each fixed $\varsigma > 0$, we have a linear programming problem to check whether such numbers $z_s, s = 1, \ldots, n$ exist. Then by bisection method we can approximate the minimal ς with this property, which gives the solution to Problem (4). And it is already known that the computational cost of linear programming is $O(mn^2)$ with m and n being the number of the constrain inequalities and unknown parameters, respectively.

3.2. Coordinate transformations of type 3

In [9], for a given set of matrices $\mathbf{A}$, a calculation procedure is presented to check the least μ_1 measure obtained by the coordinate transformation matrix

$$T = \prod_{k=1}^{n} \prod_{l=1, l\neq k}^{n} T_k^l(z_k^l).$$

That is, solve the problem

$$\inf_T \mu_1(\zeta_T A_1, \zeta_T A_2) = \inf_T \max_{j \in \bar{\mathbf{n}}} \{\psi_j(\zeta_T A_1), \psi_j(\zeta_T A_2)\}, \tag{16}$$

where $z_k^l, k, l \in \bar{\mathbf{n}}, l \neq k$, are free parameters to be determined.

It is recognized that the calculation procedure is explicitly based on repeatly solving the problem

$$\min_{z\in\mathbf{R}}\max_{j\in\bar{\mathbf{n}}}\{\psi_j(\zeta_{T_k^l(z)}P_1),\psi_j(\zeta_{T_k^l(z)}P_2)\},\forall k,l\in\bar{\mathbf{n}},l\neq k, \tag{17}$$

with P_1,P_2 are given matrices. In the following, a new iterative algorithm is proposed to get the optimal solution of Problem (17).

Algorithm 2: The iterative procedure for solving Problem (17)

Step 1. Let $r:=0$ and $\mathbf{B}_1=\mathbf{B}_2:=\bar{\mathbf{n}}$. Set

$$\bar{\gamma}_i(z_i):=\max_{j\in\bar{\mathbf{n}}}\{\psi_j(\zeta_{T_k^l(z_i)}P_1),\psi_j(\zeta_{T_k^l(z_i)}P_2)\},z_i\in\mathbf{R}.$$

Step 2. Set

$$\begin{aligned}\mathbf{D}_1 \;:=\; &\{c_s|c_s \text{ is a point of nondifferentiability of } \psi_j(\zeta_{T_k^l(z_i)}P_m) \text{ and}\\ &\psi_j(\zeta_{T_k^l(c_s)}P_m)=\bar{\gamma}_i(c_s),\forall(j,m)\in(\bar{\mathbf{n}},\bar{\mathbf{2}})\}.\end{aligned}$$

$$\begin{aligned}\mathbf{D}_2 \;:=\; &\{y_s|\psi_k'(\zeta_{T_k^l(z_i)}P_m)|_{z_i=y_s}=0,\ \psi_k''(\zeta_{T_k^l(z_i)}P_m)|_{z_i=y_s}>0,\text{ and}\\ &\psi_l(\zeta_{T_k^l(y_s)}P_m)=\bar{\gamma}_i(y_s),m\in\{1,2\}\}.\end{aligned}$$

Let $c_{s_0}:=\arg\min_{c_s\in\mathbf{D}_1\cup\mathbf{D}_2}\bar{\gamma}_i(c_s)$. If $\bar{\gamma}_i(0)<\bar{\gamma}_i(c_{s_0})$, set $y_0:=0$; Otherwise, set $y_0:=c_{s_0}$.

Step 3. If $\bar{\gamma}_i(y_i)>\max_{m=1,2}\psi_k(\zeta_{T_k^l(y_i)}P_m)$ and $0\in\partial\bar{\gamma}_i(y_i)$, then Stop; otherwise, find a pair $(j_0,m_0)\in(\mathbf{B}_{m_0},\bar{\mathbf{2}})$ such that $\psi_{j_0}(\zeta_{T_k^l(y_i)}P_{m_0})=\bar{\gamma}_i(y_i)$.

Step 4. Get the set

$$\begin{aligned}\mathbf{W}_r \;:=\; &\{q_s|\psi_{j_0}(\zeta_{T_k^l(q_s)}P_{m_0})=\psi_{j_1}(\zeta_{T_k^l(q_s)}P_{m_1})=\bar{\gamma}_i(q_s),\\ &\forall(j_1,m_1)\in(\mathbf{B}_{m_1},\bar{\mathbf{2}}),(j_1,m_1)\neq(j_0,m_0)\}.\end{aligned} \tag{18}$$

Find $y_r:=\arg\min_{q_s\in\mathbf{W}_r}\bar{\gamma}_i(q_s)$. If $m_0=1$, set $\mathbf{B}_1:=\mathbf{B}_1\backslash\{j_0\}$; Otherwise, $\mathbf{B}_2:=\mathbf{B}_2\backslash\{j_0\}$.

Step 5. If $k\notin\mathbf{B}_1\cup\mathbf{B}_2$ and $\min_{s=1,\cdots,i-1}\bar{\gamma}_i(y_s)\leq\bar{\gamma}_i(y_i)$, and then STOP; otherwise, set $i:=i+1$, and go to Step 3.

Theorem 3.12. Procedure 3 terminates after finite number of iterations and produces the optimal value, h_i, to Problem (17).

Proof. If

$$\bar{\gamma}_i(y_i)>\max_{m=1,2}\psi_k(\zeta_{T_k^l(y_s)}P_m)\quad\text{ and }\quad 0\in\partial\bar{\gamma}_i(y_i),$$

y_i is a local minimum. Suppose, for instance,

$$\partial_-\bar{\gamma}_i(y_i) = \partial_-\psi_{j_0}(\zeta_{T_k^l(y_i)}P_{r_0}), \quad \partial_+\bar{\gamma}_i(y_i) = \partial_+\psi_{j_1}(\zeta_{T_k^l(y_i)}P_{r_1}).$$

From the expressions of $\psi_j(\zeta_{T_k^l(z_i)}P_r, (j,r) \in (\bar{\mathbf{n}}, \bar{\mathbf{2}})$ and the fact $j_0, j_1 \neq k$, we obtain that $\psi_{j_0}(\zeta_{T_k^l(z_i)}P_{r_0})$ is decreasing when $z_i \in (-\infty, y_i]$ and $\psi_{j_1}(\zeta_{T_k^l(z_i)}P_{r_1})$ is increasing when $z_i \in [y_i, +\infty)$. Hence, the following inequalities hold true,

$$\bar{\gamma}_i(z_i) \geq \psi_{j_0}(\zeta_{T_k^l(z_i)}P_{r_0}) \geq \psi_{j_0}(\zeta_{T_k^l(y_i)}P_{r_0}), \quad \forall\ z_i \in (-\infty, y_i],$$

and

$$\bar{\gamma}_i(z_i) \geq \psi_{j_1}(\zeta_{T_k^l(z_i)}P_{r_1}) \geq \psi_{j_1}(\zeta_{T_k^l(y_i)}P_{r_1}), \quad \forall\ z_i \in [y_i, +\infty).$$

Therefore, y_i is a solution to Problem (17).

Otherwise, if $\bar{\gamma}_i(y_i) \leq \max_{m=1,2}\psi_k(\zeta_{T_k^l(y_i)}P_m)$ or $0 \notin \partial\bar{\gamma}_i(y_i)$, by checking the rules in Steps 3 & 4, we get $y_r = \arg\min_{q_s \in \mathbf{W}_r}\bar{\gamma}_i(q_s)$. After that, the function $\psi_{j_0}(\zeta_{T_k^l(y_i)}P_{m_0})$ is not taken into account from further iterations. If $\min_{s=1,\cdots,i-1}\bar{\gamma}_i(y_s) \leq \bar{\gamma}_i(y_i)$ and $k \notin \mathbf{B}_1 \cup \mathbf{B}_2$, from the continuousness of the function $\bar{\gamma}_i(z_i)$ and the convex of the functions $\psi_j(\zeta_{T_k^l(z_i)}P_r), j \neq k, r = 1, 2$, it can be seen that, y_{i-1} is a solution to Problem (17). Otherwise, after at most $2n-1$ iterations, there is only one function left, and so we can get $\arg\min_{y_s \in \{y_0,\ldots,y_{2n-1}\}}\bar{\gamma}_i(y_s)$, which is a solution to Problem (17). □

Remark 3.13. It is straightforward to see that Algorithm 2 is constructed which calculates, in a finite number of steps, the minimum of matrix set measure after each transformation. As shown in Figure 2, from Algorithm 2, we obtain y_0, y_1 and y_2, and then from Step 5, the minimum of matrix set measure h_1 can be obtained.

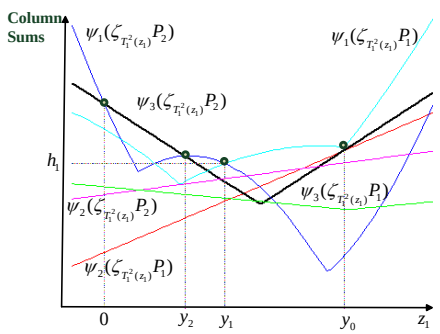

Fig. 2. A graphical explanation of Theorem 3.12.

Remark 3.14. According to Proposition 2 in [9], the sequence of the minimum of matrix set measure $\{h_i\}$ is convergent. For the matrix T defined in Problem (16), noting that neither the function $\mu_1(\zeta_T A_1, \zeta_T A_2)$ nor $\mu_1(A_1T - TP_1^*, A_2T - TP_2^*)$ is convex, the method used to obtain the needed steps presented in Proposition 3.8 fails. It seems notoriously hard to have a priori knowledge of the number of iteration steps to ensure that $h_i - h^*$ is smaller than our desired tolerance. We leave it open for our future research.

Remark 3.15. We next estimate the computational cost of the algorithms for the extended systems. Let $\mathbf{B}$ denote the set of nonzero diagonal elements of matrices $\mathbf{A}$. Let $\eta = \min_{x\in\mathbf{B}} |x|$ and $C = n(\mu_1(\mathbf{A}) + \max_{x\in\mathbf{B}} |x|)^2$. It is a well-known fact that the number of Newton iterations required to solve a quadratic equation within an accuracy of ϵ is not more than $\log_2[\frac{\ln M\epsilon}{\ln M\epsilon_0}]$, where $M = \frac{\sqrt{C}}{2\eta\sqrt{n}}$, $\epsilon_0 \neq \epsilon$ is a constant and $M\epsilon_0 < 1$. Following Remark 3.5 in [9], for the extended system $\mathbf{A}$, the computational algorithm involves solving $2m(n-1)$ quadratic equations in one variable independently at each iteration. Therefore, the proposed algorithm has the computational cost $2m(n-1)\log_2[\frac{\ln M\epsilon}{\ln M\epsilon_0}]$ for each transformation of type 2. While in this paper, according to Theorem 3.3, the number of equations we need to solve is $m(n-1)+1$. In Algorithm 2, we do not always have to compute all the elements of

$$\begin{aligned}\mathbf{W_1} \quad := \quad & \{y|\psi_{j_0}(\zeta_{T_k^l(y)}P_{m_0}) = \psi_{j_1}(\zeta_{T_k^l(y)}P_{m_1}) = \bar{\gamma}_i(y), \\ & \forall j_1, j_0 \in \bar{\mathbf{n}}, m_0, m_1 \in \bar{\mathbf{2}}, (j_1, m_1) \neq (j_0, m_0)\},\end{aligned} \tag{19}$$

which is obtained by solving $\frac{mn(mn-1)}{2}$ quadratic equations. Compared with Theorem 3.10 in [9], the number of quadratic equations we need to solve is less than $\frac{mn(mn-1)}{2}$. Therefore, Algorithm 2 has much reduced the computational cost, especially for the case $d_i = 0$.

4. NUMERICAL EXAMPLE

In this section, we give an example to show the performance of the algorithms for solving (4) and (16), respectively. Consider the following 4th-order switched linear system consisting of two subsystems described by

$$A_1 = \begin{bmatrix} -1.7672 & -0.0715 & 1.1008 & 0.5852 \\ 0.2073 & -2.1981 & 0.2368 & 0.6913 \\ 0.7711 & 1.1407 & -2.9919 & 0.3340 \\ 1.2187 & -0.2046 & 0.3339 & -1.1840 \end{bmatrix},$$

and

$$A_2 = \begin{bmatrix} -2.9930 & 0.4460 & 0.7098 & -0.2002 \\ 0.4367 & -4.1214 & 0.1275 & 0.1464 \\ 0.7786 & 0.8034 & -3.9662 & -0.3073 \\ -0.0371 & 0.6581 & 0.0611 & -3.4581 \end{bmatrix}.$$

It is not hard to obtain that $\lambda(A_1, A_2) = -0.2204$, $\mu_1(A_1, A_2) = 0.4299$ and $k = 3$.

Solving (5) in Algorithm 1 gives the minimum of matrix set measure $h_1 = 0.2640$, the corresponding transformation matrix

$$T_1(d_1) = \mathrm{diag}(1, 1, d_1, 1) = \mathrm{diag}(1, 1, 1.9479, 1),$$

and $k = 2$. Substituting $T_1(d_1)$ into (6) leads to $P_r^2 = \zeta_{T_1(d_1)}A_r, r \in \{1, 2\}$. Accordingly, by solving

$$\min_{z_2\neq 0} \max_{j\in\bar{\mathbf{n}}} (\psi_j(\zeta_{T_2(z_2)}P_1^2), \psi_j(\zeta_{T_2(z_2)}P_2^2),$$

$i = 1, \ldots, 6$			$i = 7, \ldots, 12$			$i = 13, \ldots, 16$		
k	d_i	h_i	k	d_i	h_i	k	d_i	h_i
2	1.9479	0.2640	1	1.0147	-0.0955	2	1.0004	-0.0993
4	2.5332	-0.0152	3	1.0056	-0.0965	4	1.0002	-0.0993
1	1.0926	-0.0591	2	1.0044	-0.0983	1	1.0001	-0.0993
3	1.0652	-0.0702	4	1.0021	-0.0985	3	1.0000	-0.0994
2	1.0411	-0.0869	1	1.0012	-0.0991			
4	1.0195	-0.0885	3	1.0005	-0.0991			

Tab. 1. Sequence $\{h_i\}$, $\{d_i\}$ and $\{k\}$ via transformations of type 2.

z_1	z_2	z_3	z_4	ς	α	β
0.9432	0.8832	0.7255	0.8705	0.4299	-0.2204	0.4299
0.9297	0.6242	0.5074	0.9422	0.1047	-0.2204	0.1047
0.8993	0.3758	0.4890	0.8638	-0.0578	-0.2204	-0.0578
0.0000	0.0000	0.0000	0.0000	-0.1391	-0.1391	-0.0578
0.8698	0.3609	0.4573	0.8396	-0.0985	-0.1391	-0.0985
0.0000	0.0000	0.0000	0.0000	-0.1188	-0.1188	-0.0985
0.0000	0.0000	0.0000	0.0000	-0.1086	-0.1086	-0.0985
0.0000	0.0000	0.0000	0.0000	-0.1036	-0.1036	-0.0985
0.0000	0.0000	0.0000	0.0000	-0.1010	-0.1010	-0.0985
0.0000	0.0000	0.0000	0.0000	-0.0997	-0.0997	-0.0985
0.8562	0.3547	0.4502	0.8265	-0.0991	-0.0997	-0.0991
0.0000	0.0000	0.0000	0.0000	-0.0994	-0.0994	-0.0991
0.8472	0.3508	0.4455	0.8178	-0.0993	-0.0994	-0.0993
0.8448	0.3498	0.4443	0.8156	-0.0994	-0.0994	-0.0994

Tab. 2. Solutions for Problem (15) by bisection method.

one obtains $h_2 = -0.0152$, $d_2 = 2.5332$ and $k = 4$. Using a similar idea, it then follows from Algorithm 1 that the sequences $\{h_i\}$ and $\{d_i\}$ are derived. Table 1 shows the numerical result obtained by Algorithm 1. Since $d_{16} = 1$, it follows from Proposition 3.5 that the least μ_1 measure is reached, i. e., $\mu_1(P_1^{16}, P_2^{16}) = -0.0994$. This indicates that the spectral abscissa lies in the interval $[-0.2204, -0.0994]$.

Since $\lambda(A_1, A_2) = -0.2204$, $\mu_1(A_1, A_2) = 0.4299$, let the initial interval of the bisection method be [-0.2204,0.4299]. Then, according to Remark 3.11, combining the bisection method and the linear programming tool for each fixed ς, we get the least μ_1 measure $\mu_1(\zeta_{T^{-1}} A_1, \zeta_{T^{-1}} A_2) = -0.0994$ with $z_1 = 0.8448, z_2 = 0.3498, z_3 = 0.4443, z_4 = 0.8156$.

$i = 1, \ldots, 26$		$i = 27, \ldots, 52$		$i = 53, \ldots, 78$	
d_i	h_i	d_i	h_i	d_i	h_i
-0.0033	0.4285	0.0000	-0.0544	0.0000	-0.1131
0.4469	0.4098	0.0233	-0.0588	0.0063	-0.1146
-0.1242	0.2396	0.0000	-0.0588	0.0000	-0.1146
0.6182	0.0974	0.0000	-0.0588	0.0003	-0.1146
-0.0391	0.0884	0.2763	-0.0713	0.0000	-0.1146
0.0003	0.0883	0.0000	-0.0713	0.0002	-0.1148
0.0000	0.0883	0.2344	-0.0713	0.0017	-0.1159
0.0000	0.0883	0.0000	-0.0713	0.0011	-0.1166
0.0000	0.0883	0.0000	-0.0713	-0.0017	-0.1167
0.1575	-0.0065	0.0000	-0.0713	0.0005	-0.1167
0.0000	-0.0065	0.0095	-0.0721	-0.0004	-0.1167
0.0416	-0.0315	0.0000	-0.0721	0.0000	-0.1167
0.0849	-0.0400	0.0000	-0.0721	0.0000	-0.1167
0.0000	-0.0400	0.1807	-0.1051	0.0000	-0.1167
0.0000	-0.0400	0.0000	-0.1051	0.0000	-0.1167
0.0051	-0.0410	0.0193	-0.1087	0.0000	-0.1167
0.0000	-0.0410	0.0000	-0.1087	0.0000	-0.1167
0.0000	-0.0410	0.0161	-0.1096	0.0000	-0.1167
0.0503	-0.0433	0.0000	-0.1096	0.0010	-0.1174
0.0000	-0.0433	0.0025	-0.1130	0.0003	-0.1176
0.0000	-0.0433	0.0000	-0.1130	-0.0008	-0.1176
0.0000	-0.0433	0.0001	-0.1131	0.0002	-0.1176
0.0077	-0.0541	-0.0003	-0.1131	-0.0002	-0.1176
0.0003	-0.0544	0.0002	-0.1131	0.0000	-0.1176
0.0001	-0.0544	-0.0002	-0.1131	0.0000	-0.1176
0.0000	-0.0544	0.0000	-0.1131	0.0000	-0.1176

Tab. 3. Sequence $\{h_i\}$ and $\{d_i\}$ via transformations of type 3.

The least μ_1 measure is exactly the same as the result obtained by Algorithm 1. Let α and β be the end points of each interval respectively. The detail numerical result is presented in Table 2.

Next, applying Algorithm 2 to solve

$$\min_{z_i \in \mathbf{R}} \max_{j \in \bar{\mathbf{n}}} \{\psi_j(\zeta_{T_1^2(z_i)} A_1), \psi_j(\zeta_{T_1^2(z_i)} A_2)\}, \tag{20}$$

it can be seen that $d_1 = 0.4285$. By using Procedure 2 in [9], it can be found that the

least μ_1 measure is reached: $\mu_1(P_1^{76}, P_2^{76}) = -0.1176$. This exhibits that the spectral abscissa lies in the interval $[-0.2204, -0.1176]$. This result not only characterizes the stability of this system, but also indicates that the largest divergence rate of this system is not more than -0.1176. The numerical result is presented in Table 3.

5. CONCLUSION

In this work, the coordinate transformation method has been presented for computing the spectral abscissa for continuous-time switched linear systems. A computational algorithm based on the coordinate transformations of type 2 was presented to obtain the least μ_1 measure. Furthermore, we also proposed the need steps to get the least μ_1 matrix set measure with a desired accuracy. For the coordinate transformations of type 3, we gave a new detailed procedure to obtain the minimum of matrix set μ_1 measure after each transformation. And then a convergent sequence of the minima of matrix set μ_1 measure is generated. The limit of this sequence can be used as an upper bound estimate of the spectral abscissa. We also examined a numerical simulation to show the performance of these algorithms. However, it is worthwhile mentioning that the method proposed in this paper can give the acceptable but not arbitrarily sharp estimation of the spectral abscissa, since the size of the matrices is not increased.

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