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CONVERSE THEOREM FOR PRACTICAL STABILITY OF NONLINEAR IMPULSIVE SYSTEMS AND APPLICATIONS

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The Lyapunov's second method is one of the most famous techniques for studying the stability properties of dynamic systems. This technique uses an auxiliary function, called Lyapunov function, which checks the stability properties of a specific system without the need to generate system solutions. An important question is about the reversibility or converse of Lyapunov's second method; i. e., given a specific stability property does there exist an appropriate Lyapunov function? The main result of this paper is a converse Lyapunov Theorem for practical asymptotic stable impulsive systems. Applying our converse Theorem, several criteria on practical asymptotic stability of the solution of perturbed impulsive systems and cascade impulsive systems are established. Finally, some examples are given to show the effectiveness of the derived results.

Keywords: converse Lyapunov theorem, practical asymptotic stability, impulsive systems, cascade systems, perturbed systems

Classification: 34A37, 34D20

1. INTRODUCTION

It is well-known that the second method of Lyapunov is considered as the most widely-used tools for stability analysis of various types of mathematically described dynamical systems, including continuous and discrete differential equations, impulsive differential equations, and many others.

Classical results in this direction are presented in the seminal books [13, 21, 31, 32]. Extensions of this theory can be found e. g. in [2, 18, 19, 22], and the references therein.

The strength of Lyapunov's second method is that it is possible to ascertain stability without solving the underlying differential equation. The second method states that if one can find an appropriate Lyapunov function, then the system has some stability property. However, the main draw-back of this method is the need to find a Lyapunov function, which is frequently a difficult task. In many cases, Lyapunov theory provides an affirmative answer to this problem. The answer takes the form of converse Lyapunov Theorems. Unfortunately, these converse Theorems are proven by actually constructing auxiliary functions which always assume the knowledge of the solution of the differential

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equation. Therefore, these converse Theorems cannot really be used to construct an explicit formula for Lyapunov function, except in special cases (e. g. linear systems). However, these Theorems prove at least that such a function exists, and on the other hand, they can be used to study the stability of perturbed systems.

Over the years, many converse Theorems have been established for different kinds of stability, firstly, for continuous systems see [12, 22, 24]. A recent review of converse Theorems is given in [15] and the references therein, and later for discrete systems see[1, 17], for hybrid systems [7, 23] and impulsive systems [2, 30].

Recently, a converse stability Theorem for practical exponential stability has been obtained in [9]. This Theorem is used to analyze the practical exponential stability of the zero solution of perturbed impulsive systems and cascade impulsive systems. However; in the present paper, the contribution is to establish a converse stability Theorem for practical asymptotic stability and its applications, which generalizes an inverse Theorem for practical exponential stability in [9].

The practical stability, in the sense introduced in [8, 16], is very important and very useful for analyzing the stability or for designing practical controllers of dynamical systems since controlling a system to an idealized point is either expensive or impossible in the presence of disruptions and the best which we can hope in such situations is to use practical stability. The practical stability only needs to stabilize a system into a region of phase space, namely the system may oscillate close to the state, in which the performance is still acceptable. In the past decades, practical stability has been studied by many researchers such as cited in [3, 4, 5, 6, 9, 10].

This paper is organized as follows: In Section 2 some definitions and notations are given and a concept of practical asymptotic stability for impulsive systems is presented. Criteria for asymptotic practical stability for impulsive systems are the focus of Section 3. However; in section 4, this Theorem is used to study the practical asymptotic stability of perturbed impulsive systems and cascade impulsive systems. In the last section, some examples are worked out to illustrate our results.

2. PRELIMINARY NOTES

Let $\mathbb{R}_+ = [0, +\infty[$, $\mathbb{R}^n$, $n \in \mathbb{N}^*$ be the n-dimensional Euclidean space with elements $x = col(x_1, x_2, \dots, x_n)$ and the Euclidian norm $\|x\| = \left(\sum_{i=1}^n x_i^2\right)^{\frac{1}{2}}$. Moreover, we shall use the notation $\mathcal{B}_r^n = \{x \in \mathbb{R}^n;\ \|x\| \le r\}$, the closed ball of $\mathbb{R}^n$ of radius $r > 0$.
Let us consider the following impulsive system:

$$\begin{aligned} \dot{x}(t) &= f(t, x(t)), \qquad\qquad t \neq \tau_k,\ t \in \mathbb{R}_+ \\ \Delta x(\tau_k) &= I_k(x(\tau_k^-)), \quad t = \tau_k,\ k = 1, 2, \dots \\ x(t_0^+) &= x_0, \end{aligned} \tag{1}$$

in which $x(t), x_0 \in \mathbb{R}^n$, are respectively the system's state and an initial condition; $f : \mathbb{R}_+ \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ and $I_k : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ are two functions and $\Delta x(\tau_k) \triangleq x({\tau_k}^+) - x({\tau_k}^-)$, with $x({\tau_k}^+) = \lim_{h\to 0^+} x(\tau_k + h)$ and $x({\tau_k}^-) = \lim_{h\to 0^+} x(\tau_k - h)$, where $x({\tau_k}^-) = x(\tau_k)$. To ensure the existence and the uniqueness of the solutions to the impulsive system (1), we suppose that the following classical conditions are satisfied [2].

(a) The function f is a continuous in each domain $G_k^n :=]\tau_{k-1}, \tau_k] \times \mathcal{B}_r^n$, $(k \in \mathbb{N}^*)$ and it satisfies the following properties

(i) There exist a positive constant f_0 such that, for all $t \geq 0$, we have $\|f(t,0)\| \leq f_0$.

(ii) Lipschitz condition: for all $r > 0$, there exists a positive constant L_r, such that, for all $x, y \in \mathcal{B}_r^n$ and $t \geq 0$, we have

$$\|f(t,x) - f(t,y)\| \leq L_r \|x - y\|.$$

(iii) For any $k \in \mathbb{N}^*$ and $x \in \mathcal{B}_r^n$, the function $f(t,x)$ admits a finite limits at (τ_k, x), i. e.

$$\lim_{t \to \tau_k^-} f(t,x) = f(\tau_k, x), \quad \lim_{t \to \tau_k^+} f(t,x) = f(\tau_k^+, x).$$

(b) The incremental change of the state $I_k : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ at the time τ_k, $k = 1, 2, \ldots$ are such that

- There exists a positive constant H, such that, if $\|x\| < H$ and $I_k(x) \neq 0$, then $\|x + I_k(x)\| < H$.
- Lipschitz condition: There exists a positive constant M_k, such that

$$\|I_k(x) - I_k(y)\| \leq M_k \|x - y\|,$$

for all x, $y \in \mathbb{R}^n$.

(c) The times of the impulsive effects $\tau_k, k \in \mathbb{N}^*$ are fixed and an unbounded increasing sequence with $\lim_{k \to +\infty} \tau_k = +\infty$. Moreover, there exist a constants $\theta_1, \theta_2 > 0$, such that

$$\theta_1 \leq \inf_{k \in \mathbb{N}^*} \{s_k\} < \sup_{k \in \mathbb{N}^*} \{s_k\} \leq \theta_2$$

where $s_k = \tau_{k+1} - \tau_k$, $\forall\, k \in \mathbb{N}^*$. The condition (c) says that, as k towards to $+\infty$, the spending time s_k between two consecutive impulses, does not became infinitely long $s_k \to +\infty$ or infinitely short $s_k \to 0$.

Let $(t_0, x_0) \in \mathbb{R}_+ \times \mathbb{R}^n$ be an initial condition. Since the conditions (a), (b) and (c) are satisfied, then there exists an unique solution $x(t) \triangleq x(t, t_0, x_0)$ of system (1), which starts at time t_0 at the point x_0. Moreover, any solution $x(t, t_0, x_0)$ of (1) is defined on the interval $[t_0, +\infty[$, and is continuously differentiable for all $t \neq \tau_k$ with points of discontinuity of the first kind at $t = \tau_k$, $k = 1, 2, \ldots$ [2].

Let us define the following class of functions and specify the notion of practical asymptotic stability mode of solutions of the system (1).

- $\mathcal{F}(\mathbb{R}_+)$ the set of all functions from $\mathbb{R}_+$ to $\mathbb{R}_+$. If $\varphi \in \mathcal{F}(\mathbb{R}_+)$, we denote by $I_{a,b}(\alpha, \varphi)$ the following expression $I_{a,b}(\alpha, \varphi) := \int_a^b e^{-\alpha(b-z)} \varphi(z)\, \mathrm{d}z$, where $\alpha > 0$ and $a, b \in \mathbb{R}_+$ such that $a \leq b$.

- $\mathcal{M}(I,\mathbb{C})$ the set of all measurable functions from I to $\mathbb{C}$, where I is an interval of $\mathbb{R}$.
- $\mathcal{F}(I,J)$ (resp $\mathcal{C}(I,J)$) be the set of all functions (resp all continuous functions) from I into J.
- $\mathcal{C}^1(I,J)$ the set of continuously differentiable functions from I into J.
- $P\mathcal{C}(\mathbb{R}_+,\mathbb{R})$, the set of functions $\psi:\mathbb{R}_+\to\mathbb{R}$ which are continuous for $t\in[\tau_k,\tau_{k+1}[, k\in\mathbb{N}^*$ such that $\psi(\tau_k^-)$, $\psi(\tau_k^+)$ exist and $\psi(\tau_k^-)=\psi(\tau_k)$ for all $,k\in\mathbb{N}^*$.
- $P\mathcal{C}^1(\mathbb{R}_+,\mathbb{R})$ the set of functions $\psi:\mathbb{R}_+\to\mathbb{R}$ which are piecewise continuous differentiable.
- $L^p(\mathcal{I})=\Big\{\varphi\in\mathcal{M}(\mathcal{I},\mathbb{C});\ \|\varphi\|_p<+\infty\Big\}$, where $\|\varphi\|_p=\Big[\int_{\mathcal{I}}|\varphi(s)|^p\,\mathrm{d}s\Big]^{\frac{1}{p}}$, and $p\in[1,+\infty[$.
- $\mathcal{B}(\mathbb{R}_+)$ is the set of all bounded functions from $\mathbb{R}_+$ to $\mathbb{R}_+$,

$$\mathcal{B}(\mathbb{R}_+)=\Big\{\varphi\in\mathcal{F}(\mathbb{R}_+);\exists\,m_\varphi\ge0,\quad 0\le\varphi(t)\le m_\varphi,\ \forall\,t\in\mathbb{R}_+\Big\}.$$

- $\mathcal{A}=\Big\{\varphi\in\mathcal{F}(\mathbb{R}_+);\ \exists\ r:=\nu_\varphi,M_\varphi\ge0;\ I_{a,b}(\alpha,\varphi)\le\frac{M_\varphi}{\alpha^{\nu_\varphi}};\,\forall\,b\ge a\ge0,\alpha>0\Big\}$.

As a first step, we give some properties of the set $\mathcal{A}$.

Remark 2.1. It is not difficult to establish the following useful properties

- For all $p\ge1$, we have $L^p(\mathbb{R}_+)\subset\mathcal{A}$ and if $\varphi\in L^p(\mathbb{R}_+)$, we obtain

$$\forall p\in[1,+\infty],\ M_\varphi=\|\varphi\|_p,\ \text{and}\ \nu_\varphi=1-\frac{1}{p}.$$

- $\mathcal{B}(\mathbb{R}_+)\subset\mathcal{A}$ and if $\varphi\in\mathcal{B}(\mathbb{R}_+)$, we obtain

$$M_\varphi=m_\varphi\ \text{and}\ \nu_\varphi=1.$$

- If $\varphi_1\in\mathcal{A},\varphi_2\in\mathcal{B}(\mathbb{R}_+)$, then $\varphi_1\varphi_2\in\mathcal{A}$ and we have

$$M_{\varphi_1\varphi_2}=m_{\varphi_2}M_{\varphi_1},\ \text{and}\ \nu_{\varphi_1\varphi_2}=\nu_{\varphi_1}.$$

- If $\varphi_1,\varphi_2\in\mathcal{A}$, then $\varphi_1+\varphi_2\in\mathcal{A}$. Moreover, we have

$$M_{\varphi_1+\varphi_2}=2\max(M_{\varphi_1},M_{\varphi_2})\ \text{and}\ \nu_{\varphi_1+\varphi_2}\in\Big\{\nu_{\varphi_1},\nu_{\varphi_2}\Big\}.$$

- If $\varphi\in\mathcal{B}(\mathbb{R}_+)$ such that $\varphi'\ge0$, then $\varphi'\in\mathcal{A}$ with $M_{\varphi'}=m_\varphi$ and $\nu_{\varphi'}=0$.

In control theory, it is often required to check if a nonautonomous system is stable or not. To cope with this, it is necessary to use some special comparison functions. Class $\mathcal{K}$ and $\mathcal{KL}$ functions belong to this family:

Definition 2.1. (Khalil [18]) A continuous function $\alpha : [0,a) \to [0,b)$ is said to belong to class $\mathcal{K}$ if it is strictly increasing and $\alpha(0) = 0$. It is said to belong to class $\mathcal{K}_\infty$ if $a = b = +\infty$ and $\alpha(r) \to +\infty$ as $r \to +\infty$.

Definition 2.2. (Khalil [18]) A continuous function $\beta : [0,a) \times [0,+\infty[\to [0,+\infty[$ is said to belong to class $\mathcal{KL}$ if, for each fixed s, the mapping $\beta(r,s)$ belongs to class $\mathcal{K}$ with respect to r and, for each fixed r, the mapping $\beta(r,s)$ is decreasing with respect to s and $\beta(r,s) \to 0$ as $s \to +\infty$.

In the following, we give some definitions which will be used in this paper.

Definition 2.3. (Jiang et al. [16]) System (1) is uniformly practically asymptotically stable (U.P.A.S), if a function β of class $\mathcal{KL}$ and a nonnegative constant ρ exist such that, for each initial condition (t_0, x_0) with $x_0 \in \mathcal{B}_r^n$, we have

$$\|x(t,t_0,x_0)\| \leq \rho + \beta\Big(\|x_0\|, t - t_0\Big), \ \ \forall\, t \geq t_0. \tag{2}$$

Remark 2.2. Recall that, if (2) is satisfied with $\beta(r,s) = kre^{-\lambda s}$, $k, \lambda > 0$, then the system (1) is said to be uniformly practically exponentially stable (U.P.E.S) with rate λ and region of attraction $\mathcal{B}_r^n$ ([8],[9]). It is well known that if the system (1) is uniformly practically exponentially stable, then it is uniformly practically asymptotically stable. However, the converse is false. In our recent result [9], we have derived a converse Lyapunov Theorem for a practical exponential stable impulsive system. In the present work, we generalize our cited paper for practical asymptotic stable impulsive systems. We point out, that the proposed Lyapunov function in this paper (see (8)) is different from the one given in [9].

Definition 2.4. (Yang [30]) The function $V : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}_+$ is said to belong class $\mathcal{V}_2$ if $V(t,x)$ is positive definite, locally Lipschitz in x, continuous everywhere except possibly at a sequence of points $\{\tau_k\}$ at which $V(t,x)$ is left continuous and the right limit $V(\tau_k^+, x)$ exists for all $x \in \mathbb{R}^n$.

Definition 2.5. (Yang [30]) The Dini derivative or the upper right-hand generalized derivative of a function $V(t,x)$ along the solutions of system (1) is defined by:

$$D^+_{(1)}V(t,x) = \limsup_{h\to 0} \frac{1}{h}\big[V(t+h, x+hf(t,x)) - V(t,x)\big]. \tag{3}$$

To prove of the main results, we shall use the following lemmas.

Lemma 2.6. If $\alpha \in \mathcal{K}_\infty$ then, for all $a, b \geq 0$, we have

$$\alpha(a+b) \leq \alpha(2a) + \alpha(2b).$$

Lemma 2.7. (Lakshmikantham et al. [20]) Let $u, v \in PC(\mathbb{R}_+, \mathbb{R})$ a nonnegative functions satisfying,

$$u(t) \leq c + \int_{t_0}^{t} v(s)u(s)\,\mathrm{d}s + \sum_{t_0 \leq \tau_i < t} b_i u(\tau_i), \tag{4}$$

for all $t \geq t_0 \geq 0$, where the constants c and b_i are positive. Then, we have

$$u(t) \leq c \prod_{t_0 \leq \tau_i < t} (1 + b_i) \exp\Big(\int_{t_0}^{t} v(s)\, \mathrm{d}s \Big). \tag{5}$$

Lemma 2.8. (Lakshmikantham et al. [20]) Assume that $v \in PC^1(\mathbb{R}_+, \mathbb{R})$ and $v(t)$ is left-continuous at $\tau_k,\ k = 1, 2, \ldots$, satisfying:

$$\begin{aligned} \dot{v}(t) &\leq a(t)v(t) + b(t),\ t \neq \tau_k, \\ v(\tau_k^+) &\leq c_k v(\tau_k) + d_k,\ t = \tau_k,\ k = 1, 2, \ldots \\ v(t_0^+) &= v_0, \end{aligned} \tag{6}$$

where the functions $a, b \in \mathcal{C}(\mathbb{R}_+, \mathbb{R})$, $c_k \geq 0$ and $d_k \in \mathbb{R}$ are constants. Then, v satisfies the following inequality

$$\begin{aligned} v(t) \leq\ & v_0 \prod_{t_0 < \tau_k < t} c_k e^{A(t,t_0)} + \sum_{t_0 < \tau_k < t} \left(\prod_{\tau_k < \tau_j < t} c_j \right) e^{A(t,\tau_k)} d_k \\ & + \int_{t_0}^{t} \left(\prod_{s < \tau_k < t} c_k \right) e^{A(t,s)} b(s)\, \mathrm{d}s, \end{aligned}$$

where $A(t, s) = \int_s^t a(s)\, \mathrm{d}s$, for all $t \geq s \geq 0$.

3. MAIN RESULTS

This section is dedicated to start the main results of this paper. In the following subsection, we introduce a sufficient conditions satisfied by a Lyapunov function that guaranteed the practical asymptotic stability of the nonlinear impulsive system (1). Then, we establish a converse Lyapunov Theorem for a practical asymptotic stable impulsive system (1).

3.1. Practical asymptotic stability

In the following Theorem, we give some sufficient conditions to ensure the practical asymptotic stability of the nonlinear impulsive systems (1).

Theorem 3.1. Assume that the impulsive system (1) satisfies the conditions $(a), (b)$ and (c). Furthermore, suppose that there exist a function $V \in \mathcal{V}_2$, a function $L \in PC(\mathbb{R}_+, \mathbb{R}_+)$, a function $r \in \mathcal{A}$, positive scalars constants a, λ, c_k, d_k $(k \in \mathbb{N}^*)$ and $\alpha_1, \alpha_2 \in \mathcal{K}$ which satisfy:

1. $\alpha_1(\|x\|) \leq V(t, x) \leq \alpha_2(\|x\|) + a, \forall\, x \in \mathcal{B}_r^n,\ t \geq 0,$
2. $D^+V(t, x) \leq -\lambda V(t, x) + r(t), \forall\, x \in \mathcal{B}_r^n,\ t \neq \tau_k,$
3. $|V(t, x) - V(t, y)| \leq L(t)\|x - y\|,\ \forall\, x, y \in \mathcal{B}_r^n,$

4. $V(\tau_k^+, x(\tau_k^+)) \leq c_k V(\tau_k, x(\tau_k)) + d_k$, $\forall\, k \in \mathbb{N}^*$,

where $\prod_{k=1}^{+\infty} c_k < +\infty$ and $\sum_{k\geq 1} d_k e^{-\alpha s_k}$ converges for some positive constant α. Then, the impulsive system (1) is uniformly practically asymptotically stable.

Proof. Let us consider $t \geq t_0$ and $m_0, m \in \mathbb{N}^*$ such that $t_0 \in [\tau_{m_0}, \tau_{m_0+1}[$ and $t \in [\tau_m, \tau_{m+1}[$. Since the infinite product $\prod_{k\geq 1} c_k$ converges, then there exists a positive constant $P > 0$, such that $P_n = \prod_{k=1}^{n} c_k \leq P$, for all $n \geq 1$. Conditions 2 and 4 of Theorem 3.1 implies that the function $V(t,x)$ satisfies Lemma 2.8. Then, we obtain the following estimation for all $t \geq t_0$,

$$\begin{aligned} V(t,x(t)) \leq V(t_0^+, x_0) \left(\prod_{t_0<\tau_k<t} c_k \right) e^{-\lambda(t-t_0)} + \sum_{t_0<\tau_k<t} \left(\prod_{\tau_k<\tau_j<t} c_j \right) e^{-\lambda(t-\tau_k)} d_k \\ + \int_{t_0}^{t} \left(\prod_{s<\tau_k<t} c_k \right) e^{-\lambda(t-s)} r(s)\, \mathrm{d}s. \end{aligned} \tag{7}$$

Then, from (7), one get

$$\begin{aligned} V(t,x(t)) \leq V(t_0^+, x_0) P e^{-\lambda(t-t_0)} + P e^{\alpha\theta_2 - \lambda\theta_1} \sum_{k=m_0+1}^{m} d_k e^{-\alpha s_k} \\ + P \int_{t_0}^{t} e^{-\lambda(t-s)} r(s)\, \mathrm{d}s. \end{aligned}$$

It follows that,

$$V(t,x(t)) \leq V(t_0^+, x_0) P e^{-\lambda(t-t_0)} + R,$$

where $R = P\big(e^{\alpha\theta_2}(1 + e^{-\lambda\theta_1}) \sum_{k=1}^{+\infty} d_k e^{-\alpha s_k} + \frac{M_r}{\lambda^{\nu_r}}\big)$. Using the condition 1 of Theorem 3.1 and Lemma 2.6, we can easily see that for all $x_0 \in \mathcal{B}_r^n$,

$$\begin{aligned} \|x(t,t_0,x_0)\| &\leq \alpha_1^{-1}\Big(V(t_0^+, x_0) P e^{-\lambda(t-t_0)} + R\Big), \\ &\leq \alpha_1^{-1}\big((\alpha_2(\|x_0\|) + a) P e^{-\lambda(t-t_0)} + R\big), \\ &\leq \alpha_1^{-1}\big(P\alpha_2(\|x_0\|) e^{-\lambda(t-t_0)} + aP + R\big), \\ &\leq \alpha_1^{-1}\big(2P\alpha_2(\|x_0\|) e^{-\lambda(t-t_0)}\big) + \alpha_1^{-1}\big(2aP + 2R\big), \\ &\leq \beta(\|x_0\|, t-t_0) + \rho, \end{aligned}$$

where

$$\beta(r,s) = \alpha_1^{-1}\big(2P\alpha_2(r) e^{-\lambda s}\big) \in \mathcal{KL}, \quad \rho = \alpha_1^{-1}\big(2aP + 2R\big).$$

This finishes and shows the proof of Theorem 3.1. □

Remark 3.1. Note that, if $r(t) = 0$, $a = 0$ and $d_k = 0$ for all $k \in \mathbb{N}^*$, we obtain the uniform asymptotic stability of the impulsive system (1)(see [2]). In the above Theorem, we have established some sufficient conditions that guaranteed the practical asymptotic stability of system (1) using Lyapunov function. Our result is different from other many Lyapunov Theorem for nonlinear impulsive systems due to special structure of the set $\mathcal{A}$. Obviously, our Theorem does not generalize other Lyapunov Theorems for impulsive systems, but it is a simple variant which gives different class of Lyapunov Theorem by introducing the set $\mathcal{A}$ which is a very large set as pointed out in Remark 2.1. The importance the above Theorem does not became only from its results, but it is a consequence from the fact that the sufficient conditions satisfied by the Lyapunov function are also necessary for any U.P.A.S. impulsive system (1). In the following, we give a positive answer to the following natural question: if the impulsive system (1) is U.P.A.S., is there a Lyapunov function $V(t,x)$ which satisfies the conditions *1-4* of the above Theorem? In fact, we prove in the following subsection that a U.P.A.S. impulsive system (1) admits a Lyapunov function satisfying the conditions $1-4$, with $r \in \mathcal{A}$.

3.2. A converse stability Theorem

In this subsection, a so-called converse Lyapunov Theorem for practical asymptotic stable impulsive system (1) is stated and proved. This Theorem will be applied in the sequel to two problems in control theory, and it will be shown that it leads to an elegant solutions to each of these problems.

Theorem 3.2. Assume that the impulsive system (1) satisfies the conditions $(a), (b)$ and (c). Moreover, we suppose that the impulsive system (1) is uniformly practically asymptotically stable. Then, there exist $\alpha_1, \alpha_2 \in \mathcal{K}$, a positive constants a, λ, c_k, d_k, a functions $L \in PC(\mathbb{R}_+, \mathbb{R}_+)$, $r \in \mathcal{A}$, and a Lyapunov function $V \in \mathcal{V}_2$ that satisfy:

1. $\alpha_1(\|x\|) \leq V(t,x) \leq \alpha_2(\|x\|) + a, \forall\, x \in \mathcal{B}_r^n,\ t \geq 0,$
2. $D^+V(t,x) \leq -\lambda V(t,x) + r(t), \forall\, x \in \mathcal{B}_r^n,\ t \neq \tau_k,$
3. $|V(t,x) - V(t,y)| \leq L(t)\|x-y\|,\ \forall\, x, y \in \mathcal{B}_r^n,$
4. $V(\tau_k^+, x(\tau_k^+)) \leq c_k V(\tau_k, x(\tau_k)) + d_k,\ \forall\, k \in \mathbb{N}^*,$

where $\prod_{k=1}^{+\infty} c_k < +\infty$ and $\sum_{k\geq 1} d_k e^{-\alpha s_k}$ converges for some positive constant α.

P r o o f. For $(t,x) \in \mathbb{R}_+ \times \mathcal{B}_r^n$, let $V(t,x)$ be the function given by:

$$\begin{aligned} V(t,x) &= \sup_{s\geq 0}\Big\{\varphi\big(\|x(t+s,t,x)\|\big)\frac{1+\theta(t)s}{1+s}\Big\},\ t \neq \tau_k,\\ V(\tau_k,x) &= \lim_{t\longrightarrow\tau_k^-} V(t,x) = V(\tau_k^-,x),\ k \in \mathbb{N}^*, \end{aligned} \tag{8}$$

where the function $\varphi \in \mathcal{K}_\infty \cap \mathcal{C}^1(\mathbb{R}_+, \mathbb{R}_+)$ and $\theta \in \mathcal{B}(\mathbb{R}_+)$, such that $\theta'(t) \geq 0$ and $\lambda := \inf_{t\geq 0}\{\theta(t)\} - 1 > 0$. Obviously, the function $V(t,x)$ is well defined since the function

$$s \mapsto \varphi\big(\|x(t+s,t,x)\|\big)\frac{1+\theta(t)s}{1+s}$$

is bounded on $\mathbb{R}_+$. We prove in Appendix A, that the Lyapunov function candidate V belong to the class $\mathcal{V}_2$. The details of the proof are given in appendix. The different stages of the proof are inspired by the recent work [9]. □

4. APPLICATIONS OF CONVERSE THEOREM

The converse Lyapunov Theorem derived in the above section is used in this section to solve two problems in stability analysis and control theory. The first application is about the practical stability of perturbed impulsive system, while the second one concerns the practical stability of cascaded impulsive systems.

4.1. Practical asymptotic stability of perturbed impulsive system

Let us consider the perturbed impulsive system having the form

$$\begin{array}{rcl} \dot{x}(t) & = & f(t,x(t)) + g(t,x(t)), \;\; t \neq \tau_k, \;\; k = 1,2,\dots \\ \Delta x & = & I_k(x) + J_k(x), \;\; t = \tau_k, \;\; k = 1,2,\dots \\ x(t_0^+) & = & x_0, \end{array} \tag{9}$$

where $x(t), x_0 \in \mathbb{R}^n$ and $f,\, g : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ are two continuous functions in each domain G_k^n and for any $k \in \mathbb{N}^*$ and $x \in \mathbb{R}^n$ there exist the finite limits

$$\lim_{t\to\tau_k^-} f(t,x) = f(\tau_k, x), \;\; \lim_{t\to\tau_k^+} f(t,x) = f(\tau_k^+, x),$$

and

$$\lim_{t\to\tau_k^-} g(t,x) = g(\tau_k, x), \;\; \lim_{t\to\tau_k^+} g(t,x) = g(\tau_k^+, x).$$

In addition, suppose that f is a locally Lipschitz function with respect the variable x. The nominal system associated at the perturbed system (9) is the impulsive system (1) where $g(t,x)$ and $J_k(x)$ are the perturbations that affect the impulsive system under consideration.

Consider the following hypotheses:

(H1): The nominal system (1) is assumed to be U.P.A.S and the perturbation term $g(t,x)$ satisfies,

$$\|g(t,x)\| \leq \phi(t), \tag{10}$$

for all $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^n$, where $\phi \in \mathcal{A}$.

(H2): The incremental changes $I_k(.)$ and $J_k(.)$ of the state at the time τ_k satisfy,

i) For all $k \in \mathbb{N}^*$, $\forall\, x, y \in \mathbb{R}^n$, $\|I_k(x) - I_k(y)\| \leq M_k \|x - y\|$, such that $\sum_{k=1}^{+\infty} M_k < +\infty$,

ii) For all $k \in \mathbb{N}^*$, $\forall\, x \in \mathbb{R}^n$, $\|J_k(x)\| \leq \mu_k$, and there exist $\beta > 0$, such

$$\sum_{k=1}^{+\infty} \mu_k e^{-\beta s_k} < +\infty.$$

Then, one can state the following Theorem.

Theorem 4.1. Assume that **(H1)** and **(H2)** hold. Then, the perturbed impulsive system (9) is uniformly practically asymptotically stable.

P r o o f. By assumption **(H1)** there exists a Lyapunov function $V(t,x)$ which satisfying conditions $1-4$ of Theorem 3.2. Since, the series $\sum_{k\geq 1} M_k$ converges, then the function $L(t)$, given by the formula (35), is bounded by a positive constant l, i. e. $L(t) \leq l$, for all $t \geq 0$.
If $t = \tau_k$, we have

$$\begin{aligned}
V\big(\tau_k^+, x(\tau_k^+)\big) &= V\big(\tau_k^+, x(\tau_k) + I_k(x(\tau_k)) + J_k(x(\tau_k))\big),\\
&= V\big(\tau_k^+, x(\tau_k) + I_k(x(\tau_k)) + J_k(x(\tau_k))\big) - V\big(\tau_k^+, x(\tau_k) + I_k(x(\tau_k))\big)\\
&+ V\big(\tau_k^+, x(\tau_k) + I_k(x(\tau_k))\big),\\
&\leq c_k V\big(\tau_k, x(\tau_k)\big) + d_k + l\|J_k\big(x(\tau_k)\big)\|,\\
&\leq c_k V(\tau_k, x(\tau_k)) + \ell(d_k + \mu_k),
\end{aligned}$$

where $\ell = \max(1,l)$. Other, if $t \neq \tau_k$, we get

$$\begin{aligned}
D^+_{(9)}V(t,x(t)) &= \limsup_{h\to 0^+} \frac{1}{h}\big[V(t+h, x+hf(t,x)+hg(t,x)) - V(t,x)\big],\\
&= \limsup_{h\to 0^+} \frac{1}{h}\big[V(t+h, x+hf(t,x)+hg(t,x)) - V(t+h, x+hf(t,x))\\
&+ V(t+h, x+hf(t,x)) - V(t,x)\big],\\
&= \limsup_{h\to 0^+} \frac{1}{h}\big[V(t+h, x+hf(t,x)+hg(t,x)) - V(t+h, x+hf(t,x))\big]\\
&+ D^+_{(1)}V(t,x),\\
&\leq D^+_{(1)}V(t,x) + l\|g(t,x)\|,\\
&\leq -\lambda V(t,x) + r(t) + l\phi(t) \leq -\lambda V(t,x) + \ell(r(t)+\phi(t)).
\end{aligned}$$

Then, by Theorem 3.1, the solution $x(t,t_0,x_0)$ of the perturbed impulsive system (9) satisfies the following estimation,

$$\|x(t,t_0,x_0)\| \leq \rho + \beta(\|x_0\|, t-t_0),$$

for all $t \geq t_0 \geq 0$, and $x_0 \in \mathcal{B}^n_r$, where,

$$\beta(r,s) = \alpha_1^{-1}\big(2P\alpha_2(r)e^{-\lambda s}\big) \in \mathcal{KL} \text{ and } \rho = \alpha_1^{-1}(2M),$$

with,

$$M = \ell P\Big[\frac{a}{\ell} + (1+e^{-\lambda\theta_1})\big(e^{\alpha\theta_2}\sum_{k=1}^{+\infty} d_k e^{-\alpha s_k} + e^{\beta\theta_2}\sum_{k=1}^{+\infty}\mu_k e^{-\beta s_k}\big) + \frac{M_\phi}{\lambda^{\nu_\phi}} + \frac{M_r}{\lambda^{\nu_r}}\Big].$$

□

4.2. Practical asymptotic stability of cascade impulsive systems

In this part, we establish sufficient conditions for uniform practical asymptotic stability of cascade impulsive systems. Let us consider the following cascade impulsive system:

$$\begin{array}{rcll} \dot{x}_1 & = & f(t,x_1)+g(t,x), & t\neq\tau_k,\ \ k=1,2,\ldots \\ \dot{x}_2 & = & h(t,x_2), & t\neq\tau_k,\ \ k=1,2,\ldots \\ \triangle x_1 & = & I_k(x_1)+J_k(x_1,x_2), & t=\tau_k,\ \ k=1,2,\ldots \\ \triangle x_2 & = & L_k(x_2), & t=\tau_k,\ \ k=1,2,\ldots \\ x_1(t_0^+) & = & x_0,\ x_2(t_0^+)=x_{20} & \end{array} \tag{11}$$

where $x_1\in\mathbb{R}^n$, $x_2\in\mathbb{R}^m$, $x:=col(x_1,x_2)$, and $f:\mathbb{R}_+\times\mathbb{R}^n\to\mathbb{R}^n$, $h:\mathbb{R}_+\times\mathbb{R}^m\to\mathbb{R}^m$, $g:\mathbb{R}_+\times\mathbb{R}^{n+m}\to\mathbb{R}^n$ are three continuous functions in each domains G_k^n, G_k^m and G_k^{n+m} respectively such that for any $k\in\mathbb{N}^*$ there exist the finite limits

$$\lim_{t\to\tau_k^-} f(t,x_1)=f(\tau_k,x_1),\ \lim_{t\to\tau_k^+} f(t,x_1)=f(\tau_k^+,x_1),\ \forall\, x_1\in\mathbb{R}^n,$$

$$\lim_{t\to\tau_k^-} g(t,x)=g(\tau_k,x),\ \lim_{t\to\tau_k^+} g(t,x)=g(\tau_k^+,x),\ \forall\, x\in\mathbb{R}^{n+m},$$

and

$$\lim_{t\to\tau_k^-} h(t,x_2)=h(\tau_k,x_2),\ \lim_{t\to\tau_k^+} h(t,x_2)=h(\tau_k^+,x_2),\ \forall\, x_2\in\mathbb{R}^m.$$

Denoted by $x_0:=col(x_{10},x_{20})$ and $x(.,t_0,x_0)=(x_1(.,t_0,x_0),x_2(.,t_0,x_{20}))$ the solution of the system (11) starting from x_0 at $t=t_0$, i. e. $x(t_0,t_0,x_0)=(x_{10},x_{20})$.

Consider now the following two subsystems:

$$\begin{array}{rcll} \dot{x}_1 & = & f(t,x_1), & t\neq\tau_k,\ \ k=1,2,\ldots \\ \triangle x_1 & = & I_k(x_1), & t=\tau_k,\ \ k=1,2,\ldots \\ x_1(t_0^+) & = & x_{10}, & \end{array} \tag{12}$$

and

$$\begin{array}{rcll} \dot{x}_2 & = & h(t,x_2), & t\neq\tau_k,\ \ k=1,2,\ldots \\ \triangle x_2 & = & L_k(x_2), & t=\tau_k,\ \ k=1,2,\ldots \\ x_2(t_0^+) & = & x_{20}. & \end{array} \tag{13}$$

These two subsystems satisfy conditions $(a),(b)$ and (c). In addition, suppose the following hypotheses hold:

(H3): Systems (12) and (13) are assumed to be U.P.A.S and the interconnection term $g(t,x,y)$ satisfies

$$\|g(t,x,y)\|\leq\psi(t),\ \forall\, t\geq 0,\ (x,y)\in\mathbb{R}^n\times\mathbb{R}^m,$$

with $\psi\in\mathcal{A}$.
(H4): Functions $I_k(.)$ and $J_k(.)$ satisfy

i) $\forall\, k\in\mathbb{N}^*$, $\forall\, x,y\in\mathbb{R}^n$, $\|I_k(x)-I_k(y)\|\leq M_k\|x-y\|$, such that $\sum_{k\geq 1} M_k$ converges.

ii) For all $k \geq 1$ and $(x, y) \in \mathbb{R}^n \times \mathbb{R}^m$ $\|J_k(x, y)\| \leq \delta_k$ and there exist $\delta > 0$, such that $\sum_{k=1}^{+\infty} \delta_k e^{-\delta s_k} < +\infty$.

In the following Theorem, we establish the practical uniform asymptotic stability of cascade system (11) under the above hypotheses.

Theorem 4.2. Suppose that assumptions **(H3)** and **(H4)** hold. Then, the cascade impulsive system (11) is uniformly practically asymptotically stable.

Proof. By assumption **(H3)** there exists a Lyapunov function $V(t, x)$ which satisfies Theorem 3.2 and by using the fact that $\sum_{k\geq 1} M_k$ converges, then as in the proof of Theorem 4.1, we obtain

$$|V(t, x) - V(t, y)| \leq l\|x - y\|, \; x, y \in \mathbb{R}^n, \; t \geq 0$$

for some $l > 0$.
For $t \neq \tau_k$, we have

$$\begin{aligned}
D^+V_{(11)}(t, x_1) &= \limsup_{h\to 0^+} \frac{1}{h}\big[V\big(t + h, x_1 + h(f(t, x_1) + g(t, x)\big) - V(t, x_1)\big], \\
&\leq \limsup_{h\to 0^+} \frac{1}{h}\big[V\big(t + h, x_1 + h(f(t, x_1) + g(t, x))\big) - V\big(t + h, x_1 + hf(t, x_1)\big)\big] \\
&+ \limsup_{h\to 0^+} \frac{1}{h}\big[V\big(t + h, x_1 + hf(t, x_1)\big) - V(t, x_1)\big], \\
&\leq D^+V_{(12)}(t, x_1) + l\|g(t, x)\|, \\
&\leq D^+V_{(12)}(t, x_1) + l\psi(t).
\end{aligned}$$

Then, we obtain the following inequality,

$$D^+V_{(11)}(t, x_1) \leq -\lambda V(t, x_1) + r(t) + l\psi(t) \leq -\lambda V(t, x_1) + \ell(r(t) + \psi(t)) \tag{14}$$

with $\ell = \max(1, l)$.
If $t = \tau_k$, we get

$$\begin{aligned}
V(\tau_k^+, x_1(\tau_k^+)) &= V\big(\tau_k^+, x_1(\tau_k) + I_k(x_1(\tau_k)) + J_k(x_1(\tau_k), x_2(\tau_k))\big), \\
&\leq V\big(\tau_k^+, x_1(\tau_k) + I_k(x_1(\tau_k)) + J_k(x_1(\tau_k), x_2(\tau_k)))\big) \\
&- V\big(\tau_k^+, x_1(\tau_k) + I_k(x_1(\tau_k))\big) + V\big(\tau_k^+, x_1(\tau_k) + I_k(x_1(\tau_k))\big), \\
&\leq V\big(\tau_k^+, x_1(\tau_k) + I_k(x_1(\tau_k))\big) + l\|J_k(x_1(\tau_k), x_2(\tau_k))\|, \\
&\leq V\big(\tau_k^+, x_1(\tau_k) + I_k(x_1(\tau_k))\big) + l\delta_k.
\end{aligned}$$

It follows that,

$$\begin{aligned}
V(\tau_k^+, x_1(\tau_k^+)) &\leq c_k V(\tau_k, x_1(\tau_k)) + l\delta_k + d_k, \\
&\leq c_k V(\tau_k, x_1(\tau_k)) + \ell(\delta_k + d_k).
\end{aligned} \tag{15}$$

So, by using the inequalities (14) and (15), we conclude that the Lyapunov function $V(t,x_1)$ verifies the conditions of Theorem 3.1. Then, the solution $x_1(t,t_0,x_0)$ of the cascade system (11) satisfies, for all $x_0 \in \mathcal{B}_r^n$,

$$\|x_1(t,t_0,x_0)\| \leq \rho + \beta(\|x_0\|, t-t_0)$$

with,

$$\beta(r,s) = \alpha_1^{-1}\big(2P\alpha_2(r)e^{-\lambda s}\big) \text{ and } \rho = \alpha_1^{-1}(2M),$$

where,

$$M = \ell P\Big[\frac{a}{\ell} + (1+e^{-\lambda\theta_1})\big(e^{\alpha\theta_2}\sum_{k=1}^{+\infty} d_k e^{-\alpha s_k} + e^{\delta\theta_2}\sum_{k=1}^{+\infty}\delta_k e^{-\delta s_k}\big) + \frac{M_\psi}{\lambda^{\nu_\psi}} + \frac{M_r}{\lambda^{\nu_r}}\Big].$$

Thus, we achieve the proof of Theorem 4.2. □

5. NUMERICAL EXAMPLES

In this section we give some examples to illustrate the main result. Our first example shows the applicability of Theorem 4.1 for a simple class of perturbed systems. The second and third examples illustrate the uniform practical asymptotic stability of a cascade impulsive system.

Example 5.1. Impulsive neural network system
Consider the perturbed impulsive neural network system:

$$\begin{array}{rcl} \dot{x} & = & f(t,x) + g(t,x), \ \ t \neq \tau_k, \ k = 1,2,\dots \\ \Delta x & = & I_k(x) + J_k(x), \ \ t = \tau_k, \ k = 1,2,\dots \\ x(t_0^+) & = & x_0, \end{array} \tag{16}$$

where, $x = [x_1, x_2]^T \in \mathbb{R}^2$ and $x_0 \in \mathbb{R}^2$

$$f(t,x) = \begin{bmatrix} f_1(t,x) \\ f_2(t,x) \end{bmatrix} = \begin{bmatrix} -2x_1 + \tanh(x_1) + \tanh(x_2) - 4\pi\sin^3(t)\cos(t) \\ \\ -52x_2 - 100\tanh(x_1) + 2\tanh(x_2) - \cos(t)\sin^3(t) \end{bmatrix}$$

and the perturbation term $g(t,x)$ is given by

$$g(t,x) = \begin{bmatrix} g_1(t,x) \\ g_2(t,x) \end{bmatrix} = \begin{bmatrix} 1 + \frac{\pi}{13}\exp(-\pi x_1^2 - 2\pi x_2^2)\rho(t) \\ \\ -\frac{\pi}{27} - 2\pi\exp(-3t)\exp(-x_1^2 - x_2^2)\rho(t) \end{bmatrix}$$

for all $t \geq 0$ and $x \in \mathbb{R}^2$.

The impulsive jumps $\Delta(x) = \Delta(x_1,x_2)^T$ are characterized by:

$$\Delta(x_1,x_2)^T = I_k(x_1,x_2) + J_k(x_1,x_2),$$

where

$$I_k(x_1,x_2) = \left[\pi^3 \frac{(-1)^k}{3^k} x_1, 2\pi(-1)^{k+1}\cos(k)\exp(-k^2)x_2\right]^T$$

and

$$J_k(x_1,x_2) = \left[2\pi\frac{\cos(k)}{k^2}, -\pi\frac{\sin(k)}{2^k}\right]^T$$

such that ρ is the integrable unbounded function given by

$$\rho(t) = \begin{cases} 0, & t \in [0, 2-\frac{2}{8}] \\ n^4 t + n - n^5, & t \in [n - \frac{1}{n^3}, n], n \geq 2 \\ -n^4 t + n + n^5, & t \in [n, n + \frac{1}{n^3}], n \geq 2 \\ 0, & t \in [n + \frac{1}{n^3}, n+1 - \frac{1}{(n+1)^3}]. \end{cases}$$

Now, by choosing $V(t,x) = x_1^2 + x_2^2$, all assumptions of Theorem 4.1 are satisfied with

$$\lambda = 1, \;\; a = 0, \;\; r(t) = 103 + 2(1+2\pi)^2 \;\; \text{and} \;\; c_k = 1 + \frac{\pi^3}{3^k}(2 + \frac{\pi^3}{3^k}) + 4\pi e^{-k^2}(1 + \pi e^{-k^2}), \;\; k \geq 1.$$

It follows that, the nominal system of (16)

$$\begin{array}{lcl} \dot{x} & = & f(t,x), \;\; t \neq \tau_k, \;\; k = 1,2,\ldots \\ \Delta x & = & I_k(x), \;\; t = \tau_k, \;\; k = 1,2,\ldots \\ x(t_0^+) & = & x_0, \end{array} \tag{17}$$

is U.P.A.S. Moreover, by using the fact that perturbation term $g(t,x) = [g_1(t,x), g_2(t,x)]^T$ satisfies (10) with $\phi(t) = \sqrt{2 + \frac{2\pi^2}{27^2} + 4\rho^2(t)}$ and the assumption **(H2)** is satisfied with $M_k = \frac{\pi^3}{3^k} + 2\pi e^{-k^2}$ and $\mu_k = \sqrt{\frac{4\pi^2}{k^4} + \frac{\pi^2}{4^k}}$. Then, all conditions of Theorem 4.1 are satisfied and we can conclude the U.P.A.S of perturbed system (16). For the simulation, we select $x_0 = (-4.4, 10.4)$ as an initial condition. Then we obtain the following result: The convergence dynamics of the network system (16) with impulses are shown in figure 1. The figure 1 shows the evolution of the solution of the nominal system (17) and the perturbed impulsive system (16) over time.

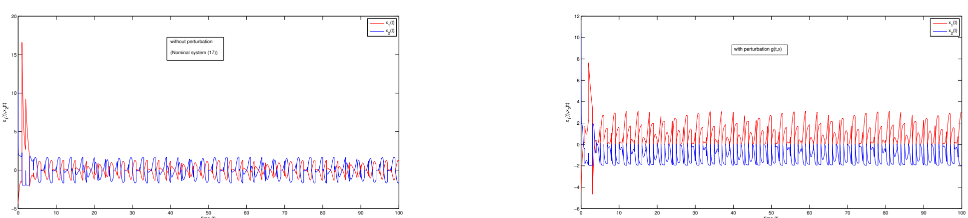

Fig. 1. Trajectories of systems (16) and (17) with impulses.

Example 5.2. Application in robot control: Impulsive Mechanical Systems.
Consider the Lagrangian formulation of the dynamics of an $n-$degree-of-freedom mechanical system

$$D(\theta)\ddot{\theta} + C(\theta,\dot{\theta})\dot{\theta} + G(\theta) = B(\theta)\tau, \tag{18}$$

where $\theta \in \mathbb{R}^n$ is the vector of generalized coordinates, $D(\theta)$ is the inertia matrix, $C(\theta,\dot{\theta})$ is the matrix of Coriolis and centrifugal terms, $G(\theta)$ is the gradient of the potential energy field, $\tau \in \mathbb{R}^m$, $(m < n)$ is the input generalized force and $B(\theta) \in \mathbb{R}^{n\times m}$ has full rank for all θ, describes the effects of actuators on the generalized coordinates. We partition the vector $\theta \in \mathbb{R}^n$ of generalized coordinates as $\theta^T = [\theta_1^T, \theta_2^T]$, where $\theta_1 \in \mathbb{R}^{n-m}$ represents the unactuated (passive) joints and $\theta_2 \in \mathbb{R}^m$ represents the actuated (active) joints. The Euler-Lagrange equation of motion of such a system are then given by [26]

$$M_{11}(\theta)\ddot{\theta}_1 + M_{12}(\theta)\ddot{\theta}_2 + h_1(\theta_1,\dot{\theta}_1,\theta_2,\dot{\theta}_2) + \phi_1(\theta_1,\theta_2) = 0 \tag{19}$$
$$M_{12}(\theta)\ddot{\theta}_1 + M_{22}(\theta)\ddot{\theta}_2 + h_2(\theta_1,\dot{\theta}_1,\theta_2,\dot{\theta}_2) + \phi_2(\theta_1,\theta_2) = \Lambda(\theta_1,\theta_2)\tau \tag{20}$$

where h_i includes Coriolis and centrifugal terms and ϕ_i contains the terms derived from the potential energy, such as gravitational and elastic generalized forces. The $m \times m$ matrix $\Lambda(\theta_1,\theta_2)$ is assumed to be invertible.

An interesting property that holds for the entire class of underactuated mechanical system is the so-called collocated partial feedback linearization property [27], which is a consequence of positive definiteness of the inertia matrix. The advantage of the collocated partial feedback linearization is both a conceptual and a structural simplification of the control problem. We can write the system (19) – (20) under consideration after the stage collocated partial feedback linearization, as

$$\dot{\eta} = \varphi(t,\eta,x,u) := w(t,\eta) + h(t,\eta,x) + g(t,\eta,x)u, \tag{21}$$
$$\dot{x} = Ax + Bu, \tag{22}$$

with suitable definitions of all quantities, such that $h(t,\eta,0) = 0$ and u is a new control input to be determined. The pair (A,B) is controllable since the linear system is a set of m double integrator and the expression

$$\dot{\eta} = w(t,\eta) \tag{23}$$

represents the zero dynamics [14].

The Lagrangian formulation of the dynamics of an n-degree-of-freedom mechanical system can now be expressed as a system with impulse effects [25](e. g., for a walking robot, states at which the foot of the swing leg hits the ground, and a new step begins). Assuming that the system trajectories possess finite left and right limits, the model is then in cascade form.

The practical stability is very useful for designing practical controllers because in many cases, control a system to an idealized point is either expensive or impossible because of the finite measuring accuracy of sensors and actuators. We move away from the paradigm of asymptotic stabilization. Instead, we focus on practical stabilization. More specifically, we consider the following practical stabilization problem. Find a smooth function $u : \mathbb{R}\times\mathbb{R}^n \to \mathbb{R}^n$, $(t,x) \mapsto u(t,x)$, such that, the system (21) – (22) is

U.P.A.S. If the control term u is chosen to be a function only of x ant t, for example, $u(t,x) = -Kx + v(t)$, with $\|v\| \in \mathcal{A}$, then the system (21)−(22) will be in a cascade impulsive form

$$\begin{array}{rcll} \dot{\eta} &=& w(t,\eta) + \overline{g}(t,\eta,x), & t \neq \tau_k,\ k = 1,2,\dots, \\ \dot{x} &=& \overline{A}x + Bv(t), & t \neq \tau_k,\ k = 1,2,\dots, \\ \Delta(\eta) &=& I_k(\eta) + J_k(\eta,x), & t = \tau_k,\ k = 1,2,\dots, \\ \Delta(x) &=& L_k(x), & t = \tau_k,\ k = 1,2,\dots, \\ \eta(t_0^+) &=& \eta_0,\ \ x(t_0^+) = x_0, & \end{array} \tag{24}$$

with $\overline{A} := A - BK$ is a Hurwitz matrix and $\overline{g}(t,\eta,x) := h(t,\eta,x) + g(t,\eta,x)(v(t) - Kx)$. Let us consider the following subsystems

$$\begin{array}{rcll} \dot{\eta} &=& w(t,\eta), & t \neq \tau_k,\ k = 1,2,\dots, \\ \Delta(\eta) &=& I_k(\eta), & t = \tau_k,\ k = 1,2,\dots, \\ \eta(t_0^+) &=& \eta_0, & \end{array} \tag{25}$$

and

$$\begin{array}{rcll} \dot{x} &=& \overline{A}x + Bv(t), & t \neq \tau_k,\ k = 1,2,\dots, \\ \Delta(x) &=& L_k(x), & t = \tau_k,\ k = 1,2,\dots, \\ x(t_0^+) &=& x_0. & \end{array} \tag{26}$$

Let us assume that

1. Systems (25) and (26) are assumed to be U.P.A.S,
2. $\|g(t,\eta,x)\| \le \frac{\chi(t)}{\|v\| + \|K\|\|x\|}$ where $\chi \in \mathcal{A}$,
3. $\|h(t,\eta,x)\| \le \psi(t)$ such that $\psi \in \mathcal{A}$,
4. $\|I_k(\eta_1) - I_k(\eta_2)\| \le M_k\|\eta_1 - \eta_2\|$, such that $\sum_{k\ge1} M_k$ converges,
5. $\|J_k(\eta,x)\| \le \delta_k$ and there exists a constant $\delta > 0$, such that $\sum_{k\ge1} \delta_k e^{-\delta s_k}$ converges.

Then all conditions of Theorem 4.2 are satisfied and cascade system (24) is U.P.A.S. The planar compass-gait biped of Figure 2 is one of the classic examples and yet it still holds some interesting challenges from the standpoint of global nonlinear control. Referring to Figure 2, the single-support phase dynamics are represented by the continuous 2nd-order differential equation

$$D(\theta)\ddot{\theta} + C(\theta,\dot{\theta})\dot{\theta} + G(\theta) = \tau, \ \ \theta = (\theta_s, \theta_{ns})^T \in \mathcal{D} \subset \mathbb{R}^2, \tag{27}$$

where the matrices D, C and G are given symbolically by

$$D(\theta) = \begin{bmatrix} \frac{\ell^2}{4}(5m + 4m_h) & -\frac{\ell^2 m}{2}\cos(\theta_s - \theta_{ns}) \\ -\frac{\ell^2 m}{2}\cos(\theta_s - \theta_{ns}) & \frac{\ell^2 m}{4} \end{bmatrix},$$

$$C(\theta,\dot{\theta}) = \frac{\ell^2 m}{2}\begin{bmatrix} 0 & -\dot{\theta}_{ns}\sin(\theta_s - \theta_{ns}) \\ \dot{\theta}_{ns}\sin(\theta_s - \theta_{ns}) & 0 \end{bmatrix}$$

and $G(\theta) = \begin{bmatrix} -g\ell(3m+2m_h)\frac{\sin(\theta_s)}{2} \\ g\ell(m)\frac{\sin(\theta_{ns})}{2} \end{bmatrix}$. The continuous-time single-support phase is defined by constraint $h(\theta) \geq 0$, where scalar

$$h(\theta) = \ell\big((\cos(\theta_s) - \cos(\theta_{ns})) + (\sin(\theta_s) - \sin(\theta_{ns}))\tan(\gamma)\big)$$

gives the height of the swing foot above ground with slope angle γ. The instantaneous impact event from foot-ground strike is indicated by the guard condition/switching surface

$$\mathcal{G} = \{(\theta, \dot{\theta}) | h(\theta) = 0,\, \dot{h} = (\nabla_\theta h)\dot{\theta} < 0\}.$$

We model these impulsive events as perfectly plastic (inelastic) collisions, so any solution trajectory $x(t) = (\theta(t), \dot{\theta}(t))$ intersecting the ground plane is subjected to the discontinuous impact map $\Delta : \mathcal{G} \longrightarrow T\mathcal{Q}$. Thus, we have the impulsive dynamical system

$$\begin{aligned} D(\theta)\ddot{\theta} + C(\theta,\dot{\theta})\dot{\theta} + G(\theta) &= \tau, \qquad (\theta,\dot{\theta}) \notin \mathcal{G},\, t \notin \mathcal{T}, && (28)\\ (\theta^+,\dot{\theta}^+) &= \Delta(\theta^-,\dot{\theta}^-),\ (\theta,\dot{\theta}) \in \mathcal{G},\, t \in \mathcal{T}, && (29) \end{aligned}$$

where $\mathcal{T} \subset]0,+\infty[$. The $-$ and the $+$ signs denote the state variables, respectively, before and after the collision. For brevity we defer the details regarding map Δ. In the literature, the resetting set $\mathcal{T} \times \mathcal{G}$ is defined in terms of a countable number of functions $\tau_k : \mathcal{D} \longrightarrow]0,+\infty[$, and is given by

$$\mathcal{T} \times \mathcal{G} = \bigcup_{k\in\mathbb{N}^*} \Big\{(\tau_k(\theta),\theta),\ \theta \in \mathcal{D}\Big\}.$$

The time-dependent impulsive dynamical systems can be written as (28) and (29) with $\mathcal{T} \times \mathcal{G}$ defined as $\mathcal{T} \times \mathcal{G} \triangleq \{\tau_1, \tau_2, \ldots\} \times \mathcal{D}$ where $0 \leq \tau_1 < \tau_1 < \cdots < \tau_k <$ are prescribed resetting times.

Now, (28) and (29) can be rewritten in the form of the time-dependent impulsive dynamical system

$$\begin{aligned} D(\theta)\ddot{\theta} + C(\theta,\dot{\theta})\dot{\theta} + G(\theta) &= \tau, \qquad t \neq \tau_k,\, k \in \mathbb{N}^*, && (30)\\ (\theta^+,\dot{\theta}^+) &= \Delta(\theta^-,\dot{\theta}^-),\ t = \tau_k,\, k \in \mathbb{N}^*. && (31) \end{aligned}$$

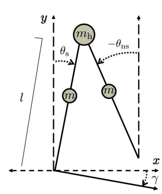

Fig. 2. Model diagram for the planar compass-gait biped.

Next, we give another example of cascade impulsive system, in three dimension with a simulation.

Example 5.3. Consider the cascade impulsive system having the following form:

$$\begin{array}{rcll} \dot{x} & = & f(t,x)+g(t,x,y), & t\neq\tau_k,\ \ k=1,2,\dots \\ \dot{y} & = & h(t,y), & t\neq\tau_k,\ \ k=1,2\dots \\ \triangle x & = & I_k(x)+J_k(x,y), & t=\tau_k,\ \ k=1,2\dots \\ \triangle y & = & L_k(y), & t=\tau_k,\ \ k=1,2\dots \\ x(t_0^+) & = & x_0,\ y(t_0^+)=y_0 & \end{array} \tag{32}$$

where $x=[x_1,x_2]^T\in\mathbb{R}^2$ and $y\in\mathbb{R}$. The functions f, h are given by

$$f(t,x)=\begin{bmatrix} f_1(t,x) \\ \\ f_2(t,x)\end{bmatrix}=\begin{bmatrix} -a_1x_1+\alpha_1(x_1)+x_1\alpha_2(x_2)-50\pi^2\dfrac{\sin(t)}{t} \\ -a_2x_2+x_2\beta_1(x_1)+\beta_2(x_2)-20\pi^3\frac{\cos(t)}{1+t}\end{bmatrix} \tag{33}$$

and

$$h(t,y)=-a_3y+\Psi(y)-5\pi\frac{\cos^3(t)}{1+t}+1 \tag{34}$$

such that

- $a_i>\frac{1}{2}$, for $i=1,2,3$.
- For all $s\in\mathbb{R}$, we have $s(2\Psi(s)-1)\leq 0$, $s\alpha_1(s)\leq 0$, $s\beta_2(s)\leq 0$, $\alpha_2(s)\leq 0$ and $\beta_1(s)\leq 0$.

The interconnection term $g(t,x,y)$ is described by

$$g(t,x,y)=\begin{bmatrix} g_1(t,x,y) \\ \\ g_2(t,x,y)\end{bmatrix}=\begin{bmatrix} 10^5\rho(t)^{\frac{1}{3}}x_2e^{-x_2^2-y^2}-\dfrac{\rho(t)x_1}{1+\|x\|^2} \\ -10^2\rho(t)^{\frac{1}{3}}x_1e^{-2x_1^2-4x_2^2-y^2}-\dfrac{\pi\rho(t)x_2}{2+\|x\|^2}\end{bmatrix},$$

where $\rho(t)$ is the integrable unbounded function defined in Example 5.1.
The impulsive jumps are characterized by

$$\begin{array}{rcl} \Delta(x) & = & \Delta(x_1,x_2)=I_k(x)+J_k(x,y), \\ \Delta y & = & -\dfrac{\cos^2(k)}{k^2}\dfrac{y}{1+y^2}, \end{array}$$

where $I_k(x)=-\frac{2}{(k\pi)^k}[x_1,x_2]^T$ and $J_k(x,y)=-\pi\frac{\cos^2(k)}{k^3}\frac{1}{(1+x_1^4+x_2^4+y^4))}[x_1,x_2]^T$.

Using $V(t,x)=x_1^2+x_2^2$ and $W(t,y)=y^2$ as Lyapunov functions for the isolated sub-systems (12) and (13), with f and h are given by (33) and (34) respectively, we can show by Theorem 3.1 that they are U.P.A.S. Moreover, the interconnection term $g(t,x,y)$ satisfies **(H3)** with

$$\psi(t)=10^2e^{-\frac{1}{2}}\sqrt{1+10^6}\rho(t)^{\frac{1}{3}}+\sqrt{2(\pi^2+1)}\rho(t),$$

and the assumption **(H4)** is satisfied with

$$M_k = \frac{2}{(\pi k)^k}, \quad \mu_k = \frac{\cos^2(k)}{k^3}, \forall\, k \in \mathbb{N}^*.$$

Finally we can deduce that all assumptions of Theorem 4.2 are satisfied. Then the cascade system (32) is U.P.A.S. The convergence dynamics of the impulsive cascade system (32) is shown in Figure 3.

For simulation, we take, $\Psi(s) = -s^5 + \frac{1}{2}$, $\alpha_1(s) = \beta_2(s) = -s^3$, $\alpha_2(s) = \beta_1(s) = -s^4$, $a_1 = a_2 = a_3 = 1$ and $(x_1(0), x_2(0), x_3(0)) = (-4.4, 2.4, \pi)$, we obtain the dynamics presented in Figures 3 and 4.

Fig. 3. Trajectories of cascade system (32).

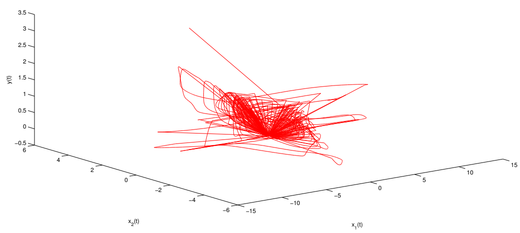

Fig. 4. Boundedness of solution of cascade system (32).

6. CONCLUSION

In this paper, we have established some sufficient conditions for uniform practical asymptotic stability of nonlinear impulsive systems using Lyapunov function. We have proved a converse Lyapunov Theorem for uniform practical asymptotic stability of nonlinear impulsive systems, which states that an U.P.A.S. nonlinear impulsive system admits a Lyapunov function satisfying all mentioned sufficient conditions. Some applications of our converse Lyapunov Theorem for uniform practical asymptotic stability of perturbed impulsive systems and cascade impulsive systems are given. Furthermore, a numerical examples have been shown, to verify our theoretical results and their effectiveness. Also, Some simulation results are given to illustrate the applicability of our results.

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Appendix

A. PROOF OF THEOREM 3.3

The proof of Theorem 3.2 is inspired by our recent work [9] and it is divided into five stages. Assume that the impulsive system (1) is U.P.A.S. Then, there exist a class $\mathcal{KL}$ function $\beta(\cdot,\cdot)$ and a positive constant ρ such that, for all initial condition $(t,x)\in\mathbb{R}_+\times\mathbb{R}^n$, we have $\|x(s,t,x)\|\leq\rho+\beta(\|x\|,s-t)$, for all $s\geq t\geq 0$ and $x\in\mathcal{B}_r^n$.

1. **First step:** First, it is not difficult to establish that
$$\alpha_1(\|x\|)\leq V(t,x)\leq\alpha_2(\|x\|)+a,$$
where $\alpha_1(s)=\varphi(s)$, $\alpha_2(s)=\frac{1}{2}\varphi^2\big(2\beta(s,0)\big)\Big)$ and $a=\frac{1}{2}\sup_{t\geq0}\theta^2(t)+\varphi(2\rho)\sup_{t\geq0}\theta(t)\in\mathbb{R}_+$. Then, the property 1 of Theorem 3.2 is proved. Notice that for all $t\in\mathbb{R}_+\backslash\{\tau_k,k\in\mathbb{N}^*\}$ and $x\in\mathcal{B}_r^n$, we have
$$0\leq\varphi\big(\|x(t+s,t,x)\|\big)\frac{1+\theta(t)s}{1+s}\leq\varphi(\rho+\beta(r,0))\theta(t)\in\mathcal{B}(\mathbb{R}_+).$$
Then, there exists $T>0$ such that for $t\neq\tau_k$ and $x\in\mathcal{B}_r^n$
$$V(t,x)=\sup_{0\leq s\leq T}\Big\{\varphi\big(\|x(t+s,t,x)\|\big)\frac{1+\theta(t)s}{1+s}\Big\}.$$
2. **Second step:** Let $x_1=x(t+h,t,x)$, it follows that for all $x\in\mathcal{B}_r^n$ and $t\neq\tau_k$, we have
$$\begin{aligned}
V(t+h,x_1)&=\sup_{s\geq0}\Big\{\varphi\big(\|x(t+h+s,t+h,x_1)\|\big)\frac{1+\theta(t+h)s}{1+s}\Big\},\\
&=\sup_{\tau\geq h}\Big\{\varphi\big(\|x(t+\tau,t,x)\|\big)\frac{1+(\tau-h)\theta(t+h)}{1+\tau-h}\Big\},\\
&=\frac{1+h}{1+h\theta(t)}\sup_{\tau\geq h}\Big\{\varphi\big(\|x(t+\tau,t,x)\|\big)\frac{1+h\theta(t)}{1+h}\frac{1+(\tau-h)\theta(t+h)}{1+\tau-h}\Big\},\\
&\leq\frac{1+h}{1+h\theta(t)}\sup_{\tau\geq h}\Big\{\varphi\big(\|x(t+\tau,t,x)\|\big)\frac{1+h\theta(t)}{1+h}\frac{1+\theta(t+h)\tau}{1+\tau}\Big\},\\
&=\frac{1+h}{1+h\theta(t)}\sup_{\tau\geq h}\Big\{\varphi\big(\|x(t+\tau,t,x)\|\big)\frac{1+h\theta(t)}{1+h}\Big[\frac{1+\theta(t)\tau}{1+\tau}\\
&+\frac{(\theta(t+h)-\theta(t))\tau}{1+\tau}\Big]\Big\},\\
&\leq\frac{1+h}{1+h\theta(t)}\sup_{\tau\geq h}\Big\{\varphi\big(\|x(t+\tau,t,x)\|\big)\frac{1+h\theta(t)}{1+h}\Big[\frac{1+\theta(t)\tau}{1+\tau}\\
&+(\theta(t+h)-\theta(t))\Big]\Big\},\\
&\leq\frac{1+h}{1+h\theta(t)}\sup_{\tau\geq h}\Big\{\varphi\big(\|x(t+\tau,t,x)\|\big)\Big[\frac{1+\theta(t)\tau}{1+\tau}+\frac{h\theta(t)}{1+h}\frac{1+\theta(t)\tau}{1+\tau}\\
&+\frac{1+h\theta(t)}{1+h}(\theta(t+h)-\theta(t))\Big]\Big\},\\
&\leq\frac{1+h}{1+h\theta(t)}\sup_{\tau\geq h}\Big\{\varphi\big(\|x(t+\tau,t,x)\|\big)\Big[\frac{1+\theta(t)\tau}{1+\tau}+h\theta^2(t)\\
&+\frac{1+h\theta(t)}{1+h}(\theta(t+h)-\theta(t))\Big]\Big\},\\
&\leq\frac{1+h}{1+h\theta(t)}V(t,x)+C_{r,\rho}\big(\frac{h(1+h)\theta^2(t)}{1+h\theta(t)}+\theta(t+h)-\theta(t)\big),
\end{aligned}$$

with $C_{r,\rho} = \varphi(2\rho) + \varphi(2\beta(r,0))$.
Therefore,

$$\begin{aligned}
D^+V(t,x) &= \limsup_{h\to 0^+} \frac{V\Big(t+h, x(t+h,t,x)\Big) - V(t,x)}{h} \\
&\le \lim_{h\to 0^+} \Big[\frac{1}{h}\Big(\frac{1+h}{1+h\theta(t)} - 1\Big)V(t,x) \\
&+ \Big(\frac{(1+h)\theta^2(t)}{1+h\theta(t)} + \frac{\theta(t+h)-\theta(t)}{h}\Big)\Big(\varphi(2\rho)+\varphi(2\beta(r,0))\Big)\Big] \\
&= -(\theta(t)-1)V(t,x) + C_{r,\rho}\Big(\theta'(t)+\theta^2(t)\Big).
\end{aligned}$$

It follows that,

$$D^+V(t,x) \le -\lambda V(t,x) + r(t),$$

with,

$$\lambda = \inf_{t\ge 0}\{\theta(t)\} - 1 > 0 \text{ and } r(t) = C_{r,\rho}\Big(\theta'(t)+\theta^2(t)\Big) \in \mathcal{A}.$$

Which proves property 2.

3. **Third step:** Let $x,\ y \in \mathcal{B}_r^n$ and $0 < \tau \le T$. By using Lemma 2.7, we obtain the following estimation

$$\|x(t+\tau,t,x) - x(t+\tau,t,y)\| \le \|x-y\|e^{\tau L_\rho} \prod_{t<\tau_k<t+\tau}(1+M_k).$$

It follows that, if $t \ne \tau_k,\ k \in \mathbb{N}^*$

$$\begin{aligned}
|V(t,x)-V(t,y)| &= \Big|\sup_{0\le s\le T}\Big\{\varphi\Big(\|x(t+s,t,x)\|\Big)\frac{1+s\theta(t)}{1+s}\Big\} \\
&- \sup_{0\le s\le T}\Big\{\varphi\Big(\|x(t+s,t,y)\|\Big)\frac{1+s\theta(t)}{1+s}\Big\}\Big|, \\
&\le \sup_{0\le s\le T}\Big\{\Big|\varphi\Big(\|x(t+s,t,x)\|\Big) - \varphi\Big(\|x(t+s,t,y)\|\Big)\Big|\frac{1+s\theta(t)}{1+s}\Big\}, \\
&\le \sup_{z\in[0,r+\beta(\rho,0)]}|\varphi'(z)| \sup_{0\le s\le T}\Big\{\Big|\|x(t+s,t,x)\| - \|x(t+s,t,y)\|\Big|\frac{1+s\theta(t)}{1+s}\Big\}, \\
&\le \sup_{z\in[0,r+\beta(\rho,0)]}|\varphi'(z)| \sup_{0\le s\le T}\Big\{e^{L_\rho s}\frac{1+s\theta(t)}{1+s}\prod_{t<\tau_k<t+s}(1+M_k)\Big\}\|x-y\|, \\
&:= L(t)\|x-y\|,
\end{aligned}$$

where,

$$\begin{aligned}
L(t) &= \sup_{z\in[0,r+\beta(\rho,0)]}|\varphi'(z)| \sup_{0\le s\le T}\Big\{e^{L_\rho s}\frac{1+s\theta(t)}{1+s}\prod_{t<\tau_k<t+s}(1+M_k)\Big\}, \\
&\le e^{L_\rho T}\theta(t) \sup_{z\in[0,r+\beta(\rho,0)]}|\varphi'(z)| \prod_{t<\tau_k<t+T}(1+M_k), \quad t \ge 0.
\end{aligned} \tag{35}$$

Thus, property 3 is proved for $t \ne \tau_k$.

4. **Fourth step:** Let $\eta(t, t_0, x_0)$ be the solution of the impulsive system (1) starting from x_0 at $t = t_0$. Since for $\tau_{k-1} < \lambda < \tau_k < \mu < \tau_{k+1}$, and $s > \mu$ the relation
$$x(s, \mu, \eta(\mu, \tau_k, x + I_k(x))) = x(s, \lambda, \eta(\lambda, \tau_k, x))$$
holds, then
$$V(\mu, \eta(\mu, \tau_k, x + I_k(x))) \leq V(\lambda, \eta(\lambda, \tau_k, x)).$$
Passing to the limit as $\mu \to \tau_k^+$ and $\lambda \to \tau_k^-$, we obtain
$$V(\tau_k^+, x(\tau_k^+)) \leq V(\tau_k, x(\tau_k)) \leq (1 + u_k)V(\tau_k, x(\tau_k)) + d_k,$$
where $(u_k)_{k\geq 1}$, $(d_k)_{k\geq 1}$ are any positive sequences such that
   - $\sum_{k\geq 1} u_k$ converges
   - $\sum_{k\geq 1} d_k e^{-\alpha s_k}$ converges for some $\alpha > 0$ with $s_k = \tau_{k+1} - \tau_k$, $k \in \mathbb{N}^*$.

   Let us consider the numerical sequence $(\gamma_k)_{k\geq 1}$ defined by, for all $k \in \mathbb{N}^*$, $\gamma_k = d_k e^{-\alpha s_k}$, $\alpha > 0$. Note that $\sum_{k\geq 1} \gamma_k$ converges. Which proves property 4.
5. **Fifth step:** Now, let $\tau_k \in \mathbb{R}_+$ and $x \in \mathcal{B}_r^n$ be fixed and $t_i \in]\tau_k, \tau_{k+1}[$, $x_i \in \mathcal{B}_r^n$ and $u_i = x(t_i, \tau_k, x)$ for $i = 1, 2$. Then, using [9], we get
$$\Big|V(t_i, x_i) - V(t_i, u_i)\Big| \leq L(t_i)\|x_i - x\| + L(t_i)N(t_i)(f_0 + L_\rho\|x\|),\ i = 1, 2,$$
where
$$N(t_i) = t_i - \tau_k + L_\rho \int_{\tau_k}^{t_i} (s - \tau_k)e^{L_\rho(t_i - s)}\,\mathrm{d}s,\ t_i \in]\tau_k, \tau_{k+1}[.$$
Moreover, it is easy to show that
$$\lim_{t_i \to \tau_k^+} L(t_i) = L(\tau_k^+),\quad \lim_{t_i \to \tau_k^+} N(t_i) = 0.$$
Denote
$$a_t(\delta) = \sup_{s\geq\delta} \Big\{\varphi\Big(\|x(t + s, t, x)\|\Big)\frac{1 + s\theta(t)}{1 + s}\Big\}.$$
The function $a_t(\delta)$ is non-increasing for $\delta \geq 0$ and $\lim_{\delta\to 0^+} a_t(\delta) = a_t(0)$ since $\varphi\Big(\|x(t + s, t, x)\|\Big)\frac{1+s\theta(t)}{1+s}$ is a bounded and piecewise continuous function for $s \geq 0$ and is continuous in some neighborhood of $s = 0$. Then, we obtain
$$\begin{aligned}
\Big|V(t_1, u_1) - V(t_2, u_2)\Big| &= \Big|\sup_{s\geq 0}\Big\{\varphi\Big(\|x(t_1 + s, t_1, u_1)\|\Big)\frac{1 + s\theta(t_1)}{1 + s}\Big\} \\
&\quad - \sup_{s\geq 0}\Big\{\varphi\Big(\|x(t_2 + s, t_2, u_2)\|\Big)\frac{1 + s\theta(t_2)}{1 + s}\Big\}\Big|, \\
&= \Big|\sup_{\tau\geq t_1-\tau_k}\Big\{\varphi\Big(\|x(\tau_k + \tau, t_1, x(t_1, \tau_k, x))\|\Big)\frac{1 + \theta(t_1)(\tau_k + \tau - t_1)}{1 + \tau_k + \tau - t_1}\Big\} \\
&\quad - \sup_{\tau\geq t_2-\tau_k}\Big\{\varphi\Big(\|x(\tau_k + \tau, t_2, x(t_2, \tau_k, x))\|\Big)\frac{1 + \theta(t_2)(\tau_k + \tau - t_2)}{1 + \tau_k + \tau - t_2}\Big\}\Big|, \\
&= \Big|\sup_{\tau\geq t_1-\tau_k}\Big\{\varphi\Big(\|x(\tau_k + \tau, \tau_k, x)\|\Big)\frac{1 + \theta(t_1)(\tau_k + \tau - t_1)}{1 + \tau_k + \tau - t_1}\Big\} \\
&\quad - \sup_{\tau\geq t_2-\tau_k}\Big\{\varphi\Big(\|x(\tau_k + \tau, \tau_k, x))\|\Big)\frac{1 + \theta(t_2)(\tau_k + \tau - t_2)}{1 + \tau_k + \tau - t_2}\Big\}\Big|, \\
&\leq \Big|a_{\tau_k}(t_1 - \tau_k) - a_{\tau_k}(t_2 - \tau_k)\Big| + I(t_1, t_1 - \tau_k) + I(t_2, t_2 - \tau_k) \longrightarrow 0,
\end{aligned}$$

as $t_i \to \tau_k^+$, $i = 1, 2$, where

$$I(t_1, t_1 - \tau_k) = \sup_{\tau \geq t_1 - \tau_k} \Big\{\varphi\Big(\|x(\tau_k + \tau, \tau_k, x)\|\Big)\Big\}\Big(\theta(t_1) - \theta(\tau_k) + (t_1 - \tau_k)(\theta(t_1) - 1)\Big)$$

and

$$I(t_2, t_2 - \tau_k) = \sup_{\tau \geq t_2 - \tau_k} \Big\{\varphi\Big(\|x(\tau_k + \tau, \tau_k, x)\|\Big)\Big\}\Big(\theta(t_2) - \theta(\tau_k) + (t_2 - \tau_k)(\theta(t_2) - 1)\Big).$$

Thus, we obtain the following inequality

$$\Big|V(\tau_k^+, x_1) - V(\tau_k^+, x_2)\Big| \leq L(\tau_k^+)\Big(\|x_1 - x\| + \|x_2 - x\|\Big).$$

The last inequality is satisfied for all x_1, $x_2 \in \mathcal{B}_r^n$, in particular if $x_2 = x$, we get

$$\Big|V(\tau_k^+, x_1) - V(\tau_k^+, x)\Big| \leq L(\tau_k^+)\|x_1 - x\|.$$

It follows that, the mapping $x \mapsto V(\tau_k^+, x)$ is continuous on the compact set $\mathcal{B}_r^n$. Then, we conclude that $V(\tau_k^+, x)$ exists and the prove of property 1 is finished.
To prove the continuity of the function $V(t, x)$, let $x, x_1 \in \mathcal{B}_r^n$, $\tau_{k-1} < t < \tau_k$ and $\delta > 0$ be such that $t + \delta < \tau_k$. Then

$$\begin{aligned}|V(t+\delta, x_1) - V(t, x)| \leq{}& |V(t+\delta, x_1) - V(t+\delta, x)| + |V(t+\delta, x) - V(t+\delta, x(t+\delta, t, x)| \\ &+ |V(t+\delta, x(t+\delta, t, x) - V(t, x)|.\end{aligned} \tag{36}$$

Property 3 implies the estimation

$$|V(t+\delta, x_1) - V(t+\delta, x)| \leq L(t+\delta)\|x_1 - x\|,$$

and

$$|V(t+\delta, x) - V(t+\delta, x(t+\delta, t, x)| \leq L(t+\delta)\|x - x(t+\delta, t, x)\|.$$

Since, for $t \neq \tau_k$, $\lim_{\delta\to 0} L(t+\delta) = L(t)$ and $\lim_{\delta\to 0} \|x - x(t+\delta, t, x)\| = 0$, then the first two terms in the right-hand side of estimate (36) are small if $\|x - x_1\|$ and δ are small. For the third term in (36), we proceed as follows

$$\begin{aligned}0 &\leq |V(t+\delta, x(t+\delta, t, x) - V(t, x)|,\\
&= |\sup_{s\geq 0}\Big\{\varphi\Big(\|x(t+\delta+s, t+\delta, x(t+\delta, t, x))\|\Big)\frac{1 + s\theta(t+\delta)}{1+s}\Big\}\\
&- \sup_{s\geq 0}\Big\{\varphi\Big(\|x(t+s, t, x)\|\Big)\frac{1+s\theta(t)}{1+s}\Big\}\Big|,\\
&= \Big|\sup_{\tau\geq\delta}\Big\{\varphi\Big(\|x(t+\tau, t, x)\|\Big)\frac{1+\theta(t+\delta)(\tau-\delta)}{1+\tau-\delta}\Big\}\\
&- \sup_{s\geq 0}\Big\{\varphi\Big(\|x(t+s, t, x)\|\Big)\frac{1+s\theta(t)}{1+s}\Big\}\Big|,\\
&\leq |a_t(\delta) - a_t(0)| + \Big|a_t(\delta) - \sup_{\tau\geq\delta}\Big\{\varphi\Big(\|x(t+\tau, t, x)\|\Big)\frac{1+\theta(t+\delta)(\tau-\delta)}{1+\tau-\delta}\Big\}\Big|,\\
&\leq |a_t(\delta) - a_t(0)| + \sup_{\tau\geq\delta}\Big\{\varphi\Big(\|x(t+\tau, t, x)\|\Big)\Big|\frac{1+\tau\theta(t)}{1+\tau} - \frac{1+\theta(t+\delta)(\tau-\delta)}{1+\tau-\delta}\Big|\Big\},\\
&\leq |a_t(\delta) - a_t(0)| + \Big(\theta(t+\delta) - \theta(t) + \delta\theta(t+\delta)\Big)\sup_{\tau\geq\delta}\Big\{\varphi\Big(\|x(t+\tau, t, x)\|\Big)\Big\} \longrightarrow 0\end{aligned}$$

as $\delta \longrightarrow 0$. Hence, $V(t,x)$ is continuous for $x \in \mathcal{B}_r^n$ and $t \neq \tau_k$. The last two inequalities lead to the following estimation,

$$\Big|V(t_i,x_i) - V(t_i,u_i)\Big| \leq L(t_i)\|x_i - x\| + L(t_i)N(t_i)(f_0 + L_\rho\|x\|),\ i = 1,2.$$

Moreover, it is not difficult to show that

$$\lim_{t_i \to \tau_k^+} L(t_i) = L(\tau_k^+),\ \lim_{t_i \to \tau_k^+} N(t_i) = 0,$$

for $i = 1, 2$ and we have the following inequality

$$\Big|V(\tau_k^+, x_1) - V(\tau_k^+, x_2)\Big| \leq L(\tau_k^+)\Big(\|x_1 - x\| + \|x_2 - x\|\Big).$$

The last inequality is satisfied for all x_1, $x_2 \in \mathcal{B}_r^n$, in particular if $x_2 = x$, we obtain

$$\Big|V(\tau_k^+, x_1) - V(\tau_k^+, x)\Big| \leq L(\tau_k^+)\|x_1 - x\|.$$

It follows that, the mapping $x \mapsto V(\tau_k^+, x)$ is continuous on the compact set $\mathcal{B}_r^n$. Then, we conclude that $V(\tau_k^+, x)$ exists. Similarly, the existence of the limit $V(\tau_k^-, x)$ can be proved and since the equality $V(\tau_k^-, x) = V(\tau_k, x)$ holds by definition, then $V(t,x) \in \mathcal{V}_2$. This finishes the proof of Theorem 3.2.

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