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ON IDEALS IN DE MORGAN RESIDUATED LATTICES

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In this paper, we introduce a new class of residuated lattices called De Morgan residuated lattices, we show that the variety of De Morgan residuated lattices includes important subvarieties of residuated lattices such as Boolean algebras, MV-algebras, BL-algebras, Stonean residuated lattices, MTL-algebras and involution residuated lattices. We investigate specific properties of ideals in De Morgan residuated lattices, we state the prime ideal theorem and the pseudo-complementedness of the ideal lattice, we pay attention to prime, maximal, $\odot$-prime ideals and to ideals that are meet-irreducible or meet-prime in the lattice of all ideals. We introduce the concept of an annihilator of a given subset of a De Morgan residuated lattice and we prove that annihilators are a particular kind of ideals. Also, regular annihilator and relative annihilator ideals are considered.

Keywords: residuated lattice, De Morgan laws, filter, deductive system, ideal, $\cap$-prime, $\cap$-irreducible, annihilator

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1. INTRODUCTION

Recently, a lot of work has been done with respect to the co-annihilators in varieties of residuated lattices. For example, in 2013, C. Mureşan ([12]) investigated substructures of residuated lattices and the filter theory by the concept of co-annihilators in order to characterise Co-Stone residuated lattices. L. Leuştean ([14]) used them in order to construct the Baer extention of BL-algebras and in 2016, F. G. Maroof et al. ([9]) published a study on co-annihilators and relative co-annihilators in residuated lattices. In 2013, C. Lele et al. ([8]) constructed some examples to show that, unlike in MV-algebras, ideals and filters are dual but behave differently in BL-algebras. And in 2014, J. Rachunek and D. Salounova presented some results on ideals and involutive filters in residuated lattices (see the presentation at the meeting of SSAOS 2014, Stara Lesna, September 6-12 [16]). In 2015, D. Buşneag et al. ([5]) investigated the variety of Stonean residuated lattices and they discussed it from the view of ideal theory. In 2016, Yu Xi Zou et al. ([18]) published a study on ideals and annihilator ideals in BL-algebras. In this paper, motivated by the previous research on co-annihilators in residuated lattices

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(see [9, 12, 14]), ideals in MV-algebras and BL-algebras (see [8]), annihilator ideals in BL-algebras (see [18]) and implicative ideals in Stonean residuated lattices (see [5]), we study the notions of ideal and annihilator ideal in a new class of residuated lattices called De Morgan residuated lattices.

This paper is organized as follows: in the next section we give some preliminaries including the basic definitions, some examples of residuated lattices, rules of calculus and theorems that are needed in the sequel. In Section 3, we study ideals in residuated lattices, and it contains two subsections: in Subsection 3.1, we study in the general case of residuated lattices the relationships between various notions of ideals from the literature such as left and right ideals (see [8, 16]) and implicative ideals (see [5]) and we show that these notions coincide (Theorem 3.5), after that we pay attention to prime, maximal and $\odot$-prime ideals. We propose some characterizations for maximal ideals in residuated lattices (Theorem 3.9). In Subsection 3.2, we define $\odot$-prime ideals and we study them in order to establish the relationship between ideals and filters in residuated lattices. In Section 4, we study ideals in a new class of residuated lattices called De Morgan residuated lattices, it contains five subsections: in Subsection 4.1, we define the class of De Morgan residuated lattices (the residuated lattice L is called De Morgan if the De Morgan law $(x \wedge y)^* = x^* \vee y^*$, for all $x, y \in L$ hold), we note that De Morgan residuated lattices are not same as residuated De Morgan lattices. We give the relationship between De Morgan residuated lattices and other algebraic structures, in this sense we show that the variety of De Morgan residuated lattices includes important subvarieties of residuated lattices such as Boolean algebras, MV-algebras, BL-algebras, Stonean residuated lattices, MTL-algebras and involution residuated lattices. We note that the classes of De Morgan residuated lattices and semi-G-algebras are different. In Subsection 4.2, our goal is to study ideals in the variety of De Morgan residuated lattices, we investigate specific properties of ideals in De Morgan residuated lattices, we state the prime ideal theorem (Theorem 4.14) and the pseudo-complementedness of the ideal lattice (Theorem 4.30), we pay attention to prime, maximal and to ideals that are meet-irreducible or meet-prime in the lattice of all ideals. In Subsection 4.3, since the notion of annihilator is missing in De Morgan residuated lattices, we fill this gap by introducing the concept of annihilator for De Morgan residuated lattices, we show that annihilators are a particular kind of ideals and we put in evidence some properties of them. We get that the ideal lattice $(\mathcal{I}_i(L), \subseteq)$ is pseudo-complemented, and for any ideal I, its pseudo-complement is the annihilator $I^{\perp}$. We define $An(L)$ to be the set of all annihilators of L, then we have that $(An(L), \cap, \vee_{An(L)}, \perp, \{0\}, L)$ is a Boolean algebra. In Subsection 4.4, we introduce the notion of regular ideal and give a notation $R(I)$. We show that $(R(\mathcal{I}_i(L)), \sqcap, \sqcup, R(0), R(L))$ is a pseudo-complemented lattice, a complete Brouwerian lattice and an algebraic lattice, when L is a totally ordered De Morgan residuated lattice. In Subsection 4.5, we introduce the annihilator of a nonempty subset X of L with respect to an ideal I and study some properties of them. As an application, we show that if I and J are ideals in the De Morgan residuated lattice L, then $(J, I)^{\perp}$ is the relative pseudo-complement of J with respect to I in the ideal lattice $(\mathcal{I}_i(L), \subseteq)$.

2. PRELIMINARIES

Definition 2.1. (Galatos et al. [10]) A *residuated lattice* $(L, \vee, \wedge, \odot, \rightarrow, 0, 1)$ is an algebra of type $(2, 2, 2, 2, 0, 0)$ equipped with an order $\leq$ such that

LR_1 : $(L, \vee, \wedge, 0, 1)$ is a bounded lattice relative to $\leq$;

LR_2 : $(L, \odot, 1)$ is a commutative ordered monoid;

LR_3 : $\odot$ and $\rightarrow$ form an adjoint pair, i. e., $a \odot x \leq b$ iff $x \leq a \rightarrow b$, for all $x, a, b \in L$.

For examples of residuated lattices see [4, 10, 13, 15, 17].

In what follows by L we denote the universe of a residuated lattice (unless otherwise specified). For $x \in L$ and $n \geq 0$ we define $x^* = x \rightarrow 0$, $x^{**} = (x^*)^*$, $x^0 = 1$ and $x^n = x^{n-1} \odot x$ for $n \geqslant 1$.

For $x, y, z \in L$, we have the following rules of calculus (see [4, 10, 15, 17]):

(c_1) $x \rightarrow x = 1,\ x \rightarrow 1 = 1,\ 1 \rightarrow x = x$;
(c_2) $x \leq y$ iff $x \rightarrow y = 1$;
(c_3) If $x \leq y$, then $z \odot x \leq z \odot y,\ z \rightarrow x \leq z \rightarrow y,\ y \rightarrow z \leq x \rightarrow z$;
(c_4) If $x \leq y$, then $y^* \leq x^*,\ x^{**} \leq y^{**}$;
(c_5) $x \odot x^* = 0,\ x \leq x^{**},\ x^{***} = x^*$;
(c_6) $x \odot y = 0$ iff $x \leq y^*$;
(c_7) $x \odot (y \vee z) = (x \odot y) \vee (x \odot z)$;
(c_8) $x \vee (y \odot z) \geq (x \vee y) \odot (x \vee z)$;
(c_9) $x \rightarrow (y \wedge z) = (x \rightarrow y) \wedge (x \rightarrow z)$;
(c_{10}) $x \odot (x \rightarrow y) \leq x \wedge y$;
(c_{11}) $(x \vee y)^* = x^* \wedge y^*$;
(c_{12}) $x^* \odot y^* \leq (x \odot y)^*,\ x^{**} \odot y^{**} \leq (x \odot y)^{**}$;
(c_{13}) $(x \rightarrow y^{**})^{**} = x \rightarrow y^{**}$;
(c_{14}) $x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z) = (x \odot y) \rightarrow z,\ (x \odot y)^* = x \rightarrow y^* = y \rightarrow x^*$.

Following the above mentioned literature, we consider the identities:

(i_1) $x \wedge y = x \odot (x \rightarrow y)$ divisibility;
(i_2) $(x^* \wedge y^*)^* = [x^* \odot (x^* \rightarrow y^*)]^*$ semi-divisibility;
(i_3) $(x \rightarrow y) \vee (y \rightarrow x) = 1$ prelinearity;
(i_4) $x^* \vee x^{**} = 1$ Stone property;
(i_5) $x^2 = x$ idenpotence;
(i_6) $x = x^{**}$ involution;
(i_7) $(x^2)^* = x^*$.

Then the *residuated lattice* L is called:

Divisible if L verifies (i_1);
Semi-divisible if L verifies (i_2);
MTL-algebra if L verifies (i_3);
BL-algebra if L verifies (i_1) and (i_3);
Stonean if L verifies (i_4);
G-algebra(*Heyting algebra*) if L verifies (i_5);
Involution if L verifies (i_6);
semi-G-algebra if L verifies (i_7).

Definition 2.2. (Piciu [15]) A *filter* (or implicative filter, $\odot$-filter) is a nonempty subset F of L such that
(F_1) If $x \leq y$ and $x \in F$, then $y \in F$;
(F_2) If $x, y \in F$, then $x \odot y \in F$.

Remark 2.3.
1. F is a filter of L iff $1 \in F$ and if $x, x \to y \in F$, then $y \in F$ (that is, F is a *deductive system* of L).
2. Every filter is a lattice filter in the lattice $(L, \wedge, \vee)$, but the converse need not hold ([15, 17]).

So, if we denote by $\mathcal{F}(L)$ ($\mathcal{F}_i(L)$) the set of all lattice filters (filters) of L, then $\mathcal{F}_i(L) \subseteq \mathcal{F}(L)$. We have ([3]), $\mathcal{F}_i(L) = \mathcal{F}(L)$ iff $x \odot y = x \wedge y$ for every $x, y \in L$.

Definition 2.4. (Mureşan [12]) We say that a proper filter $P \in \mathcal{F}_i(L)$ is a *prime* filter iff, for all $x, y \in L$, if $x \vee y \in P$, then $x \in P$ or $y \in P$.

We recall that a filter M of L is called *maximal* if $M \neq L$ and M is not strictly contained in a proper filter of L. Clearly, if we denote by $Spec_F(L)$ the set of all prime filters of L, and by $Max_F(L)$ the set of all maximal filters of L, then $Max_F(L) \subseteq Spec_F(L)$.

Proposition 2.5. (Galatos et al. [10], Piciu [15], Turunen [17]) Let L be a residuated lattice and $M \in \mathcal{F}_i(L)$, $M \neq L$. The following conditions are equivalent:

(i) $M \in Max_F(L)$;

(ii) If $x \notin M$, then there is $n \geq 1$ such that $(x^n)^* \in M$.

Proposition 2.6. (Buşneag et al. [4]) If $M \in Max_F(L)$, then $x \in M$ iff $x^{**} \in M$.

Theorem 2.7. (Blyth [2]) Let $(L, \wedge, \vee)$ be a lattice and let $f : L \times L \to L$ be a closure map. Then Imf is a lattice in which the lattice operations are given by $\inf f(a, b) = a \wedge b$, $\sup f(a, b) = f(a \vee b)$.

If L is a residuated lattice, then for $x, y \in L$ we define

$$x \oplus y = (x^* \odot y^*)^* \overset{(c_{14})}{=} x^* \to y^{**}. \tag{1}$$

The operation $x \oplus y$ will be called *strong addition.*

Lemma 2.8. (Buşneag et al. [5]) Let L be a residuated lattice and $x, y, z, t \in L$. Then:
(c_{15}) $x \oplus 0 = x^{**},\ x \oplus 1 = 1, x \oplus x^* = 1$;
(c_{16}) $x \oplus y = y \oplus x,\ x, y \leq x \oplus y$;
(c_{17}) $x \oplus (y \oplus z) = (x \oplus y) \oplus z$;
(c_{18}) If $x \leq y$, then $x \oplus z \leq y \oplus z$;
(c_{19}) If $x \leq y$, $z \leq t$, then $x \oplus z \leq y \oplus t$.

Remark 2.9. By Lemma 2.8, we conclude that the operation $\oplus$ is commutative, associative and compatible with the order relation.

For $x \in L$ and $n \geq 0$, we define $0 \cdot x = 0$ and $n \cdot x = [(n-1) \cdot x] \oplus x$ for $n \geq 1$. For simplicity, we denote $nx := n \cdot x$. Inductively, we deduce:

Corollary 2.10. If $x, y \in L$ and $m, n \geq 1$, then
(c_{20}) If $m \leq n$, then $mx \leq nx$;
(c_{21}) If $x \leq y$, then $mx \leq my$.

P r o o f. (c_{20}). Since $x \overset{(c_5)}{\leq} x^{**}$, $x \leq x^* \to x^{**} = x \oplus x = 2x$, we conclude that $x \leq 2x$, this is, if $m \leq n$, then $mx \leq nx$, for any natural numbers $m, n \geq 1$.

(c_{21}). Since $x \leq y$, $y^* \overset{(c_4)}{\leq} x^*$, $y^* \odot y^* \overset{(c_3)}{\leq} x^* \odot x^*$, $(x^* \odot x^*)^* \leq (y^* \odot y^*)^*$, $x \oplus x \leq y \oplus y$, $2x \leq 2y$, we conclude that $mx \leq my$, for every natural number $m \geq 1$. $\square$

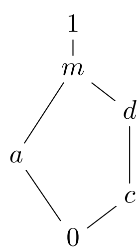
If L is a residuated lattice, then for $x, y \in L$ we define

$$x \oslash y = x^* \to y, \tag{2}$$

$$x \obslash y = y^* \to x. \tag{3}$$

The operation $x \oslash y = x^* \to y$ will be called *left addition*, and $x \obslash y = y^* \to x$ will be called *right addition* ([8, 16]). In the following examples we show that the left, right and strong additions are different in residuated lattices.

Example 2.11. In residuated lattices the left and right additions, respective the left and strong additions may not coincide. Indeed, if we consider the lattice $L = \{0, a, c, d, m, 1\}$ with $0 < a < m < 1$, $0 < c < d < m < 1$, but a incomparable with c and d.



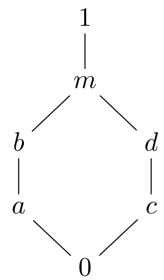
Then ([13], page 233) L is a residuated lattice with respect to the following operations:

$\to$	0	a	c	d	m	1
0	1	1	1	1	1	1
a	d	1	d	d	1	1
c	a	a	1	1	1	1
d	a	a	m	1	1	1
m	0	a	d	d	1	1
1	0	a	c	d	m	1

$\odot$	0	a	c	d	m	1
0	0	0	0	0	0	0
a	0	a	0	0	a	a
c	0	0	c	c	c	c
d	0	0	c	c	c	d
m	0	a	c	c	m	m
1	0	a	c	d	m	1

Since $a \oslash c = a^* \to c = d \to c = m$, $a \obslash c = c^* \to a = a \to a = 1$ and $a \oplus c = (a^* \odot c^*)^* = (d \odot a)^* = 0^* = 1$, we conclude that $a \oslash c \neq a \obslash c$, $a \oslash c \neq a \oplus c$, and $a \obslash c = a \oplus c$. Therefore, the left and right additions, respective the left and strong additions may not coincide.

Example 2.12. In a residuated lattices the left and right additions may not coincide with strong additions. Indeed, if we consider the lattice $L = \{0, a, b, c, d, m, 1\}$ with $0 < a < b < m < 1$, $0 < c < d < m < 1$ and elements $\{a, c\}$ and $\{b, d\}$ are pairwise incomparable.



Then ([13], page 234) L is a residuated lattice with respect to the following operations:

$\to$	0	a	b	c	d	m	1
0	1	1	1	1	1	1	1
a	d	1	1	d	d	1	1
b	d	m	1	d	d	1	1
c	b	b	b	1	1	1	1
d	b	b	b	m	1	1	1
m	0	b	b	d	d	1	1
1	0	a	b	c	d	m	1

$\odot$	0	a	b	c	d	m	1
0	0	0	0	0	0	0	0
a	0	a	a	0	0	a	a
b	0	a	a	0	0	a	b
c	0	0	0	c	c	c	c
d	0	0	0	c	c	c	d
m	0	a	a	c	c	m	m
1	0	a	b	c	d	m	1

Since $a \oslash c = a^* \to c = d \to c = m$, $a \obslash c = c^* \to a = b \to a = m$ and $a \oplus c = (a^* \odot c^*)^* = (d \odot b)^* = 0^* = 1$, we conclude that $a \oslash c = a \obslash c \neq a \oplus c$. Therefore, the left and right additions coincide and they are different than the strong addition.

3. IDEALS IN A RESIDUATED LATTICE

In residuated lattices literature ([5, 8, 16]) can be found at least two types of ideals, J. Rachunek et al. ([16]) defined the left and right ideals in the case of non-commutative residuated lattices, soon after, C. Lele et al. ([8]) published a study on left ideals in BL-algebras, and D. Buşneag et al. ([5]) defined the implicative ideals in Stonean residuated lattices. These notions of left, right and implicative ideals were defined differently. However, we will see that there are strong similarities between them. In this section, to avoid misunderstandings, we study the relationships between these notions of ideals in residuated lattices. Also, we study prime, maximal and $\odot$-prime ideals in residuated lattices.

3.1. Left, right and implicative ideals in a residuated lattice

Left and right ideals in a non-commutative residuated lattice L were defined and studied by J. Rachunek et al. ([16]). If L is a commutative residuated lattice, then $x^{\sim} = x^{-}$ (because $\rightarrow$ and $\rightsquigarrow$ coincide).

Definition 3.1. (Lele and Nganou [8], Rachunek and Salounova [16]) A nonempty subset I will be called:
(i) *a left ideal* (*L-ideal*, for short) of L if
(I_{l_1}) If $x \leq y$ and $y \in I$, then $x \in I$;
(I_{l_2}) If $x, y \in I$, then $x \oslash y \in I$.

(ii) *a right ideal* (*R-ideal*, for short) of L if
(I_{r_1}) If $x \leq y$ and $y \in I$, then $x \in I$;
(I_{r_2}) If $x, y \in I$, then $x \otimes y \in I$.

(iii) *a left-right ideal* (*LR-ideal*, for short) of L if it is both a left and right ideal.

Every L-ideal as well as every R-ideal of a residuated lattice L is a lattice ideal. Therefore, an LR-ideal is a lattice ideal. However, lattice ideals are not always LR-ideals, that we can see in Example 2.11, where $I = \{0, a, c, d, m\}$ is a lattice ideal, but $a \otimes c = c^* \rightarrow a = a \rightarrow a = 1 \notin I$.

If I is an LR-ideal of a residuated lattice L, define the binary relation θ_I on L as follows $(x, y \in L) : \langle x, y\rangle \in \theta_I$ iff $x^* \odot y \in I$ and $x \odot y^* \in I$.

Theorem 3.2. (Lele and Nganou [8], Rachunek and Salounova [16]) θ_I is a congruence on the reduct $(L, \odot, \vee, \rightarrow, 0, 1)$ of the residuated lattice L. If L is a pseudo BL-algebra, then θ_I is a congruence on L.

Definition 3.3. (Buşneag et al. [5]) A nonempty subset I will be called *an ideal* (*implicative ideal*) of L if
(I_1) If $x \leq y$ and $y \in I$, then $x \in I$;
(I_2) If $x, y \in I$, then $x \oplus y \in I$.

Remark 3.4. (Buşneag et al. [5]) Every ideal is a lattice ideal in the lattice $(L, \wedge, \vee, 0, 1)$, but the converse is not true. Moreover, the intersection of two ideals becomes an ideal.

In the following theorem we show that the notions of LR-ideals and ideals (implicative ideals) coincide.

Theorem 3.5. If I is a nonempty subset of L, then I is an ideal iff I is an LR-ideal.

Proof. "$\Rightarrow$" Let I be an ideal of L. By (c_{16}) we have $x \oplus y = y \oplus x$, that is, $x^* \rightarrow y^{**} = y^* \rightarrow x^{**}$. By ($c_5$) and ($c_3$) we have $y \leq y^{**}$, for all $y \in L$ and $x^* \rightarrow y \leq x^* \rightarrow y^{**}$, that is, $x \oslash y \leq x \oplus y \in I$. Since I is an ideal, we conclude that $x \oslash y \in I$, that is, I is an L-ideal of L.

By (c_5) and (c_3) we have $x \leq x^{**}$, for all $x \in L$ and $y^* \rightarrow x \leq y^* \rightarrow x^{**}$, that is, $x \otimes y \leq x \oplus y \in I$. Since I is an ideal, we conclude that $x \otimes y \in I$, that is, I is an R-ideal of L. Therefore, I is an LR-ideal of L.

“ $\Leftarrow$ ”. Let I be an LR-ideal of L. Then $x \oslash y = x^* \to y \in I$ and $x \otimes y = y^* \to x \in I$, for all $x, y \in I$. By residuation property we have $x^{**} \leq x^* \to y \in I$ and $y^{**} \leq y^* \to x \in I$, and since I is an LR-ideal, we conclude that $x^{**} \in I$ and $y^{**} \in I$. Since I is an LR-ideal and $x, y, x^{**}, y^{**} \in I$, then $x \oplus y = x^* \to y^{**} = x \oslash y^{**} \in I$ and $y \oplus x = y^* \to x^{**} = x^{**} \otimes y \in I$. By (c_{16}) we have $x \oplus y = y \oplus x$, hence $x \oplus y \in I$, that is, I is an ideal of L. □

We denote by $\mathcal{I}_i(L)$ the set of all ideals of L. Clearly, if $I \in \mathcal{I}_i(L)$, then $I = L$ iff $1 \in I$.

Proposition 3.6. (i) If I is an ideal of L, then $x \in I$ iff $x^{**} \in I$;

(ii) If I is an ideal of L, then $x \in I$ iff $nx \in I$, for all $n \geq 1$;

(iii) If I is an ideal of L, then $x^{**} \in I$ iff $nx \in I$, for all $n \geq 1$.

Proof. (i) Since I is an ideal and $0, x \in I$, it follows that $x^{**} = 0 \oplus x \in I$. Therefore, $x^{**} \in I$. Conversely, if $x^{**} \in I$, by (c_5), $x \leq x^{**} \in I$, we conclude that $x \in I$.

(ii) Since I is an ideal and $x \in I$, it follows that $nx \in I$. Conversely, if $nx \in I$, by (c_{20}), $x \leq nx \in I$, we conclude that $x \in I$.

(iii) It follows easily from (i) and (ii). □

For a nonempty subset S of L we denote by $(S]$ the ideal of L generated by S (that is, $(S] = \cap\{I \in \mathcal{I}_i(L) : S \subseteq I\}$) and for an element $a \in L$ by $(a]$ the ideal generated by $\{a\}$. If $I \in \mathcal{I}_i(L)$ and $a \in L$ we denote $I(a) = (I \cup \{a\}]$. Clearly, $I(a) = I$ iff $a \in I$.

Proposition 3.7. (Buşneag et al. [5]) Let L be a residuated lattice, $S \subseteq L$ a nonempty subset, $a \in L$ and $I \in \mathcal{I}_i(L)$. Then:

(i) $(S] = \{x \in L : x \leq s_1 \oplus \cdots \oplus s_n,$ for some $n \geq 1$ and $s_1, \ldots, s_n \in S\}$;

(ii) $(a] = \{x \in L : x \leq na$ for some $n \geq 1\}$;

(iii) $I(a) = \{x \in L : x \leq i \oplus na$ for some $i \in I$ and $n \geq 1\}$;

(iv) $(\mathcal{I}_i(L), \subseteq)$ is a complete lattice, where for $I_1, I_2 \in \mathcal{I}_i(L)$, $I_1 \vee I_2 = (I_1 \cup I_2] = \{x \in L : x \leq i_1 \oplus i_2$ with $i_1 \in I_1$ and $i_2 \in I_2\}$.

Proposition 3.8. If Λ is an index set and $(I_i)_{i\in\Lambda}$ is a family of ideals of L, then the infimum of this family is $\wedge_{i\in\Lambda} I_i = \cap_{i\in\Lambda} I_i$ and the supremum is $\vee_{i\in\Lambda} I_i = (\cup_{i\in\Lambda} I_i] = \{x \in L : x \leq x_{i_1} \oplus x_{i_2} \oplus \cdots \oplus x_{i_m},$ where $i_1, \ldots, i_m \in \Lambda,\ x_{i_j} \in I_{i_j},\ 1 \leq j \leq m,$ for some $m \geq 1\}$.

Proof. Let Λ be an index set and $(I_i)_{i\in\Lambda}$ a family of ideals of L. The identity $\wedge_{i\in\Lambda} I_i = \cap_{i\in\Lambda} I_i$ follows from the definition of ideal. In order to prove the second identity, for simplicity, we denote by $X := \cup_{i\in\Lambda} I_i$ and $\overline{X} := \{x \in L : x \leq x_{i_1} \oplus x_{i_2} \oplus \cdots \oplus x_{i_m},$ where $i_1, \ldots, i_m \in \Lambda, x_{i_j} \in I_{i_j}, 1 \leq j \leq m,$ for some $m \geq 1\}$. Clearly, X is a nonempty set and by Proposition 3.7(i), we get that $(X]$ is a nonempty set, too. By Lemma 2.8, we have that the operation $\oplus$ is commutative, associative and compatible with the order relation, so it is immediate that $\overline{X}$ is an ideal which contains the set X. Since $(X]$ is the ideal of L generated by X (that is, $(X] = \cap\{I \in \mathcal{I}_i(L) : X \subseteq I\}$), it follows that $(X] \subseteq \overline{X}$. Let $x \in \overline{X}$, and let I be an arbitrary ideal of L including X. Since $x \in \overline{X}$, it follows that for some $m \geq 1$ there are $x_{i_j} \in I_{i_j} \subseteq X$ such that $x \leq x_{i_1} \oplus x_{i_2} \oplus \cdots \oplus x_{i_m}$, where $i_1, \ldots, i_m \in \Lambda$ and $1 \leq j \leq m$. Since I is an ideal of L and $x_{i_1}, x_{i_2}, \ldots, x_{i_m} \in X \subseteq I$, it follows that $x_{i_1} \oplus x_{i_2} \oplus \cdots \oplus x_{i_m} \in I$, hence $x \in I$. Therefore, $\overline{X} \subseteq I$. As I was arbitrary, we conclude that $\overline{X} \subseteq (X]$. $\square$

An ideal $I \in \mathcal{I}_i(L)$ is called *proper* if $I \neq L$. A proper ideal $I \in \mathcal{I}_i(L)$ is called *prime* iff, for all $a, b \in L$, if $a \wedge b \in I$, then $a \in I$ or $b \in I$. We denote by $Spec_{Id}(L)$ the set of all prime ideals of L.

An ideal $M \in \mathcal{I}_i(L)$ is called *maximal* if M is not strictly contained in a proper ideal of L. We denote by $Max_{Id}(L)$ the set of all maximal ideals of L.

The following result is a characterization of maximal ideals in residuated lattices:

Theorem 3.9. If M is a proper ideal of L, then the following conditions are equivalent:
(i) $M \in Max_{Id}(L)$;
(ii) For any $x \notin M$, there exist $d \in M, n \geq 1$ such that $d \oplus (nx) = 1$;
(iii) For any $x \in L$, $x \notin M$ iff $(nx)^* \in M$, for some $n \geq 1$.

Proof. $(i) \Rightarrow (ii)$ Let $M \in Max_{Id}(L)$. If $x \notin M$, then $M \subset M(x)$, by the maximality of M we conclude that $M(x) = L$, hence $1 \in M(x)$. By Proposition 3.7(iii), there exist $d \in M$ and $n \geq 1$ such that $1 \leq d \oplus (nx)$, that is, $1 = d \oplus (nx)$.

$(ii) \Rightarrow (iii)$ Let $x \in L \setminus M$ be such that $d \oplus (nx) = 1$, for some $d \in M$ and $n \geq 1$. By Proposition 3.6(i) we have $d \in M$ iff $d^{**} \in M$. By residuation property and (c_4), we obtain successively $d \oplus (nx) = 1$, $1 \leq d \oplus (nx)$, $1 \leq d^* \to (nx)^{**}$, $1 \odot d^* \leq (nx)^{**}$, $d^* \leq (nx)^{**}$, $(nx)^{***} \leq d^{**}$, $(nx)^* \leq d^{**} \in M$, hence $(nx)^* \in M$. Since $(nx) \oplus (nx)^* = 1 \notin M$ and $(nx)^* \in M$, it follows that $nx \notin M$, and by Proposition 3.6(ii), we get that $x \notin M$.

$(iii) \Rightarrow (i)$ Assume there is a proper ideal M' such that $M \subset M'$. Then there exists an element $x \in M'$ such that $x \notin M$. By the hypothesis, there exists $n \geq 1$ such that $x \notin M$ iff $(nx)^* \in M \subset M'$. Hence $(nx)^* \in M'$. By Proposition 3.6(ii) we have $x \in M'$ iff $nx \in M'$. Since $nx \in M'$, $(nx)^* \in M'$, but $1 = (nx) \oplus (nx)^* \in M'$, it follows that $M' = L$, a contradiction. $\square$

Let L, L' be residuated lattices. On $L \times L'$ we consider the order relation $(x, y) \leq (x', y')$ iff $x \leq x'$ and $y \leq y'$ and the operations
$(x, y) \wedge (x', y') = (x \wedge x', y \wedge y')$,
$(x, y) \vee (x', y') = (x \vee x', y \vee y')$,
$(x, y) \odot (x', y') = (x \odot x', y \odot y')$,
$(x, y) \to (x', y') = (x \to x', y \to y')$ for all $x, y \in L$ and $x', y' \in L'$.

The $L \times L'$ with the above operations is a residuated lattice called the *direct product* of L and L'.

Theorem 3.10. Let L, L' be residuated lattices. Then K is an ideal of $L \times L'$ iff there exist $P \in \mathcal{I}_i(L)$ and $Q \in \mathcal{I}_i(L')$ such that $K = P \times Q$.

Proof.

"$\Rightarrow$" If $K \in \mathcal{I}_i(L \times L')$, we consider $P = \{x \in L : (x, x') \in K \text{ for some } x' \in L'\}$ and $Q = \{x' \in L' : (x, x') \in K \text{ for some } x \in L\}$.

Clearly, $K = P \times Q$. Since $(0, 0) \in K$ we get $0 \in P$. Let $x, y \in P$. Then there exist $x', y' \in L'$ such that $(x, x'), (y, y') \in K$. Thus, $(x, x') \oplus (y, y') = (x \oplus y, x' \oplus y') \in K$, so $x \oplus y \in P$.

Let $x \leq y$ and $y \in P$. Then there exists $x' \in L'$ such that $(x, x') \in K$. Since $(x, x') \leq (y, x')$, it follows that $x \in P$. So $P \in \mathcal{I}_i(L)$. Similarly, $Q \in \mathcal{I}_i(L')$.

"$\Leftarrow$". Let $K = P \times Q$ for some $P \in \mathcal{I}_i(L)$ and $Q \in \mathcal{I}_i(L')$. Clearly, $K \subseteq L \times L'$. We consider $(x, y), (p, q) \in K$. Then $x, p \in P$ and $y, q \in Q$, that is $x \oplus p \in P$, $y \oplus q \in Q$. Therefore, $(x, y) \oplus (p, q) = (x \oplus p, y \oplus q) \in P \times Q = K$.

Now, let $(p, q) \in K$ be such that $(x, y) \leq (p, q)$. Then $x \leq p$ with $p \in P$ and $y \leq q$ with $q \in Q$. Since $P \in \mathcal{I}_i(L)$ and $Q \in \mathcal{I}_i(L')$, it follows that $x \in P$ and $y \in Q$, that is $(x, y) \in K$. Hence $K \in \mathcal{I}_i(L \times L')$. $\square$

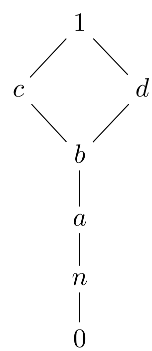
Corollary 3.11. Let $\prod_{i=1}^{n} L_i$ be a finite direct product of residuated lattices. Then K is an ideal of $\prod_{i=1}^{n} L_i$ iff there exist $P_i \in \mathcal{I}_i(L_i)$ such that $K = \prod_{i=1}^{n} P_i$, for every $1 \leq i \leq n$.

3.2. $\odot$-prime ideals in residuated lattices

In this section, in order to establish the relationship between ideals and filters in residuated lattices we define the subset of prime ideals called $\odot$-prime ideals.

Definition 3.12. A proper ideal P is called *$\odot$-prime* if it satisfies the condition: if $x \odot y \in P$, then $x \in P$ or $y \in P$.

Example 3.13. Let $L = \{0, n, a, b, c, d, 1\}$ with $0 < n < a < b < c, d < 1$, but c and d are incomparable.



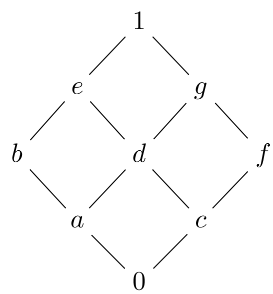
Then ([13], page 229) L is a distributive residuated lattice with respect to the following operations:

$\rightarrow$	0	n	a	b	c	d	1
0	1	1	1	1	1	1	1
n	d	1	1	1	1	1	1
a	n	n	1	1	1	1	1
b	n	n	a	1	1	1	1
c	0	n	a	d	1	d	1
d	n	n	a	c	c	1	1
1	0	n	a	b	c	d	1

$\odot$	0	n	a	b	c	d	1
0	0	0	0	0	0	0	0
n	0	0	0	0	n	0	n
a	0	0	a	a	a	a	a
b	0	0	a	b	b	b	b
c	0	n	a	b	c	b	c
d	0	0	a	b	b	d	d
1	0	n	a	b	c	d	1

In the residuated lattice L from Example 3.13, if $P = (n] = \{0, n\}$ then P is an $\odot$-prime ideal.

Example 3.14. Let $L = \{0, a, b, c, d, e, f, g, 1\}$ with $0 < a < b < e < 1$, $0 < a < d < e < 1$, $0 < a < d < g < 1$, $0 < c < d < e < 1$, $0 < c < d < g < 1$, $0 < c < f < g < 1$ and elements $\{a, c\}$, $\{b, d\}$, $\{d, f\}$, $\{e, g\}$ and $\{b, f\}$ are pairwise incomparable.



Then ([13], page 166) L is a residuated lattice with respect to the following operations:

$\rightarrow$	0	a	b	c	d	e	f	g	1
0	1	1	1	1	1	1	1	1	1
a	g	1	1	g	1	1	g	1	1
b	f	g	1	f	g	1	f	g	1
c	e	e	e	1	1	1	1	1	1
d	d	e	e	g	1	1	g	1	1
e	c	d	e	f	g	1	f	g	1
f	b	b	b	e	e	e	1	1	1
g	a	b	b	d	e	e	g	1	1
1	0	a	b	c	d	e	f	g	1

$\odot$	0	a	b	c	d	e	f	g	1
0	0	0	0	0	0	0	0	0	0
a	0	0	a	0	0	a	0	0	a
b	0	a	b	0	a	b	0	a	b
c	0	0	0	0	0	0	c	c	c
d	0	0	a	0	0	a	c	c	d
e	0	a	b	0	a	b	c	d	e
f	0	0	0	c	c	c	f	f	f
g	0	0	a	c	c	d	f	f	g
1	0	a	b	c	d	e	f	g	1

Proposition 3.15. Every $\odot$-prime ideal is a prime ideal. The converse may not hold.

Proof. Let P be an $\odot$-prime. If, on the contrary, we assume that P is not a prime ideal. For every $x, y \in L$, $x \wedge y \in P$, then $x \notin P$ and $y \notin P$. Since $x \odot y \leq x \wedge y \in P$, it follows that $x \odot y \in P$. Since $x \odot y \in P$ and P is an $\odot$-prime ideal of L, we get $x \in P$ or $y \in P$, a contradiction. Therefore, P is a prime ideal of L.

In the residuated lattice L from Example 3.14, if we consider $P = (b] = \{0, a, b\}$, then it is easy to ascertain that P is a prime ideal of L. Moreover, P is a maximal ideal of L. Since $e \odot d = a \in P$ and $e \notin P, d \notin P$, it follows that P is not $\odot$-prime. $\square$

If we denote by $\odot$-$Spec_{Id}(L)$ the set of all $\odot$-prime ideals of L, then from Proposition 3.15 we conclude that $\odot$-$Spec_{Id}(L) \subseteq Spec_{Id}(L)$. In Stonean residuated lattices the notions of $\odot$-prime and prime ideals coincide (see Proposition 12, [5]). However, in Example 2.11, $(a] = \{0, a\}$, $(d] = \{0, c, d\}$ are prime and $\odot$-prime ideals, and L is not Stonean ($a^* \vee a^{**} = d \vee a = m \neq 1$).

Proposition 3.16. An ideal P is prime iff it is $\odot$-prime and, for all $x, y \in L$, $x \wedge y \in P$ whenever $x \odot y \in P$.

Proof. Let P be a prime ideal and $x, y \in L$. By hypothesis, if $x \odot y \in P$, then $x \wedge y \in P$, and so $x \in P$ or $y \in P$ (as P is a prime ideal). Therefore, P is $\odot$-prime. Conversely, if $x \odot y \in P$, and by hypothesis P is $\odot$-prime, then $x \in P$ or $y \in P$. Since $x \wedge y \leq x, y$ and $x \in P$ or $y \in P$, it follows that $x \wedge y \in P$. $\square$

For $S \subseteq L$ we denote $\overline{S} = L \setminus S$. The following result represents the relationship between ideals and filters in residuated lattices.

Theorem 3.17. For M an ideal of L. Then M is a $\odot$-prime ideal iff $\overline{M}$ is a maximal filter of L.

Proof. " $\Rightarrow$ " Let M be an ideal of L. Assume that M is a $\odot$-prime ideal. By Proposition 3.15, we get M is a prime ideal of L, and by definition, M is proper (that is, $M \neq L$). From $0 \in M$, we get $0 \notin \overline{M}$, that is $\overline{M} \neq L$. From $M \neq L$, it follows that $1 \notin M$, hence $1 \in \overline{M}$.

For $x, y \in L$ with $x \leq y$, assume $x \in \overline{M}$, that is, $x \notin M$. Clearly, $y \notin M$, that is, $y \in \overline{M}$. We get $x, y \in \overline{M}$, that is, $x, y \notin M$. To prove $\overline{M}$ is a filter, by contrary, we assume that $x \odot y \notin \overline{M}$, that is, $x \odot y \in M$. Since M is $\odot$-prime ideal, and $x \odot y \in M$, then $x \in M$ or $y \in M$, a contradiction.

Now, we prove $\overline{M}$ is a prime filter. Let $x, y \in L$ such that $x \vee y \in \overline{M}$, that is, $x \vee y \notin M$. By contrary, we assume that $x \notin \overline{M}$ and $y \notin \overline{M}$, so $x \in M$ and $y \in M$. From $x \vee y \notin M$, since M is $\odot$-prime ideal and $(x \vee y) \odot (x \vee y)^* = 0 \in M$, we conclude that $(x \vee y)^* \in M$. By (c_{11}), $(x \vee y)^* = x^* \wedge y^*$, it follows that $x^* \wedge y^* \in M$. Since $x^* \odot y^* \leq x^* \wedge y^*$, we get $x^* \odot y^* \in M$, hence $x^* \in M$ or $y^* \in M$ (as M is a $\odot$-prime ideal). For $x^* \in M$, since $x \in M$, we have $1 = x \oplus x^* \in M$, hence $M = L$, a contradiction. Analogously, for $y^* \in M$. So, $\overline{M}$ is a prime filter.

Now, we prove $\overline{M}$ is a maximal filter. By contrary, if there is a proper filter Q of L such that $\overline{M} \subset Q$, that is, there is an element $x \in Q \setminus \overline{M}$. We get $x \in Q \cap M$, and so $x \in M$. Clearly, $x^* \in \overline{M} \subset Q$, and so $x^* \in Q$. Since $x, x^* \in Q$ and $x \odot x^* = 0 \in Q$, then $Q = L$, a contradiction. Therefore, $\overline{M}$ is not strictly included in a proper filter of L, that is, $\overline{M}$ is a maximal filter of L.

" $\Leftarrow$ ". Let M be an ideal of L. We assume that $\overline{M}$ is a maximal filter of L. Since $1 \in \overline{M}$, then $1 \notin M$, that is, M is a proper ideal.

Now, we prove M is $\odot$-prime ideal. Let $x, y \in L$ be such that $x \odot y \in M$, that is, $x \odot y \notin \overline{M}$. Since $\overline{M}$ is a maximal filter, by Proposition 2.5, $[(x \odot y)^n]^* \in \overline{M}$ for some $n \geq 1$. If, on the contrary, $x, y \notin M$, that is, $x, y \in \overline{M}$, then $x \odot y \in \overline{M}$, hence $(x \odot y)^n \in \overline{M}$. Since $(x \odot y)^n, [(x \odot y)^n]^* \in \overline{M}$ and $0 = (x \odot y)^n \odot [(x \odot y)^n]^* \in \overline{M}$, we conclude that $\overline{M} = L$, a contradiction. □

Corollary 3.18. In L, every $\odot$-prime ideal is contained in an unique maximal ideal.

Proof. Let P be an $\odot$-prime ideal of L. From Proposition 3.15, we get that P is a prime ideal of L. Since P is a prime ideal of L, it follows that P is a proper ideal of L. Using Zorn's Lemma we conclude that P is contained in a maximal ideal. Assume that there are two distinct maximal ideals M_1 and M_2 such that $P \subseteq M_1$ and $P \subseteq M_2$. Since $M_1 \neq M_2$, there is $a \in M_1$ such that $a \notin M_2$. By Theorem 3.9(iii), there is $n \geq 1$ such that $(na)^* \in M_2$. Then $(na)^{**} \notin M_2$, hence $(na)^{**} \notin P$. Since $a \in M_1$, then $na \in M_1$, hence $(na)^* \notin M_1$ and $(na)^* \notin P$. Since $(na)^* \odot (na)^{**} = 0 \in P$, it follows that $(na)^* \in P$ or $(na)^{**} \in P$ (as P is an $\odot$-prime ideal) a contradiction. □

4. DE MORGAN RESIDUATED LATTICES

In this section we study a special class of residuated lattices called De Morgan residuated lattices.

4.1. General information

Definition 4.1. A residuated lattice L will be called *De Morgan* if it satisfies the identity $(x \wedge y)^* = x^* \vee y^*$, for all $x, y \in L$.

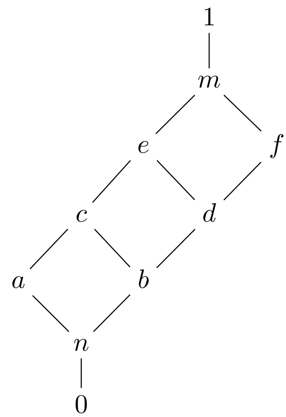
In the residuated lattice L from Example 2.11 we have $(a \wedge d)^* = 0^* = 1$, $a^* \vee d^* = d \vee a = m$, so $(a \wedge d)^* \neq a^* \vee d^*$, that is L is not a De Morgan residuated lattice.

Examples of De Morgan residuated lattices are Boolean algebras, MV-algebras and BL-algebras (see [14, 18]), MTL-algebras (see [7, 11]) and Stonean residuated lattices (see [5]).

Every involution residuated lattice L is De Morgan. Indeed, if L has the involution property and by (c_{11}), then $(x \wedge y)^* = (x^{**} \wedge y^{**})^* = [(x^* \vee y^*)^*]^* = (x^* \vee y^*)^{**} = x^* \vee y^*$.

In the following examples we show that the class of De Morgan residuated lattices is a larger class than Boolean algebras, BL-algebras, Stonean residuated lattices, MTL-algebras and involution residuated lattices.

Example 4.2. Let L=$\{0, n, a, b, c, d, e, f, m, 1\}$ with $0 < n < a < c < e < m < 1$, $0 < n < b < d < f < m < 1$ and the elements $\{a, b\}$, $\{c, d\}$, $\{e, f\}$ are pairwise incomparable.



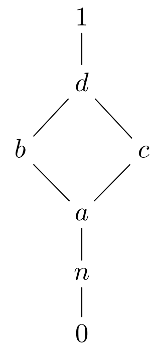
Then L is a residuated lattice with respect to the following operations:

$\rightarrow$	0	n	a	b	c	d	e	f	m	1
0	1	1	1	1	1	1	1	1	1	1
n	m	1	1	1	1	1	1	1	1	1
a	f	f	1	f	1	f	1	f	1	1
b	e	e	e	1	1	1	1	1	1	1
c	d	d	e	f	1	f	1	f	1	1
d	c	c	c	e	e	1	1	1	1	1
e	b	b	c	d	e	f	1	f	1	1
f	a	a	a	c	c	e	e	1	1	1
m	n	n	a	b	c	d	e	f	1	1
1	0	n	a	b	c	d	e	f	m	1

$\odot$	0	n	a	b	c	d	e	f	m	1
0	0	0	0	0	0	0	0	0	0	0
n	0	0	0	0	0	0	0	0	0	n
a	0	0	a	0	a	0	a	0	a	a
b	0	0	0	0	0	0	0	b	b	b
c	0	0	a	0	a	0	a	b	c	c
d	0	0	0	0	0	b	b	d	d	d
e	0	0	a	0	a	b	c	d	e	e
f	0	0	0	b	b	d	d	f	f	f
m	0	0	a	b	c	d	e	f	m	m
1	0	n	a	b	c	d	e	f	m	1

It is easy to ascertain that L is a De Morgan residuated lattice. Since $a \wedge b = n$ and $a \odot (a \rightarrow b) = a \odot f = 0$, it follows that $a \wedge b \neq a \odot (a \rightarrow b)$, consequently, L is not a divisible residuated lattice, so L is not a BL-algebra. Since $a^* \vee a^{**} = f \vee a = m \neq 1$, it follows that L is not a Stonean residuated lattice. Since $(a \rightarrow b) \vee (b \rightarrow a) = f \vee e = m \neq 1$, it follows that L is not a MTL-algebra. Since $b^2 = b \odot b = 0 \neq b$, it follows that L is not a G-algebra.

Example 4.3. Since the De Morgan residuated lattice from Example 4.2 is an involution residuated lattice, in this example we present a De Morgan residuated lattice without the involution property: let $L = \{0, n, a, b, c, d, 1\}$ with $0 < n < a < b, c < d < 1$, and b and c are incomparable.



Then ([13], page 247) L is a residuated lattice with respect to the following operations:

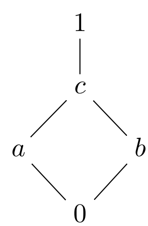
$\to$	0	n	a	b	c	d	1
0	1	1	1	1	1	1	1
n	0	1	1	1	1	1	1
a	0	d	1	1	1	1	1
b	0	c	c	1	c	1	1
c	0	b	b	b	1	1	1
d	0	a	a	b	c	1	1
1	0	n	a	b	c	d	1

$\odot$	0	n	a	b	c	d	1
0	0	0	0	0	0	0	0
n	0	n	n	n	n	n	n
a	0	n	n	n	n	n	a
b	0	n	n	b	n	b	b
c	0	n	n	n	c	c	c
d	0	n	n	b	c	d	d
1	0	n	a	b	c	d	1

It is easy to ascertain that L is a De Morgan residuated lattice. Since $n^{**} = 1 \neq n$, it follows that L is not an involution residuated lattice, however, L is Stonean.

Divisible residuated lattices are not always De Morgan as we can see in the following example.

Example 4.4. Let $L = \{0, a, b, c, 1\}$ with $0 < a, b < c < 1$, and a and b are incomparable.



Then ([13], page 187) L is a residuated lattice with respect to the following operations:

$\to$	0	a	b	c	1
0	1	1	1	1	1
a	b	1	b	1	1
b	a	a	1	1	1
c	0	a	b	1	1
1	0	a	b	c	1

$\odot$	0	a	b	c	1
0	0	0	0	0	0
a	0	a	0	a	a
b	0	0	b	b	b
c	0	a	b	c	c
1	0	a	b	c	1

It easy to see that L is a G-algebra, and so divisible. L is not a BL-algebra because $(a \to b) \vee (b \to a) = b \vee a = c \neq 1$. Since $(a \wedge b)^* = 0^* = 1$ and $a^* \vee b^* = b \vee a = c$, it follows that $(a \wedge b)^* \neq a^* \vee b^*$, hence L is not De Morgan.

In conclusion, the class of De Morgan residuated lattices includes important subclasses of residuated lattices such as Boolean algebras, MV-algebras, BL-algebras, Stonean residuated lattices, MTL-algebras and involution residuated lattices.

Remark 4.5. In Example 4.4 the residuated lattice L is a G-algebra, and so L is a semi-G-algebra, but L is not De Morgan. Also, the residuated lattice L from Example 4.2 is a De Morgan residuated lattice, but L is not a semi-G-algebra (because $(b^2)^* = 0^* = 1 \neq e = b^*$). Therefore, the classes of semi-G-algebras and De Morgan residuated lattices are different.

In what follows (unless otherwise specified) by L we denote a *De Morgan residuated lattice.*

Lemma 4.6. If $x, y, z \in L$, then
(c_{22}) $(x \wedge y)^{**} = x^{**} \wedge y^{**}$;
(c_{23}) $x \oplus (y \wedge z) = (x \oplus y) \wedge (x \oplus z)$;
(c_{24}) $x \wedge (y \oplus z) \leq (x \wedge y) \oplus (x \wedge z)$.

Proof. (c_{22}). We have $(x \wedge y)^{**} = [(x \wedge y)^*]^* = (x^* \vee y^*)^* \overset{(c_{11})}{=} x^{**} \wedge y^{**}$.

(c_{23}). Since $y \wedge z \leq y, z$, by (c_{18}) we conclude that $x \oplus (y \wedge z) \leq x \oplus y, x \oplus z$. Consider $t \in L$ be such that $t \leq x \oplus y, x \oplus z$. Then $t \leq (x \oplus y) \wedge (x \oplus z) = (x^* \to y^{**}) \wedge (x^* \to z^{**}) \overset{(c_9)}{=} x^* \to (y^{**} \wedge z^{**}) = x^* \to (y \wedge z)^{**} = x \oplus (y \wedge z)$.

(c_{24}). We have $x \wedge (y \oplus z) = x \wedge (y^* \odot z^*)^* \leq x^{**} \wedge (y^* \odot z^*)^* = (x^*)^* \wedge (y^* \odot z^*)^* \overset{(c_{11})}{=} [x^* \vee (y^* \odot z^*)]^* \overset{(c_8)}{\leq} [(x^* \vee y^*) \odot (x^* \vee z^*)]^* = ((x \wedge y)^* \odot (x \wedge z)^*)^* = (x \wedge y) \oplus (x \wedge z)$. □

Corollary 4.7. If $x, y \in L$ and $n \geq 2$, then
(c_{25}) $x \wedge (ny) \leq n(x \wedge y)$.

Proof. Mathematical induction relative to n, using (c_{24}). □

Corollary 4.8. If $x, y \in L$ and $m \geq 2$ or $n \geq 2$, then
(c_{26}) $(mx) \wedge (ny) \leq (mn)(x \wedge y)$.

Proof. Assume $m \geq 2$. If $n = 0$ in (c_{26}) we have equality. If $n = 1$, (c_{26}) follows from (c_{25}).

If $n \geq 2$, by (c_{25}) we conclude that $(mx) \wedge (ny) \leq n[(mx) \wedge y] \leq n[m(x \wedge y)] = (mn)(x \wedge y)$.

Analogously, if $n \geq 2$. □

4.2. Ideals and prime ideals

Proposition 4.9. Let $I \in \mathcal{I}_i(L)$ and $a, b \in L$ such that $a \wedge b \in I$. Then $I(a) \cap I(b) = I$.

Proof. Clearly, $I \subseteq I(a) \cap I(b)$. To prove the converse inclusion, let $x \in I(a) \cap I(b)$. Then there are $d_1, d_2 \in I$ and $m, n \geq 1$ such that $x \leq d_1 \oplus (ma)$ and $x \leq d_2 \oplus (nb)$. Since $a \wedge b \in I$, I is an ideal of L, and, by Proposition 3.6, $(c_{23}), (c_{25})$ and (c_{26}), we obtain successively $x \leq [d_1 \oplus (ma)] \wedge [d_2 \oplus (nb)] \leq (d_1 \wedge d_2) \oplus [(ma) \wedge (nb)] \oplus [d_1 \wedge (nb)] \oplus [d_2 \wedge (ma)] \in I$, hence $x \in I$. Therefore, $I(a) \cap I(b) = I$. $\square$

Corollary 4.10. For $I \in \mathcal{I}_i(L)$ the following conditions are equivalent:

(i) If $I = I_1 \cap I_2$ with $I_1, I_2 \in \mathcal{I}_i(L)$, then $I = I_1$ or $I = I_2$;

(ii) For $a, b \in L$, if $a \wedge b \in I$, then $a \in I$ or $b \in I$.

Proof. $(i) \Rightarrow (ii)$ If $a, b \in L$ are such that $a \wedge b \in I$, then by Proposition 4.9, $I(a) \cap I(b) = I$, hence $I = I(a)$ or $I = I(b)$, so $a \in I$ or $b \in I$.

$(ii) \Rightarrow (i)$ Let $I_1, I_2 \in \mathcal{I}_i(L)$ such that $I = I_1 \cap I_2$. If, on the contrary, $I \neq I_1$ and $I \neq I_2$, then there are $a \in I_1 \setminus I$ and $b \in I_2 \setminus I$. Since I is an ideal of L and $a \wedge b \leq a, b$, it holds that $a \wedge b \in I_1 \cap I_2 = I$, so $a \in I$ or $b \in I$, a contradiction. Therefore, $I = I_1$ or $I = I_2$. $\square$

Proposition 4.11. (i) If $I \in \mathcal{I}_i(L)$ and $x, y \in L$, then $I(x) \cap I(y) \subseteq I(x \wedge y)$;

(ii) The lattice $(\mathcal{I}_i(L), \subseteq)$ is distributive.

Proof. (i). If $z \in I(x) \cap I(y)$, following Proposition 3.7(iii), $z \leq i \oplus (mx), j \oplus (ny)$ with $i, j \in I$ and $m, n \geq 1$.

Let $k = i \oplus j \in I$, then $z \leq k \oplus (mx), k \oplus (ny)$, so we obtain successively $z \leq (k \oplus (mx)) \wedge (k \oplus (ny)) \overset{(c_{23})}{=} k \oplus ((mx) \wedge (ny)) \overset{(c_{26})}{\leq} k \oplus [(mn)(x \wedge y)] \in I(x \wedge y)$, consequently $z \in I(x \wedge y)$. Therefore, $I(x) \cap I(y) \subseteq I(x \wedge y)$.

(ii). Consider $I, I_1, I_2 \in \mathcal{I}_i(L)$. We have that $I_1 \vee I_2 = \{z \in L : z \leq i_1 \oplus i_2 \text{ with } i_1 \in I_1, \text{ and } i_2 \in I_2\}$.

Clearly, $(I \cap I_1) \vee (I \cap I_2) \subseteq I \cap (I_1 \vee I_2)$.

To prove $I \cap (I_1 \vee I_2) \subseteq (I \cap I_1) \vee (I \cap I_2)$, consider $z \in I \cap (I_1 \vee I_2)$. Then $z \in I$ and $z \leq i_1 \oplus i_2$ with $i_1 \in I_1$, and $i_2 \in I_2$.

We have $z = z \wedge (i_1 \oplus i_2) \overset{(c_{24})}{\leq} (z \wedge i_1) \oplus (z \wedge i_2)$.

Since $z \wedge i_1 \in I_1$ and $z \wedge i_2 \in I_2$ we conclude that $z \in (I \cap I_1) \vee (I \cap I_2)$, hence $I \cap (I_1 \vee I_2) \subseteq (I \cap I_1) \vee (I \cap I_2)$. Therefore, the lattice $(\mathcal{I}_i(L), \subseteq)$ is distributive. $\square$

Proposition 4.12. The lattice $(\mathcal{I}_i(L), \subseteq)$ is a complete Brouwerian lattice.

Proof. Let Λ be an index set. By Proposition 3.7(iv), we have that the lattice $(\mathcal{I}_i(L), \subseteq)$ is complete. In order to prove that $\mathcal{I}_i(L)$ is Brouwerian we must show that for every ideal I and every family $(I_i)_{i\in\Lambda}$ of ideals, $I \wedge_{i\in\Lambda} (\vee_{i\in\Lambda} I_i) = \vee_{i\in\Lambda}(I \wedge_{i\in\Lambda} I_i)$, that is, $I \cap (\vee_{i\in\Lambda} I_i) = (\cup_{i\in\Lambda}(I \cap I_i)]$. Clearly, $(\cup_{i\in\Lambda}(I \cap I_i)] \subseteq I \cap (\vee_{i\in\Lambda} I_i)$. Now, let $x \in I \cap (\vee_{i\in\Lambda} I_i)$. Then $x \in I$ and there exist $i_1, \dots, i_m \in \Lambda, x_{i_j} \in I_{i_j}$, $(1 \leq j \leq m)$ such that $x \leq x_{i_1} \oplus x_{i_2} \oplus \cdots \oplus x_{i_m}$ (as $x \in \vee_{i\in\Lambda} I_i$). Then $x = x \wedge (x_{i_1} \oplus x_{i_2} \oplus \cdots \oplus x_{i_m}) \leq (x \wedge x_{i_1}) \oplus \cdots \oplus (x \wedge x_{i_m})$ by (c_{24}). Since $x \wedge x_{i_j} \in I \cap I_{i_j}$ for every $1 \leq j \leq m$, we conclude that $x \in (\cup_{i\in\Lambda}(I \cap I_i)]$, hence $I \cap (\vee_{i\in\Lambda} I_i) \subseteq (\cup_{i\in\Lambda}(I \cap I_i)]$, that is, $I \cap (\vee_{i\in\Lambda} I_i) = (\cup_{i\in\Lambda}(I \cap I_i)]$.

Therefore, $(\mathcal{I}_i(L), \subseteq)$ is a complete Brouwerian lattice. □

Theorem 4.13. For $P \in \mathcal{I}_i(L)$, the following conditions are equivalent:

(*i*) P is $\cap$-prime;

(*ii*) P is $\cap$-irreducible;

(*iii*) If $x, y \in L$ and $x \wedge y \in P$, then $x \in P$ or $y \in P$.

Proof. The equivalence of (*i*) and (*ii*) follows from Proposition 4.11(*ii*) due to the well known fact that, in a distributive lattice, an element is meet-irreducible iff it is meet-prime.

The equivalence of (*ii*) and (*iii*) is already stated in Corollary 4.10. □

We conclude that $P \in \mathcal{I}_i(L)$ is *prime* iff P is $\cap$-prime.

Theorem 4.14. (Prime ideal theorem in De Morgan residuated lattices)

Let L be a De Morgan residuated lattice, $I \in \mathcal{I}_i(L)$, $S \subseteq L$ a nonempty $\wedge$-closed subset of L such that $S \cap I = \varnothing$. Then there is a prime ideal P of L such that $I \subseteq P$ and $P \cap S = \varnothing$.

Proof. Consider the set $\mathcal{I}_I = \{J \in \mathcal{I}_i(L) :\ I \subseteq J \text{ and } S \cap J = \varnothing\}$. Since $I \in \mathcal{I}_I$, then $\mathcal{I}_I \neq \varnothing$. By Zorn's Lemma we conclude that in $\mathcal{I}_I$ we have a maximal element P. We want to prove that P is prime ideal. Clearly, $P \neq L$. By contrary, we assume that there are $a, b \in L$ such that $a \wedge b \in P$, but $a, b \notin P$. Since $a \wedge b \in P$, we have $P(a \wedge b) = P$. By the maximality of P we conclude that $P(a) \cap S \neq \varnothing$ and $P(b) \cap S \neq \varnothing$.

By Proposition 3.7(*iii*), we conclude that there are $s_1, s_2 \in S$ such that $s_1 \leq i \oplus ma, s_2 \leq j \oplus nb$, with $i, j \in P$ and $m, n \geq 1$. If consider $k = i \oplus j \in P$, then $s_1 \wedge s_2 \leq (k \oplus ma) \wedge (k \oplus nb) \overset{(c_{23})}{=} k \oplus (ma \wedge nb)$.

If $m \geq 2$ or $n \geq 2$, then using (c_{26}) we conclude that $s_1 \wedge s_2 \leq k \oplus [(ma) \wedge (nb)] \leq k \oplus [(mn)(a \wedge b)] \in P(a \wedge b) = P$, hence $s_1 \wedge s_2 \in P(a \wedge b) = P$.

If $m = n = 1$, then by (c_{23}) we have $s_1 \wedge s_2 \leq (k \oplus a) \wedge (k \oplus b) \leq k \oplus (a \wedge b) \in P(a \wedge b) = P$ and again we conclude that $s_1 \wedge s_2 \in P(a \wedge b) = P$. Then $s_1 \wedge s_2 \in P(a \wedge b) \cap S = P \cap S = \varnothing$, a contradiction. So, P is a prime ideal of L. □

Remark 4.15. If L is nontrivial, then any proper ideal of L can be extended to a prime ideal. In general, the set of ideals of L including prime ideals is not a chain. Indeed, in the De Morgan residuated lattice L from Example 3.14, the ideals of L are $(0] = \{0\}, (b] = \{0, a, b\}$ and $(f] = \{0, c, f\}$, but $(b] \nsubseteq (f], (f] \nsubseteq (b]$, so $\mathcal{I}_i(L) = \{(0], (b], (f]\}$ is not a chain.

The following result is a consequence of Theorem 4.14:

Corollary 4.16. Let $I \in \mathcal{I}_i(L)$ and $a \in L \setminus I$. Then:
(i) There is $P \in Spec_{Id}(L)$ such that $I \subseteq P$ and $a \notin P$;
(ii) I is the intersection of those prime ideals which contain I;
(iii) $\cap Spec_{Id}(L) = \{0\}$.

In the following theorem we show that every maximal ideal is prime in De Morgan residuated lattices.

Proposition 4.17. $Max_{Id}(L) \subseteq Spec_{Id}(L)$.

Proof. We prove $Max_{Id}(L) \subseteq Spec_{Id}(L)$. Let $M \in Max_{Id}(L)$. If there exist two proper ideals $N, P \in \mathcal{I}_i(L)$ such that $M = N \cap P$, then $M \subseteq N$ and $M \subseteq P$, by the maximality of M we conclude that $M = N = P$, that is, M is an $\cap$-irreducible, so prime element in the lattice of ideals $(\mathcal{I}_i(L), \subseteq)$, by Theorem 4.13. □

Example 4.18. This example shows that in general in de Morgan residuated lattices, if I is a prime ideal, then I is not always maximal.

It is easy to ascertain that the residuated lattice L from Example 3.13 is a De Morgan residuated lattice. Clearly, $(0] = \{0\}$ is a prime ideal of L, $(a] = (b] = (c] = (d] = (1] = L$ and $(n] = \{0, n\}$ is a maximal (obviously, prime) ideal of L. Therefore, $(0] \subset (n]$, so $(0]$ is a prime ideal, but not maximal.

Corollary 4.19. If L is a semi-G-algebra, then $Max_{Id}(L) = Spec_{Id}(L)$.

Proof. Let L be a De Morgan residuated lattice that is a semi-G-algebra. By Proposition 4.17 we have $Max_{Id}(L) \subseteq Spec_{Id}(L)$. By Proposition 9, ([5]) a residuated lattice L is semi-G-algebra iff $x \wedge x^* = 0$ for every $x \in L$.

Now, we prove that $Spec_{Id}(L) \subseteq Max_{Id}(L)$. Let $I \in Spec_{Id}(L)$ be a prime ideal and, if there is M a proper ideal such that $I \subset M$. Then there is an element $x \in M \setminus I$. Since $x \notin I$, I is prime ideal, and $x \wedge x^* = 0 \in I$, it follows that $x^* \in I$. Since $x^* \in I \subset M$, we get $x^* \in M$. We get $x, x^* \in M$ and $x \oplus x^* = 1 \in M$, consequently $M = L$, a contradiction. Therefore, I is not strictly contained in a proper ideal of L, that is, I is a maximal ideal of L. Therefore, $Spec_{Id}(L) = Max_{Id}(L)$. □

Example 4.20. The converse of Corollary 4.19, may not hold as we can see in this example.

It is easy to ascertain that the residuated lattice L from Example 3.14 is a De Morgan residuated lattice. Since $(d^2)^* = 0^* = 1$, $d^* = d$, we conclude that $(d^2)^* \neq d^*$, so L is not a semi-G-algebra. The proper ideals of L are $(0] = \{0\}$, $(b] = \{0, a, b\}$, and $(f] = \{0, c, f\}$, and $Max_{Id}(L) = Spec_{Id}(L) = \{(b], (f]\}$. Therefore, $Max_{Id}(L) = Spec_{Id}(L)$ and L is not a semi-G-algebra. Moreover, since $a^* \vee a^{**} = g \vee a = g \neq 1$, it follows that L is not Stonean.

We recall (see Theorem 3.2) that: If I is an ideal of a residuated lattice L, then the binary relation θ_I on L ($(x,y) \in \theta_I$ iff $x^* \odot y \in I$ and $x \odot y^* \in I$) is a congruence on the reduct $(L, \odot, \vee, \to, 0, 1)$ of the residuated lattice L. Moreover, if L is a pseudo BL-algebra, then θ_I is a congruence on L. In the following result, we show: if L is a De Morgan residuated lattice, then θ_I is a congruence on L.

Theorem 4.21. In L, θ_I is a congruence.

P r o o f. By Theorem 3.2 we have θ_I is a congruence on the reduct $(L, \odot, \vee, \to, 0, 1)$ of the residuated lattice L. It remains to prove that θ_I is compatible with the operation $\wedge$. Let $x, y, z, t \in L$, we assume that $(x,y) \in \theta_I$ and $(z,t) \in \theta_I$ and we must to prove that $(x \wedge z, y \wedge t) \in \theta_I$. We have $(x,y) \in \theta_I$ iff $x^* \odot y \in I$ and $x \odot y^* \in I$, and $(z,t) \in \theta_I$ iff $z^* \odot t \in I$ and $z \odot t^* \in I$. Since $x \wedge z \leq z$ and $z \odot t^* \in I$, we obtain $(x \wedge z) \odot t^* \leq z \odot t^* \in I$, and then $(x \wedge z) \odot t^* \in I$. Since $x \wedge z \leq x$ and $x \odot y^* \in I$, we obtain $(x \wedge z) \odot y^* \leq x \odot y^* \in I$, and then $(x \wedge z) \odot y^* \in I$. Since I is an ideal and $(x \wedge z) \odot t^* \in I$, $(x \wedge z) \odot y^* \in I$, it follows that $[(x \wedge z) \odot t^*] \vee [(x \wedge z) \odot y^*] \overset{(c_{16})}{\leq} [(x \wedge z) \odot t^*] \oplus [(x \wedge z) \odot y^*] \in I$, and so $[(x \wedge z) \odot t^*] \vee [(x \wedge z) \odot y^*] \in I$. By (c_7), we get $(x \wedge z) \odot (y^* \vee t^*) \overset{(c_7)}{=} [(x \wedge z) \odot t^*] \vee [(x \wedge z) \odot y^*] \in I$. Since $(y^* \vee t^*) = (y \wedge t)^*$, it holds that $(x \wedge z) \odot (y \wedge t)^* \in I$.

Since $y \wedge t \leq y$ and $y \odot x^* \in I$, likewise $(x \wedge z)^* \odot (y \wedge t) \in I$.

Therefore, $(x \wedge z) \odot (y \wedge t)^* \in I$ and $(x \wedge z)^* \odot (y \wedge t) \in I$ iff $(x \wedge z, y \wedge t) \in \theta_I$. $\square$

For $x \in L$ we denote by x/I the congruence class of x modulo θ_I and $L/I = \{x/I : x \in L\}$. Define the binary operations $\vee, \wedge, \odot$ and $\to$ on L/I by $(x/I) \vee (y/I) = (x \vee y)/I$, $(x/I) \wedge (y/I) = (x \wedge y)/I$, $(x/I) \odot (y/I) = (x \odot y)/I$ and $(x/I) \to (y/I) = (x \to y)/I$ for all $x, y \in L$. Then $(L/I, \vee, \wedge, \odot, \to, \mathbf{0}, \mathbf{1})$ is a De Morgan residuated lattice, which is called the *quotient De Morgan residuated lattice of* L *with respect to* I, where $\mathbf{0} = 0/I$ and $\mathbf{1} = 1/I$. The order relation on L/I is defined by $(x/I) \leq (y/I)$ iff $(x \to y)^* \in I$. For a nonempty subset S of L we denote by $S/I = \{x/I : x \in S\}$. Clearly, for $x \in L$, $x/I = \mathbf{1}$ iff $x^* \in I$ and $x/I = \mathbf{0}$ iff $x \in I$.

Corollary 4.22. $(L/I, \vee, \wedge, \odot, \to, \mathbf{0}, \mathbf{1})$ is an involution residuated lattice.

P r o o f. Let I be an ideal of L. Since for all $x \in L$, $(x, x^{**}) \in \theta_I$ iff $x^* \odot x^{**} = 0 \in I$ and $x \odot x^{***} = x \odot x^* = 0 \in I$, then $x/I = x^{**}/I$. Therefore, L/I is an involution residuated lattice. $\square$

4.3. Annihilators

We recall that by L we denote a De Morgan residuated lattice (unless otherwise specified). In this section some results may hold in residuated lattices, we will use comments to specify this fact.

Definition 4.23. Let S be a nonvoid subset of L, then we say the set $S^{\perp} = \{x \in L : a \wedge x = 0 \text{ for all } a \in S\}$ is an *annihilator* of S.

In Example 3.14, if $S = (f] = \{0, c, f\}$ then it is easy to check that $S^{\perp} = \{0, a, b\} = (b]$.

Proposition 4.24. Let S be a subset of L. Then $S^{\perp}$ is an ideal of L. Moreover, if $S \neq \{0\}$, then $S^{\perp}$ is a proper ideal of L.

Proof. For every $a \in S$, we have $a \wedge 0 = 0$, hence $0 \in S^{\perp}$, which implies $S^{\perp}$ is nonempty.

Assume that $y \in S^{\perp}$, $x \leq y$, since for all $a \in S$, $x \wedge a \leq y \wedge a = 0$, then we have $x \wedge a = 0$, that is, $x \in S^{\perp}$. Hence, $S^{\perp}$ is a down set.

Assume that $x, y \in S^{\perp}$. Let $a \in S$, then $a \wedge x = a \wedge y = 0$. Then we obtain successively $a \wedge (x \oplus y) \overset{(c_{24})}{\leq} (a \wedge x) \oplus (a \wedge y) = 0 \oplus 0 = 0$, hence $a \wedge (x \oplus y) = 0$, that is, $x \oplus y \in S^{\perp}$. Therefore, $S^{\perp}$ is an ideal of L.

If $S \neq \{0\}$, then there is $a \in S$ such that $a \neq 0$, so $1 \wedge a = a \neq 0$, then we have $1 \notin S^{\perp}$. Therefore, $S^{\perp}$ is proper. $\square$

From Proposition 4.24 we notice that the annihilators of nonvoid subsets of any residuated lattice L are down sets. Moreover, if L is a De Morgan residuated lattice, it follows that the annihilators of nonvoid subsets of L are special kind of ideals.

Proposition 4.25. Let L be a residuated lattice. For all $a, b, x, y \in L$, the following assertions hold:

(1) $\{1\}^{\perp} = \{0\}$ and $\{0\}^{\perp} = L$;
(2) if $a \leq b$, then $\{b\}^{\perp} \subseteq \{a\}^{\perp}$ and $\{a\}^{\perp\perp} \subseteq \{b\}^{\perp\perp}$;
(3) if L is distributive, then $\{a\}^{\perp} \cap \{b\}^{\perp} = \{a \vee b\}^{\perp}$;
(4) $\{a\}^{\perp} \cup \{b\}^{\perp} \subseteq \{a \wedge b\}^{\perp}$;
(5) if $x \in \{a\}^{\perp}$, then $a \leq x^*$ and $x \leq a^*$;
(6) if $x \in \{a\}^{\perp}$, $y \in \{b\}^{\perp}$, then $x \odot y$, $x \wedge y \in \{a \wedge b\}^{\perp}$;
(7) if $x \in \{a\}^{\perp}$, $y \in \{a \vee b\}^{\perp}$, then $x \wedge y \in \{a \wedge b\}^{\perp}$;
(8) if $x \in \{a\}^{\perp}$, $y \in \{a \to b\}^{\perp}$, then $x \wedge y \in \{a \wedge b\}^{\perp}$;
(9) if $x \in \{a\}^{\perp}$, $y \in \{b\}^{\perp}$, then $x \odot y$, $x \wedge y \in \{a \odot b\}^{\perp}$.

Proof. (1) For all $x \in \{1\}^{\perp}$, $x = x \wedge 1 = 0$, so $x = 0$, which implies $\{1\}^{\perp} = \{0\}$. For all $x \in L$, since $x \wedge 0 = 0$, we have $L \subseteq \{0\}^{\perp}$, and evidently, $\{0\}^{\perp} \subseteq L$, so $\{0\}^{\perp} = L$.

(2) For all $x \in \{b\}^{\perp}$, we have $a \wedge x \leq b \wedge x = 0$, so $a \wedge x = 0$. Therefore, $x \in \{a\}^{\perp}$. The rest is clear.

(3) By distributivity of L and (c_7), we have successively $x \in \{a\}^{\perp} \cap \{b\}^{\perp}$ iff $x \in \{a\}^{\perp}$ and $x \in \{b\}^{\perp}$ iff $x \wedge a = 0$ and $x \wedge b = 0$ iff $(x \wedge a) \vee (x \wedge b) = 0$ iff $x \wedge (a \vee b) = 0$ iff $x \in \{a \vee b\}^{\perp}$.

(4) If $x \in \{a\}^{\perp} \cup \{b\}^{\perp}$, then $x \in \{a\}^{\perp}$ or $x \in \{b\}^{\perp}$, so $x \wedge a = 0$ or $x \wedge b = 0$. Hence $x \wedge a \wedge b = 0$. Therefore, $x \in \{a \wedge b\}^{\perp}$.

(5) If $x \in \{a\}^{\perp}$, we get $a \odot x \leq a \wedge x = 0$, so $a \odot x \leq 0$ and by residuation property we get $a \leq x^*$ and $x \leq a^*$.

(6) Since $x \in \{a\}^{\perp}$, $y \in \{b\}^{\perp}$ and $a \wedge b \leq a, b$, by item (2), we have $\{a\}^{\perp} \subseteq \{a \wedge b\}^{\perp}$, $\{b\}^{\perp} \subseteq \{a \wedge b\}^{\perp}$, so $x, y \in \{a \wedge b\}^{\perp}$. Since $\{a \wedge b\}^{\perp}$ is a down set, we get $x \odot y, x \wedge y \in \{a \wedge b\}^{\perp}$.

(7) Since $y \in \{a \vee b\}^{\perp}$ and $b \leq a \vee b$, by item (2), we have $\{a \vee b\}^{\perp} \subseteq \{b\}^{\perp}$, so $y \in \{b\}^{\perp}$, since $\{b\}^{\perp}$ is a down set, we get $x \wedge y \in \{b\}^{\perp} \subseteq \{a \wedge b\}^{\perp}$.

(8) Since $y \in \{a \to b\}^{\perp}$ and $b \leq a \to b$, by item (2), we get $\{a \to b\}^{\perp} \subseteq \{b\}^{\perp}$, so $y \in \{b\}^{\perp}$. Since $\{b\}^{\perp}$ is a down set, we get $x \wedge y \in \{b\}^{\perp} \subseteq \{a \wedge b\}^{\perp}$.

(9) Since $x \in \{a\}^{\perp}$, $y \in \{b\}^{\perp}$ and $a \odot b \leq a, b$, by item (2), we have $\{a\}^{\perp} \subseteq \{a \odot b\}^{\perp}$, $\{b\}^{\perp} \subseteq \{a \odot b\}^{\perp}$, so $x, y \in \{a \odot b\}^{\perp}$, and since $\{a \odot b\}^{\perp}$ is a down set, we get $x \odot y, x \wedge y \in \{a \odot b\}^{\perp}$.

Alternatively, the items $(7), (8)$ and (9) follow from (6). □

In Proposition 4.25 we notice that the items $(1), (2)-(9)$ may hold in residuated lattices because in their proofs we used the fact that an annihilator is a down set.

Proposition 4.26. For all $a, b, x, y \in L$, the following assertions hold:

(1) if $x \in \{a\}^{\perp}$, $y \in \{b\}^{\perp}$, then $x \vee y$, $x \oplus y \in \{a \odot b\}^{\perp}$;

(2) if $x \in \{a\}^{\perp}$, $y \in \{b\}^{\perp}$, then $x \vee y$, $x \oplus y \in \{a \wedge b\}^{\perp}$.

Proof. (1) Since $x \in \{a\}^{\perp}$, $y \in \{b\}^{\perp}$ and $a \odot b \leq a, b$, likewise in the proof of Proposition 4.25 (9), we have $x, y \in \{a \odot b\}^{\perp}$, and so $x \vee y \in \{a \odot b\}^{\perp}$, $x \oplus y \in \{a \odot b\}^{\perp}$, as $\{a \odot b\}^{\perp}$ is an ideal of L.

(2) Since $x \in \{a\}^{\perp}$, $y \in \{b\}^{\perp}$ likewise in in the proof of Proposition 4.25 (6) we get $x, y \in \{a \wedge b\}^{\perp}$, and so $x \vee y, x \oplus y \in \{a \wedge b\}^{\perp}$, as $\{a \wedge b\}^{\perp}$ is an ideal of L. □

In Proposition 4.26 we notice that the items (1) and (2) hold in De Morgan residuated lattices because we used the fact that an annihilator is an ideal.

Remark 4.27. In [11] an open problem was proposed: find the necessary conditions for a residuated lattice to be distributive. The assertion from Proposition 4.25, (3) represent a necessary condition for distributivity, but is not a necessary and sufficient condition as we can see in what follows.

The residuated lattice L from Example 2.11 is not distributive because $c \vee (a \wedge d) = c \vee 0 = c$, $(c \vee a) \wedge (c \vee d) = m \wedge d = d$ and $c \neq d$. But $\{a\}^{\perp} = (d] = \{0, c, d\}$, $\{d\}^{\perp} = (a] = \{0, a\}$, $\{a\}^{\perp} \cap \{d\}^{\perp} = \{0\}$ and $\{a \vee d\}^{\perp} = \{m\}^{\perp} = \{0\}$, hence $\{a\}^{\perp} \cap \{d\}^{\perp} = \{a \vee d\}^{\perp}$. Therefore, the converse of Proposition 4.25(3) may not hold.

In Example 3.14 we have $\{b\}^{\perp} = \{0, c, f\}$, $\{f\}^{\perp} = \{0, a, b\}$ and $b \wedge f = 0$, so $\{b\}^{\perp} \cup \{f\}^{\perp} = \{0, a, b, c, f\}$. Hence $\{0\}^{\perp} = L \nsubseteq \{b\}^{\perp} \cup \{f\}^{\perp}$. Therefore, the inclusion in Proposition 4.25, (4) is proper.

Proposition 4.28. For $X, Y \subseteq L$, the following assertions hold:

(1) $X^{\perp} = \cap_{x \in X} \{x\}^{\perp}$;

(2) If $X \neq \emptyset$, then $(X] \cap X^{\perp} = \{0\}$;

(3) If $X \subseteq Y$, then $Y^{\perp} \subseteq X^{\perp}$;
(4) If L is distributive and X is a linear ideal of L (which means that X is totally ordered). Then $X^{\perp}$ is prime.
(5) $X \subseteq X^{\perp\perp}$;
(6) $X^{\perp} = X^{\perp\perp\perp}$;
(7) $X^{\perp} = (X]^{\perp}$;
(8) $L^{\perp} = \{0\}$;
(9) $X^{\perp} \cap X^{\perp\perp} = \{0\}$;
(10) $(X \cup Y)^{\perp} = X^{\perp} \cap Y^{\perp}$;
(11) $X^{\perp} \cup Y^{\perp} \subseteq (X \cap Y)^{\perp}$;
(12) For all $a, b \in L$, if $a \in X^{\perp}$, $b \in Y^{\perp}$, then $a \wedge b \in (X \cup Y)^{\perp}$ and $a \vee b \in (X \cap Y)^{\perp}$;
(13) $X^{\perp} \cap Y^{\perp} = \{0\}$ iff $X^{\perp} \subseteq Y^{\perp\perp}$ and $Y^{\perp} \subseteq X^{\perp\perp}$.

Proof. (1) $a \in X^{\perp}$ iff $a \wedge x = 0$, for all $x \in X$ iff $a \in \{x\}^{\perp}$, for all $x \in X$ iff $a \in \cap_{x \in X}\{x\}^{\perp}$.

(2) Assume $a \in (X] \cap X^{\perp}$. Then $a \in (X]$ and $a \in X^{\perp}$. By Proposition 3.7(i) we have $a \leq x_1 \oplus x_2 \oplus \cdots \oplus x_n$, for some $x_1, x_2, \ldots, x_n \in X$. And $a \wedge x_i = 0$, for all $i = 1, 2, \ldots, n$.
By (c_{24}) we obtain successively $a \leq a \wedge (x_1 \oplus x_2 \oplus \cdots \oplus x_n) \leq (a \wedge x_1) \oplus (a \wedge x_2) \oplus \cdots \oplus (a \wedge x_n) = 0 \oplus 0 \oplus \cdots \oplus 0 = 0$. Therefore, $a = 0$, that is, $(X] \cap X^{\perp} = \{0\}$.

(3) Let $y \in Y$. If $z \in Y^{\perp}$, we have $z \wedge y = 0$. Then for any $x \in X \subseteq Y$, $z \wedge x = 0$, and so $z \in \cap_{x \in X}\{x\}^{\perp} = X^{\perp}$ by item (1). Therefore, $Y^{\perp} \subseteq X^{\perp}$.

(4) Let L be a distributive residuated lattice. Assume that X is an ideal which is linear (totally ordered), and $x \wedge y \in X^{\perp}$, but $x, y \notin X^{\perp}$. Then there are $x', x'' \in X$, such that $x \wedge x' \neq 0$, and $y \wedge x'' \neq 0$. Set $z = x' \vee x''$. Then $z \in X$, as X is an ideal of L. By the distributivity of L we obtain $x \wedge z = x \wedge (x' \vee x'') = (x \wedge x') \vee (x \wedge x'') \neq 0$. Similarly, we have $y \wedge z \neq 0$. Since $x \wedge z \leq z$, $y \wedge z \leq z$, we conclude $x \wedge z, y \wedge z \in X$, as $z \in X$ and X is an ideal.
As X is linear (totally ordered), we may assume that $x \wedge z \leq y \wedge z$. Since $x \wedge y \in X^{\perp}$ and $z \in X$, it follows that $0 = (x \wedge y) \wedge z = x \wedge (y \wedge z) \geq x \wedge (x \wedge z) = x \wedge z$, so $x \wedge z = 0$, that is, $x \in X^{\perp}$, a contradiction. Therefore, $X^{\perp}$ is prime.

(5) By the definition of annihilator, we have $X^{\perp\perp} = \{a \in L : a \wedge x = 0 \text{ for all } x \in X^{\perp}\}$. So, for all $x \in X^{\perp}$, if $b \in X$, then $b \wedge x = 0$, that is, $b \in X^{\perp\perp}$.

(6) By item (5), tacking $X = X^{\perp}$, we have $X^{\perp} \subseteq X^{\perp\perp\perp}$. Conversely, by item (5) we have $X \subseteq X^{\perp\perp}$, and by item (2) we conclude that $X^{\perp\perp\perp} \subseteq X^{\perp}$. Therefore, $X^{\perp} = X^{\perp\perp\perp}$.

(7) Since $X \subseteq (X]$, by item (2), we have $(X]^{\perp} \subseteq X^{\perp}$. Now, we prove that $X^{\perp} \subseteq (X]^{\perp}$. Let $a \in X^{\perp}$. So for any $x \in X$ we have $a \wedge x = 0$. For any $z \in (X]$, there are $x_1, x_2, \ldots, x_n \in X$ such that $z \leq x_1 \oplus x_2 \oplus \cdots \oplus x_n$. By (c_{24}), we obtain successively $a \wedge z \leq a \wedge (x_1 \oplus x_2 \oplus \cdots \oplus x_n) \leq (a \wedge x_1) \oplus (a \wedge x_2) \oplus \cdots \oplus (a \wedge x_n) = 0 \oplus 0 \oplus \cdots \oplus 0 = 0$. Therefore, $a \wedge z = 0$, that is, $a \in (X]^{\perp}$, and so $X^{\perp} \subseteq (X]^{\perp}$. Therefore, $X^{\perp} = (X]^{\perp}$.

(8) If $a \in L^{\perp}$, then $a = 1 \wedge a = 0$ for $1 \in L$. Therefore, $L^{\perp} = \{0\}$.

(9) Clearly, $\{0\} \subseteq X^{\perp} \cap X^{\perp\perp}$. Conversely, if $x \in X^{\perp} \cap X^{\perp\perp}$, by the definition of annihilators we have $x = x \wedge x = 0$. So $X^{\perp} \cap X^{\perp\perp} \subseteq \{0\}$. Therefore, $X^{\perp} \cap X^{\perp\perp} = \{0\}$.

(10) Since $X \subseteq X \cup Y$ and $Y \subseteq X \cup Y$, it follows that $(X \cup Y)^{\perp} \subseteq X^{\perp}$ and $(X \cup Y)^{\perp} \subseteq Y^{\perp}$, so $(X \cup Y)^{\perp} \subseteq X^{\perp} \cap Y^{\perp}$. Conversely, for any $a \in X^{\perp} \cap Y^{\perp}$, we have $a \in X^{\perp}$ and $a \in Y^{\perp}$, that is, for any $x \in X$, $y \in Y$, we have $a \wedge x = 0$ and $a \wedge y = 0$. So for any $t \in X \cup Y$, we always have $a \wedge t = 0$, hence $a \in (X \cup Y)^{\perp}$. Therefore, $(X \cup Y)^{\perp} = X^{\perp} \cap Y^{\perp}$.

(11) Since $X \cap Y \subseteq X$ and $X \cap Y \subseteq Y$, we have $X^{\perp} \subseteq (X \cap Y)^{\perp}$ and $Y^{\perp} \subseteq (X \cap Y)^{\perp}$. Therefore, $X^{\perp} \cup Y^{\perp} \subseteq (X \cap Y)^{\perp}$.

(12) If $a \in X^{\perp}$, $b \in Y^{\perp}$, then $a \wedge b \in X^{\perp}$ and $a \wedge b \in Y^{\perp}$, so $a \wedge b \in X^{\perp} \cap Y^{\perp} = (X \cup Y)^{\perp}$.
If $a \in X^{\perp}$, $b \in Y^{\perp}$, then $a \in X^{\perp} \subseteq (X \cap Y)^{\perp}$ and $b \in Y^{\perp} \subseteq (X \cap Y)^{\perp}$, so $a \vee b \in (X \cap Y)^{\perp}$.

(13) For all $a \in X^{\perp}$, $b \in Y^{\perp}$, we have $a \wedge b \in X^{\perp} \cap Y^{\perp}$, since $X^{\perp} \cap Y^{\perp} = \{0\}$, we get $a \wedge b = 0$, by definition of annihilator, we get $a \in Y^{\perp\perp}$ and $b \in X^{\perp\perp}$, so $X^{\perp} \subseteq Y^{\perp\perp}$ and $Y^{\perp} \subseteq X^{\perp\perp}$. Conversely, if $X^{\perp} \subseteq Y^{\perp\perp}$ and $Y^{\perp} \subseteq X^{\perp\perp}$, then $X^{\perp} \cap Y^{\perp} \subseteq Y^{\perp\perp} \cap Y^{\perp} = \{0\}$, so $X^{\perp} \cap Y^{\perp} \subseteq \{0\}$. Clearly, $\{0\} \subseteq X^{\perp} \cap Y^{\perp}$. Therefore, $X^{\perp} \cap Y^{\perp} = \{0\}$. □

Remark 4.29. The residuated lattice L from Example 2.12 is not distributive because $c \vee (a \wedge d) = c \vee 0 = c$, $(c \vee a) \wedge (c \vee d) = m \wedge d = d$ and $c \neq d$. But $\{b\}^{\perp} = (d] = \{0, c, d\}$, $\{d\}^{\perp} = (b] = \{0, a, b\}$ are prime ideals of L and they are linear (totally ordered), too. Therefore, the assertion from Proposition 4.28(3) represent a necessary condition for distributivity, but is not a necessary and sufficient condition, as we can see the converse may not hold.
In Example 2.12. If $X = \{0, a, b\}$, $Y = \{0, c, d\}$, then $X \cap Y = \{0\}$, so we have $X^{\perp} = Y$, $Y^{\perp} = X$, and $(X \cap Y)^{\perp} = \{0\}^{\perp} = L$. Hence $L = (X \cap Y)^{\perp} \nsubseteq X^{\perp} \cup Y^{\perp} = Y \cup X = L \setminus \{m, 1\}$. Therefore, the inclusion in Proposition 4.28, (11) is proper.

Theorem 4.30. The ideal lattice $\mathcal{I}_i(L)$ is pseudo-complemented and for any ideal I of L, its pseudo-complement is $I^{\perp}$.

Proof. By Proposition 4.28(1), we have $I \cap I^{\perp} = \{0\}$. Let G be an ideal of L such that $I \cap G = \{0\}$, we shall prove that $G \subseteq I^{\perp}$. Let $a \in G$, for any $x \in I$, then we have $x \wedge a \leq x \in I$, $x \wedge a \leq a \in G$, so $x \wedge a \in I \cap G = \{0\}$. Hence $x \wedge a = 0$, for any $x \in I$, then we have $a \in I^{\perp}$. So $I^{\perp}$ is the largest ideal such that $I \cap G = \{0\}$. Therefore, $I^{\perp}$ is the pseudo-complement of I. □

By $An(L) = \{X^{\perp} : X \subseteq L\}$ we denote the *set of all annihilators* of L. Since $X^{\perp} = (X]^{\perp}$, we get that $An(L) = \{I^{\perp} : I \in \mathcal{I}_i(L)\}$. Hence $An(L)$ is the set of pseudo-complements of the pseudo-complemented lattice $\mathcal{I}_i(L)$.

Proposition 4.31. Let $I, J \in \mathcal{I}_i(L)$. Then:

(1) $\{0\}, L, I^{\perp} \in An(L)$;

(2) $I \in An(L)$ iff $I = I^{\perp\perp}$;

(3) $\perp\perp : X \to X^{\perp\perp}$ is a closure map;

(4) $I \cap (I \cap J)^{\perp} = I \cap J^{\perp}$;

(5) $(I \cap J)^{\perp\perp} = I^{\perp\perp} \cap J^{\perp\perp}$;

(6) If $I, J \in An(L)$, then $I \wedge_{An(L)} J = I \cap J$;

(7) $(I \vee J)^{\perp} = I^{\perp} \cap J^{\perp}$;

(8) If $I, J \in An(L)$, then $I \vee_{An(L)} J = (I \vee J)^{\perp\perp} = (I^{\perp} \cap J^{\perp})^{\perp}$.

Proof. (1) By Propositions 4.25 and 4.28.

(2) Assume that $I \in An(L)$, then there exists $X \subseteq L$ such that $X^{\perp} = I$, so we get $I^{\perp\perp} = X^{\perp\perp\perp} = X^{\perp} = I$. The converse is clear.

(3) By Proposition 4.28, we know the function $f : X \to X^{\perp\perp}$ is isotone (see item (2)) and we get that $f = f^2 \geq id_L$ (see items (4)–(7)). So, $X \to X^{\perp\perp}$ is a closure map.

(4) Since $(I \cap J) \cap (I \cap J)^{\perp} = \{0\}$. By Theorem 4.30, we get $I \cap (I \cap J)^{\perp} \subseteq J^{\perp}$ and so $I \cap (I \cap J)^{\perp} \subseteq I \cap J^{\perp}$. Conversely, by $I \cap J \subseteq J$, we get $J^{\perp} \subseteq (I \cap J)^{\perp}$, so $I \cap J^{\perp} \subseteq (I \cap J)^{\perp}$. Therefore, $I \cap (I \cap J)^{\perp} = I \cap J^{\perp}$.

(5) Since $I \cap J \subseteq I, J$, we get $(I \cap J)^{\perp\perp} \subseteq I^{\perp\perp} \cap J^{\perp\perp}$. Conversely, since $(I \cap J) \cap (I \cap J)^{\perp} = \{0\}$ and by Theorem 4.30, we obtain successively $I \cap (I \cap J)^{\perp} \subseteq J^{\perp} = J^{\perp\perp\perp}$, $I \cap J^{\perp\perp} \cap (I \cap J)^{\perp} = \{0\}$, $J^{\perp\perp} \cap (I \cap J)^{\perp} \subseteq I^{\perp} = I^{\perp\perp\perp}$, $I^{\perp\perp} \cap J^{\perp\perp} \cap (I \cap J)^{\perp} = \{0\}$, $I^{\perp\perp} \cap J^{\perp\perp} \subseteq (I \cap J)^{\perp\perp}$. Therefore, $(I \cap J)^{\perp\perp} = I^{\perp\perp} \cap J^{\perp\perp}$.

(6) By items (2), (3) and Theorem 2.7, we have $I \wedge_{An(L)} J = I^{\perp\perp} \wedge_{An(L)} J^{\perp\perp} = I^{\perp\perp} \cap J^{\perp\perp} = I \cap J$.

(7) Since $I, J \subseteq I \vee J$, by item (6) and $I^{\perp}, J^{\perp} \in Ann(L)$, we get $(I \vee J)^{\perp} \subseteq I^{\perp} \cap J^{\perp} = I^{\perp\perp\perp} \cap J^{\perp\perp\perp} = (I^{\perp} \cap J^{\perp})^{\perp\perp}$. Conversely, $I \subseteq I^{\perp\perp} \subseteq (I^{\perp} \cap J^{\perp})^{\perp}$, similarly, we have $J \subseteq J^{\perp\perp} \subseteq (I^{\perp} \cap J^{\perp})^{\perp}$, hence $(I^{\perp} \cap J^{\perp})^{\perp\perp} \subseteq (I \vee J)^{\perp}$. Therefore, $(I \vee J)^{\perp} = (I^{\perp} \cap J^{\perp})^{\perp\perp} = I^{\perp\perp\perp} \cap J^{\perp\perp\perp} = I^{\perp} \cap J^{\perp}$.

(8) By item (3) and Theorem 2.7, we have $I \vee_{An(L)} J = (I \vee J)^{\perp\perp}$, then by item (7) we have $I \vee_{An(L)} J = (I^{\perp} \cap J^{\perp})^{\perp}$. $\square$

Theorem 4.32. $(An(L), \wedge_{An(L)}, \vee_{An(L)}, \perp, \{0\}, L)$ is a Boolean algebra.

Proof. By Proposition 4.31(6), we have $I \wedge_{An(L)} J = I \cap J$. In order to prove that $An(L)$ is a distributive lattice, it suffices to prove that: for all $I, J, H \in An(L)$, $H \cap (I \vee_{An(L)} J) \subseteq (H \cap I) \vee_{An(L)} (H \cap J)$. Now, let $K = (H \cap I) \vee_{An(L)} (H \cap J)$, then $H \cap I \subseteq K \subseteq K^{\perp\perp}$. By Proposition 4.31(8), we notice that $K^{\perp\perp} = ((H \cap I) \vee_{An(L)} (H \cap J))^{\perp\perp} = ((H \cap I) \vee (H \cap J))^{\perp\perp\perp\perp} = ((H \cap I) \vee (H \cap J))^{\perp\perp} = (H \cap I) \vee_{An(L)} (H \cap J) = K$, so $K = K^{\perp\perp}$.

Since $H \cap I \cap K^{\perp} \subseteq K^{\perp} \cap K^{\perp\perp} = \{0\}$, by virtue of Theorem 4.30, we have $H \cap K^{\perp} \subseteq I^{\perp}$. Similarly, $H \cap K^{\perp} \subseteq J^{\perp}$. Therefore, $H \cap K^{\perp} \subseteq I^{\perp} \cap J^{\perp} = I^{\perp\perp\perp} \cap J^{\perp\perp\perp} =$

$(I^{\perp} \cap J^{\perp})^{\perp\perp}$. We conclude that $H \cap K^{\perp} \cap (I^{\perp} \cap J^{\perp})^{\perp} = \{0\}$ and so $H \cap (I \vee_{An(L)} J) = H \cap (I^{\perp} \cap J^{\perp})^{\perp} \subseteq K^{\perp\perp} = K = (H \cap I) \vee_{An(L)} (H \cap J)$.

Now, we prove that $An(L)$ is complemented. We notice that $L = \{0\}^{\perp} \in An(L)$ and $\{0\} = L^{\perp} \in An(L)$. By Proposition 4.28(1), we have $I \cap I^{\perp} = \{0\}$, for every $I \in An(L)$ and $I \vee_{An(L)} I^{\perp} = (I^{\perp} \cap I^{\perp\perp})^{\perp} = L$. Hence the complement of I in $An(L)$ is $I^{\perp}$. Therefore, $An(L)$ is a Boolean algebra. □

4.4. Regular ideals

In what follows (unless otherwise specified) we denote by L a De Morgan residuated lattice. For simplicity we use the notation $i^{\perp}$ instead of $\{i\}^{\perp}$, for all $i \in L$.

Definition 4.33. An ideal I of L is called *regular* iff for all $i \in I$, $i^{\perp\perp} \subseteq I$. We denote by $Reg_i(L)$ the *set of all regular ideals* of L.

Remark 4.34. In Example 3.14, if $I = (f] = \{0, c, f\}$ then it is easy to check that $0^{\perp\perp} = \{0\}$, $c^{\perp\perp} = I$ and $f^{\perp\perp} = I$. Therefore, I is a regular ideal of L.

But not all ideals are regular. Indeed, in Example 3.13, if $I = \{0, n\}$ then it is easy to check that $0^{\perp\perp} = L^{\perp} = \{0\}$, $n^{\perp} = \{0\}$ and $n^{\perp\perp} = \{0\}^{\perp} = L \nsubseteq I$.

Definition 4.35. For an ideal I of L we denote by $R(I) := \{x \in L : \exists i \in I \text{ such that } i^{\perp} \subseteq x^{\perp}\}$.

Lemma 4.36. $R(I)$ is the smallest regular ideal containing I. Moreover, if I is a proper ideal of L, then $R(I)$ is a proper ideal of L.

P r o o f. Clearly, $I \subseteq R(I)$.

Assume that $x \leq y$ and $y \in R(I)$. Since $y \in R(I)$, then there is an element $i \in I$ such that $i^{\perp} \subseteq y^{\perp}$. Since $x \leq y$, we get $y^{\perp} \subseteq x^{\perp}$, so $i^{\perp} \subseteq y^{\perp} \subseteq x^{\perp}$. Therefore, $x \in R(I)$.

Assume that $x, y \in R(L)$. We will prove that $x \oplus y \in R(L)$. Since $x, y \in R(L)$, we get there are $i, j \in I$ such that $i^{\perp} \subseteq x^{\perp}$ and $j^{\perp} \subseteq y^{\perp}$. Clearly, $i \oplus j \in I$. For any $t \in (i \oplus j)^{\perp}$ we have $t \wedge (i \oplus j) = 0$, so $t \wedge i = 0$ and $t \wedge j = 0$. Since $(i \oplus j)^{\perp} \subseteq i^{\perp} \subseteq x^{\perp}$ and $(i \oplus j)^{\perp} \subseteq j^{\perp} \subseteq y^{\perp}$, we conclude that $t \wedge x = 0$ and $t \wedge y = 0$. By (c_{24}), we have $t \wedge (x \oplus y) \leq (t \wedge x) \oplus (t \wedge y) = 0 \oplus 0 = 0$, that is, $t \in (x \oplus y)^{\perp}$, and so $(i \oplus j)^{\perp} \subseteq (x \oplus y)^{\perp}$. Hence, $x \oplus y \in R(I)$. Therefore, $R(I)$ is an ideal of L.

Now, we prove that $R(I)$ is a regular ideal of L. For any $x \in R(I)$, there is an element $i \in I$ such that $i^{\perp} \subseteq x^{\perp}$. For any $t \in x^{\perp\perp}$, by Proposition 4.28(6), we have $i^{\perp} \subseteq x^{\perp} = x^{\perp\perp\perp} \subseteq t^{\perp}$, so $i^{\perp} \subseteq x^{\perp} \subseteq t^{\perp}$, that is, $t \in R(I)$ and $x^{\perp\perp} \subseteq R(I)$. Therefore, $R(I)$ is an regular ideal of L.

Now, we prove $R(I)$ is the smallest regular ideal containing I. Let K be a regular ideal such that $I \subseteq K$. For any $x \in R(I)$, there is $i \in I$ such that $i^{\perp} \subseteq x^{\perp}$. Then by Proposition 4.28, we have $x \in (x] \subseteq (x]^{\perp\perp} = x^{\perp\perp} \subseteq i^{\perp\perp} \subseteq K$. Therefore, $R(I) \subseteq K$.

If I is a proper ideal of L, then $1 \notin I$. Since $1^{\perp} = 0$ and $1 \notin I$, by the definition of $R(I)$ we conclude that $1 \notin R(I)$. □

Remark 4.37. In Example 3.14, if $I = (f] = \{0, c, f\}$ then it is easy to check that $R(I) = I$.

Proposition 4.38. Let I, J be ideals of L. Then:

(1) $I^{\perp}$ is a regular ideal of L;
(2) $R(R(I)) = R(I)$;
(3) $R(I^{\perp}) = I^{\perp}$;
(4) If $I \subseteq J$, then $R(I) \subseteq R(J)$;
(5) $R(I)$ is the intersection of all regular ideals containing I;
(6) I is a regular ideal iff $R(I) = I$;
(7) $\cap\{I : I \text{ is a regular ideal of } L\} = \{0\}$;
(8) $R(I) = R(J)$ iff $I \subseteq R(J)$ and $J \subseteq R(I)$.

Proof. (1) For any $a \in I^{\perp}$, we have $I^{\perp\perp} \subseteq a^{\perp}$, then we get $a^{\perp\perp} \subseteq I^{\perp\perp\perp} = I^{\perp}$, so $I^{\perp}$ is a regular ideal.

(2), (3), (4), (5) and (6) By Lemma 4.36 and item (1), we can easily prove them.

(7) It is clear, since $\{0\}$ is a regular ideal of L.

(8) Let $R(I) = R(J)$, so $I \subseteq R(I) \subseteq R(J)$ and $J \subseteq R(J) \subseteq R(I)$. Therefore, $I \subseteq R(J)$ and $J \subseteq R(I)$. Conversely, since $R(I)$ is the smallest regular ideal containing I, by $I \subseteq R(J)$ we get $I \subseteq R(I) \subseteq R(J)$ and by $J \subseteq R(I)$ we get $J \subseteq R(J) \subseteq R(I)$. Therefore, $R(I) = R(J)$. □

Lemma 4.39. Let L, L' be De Morgan residuated lattices, $f : L \to L'$ be a homomorphism, $\emptyset \neq X \subseteq L$. Then $f(X^{\perp}) \subseteq (f(X))^{\perp}$.

Proof. For all $x \in f(X^{\perp})$, there is $y \in X^{\perp}$ such that $x = f(y)$. For all $z \in f(X)$, there is $t \in X$ such that $z = f(t)$. We have $x \wedge z = f(y) \wedge f(t) = f(y \wedge t) = f(0) = 0$. Therefore, $x \in (f(X))^{\perp}$. □

Lemma 4.40. Let L, L' be De Morgan residuated lattices, $f : L \to L'$ be a surjective homomorphism, $\emptyset \neq Y \subseteq L'$. Then $(f^{-1}(Y))^{\perp} \subseteq f^{-1}(Y^{\perp})$.

Proof. For all $b \in Y$, there is $a \in L$ such that $b = f(a)$, that is, $a \in f^{-1}(b) \subseteq f^{-1}(Y)$, so $x \wedge a = 0$, for any $x \in (f^{-1}(Y))^{\perp}$. Then $f(x) \wedge b = f(x) \wedge f(a) = f(a \wedge x) = f(0) = 0$. Therefore, $f(x) \in Y^{\perp}$, that is, $x \in f^{-1}(Y^{\perp})$. □

Theorem 4.41. Let L, L' be De Morgan residuated lattices, $f : L \to L'$ be an isomorphism, $\emptyset \neq X \subseteq L$, $\emptyset \neq Y \subseteq L'$. Then $f(X^{\perp}) = (f(X))^{\perp}$ and $(f^{-1}(Y))^{\perp} = f^{-1}(Y^{\perp})$.

Proof. By Lemma 4.39, we have $f(X^{\perp}) \subseteq (f(X))^{\perp}$. Now, we prove that $f(X^{\perp}) \supseteq (f(X))^{\perp}$. For any $y \in (f(X))^{\perp} \subseteq L'$, since f is surjective, there is $x \in L$, such that $f(x) = y$. For any $a \in X$, we have $f(a) \in f(X)$, so $f(x \wedge a) = f(x) \wedge f(a) = y \wedge f(a) = 0$. Since f is injective, we get $x \wedge a = 0$, so $x \in X^{\perp}$, that is $y \in f(X^{\perp})$. Therefore, $f(X^{\perp}) = (f(X))^{\perp}$.

By Lemma 4.40, we have $(f^{-1}(Y))^{\perp} \subseteq f^{-1}(Y^{\perp})$. Now, we prove that $(f^{-1}(Y))^{\perp} \supseteq f^{-1}(Y^{\perp})$. For any $x \in f^{-1}(Y^{\perp})$, we have $f(x) \in Y^{\perp}$. For all $a \in f^{-1}(Y)$, we have $f(a) \in Y$, $f(x \wedge a) = f(x) \wedge f(a) = 0$. Since f is injective, we get $x \wedge a = 0$, so $x \in (f^{-1}(Y))^{\perp}$. Therefore, $(f^{-1}(Y))^{\perp} = f^{-1}(Y^{\perp})$. □

Corollary 4.42. Let L, L' be De Morgan residuated lattices, $f : L \to L'$ be a homomorphism, $\emptyset \neq X \subseteq L$, $\emptyset \neq Y \subseteq L'$. Then:

(i) $f(X^{\perp}) = (f(X))^{\perp}$ iff $(f(X^{\perp}))^{\perp\perp} = f(X^{\perp})$ and $(f(X))^{\perp} \cap (f(X^{\perp}))^{\perp} = \{0\}$;

(ii) If f is surjective, then $(f^{-1}(Y))^{\perp} = f^{-1}(Y^{\perp})$ iff $f^{-1}(Y^{\perp}) \cap (f^{-1}(Y))^{\perp\perp} = \{0\}$.

Proof. (i). If $f(X^{\perp}) = (f(X))^{\perp}$, then $f(X^{\perp})^{\perp\perp} = (f(X))^{\perp\perp\perp} = (f(X))^{\perp} = f(X^{\perp})$. Also, $(f(X))^{\perp} \cap (f(X^{\perp}))^{\perp} = (f(X))^{\perp} \cap (f(X))^{\perp\perp} = \{0\}$. Conversely, by Lemma 4.39, we have $f(X^{\perp}) \subseteq (f(X))^{\perp}$. Now, we prove that $f(X^{\perp}) \supseteq (f(X))^{\perp}$. Since $(f(X))^{\perp} \cap (f(X^{\perp}))^{\perp} = \{0\}$, we have $(f(X))^{\perp} \subseteq (f(X^{\perp}))^{\perp\perp} = f(X^{\perp})$. Therefore, $f(X^{\perp}) = (f(X))^{\perp}$.

(ii). If $(f^{-1}(Y))^{\perp} = f^{-1}(Y^{\perp})$, then $f^{-1}(Y^{\perp}) \cap (f^{-1}(Y))^{\perp\perp} = (f^{-1}(Y))^{\perp} \cap (f^{-1}(Y))^{\perp\perp} = \{0\}$. Conversely, if $x \in f^{-1}(Y^{\perp}), y \in L$ are such that $y \leq x$, then $f(y) \leq f(x)$. Since $f(x) \in Y^{\perp}$, so $f(y) \in Y^{\perp}$, and so $y \in f^{-1}(Y^{\perp})$. Consequently, $f^{-1}(Y^{\perp})$ is a down set. Since $f^{-1}(Y^{\perp}) \cap (f^{-1}(Y))^{\perp\perp} = \{0\}$, we have $f^{-1}(Y^{\perp}) \subseteq (f^{-1}(Y))^{\perp\perp\perp} = (f^{-1}(Y))^{\perp}$. It follows that $f^{-1}(Y^{\perp}) \subseteq (f^{-1}(Y))^{\perp}$. By Lemma 4.40, we have $(f^{-1}(Y))^{\perp} \subseteq f^{-1}(Y^{\perp})$. Therefore, $(f^{-1}(Y))^{\perp} = f^{-1}(Y^{\perp})$. □

Corollary 4.43. If $f : L \to L'$ is an isomorphism, then $R(f(I)) = f(R(I))$, for any ideal I of L.

Proof. Let $z \in R(f(I))$, then there is $a \in f(I)$ such that $a^{\perp} \subseteq z^{\perp}$. Since $f : L \to L'$ is an isomorphism, it holds that there are $a_0 \in I \subseteq L$ and $z_0 \in L$ such that $a = f(a_0)$, $z = f(z_0)$. By Theorem 4.41, we have $f({a_0}^{\perp}) = (f(a_0))^{\perp} = a^{\perp} \subseteq z^{\perp} = (f(z_0))^{\perp} = f({z_0}^{\perp})$. Hence ${a_0}^{\perp} \subseteq {z_0}^{\perp}$. We conclude that $z_0 \in R(I)$ and $z = f(z_0) \in f(R(I))$. Therefore, $R(f(I)) \subseteq f(R(I))$.

Conversely, let $z \in f(R(I))$. Since f is an isomorphism, we have $z = f(z_0)$, for some $z_0 \in R(I)$. Then there is $a_0 \in I$ such that ${a_0}^{\perp} \subseteq {z_0}^{\perp}$, so $(f(a_0))^{\perp} = f({a_0}^{\perp}) \subseteq f({z_0}^{\perp}) = (f(z_0))^{\perp} = z^{\perp}$. We conclude that $(f(a_0))^{\perp} \subseteq z^{\perp}$, with $f(a_0) \in f(I)$. Hence $z \in R(f(I))$; eventualy, $f(R(I)) \subseteq R(f(I))$. Therefore, $R(f(I)) = f(R(I))$. □

Proposition 4.44. (i) If I, J are ideals of L and L', respectively, then $R(I \times J) = R(I) \times R(J)$;

(ii) Let Λ be a finite index set. If I_i are ideals of L_i, for all $i \in \Lambda$, then $R(\prod_{i\in\Lambda} I_i) = \prod_{i\in\Lambda} R(I_i)$.

Proof. (i). Let $x \in L$ and $y \in L'$, we define $(x, y)^{\perp} = x^{\perp} \times y^{\perp}$. Then $R(I \times J) = \{(x, y) : \exists (a, b) \in I \times J$ such that $(a, b)^{\perp} \subseteq (x, y)^{\perp}\} = \{(x, y) : \exists a \in I, b \in J$ such that $a^{\perp} \subseteq x^{\perp}, b^{\perp} \subseteq y^{\perp}\} = \{(x, y) : x \in R(I), y \in R(J)\} = R(I) \times R(J)$.

(ii). It follows by (i). □

Proposition 4.45. If I and J are ideals of a totally ordered De Morgan residuated lattice, then $R(I \cap J) = R(I) \cap R(J)$.

P r o o f. We note that if I is an ideal of the totally ordered De Morgan residuated lattice L, then $R(I) = L$ or $R(I) = \{0\}$. If $I \neq \{0\}$, then there exists $0 \neq a \in I$. Since L is totally ordered, we get $a^{\perp} = \{0\}$ and so $R(I) = L$. If $I = \{0\}$, then $R(I) = \{x \in L : 0^{\perp} \subseteq x^{\perp}\} = \{x \in L : x^{\perp} = L\} = \{0\}$.

Now, we prove that if I and J are ideals of the totally ordered De Morgan residuated lattice L, then $R(I \cap J) = R(I) \cap R(J)$. If $I \neq \{0\}$ and $J \neq \{0\}$, then $I \cap J \neq \{0\}$. There exist $0 \neq a \in I$ and $0 \neq b \in J$ and since L is a totally ordered we have $a \leq b$ or $b \leq a$. Assume that $a \leq b$, then $0 \neq a \in I \cap J$. So $R(I \cap J) = L, R(I) = L$ and $R(J) = L$. If $I = \{0\}$ or $J = \{0\}$, then $I \cap J = \{0\}$. Then $R(I) = \{0\}$ or $R(J) = \{0\}$ and $R(I \cap J) = \{0\}$. Therefore, $R(I \cap J) = R(I) \cap R(J)$.

Let I and J be ideals of a finite direct product $\prod_{i=1}^{n} L_i$, where L_i are totally ordered De Morgan residuated lattices, for all $1 \leq i \leq n$. By Theorem 3.10, we have $I = \prod_{i=1}^{n} I_i$, $J = \prod_{i=1}^{n} J_i$, where I_i, J_i are ideals of L_i, for all $1 \leq i \leq n$. By Proposition 4.44, it follows successively $R(I \cap J) = R(\prod_{i=1}^{n} I_i \cap \prod_{i=1}^{n} J_i) = R(\prod_{i=1}^{n}(I_i \cap J_i)) = \prod_{i=1}^{n}(R(I_i) \cap R(J_i)) = \prod_{i=1}^{n} R(I_i) \cap \prod_{i=1}^{n} R(J_i) = R(\prod_{i=1}^{n} I_i) \cap R(\prod_{i=1}^{n} J_i) = R(I) \cap R(J)$. □

We denote by $R(\mathcal{I}_i(L)) := \{R(I) : I \in \mathcal{I}_i(L)\}$. We know that for every family $(I_i)_{i \in \Lambda}$ of ideals of L we have: $\wedge_{i \in \Lambda} I_i = \cap_{i \in \Lambda} I_i$ and $\vee_{i \in \Lambda} I_i = (\cup_{i \in \Lambda} I_i]$, with Λ an index set (see Proposition 3.8).

Proposition 4.46. $(R(\mathcal{I}_i(L)), \sqcap, \sqcup, R(0), R(L))$ is a complete Brouwerian lattice, where $R(I) \sqcap R(J) = R(I \cap J)$, $R(I) \sqcup R(J) = R(I \vee J)$, and L is a totally ordered De Morgan residuated lattice.

P r o o f. By Proposition 4.45, we have $R(I \cap J) = R(I) \cap R(J)$. Hence $R(I) \sqcap R(J) = R(I) \cap R(J)$. Since $I, J \subseteq I \vee J$, by Proposition 4.38(4), we have $R(I), R(J) \subseteq R(I \vee J)$. This means that $R(I \vee J)$ is an upper bound of $R(I)$ and $R(J)$. Now let $R(I), R(J) \subseteq R(K)$, for some $K \in \mathcal{I}_i(L)$. Then $I, J \subseteq R(K)$, hence $I \vee J \subseteq R(K)$ and so $R(I \vee J) \subseteq R(R(K)) = R(K)$, by Proposition 4.38(2). Consequently, $R(I \vee J)$ is the least upper bound of $R(I)$ and $R(J)$.

For simplicity we denote by $\vee(G_i) := \vee_{i \in \Lambda} G_i$ (the join of all ideals of the family of ideals $(G_i)_{i \in \Lambda}$), and by $\sqcup(R(G_i)) := \sqcup_{i \in \Lambda}(R(G_i))$. Now, we prove that for any family of ideals $(G_i)_{i \in \Lambda}$, we have that $\sqcup(R(G_i)) = R(\vee(G_i))$. Since $R(G_i) \subseteq R(\vee(G_i))$, we get $R(\vee(G_i))$ is an upper bound of $R(G_i)$, for all $i \in \Lambda$. Also if $R(G_i) \subseteq R(K)$, for all $i \in \Lambda$, then $G_i \subseteq R(K)$, for some $K \in \mathcal{I}_i(L)$. Then $\vee(G_i) \subseteq R(K)$, hence $R(\vee(G_i)) \subseteq R(R(K)) = R(K)$, by Proposition 4.38(2). Consequently, $R(\vee G_i)$ is the least upper bound of $R(G_i)$, for all $i \in \Lambda$. So $(R(\mathcal{I}_i(L)), \sqcap, \sqcup, R(0), R(L))$ is a complete lattice. By Proposition 4.11(*ii*), we get $\vee(I \wedge G_i) = I \wedge (\vee G_i) = I \cap (\vee G_i)$. It follows successively $\sqcup(R(I) \sqcap R(G_i)) = \sqcup(R(I \cap G_i) = R(\vee(I \cap G_i)) = R(\vee(I \wedge G_i)) = R(I \cap (\vee G_i)) = R(I) \sqcap R(\vee G_i) = R(I) \sqcap (\sqcup(R(G_i)))$. Therefore, $(R(\mathcal{I}_i(L)), \sqcap, \sqcup, R(0), R(L))$ is a complete Brouwerian lattice. □

Proposition 4.47. The lattice $(R(\mathcal{I}_i(L)), \sqcap, \sqcup, R(0), R(L))$ is pseudo-complemented, where L is a totally ordered De Morgan residuated lattice.

P r o o f. Let $I, K \in \mathcal{I}_i(L)$. By Propositions 4.28(8) and 4.38(3), $\{0\} = L^{\perp}$ and $R(\{0\}) = R(L^{\perp}) = L^{\perp} = \{0\}$. So $R(I) \sqcap R(I^{\perp}) = R(I \cap I^{\perp}) = R(\{0\}) = \{0\}$. Clearly, $I \subseteq R(I)$.

Since $\{0\} = R(I) \cap (R(I))^{\perp} \supseteq I \cap (R(I))^{\perp}$, by Theorem 4.30, it follows that $(R(I))^{\perp} \subseteq I^{\perp}$. Now let $R(I) \sqcap R(K) = R(\{0\}) = \{0\}$, that is, $R(I) \cap R(K) = \{0\}$, we get $R(K) \subseteq (R(I))^{\perp} \subseteq I^{\perp} = R(I^{\perp})$ by Theorem 4.30 and Proposition 4.38(3). So, for every ideal $R(I)$, its pseudo-complement is $R(I^{\perp})$. Therefore, the lattice $(R(\mathcal{I}_i(L)), \sqcap, \sqcup, R(0), R(L))$ is pseudo-complemented. $\square$

Proposition 4.48. The lattice $(R(\mathcal{I}_i(L)), \sqcap, \sqcup, R(0), R(L))$ is an algebraic lattice, where L is a totally ordered De Morgan residuated lattice.

Proof. Let $z \in L$ and Λ be an index set. Firstly, we prove that $R((z])$ is a compact element in the lattice $R(\mathcal{I}_i(L))$. Assume that $R((z]) \subseteq \sqcup R(G_i)$, where $(G_i)_{i\in\Lambda}$ is a family of ideals. Then $z \in R((z]) \subseteq \sqcup R(G_i)$ (by the proof of Proposition 4.46, $\sqcup R(G_i) = R(\vee_{i\in\Lambda} G_i)$), so there is $a \in \vee_{i\in\Lambda} G_i$ such that $a^{\perp} \subseteq z^{\perp}$, this means that there exist $x_i \in G_i$ ($1 \le i \le n$ for some n) such that $a \le x_1 \oplus x_2 \oplus \cdots \oplus x_n$ (because $\vee_{i\in\Lambda} G_i = (\cup_{i\in\Lambda} G_i] = \{x \in L : x \le x_1 \oplus x_2 \oplus \cdots \oplus x_n,\ \text{for some } n \ge 1,\ x_i \in G_i,\ 1 \le i \le n,\ i \in \Lambda\}$). Consider $X = \{G_1, G_2, \ldots, G_n\} \subseteq \cup_{i\in\Lambda} G_i$, so $(x_1 \oplus x_2 \oplus \cdots \oplus x_n)^{\perp} \subseteq a^{\perp} \subseteq z^{\perp}$, so $z \in R(\vee_{G_i\in X} G_i)$, so we get $(z] \in R(\vee_{G_i\in X} G_i)$, and $R((z]) \in R(R(\vee_{G_i\in X} G_i)) = R(\vee_{G_i\in X} G_i) = R(G_1) \sqcup R(G_2) \sqcup \cdots \sqcup R(G_n)$, by Proposition 4.38(2). Therefore, $R((z])$ is a compact element in the lattice $R(\mathcal{I}_i(L))$. Now consider $R(I) \in R(\mathcal{I}_i(L))$. Since $I = \vee_{a\in I}(a]$, we get $R(I) = R(\vee_{a\in I}(a]) = \sqcup\{R((a]) : a \in I\}$. Therefore, $(R(\mathcal{I}_i(L)), \sqcap, \sqcup, R(0), R(L))$ is an algebraic lattice. $\square$

4.5. Relative annihilators

Definition 4.49. Let X and I be subsets of L. The annihilator of X relative to I is the set $(X, I)^{\perp} = \{a \in L : (\forall x \in X) x \wedge a \in I\}$.

In Example 3.14, if $X = \{0, c, f\}, I = \{0, a\}$, then it is easy to show that $(X, I)^{\perp} = \{0, a, b\}$. Clearly, $I \subseteq (X, I)^{\perp}$.

Lemma 4.50. If I is an ideal and $\emptyset \ne X$ a subset of L, then $(X, I)^{\perp}$ is an ideal of L. Moreover, if I is a proper ideal of L, then $(X, I)^{\perp}$ is a proper ideal of L, too.

Proof. Clearly, $0 \in (X, I)^{\perp}$, so $(X, I)^{\perp}$ is nonempty.

Now, we prove that $(X, I)^{\perp}$ is a down set. Let $a \in (X, I)^{\perp}$ and $b \in L$, such that $b \le a$. Let $x \in X$. Since $b \wedge x \le a \wedge x$ and $a \wedge x \in I$, as I is an ideal of L, we have $b \wedge x \in I$, that is, $b \in (X, I)^{\perp}$.

Now, we show that if $a, b \in (X, I)^{\perp}$, then $a \oplus b \in (X, I)^{\perp}$. Let $a, b \in (X, I)^{\perp}$, then $a \wedge x \in I$ and $b \wedge x \in I$, for all $x \in X$. By (c_{24}), we have $x \wedge (a \oplus b) \le (x \wedge a) \oplus (x \wedge b) \in I$, that is, $a \oplus b \in (X, I)^{\perp}$. Therefore, $(X, I)^{\perp}$ is an ideal of L. Moreover, it is clear that if I is a proper ideal of L, then $(X, I)^{\perp}$ is a proper ideal of L, too. $\square$

In the following proposition, we consider $(X, I)^{\perp}$ for some special cases of X and I.

Proposition 4.51. Let I, J, H be ideals of L and $\emptyset \ne X, Y \subseteq L$. Then

(1) If $I \subseteq J$, then $(X, I)^{\perp} \subseteq (X, J)^{\perp}$;

(2) If $X \subseteq Y$, then $(Y, I)^{\perp} \subseteq (X, I)^{\perp}$;
(3) $((\cup_{i\in\Lambda} X_i), I)^{\perp} = ((\cap_{i\in\Lambda} X_i), I)^{\perp}$;
(4) $(X, I)^{\perp} = \cap_{x\in X}((X, I)^{\perp})$;
(5) $(X, \cap_{i\in\Lambda}(I_i))^{\perp} = \cap_{i\in\Lambda}(X, I_i)^{\perp}$;
(6) $((X], I)^{\perp} = (X, I)^{\perp}$;
(7) $I \subseteq (X, I)^{\perp}$;
(8) $(X, I)^{\perp} = L$ iff $X \subseteq I$;
(9) $(J, I)^{\perp} \cap J \subseteq I$;
(10) $J \cap H \subseteq I$ iff $H \subseteq (J, I)^{\perp}$.

Proof. (1) Let $I \subseteq J$, and $a \in (X, I)^{\perp}$. Then $a \wedge x \in I$, for all $x \in X$. Since $I \subseteq J$, we get $a \wedge x \in J$, for all $x \in X$, that is, $a \in (X, J)^{\perp}$. Therefore, $(X, I)^{\perp} \subseteq (X, J)^{\perp}$.

(2) Let $X \subseteq Y$, and $a \in (Y, I)^{\perp}$. Then $a \wedge x \in I$, for all $x \in Y$. Since $X \subseteq Y$, we get $a \wedge x \in I$, for all $x \in X$, that is, $a \in (X, I)^{\perp}$. Therefore, $(Y, I)^{\perp} \subseteq (X, I)^{\perp}$.

(3) By item (2) we have $((\cup_{i\in\Lambda} X_i), I)^{\perp} \subseteq (X_i, I)^{\perp}$, for all $i \in \Lambda$. So $((\cup_{i\in\Lambda} X_i), I)^{\perp} \subseteq ((\cap_{i\in\Lambda} X_i), I)^{\perp}$, for all $i \in \Lambda$. Conversely, let $a \in ((\cap_{i\in\Lambda} X_i), I)^{\perp}$, then $a \in ((X_i, I)^{\perp}$, for all $i \in \Lambda$. Hence $a \wedge x_i \in I$, for all $x_i \subseteq X_i$ and $i \in \Lambda$, that is, $a \in ((\cup_{i\in\Lambda} X_i), I)^{\perp}$. Therefore, $((\cup_{i\in\Lambda} X_i), I)^{\perp} = ((\cap_{i\in\Lambda} X_i), I)^{\perp}$.

(4) It follows easily from item (3).

(5) We have successively $a \in (X, \cap_{i\in\Lambda}(I_i))^{\perp}$ iff $a \wedge x \in \cap_{i\in\Lambda}(I_i)$, for all $x \in X$ iff $a \wedge x \in I_i$, for all $x \in X$ and $i \in \Lambda$ iff $a \in (X, I_i)^{\perp}$, for all $i \in \Lambda$ iff $a \in \cap_{i\in\Lambda}(X, I_i)^{\perp}$.

(6) Since $X \subseteq (X]$, by item (2) we have $((X], I)^{\perp} \subseteq (X, I)^{\perp}$. Conversely, let $a \in (X, I)^{\perp}$ and $z \in (X]$. Then $a \wedge x \in I$, for all $x \in X$. Since $z \in (X]$, it holds that $z \leq x_1 \oplus x_2 \oplus \cdots \oplus x_n$, for some $x_1, x_2, \ldots, x_n \in X$. Clearly, $a \wedge x_i \in I$, for all $x_i \in X$. By (c_{24}), we get $a \wedge z \leq a \wedge (x_1 \oplus x_2 \oplus \cdots \oplus x_n) \leq (a \wedge x_1) \oplus (a \wedge x_2) \oplus \cdots \oplus (a \wedge x_n) \in I$. Then $a \wedge z \in I$, as I is an ideal of L, so $a \in ((X], I)^{\perp}$. Therefore, $((X], I)^{\perp} = (X, I)^{\perp}$.

(7) Let $a \in I$, then $a \wedge x \leq a \in I$, for all $x \in X$, hence $a \in (X, I)^{\perp}$. Therefore, $I \in (X, I)^{\perp}$.

(8) If $(X, I)^{\perp} = L$, then $1 \in (X, I)^{\perp}$. Then $x = x \wedge 1 \in I$, for all $x \in X$. Therefore, $X \subseteq I$. Conversely, if $X \subseteq I$, then for any $a \in L$ and for all $x \in X$ we have $a \wedge x \leq x \in X \subseteq I$, so $a \in (X, I)^{\perp}$; consequently $L \subseteq (X, I)^{\perp}$. Therefore, $(X, I)^{\perp} = L$.

(9) Let $x \in (J, I)^{\perp} \cap J$. Then $x \in (J, I)^{\perp}$ and $x \in J$. Since $x = x \wedge x \in I$, we get $(J, I)^{\perp} \cap J \subseteq I$.

(10) If $J \cap H \subseteq I$ and $x \in H$, then for any $y \in J$ we have $x \wedge y \in J \cap H$, so $x \wedge y \in I$. Therefore, $x \in (J, I)^{\perp}$, and eventually, $H \subseteq (J, I)^{\perp}$. Conversely, let $H \subseteq (J, I)^{\perp}$. By item (9) we have $J \cap H \subseteq J \cap (J, I)^{\perp} \subseteq I$. □

Theorem 4.52. $(J, I)^{\perp}$ is the relative pseudo-complement of J with respect to I in the lattice $(\mathcal{I}_i(L), \subseteq)$.

Proof. We know that $(J, I)^{\perp}$ is an ideal. By Proposition 4.51 (10), we have $J \cap H \subseteq I$ iff $H \subseteq (J, I)^{\perp}$, for every ideal H of L. Therefore, $(J, I)^{\perp}$ is the relative pseudo-complement of J with respect to I in the lattice $(\mathcal{I}_i(L), \subseteq)$. □

5. CONCLUSIONS

In the paper, we study the notion of ideal in De Morgan residuated lattices, and propose new characterisations for prime, $\odot$-prime, maximal ideals. We introduce the notion of annihilator in De Morgan residuated lattices and investigate some properties of them. We get that the ideal lattice $(\mathcal{I}_i(L)), \subseteq)$ is pseudo-complemented, and for any ideal I, its pseudo-complement is the annihilator ideal $I^{\perp}$. Also, if we define $An(L)$ to be the set of all annihilators of L, then we have that $An(L)$ is a Boolean algebra. Moreover, we give the necessary and sufficient condition under which both the image and the preimage of an annihilator under a homomorphism are annihilators. In addition, we study regular ideals and relative annihilators in De Morgan residuated lattices.

In our future work, we will continue our study of algebraic properties of ideals and annihilators in residuated lattices. We will use these ideals to define congruence relations on L and to study the properties of the quotient residuated lattice of L. It seems that the residuated lattices can be studied from ideal theory view in a very nice way.

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REFERENCES

[1] R. Balbes and Ph. Dwinger: Distributive Lattices. University of Missouri Press 1974.

[2] T. S. Blyth: Lattices and Ordered Algebraic Structures. Springer, London 2005. DOI:10.1007/b139095

[3] C. Buşneag and D. Piciu: The stable topologies for residuated lattices. Soft Computing *16* (2012), 1639–1655. DOI:10.1007/s00500-012-0849-x

[4] D. Buşneag, D. Piciu, and J. Paralescu: Divisible and semi-divisible residuated lattices. Ann. Alexandru Ioan Cuza University-Mathematics (2013), 14–45.

[5] D. Buşneag, D. Piciu, and L. C. Holdon: Some Properties of Ideals in Stonean residuated lattice. J. Multiple-Valued Logic Soft Computing *24* (2015), 5–6, 529–546.

[6] R. Cignoli: Free algebras in varieties of Stonean residuated lattices. Soft Computing *12* (2008), 315–320. DOI:10.1007/s00500-007-0183-x

[7] F. Esteva and L. Godo: Monoidal $t-$norm based logic: towards a logic for left-continnuos $t-$norms. Fuzzy Sets and Systems *124* (2001), 3, 271–288. DOI:10.1016/s0165-0114(01)00098-7

[8] C. Lele and J. B. Nganou: MV-algebras derived from ideals in BL-algebra. Fuzzy Sets and Systems *218* (2013), 103–113. DOI:10.1016/j.fss.2012.09.014

[9] F. G. Maroof, A. B. Saeid, and E. Eslami: On co-annihilators in residuated lattices. J. Intelligent Fuzzy Systems *31* (2016), 1263–1270. DOI:10.3233/ifs-162192

[10] N. Galatos, P. Jipsen, T. Kowalski, and H. Ono: Residuated Lattices: an Algebraic Glimpse at Substructural Logics. Studies in Logics and the Foundations of Mathematics, Elsevier 2007.

[11] L. C. Holdon, L. M. Niţu, and G. Chiriac: Distributive residuated lattices. Ann. University of Craiova-Mathematics and Computer Science Series *39* (2012), 100–109.

[12] C. Mureşan: Co-stone Residuated Lattices. Ann. University of Craiova-Mathematics and Computer Science Series *40* (2013), 52–75. DOI:10.1109/ismvl.2010.27

[13] A. Iorgulescu: Algebras of Logic as BCK-algebras. Academy of Economic Studies Bucharest, Romania 2008.

[14] L. Leuştean: Baer extensions of BL-algebras. J. Multiple-Valued Logic Soft Computing *12* (2006), 321–335.

[15] D. Piciu: Algebras of Fuzzy Logic. Editura Universitaria, Craiova 2007.

[16] J. Rachunek and D. Salounova: Ideals and involutive filters in residuated lattices. In: SSAOS 2014, Stara Lesna.

[17] E. Turunen: Mathematics Behind Fuzzy logic. Physica-Verlag Heidelberg, New York 1999.

[18] Y. X. Zou, X. L. Xin, and P. F. He: On annihilators in BL-algebras. Open Math. *14* (2016), 324–337. DOI:10.1515/math-2016-0029

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